Mirza Karamehmedović

Acoustic and Electromagnetic Boundary Value Problems

A Unified Functional Analytic Approach to Existence, Uniqueness and Regularity of Solutions

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We do not really deal with mathematical physics, but with physical mathematics*; not with the mathematical formulation of physical facts, but with the physical motivation of mathematical methods.*

ARNOLD SOMMERFELD[1]

[1]From the foreword of A. Sommerfeld, *Partial Differential Equations in Physics*, Academic Press Inc., New York, 1949.

Abstract

Existence, uniqueness and regularity properties of solutions to acoustic and electromagnetic boundary value problems in $\mathbb{R}^n$ are established in a unified manner, using the general framework of Clifford algebras. Problems involving smooth and Lipschitz domains are considered. In the smooth case, the analysis relies on the theory of pseudodifferential operators used in conjunction with the theory of Fredholm operators. For the analysis of the Lipschitz case, a procedure based on the Rellich energy estimates is employed. The basic variational calculus is presented. The Method of Auxiliary Sources is generalised to handle problems involving arbitrary Clifford algebra-valued field quantities, and as a novel result, it is shown that this generalised Method of Auxiliary Sources is in an infinity of special cases convergent for any Lipschitz scatterer, regardless of the particular choice of the auxiliary surface.

Keywords: acoustic and electromagnetic scattering, boundary value problems, existence, uniqueness, regularity, Method of Auxiliary Sources

Contents

List of Symbols

Unless redefined locally, the notational conventions below apply.

symbol	**description**
n	dimension of the space $\mathbb{R}^n$ of validity of boundary value problems
δ_D	the Dirac delta function
δ_K	the Kronecker delta function
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\langle \xi \rangle$	'Chinese bracket', equals $(1+\lvert\xi\rvert^2)^{1/2}$ for $\xi \in \mathbb{C}^n$
$\Phi_{k,n}$	the fundamental outgoing solution of the n-dimensional Helmholtz equation, $(\Delta + k^2 I)\Phi_{k,n}(x,y) \equiv \delta_D(x-y)\ \forall x,y \in \mathbb{R}^n$
$\mathcal{S}$	the single layer potential operator
$\mathcal{D}$	the double layer potential operator
Ω_+	the physical domain of the scatterer, $\Omega_+ \subset \mathbb{R}^n$
Ω_-	the exterior of the scatterer, $\mathbb{R}^n \backslash \overline{\Omega}_+$
$\partial\Omega$	the scatterer physical surface, $\partial\Omega_+$
$D\{\cdot\}$	domain of an operator; also, the Dirac operator
$R\{\cdot\}$	range of an operator
sgn$\{\cdot\}$	the sign of a number
$\mathcal{L}(U,V)$	space of bounded linear operators mapping U into V
$\mathcal{K}(U,V)$	space of compact operators mapping U into V
Ω°	the interior of a set Ω
$\mathbb{R}_+$, $\mathbb{R}_-$	positive and negative numbers, respectively
$\sum_I{}'$	summation with respect to *strictly increasing* multiindex I
$[A,B]$, $[A,B]_+$	commutator and anticommutator of operators A, B
σ_n	surface area of the unit ball in $\mathbb{R}^{n+1}$

Chapter 1

Introduction

Acoustic and electromagnetic boundary value problems constitute a major field of applied mathematics, finding application in numerous academic and industrial contexts. In fact, according to Reference [28], the two wave equations most frequently encountered in physics and engineering are those of acoustics and electromagnetism. The interest in the associated boundary value problems arises mainly from the fact that they provide a natural framework for addressing the problems of *acoustic and electromagnetic scattering*, processes which are at the very basis of a numerous and diverse set of technological applications. Vast literature is available on theoretical and practical aspects of acoustic and electromagnetic boundary value problems in general and scattering problems in particular.

Investigations of the well-posedness of boundary value problems can have both theoretical and practical interest. Analysis of the existence and uniqueness properties of solutions to boundary value problems involving certain 'special' geometries can provide information about the model representing a given type of wave phenomenon.[21] Moreover, numerical treatment of practical boundary value problems is typically greatly complicated by the absence of uniqueness of solution, as the obtained results may be erroneous[1] and/or may exhibit divergence. If a solution does not exist, numerical methods (or any other solution method) should of course not be attempted at all. Furthermore, knowledge of the existence and uniqueness properties can prove crucial to identifying the source of error in the numerical solution. In particular, the possibility is increased for discriminating the introduced computational error from the error induced by the ill-posedness of the problem at hand. Information on the regularity of solution can prove helpful in the implementation of numerical schemes. Indeed, both the Finite Element and the Integral Equation/Method of Moments schemes, which constitute the two most widely employed approaches to numerical solution of boundary value problems, involve approximation of the solution function by a linear combination of a finite number of pre-selected functions. Sufficient information on the space in which the unknown solution is situated can provide basis for rational choices of expansion and test vectors to be employed in a given numerical implementation. Hence, an analysis of the well-posedness of a given boundary value problem provides the insight needed for adequate implementation of numerical schemes, as well as for the correct interpretation of the results obtained. As an illustration, we elaborate on this point and suggest its

[1]A common example of the source of error in numerical solutions that is due to the lack of uniqueness are the so-called *spurious solutions* associated with the waveguide modes pertaining to the interior domain of the scatterer.

importance in practical calculations in Chapter 4, where we treat a particular numerical scheme called the Method of Auxiliary Sources and present a result concerning the convergence of the method.

The purpose of this work is to present a modern approach to the problem of establishing the existence, uniqueness and regularity properties of solutions to time-harmonic acoustic and electromagnetic boundary value problems. We wish to emphasise through a unified treatment of the acoustic and the electromagnetic cases the close relation existing between the mathematical descriptions of the two physically distinct wave phenomena. Furthermore, choosing the unified approach allows us to present and study a general 'wave model', which incorporates as special cases the linearised compressible Navier-Stokes equations and the equations of Maxwell, and which therefore transcends the physical details associated with each type of wave phenomenon. In this way, a setting is developed in which the acoustics and the electromagnetism are mutually related and put in perspective, and in which the possibility is left for modeling additional, physically distinct types of waves. This is another advantage of the unified approach.[2]

In Section 1.1 of this chapter, we present a brief account of the classical equations governing acoustic and electromagnetic phenomena. Section 1.2 introduces the general setting of a scattering problem, as considered in this text. Chapter 2 establishes the basic mathematical framework needed for the subsequent treatment of the generalised boundary value problems. In particular, we introduce the notion of a Sobolev space and define Lipschitz domains in $\mathbb{R}^n$. Moreover, the formalism of Clifford algebras is developed to an extent sufficient for our purposes, and the theory of pseudodifferential operators is introduced. Also, we define the notion of a Fredholm operator and present a number of useful results on this class of linear maps. Thereafter, the important Hodge decomposition of differential form-valued functions on compact Riemannian manifolds is discussed. Finally, we describe briefly some relevant properties of fundamental solutions to the n-dimensional Helmholtz equation. In Chapter 3, the bulk of the analysis of this text is presented. We begin by formulating a general boundary value problem comprising the acoustic and electromagnetic cases in $\mathbb{R}^n$. Thereupon, the essential theory of layer potentials is exhibited for smooth domains in $\mathbb{R}^n$. Integral representation formulae are derived for solutions of the n-dimensional Helmholtz equation for differential l-forms, whereafter the actual analysis of the existence, uniqueness and regularity properties of solutions to the general Maxwell boundary value problem is performed.

For completeness, and also in order to address the integral equation methods employed in numerical solution of scattering problems, we present in Chapter 4 the basic variational calculus. A novel generalisation of an existence result is derived for the Method of Auxiliary Sources applied to n-dimensional scattering problems involving l-forms, $n \in \mathbb{N}, l \in \{0, \ldots, n\}$.

In Chapter 5, we summarise the presented results by stating a number of conclusions. Furthermore, some topics for future work are suggested.

Finally, the Appendix contains technical computations too bulky for the main text.

We note that a substantial generalisation can be made of the presented material (see, e.g., references [35, 36]) to the framework of smooth compact Riemannian manifolds. However, for our purposes, i.e. for modeling of acoustic and electromagnetic scattering phenomena, it is sufficient to consider the spaces $\mathbb{R}^n$ with the associated standard 'flat'

[2]After all, it is the pursuit of unifying theories which characterises the efforts of modern science, and which most likely will continue in providing us with ever better models describing nature.

Laplacian operator.

1.1 The Governing Equations of Acoustics and Electromagnetism

In this section, a brief recapitulation is given of the classical systems of equations describing acoustic and electromagnetic phenomena. In Chapter 3, this matter is reconsidered and a common generalisation of these systems is presented. The purpose of the discussion in this section is merely to motivate the analysis of a certain family of boundary value problems. Accordingly, we choose not to discuss in detail the various characteristic physical aspects of acoustics and electromagnetism, typically found in specialised texts. In particular, the physical units of the various quantities are omitted in the following. References [4] and [32] provide excellent introduction to the basic physical aspects of electromagnetism.

Both the acoustic and the electromagnetic model are based upon systems of two three–dimensional coupled partial differential equations involving representations of two fundamental physical observables, the so-called *dual* observables.[3] In the acoustic case, these observables are represented by the *displacement velocity field* v and the *pressure field* p, while the equations of electromagnetism involve the *electric field intensity* and the *magnetic field intensity*, denoted E and H, respectively. The pressure field is scalar, whereas all the other mentioned field quantities are vectorial. The interaction between the fields and matter is modeled by the *constitutive parameters* assigned to materials (and, in the electromagnetic case, to the vacuum as well.) Hence, the acoustic properties of a material are described by the *density* ρ, while the electrical and the magnetical properties are expressed through the *conductivity* σ, *electric permittivity* ε and *magnetic permeability* μ. In general, the constitutive parameters are nonlinear tensor–valued complex functions of space and time. Throughout our exposition, however, we are concerned exclusively with problems comprising *lossless, time-invariant, homogeneous, isotropic linear* media (also referred to as *lossless simple* media [4]), and our attention is therefore restricted to constant real scalar constitutive parameters (with $\sigma \equiv 0$ in the electromagnetic case.)

The dynamics of a compressible gas are governed by the so-called *compressible Navier-Stokes equations* [28]. A linearisation of these equations may be effected by considering only the acoustic waves of small amplitude, i.e. involving sufficiently small displacement velocities and pressure variations in the medium. The resulting system yields a reasonably accurate model of acoustic phenomena in the mentioned amplitude range, and it is given as follows (for lossless simple ambient media)[4]:

$$\begin{aligned} \nabla p + \rho \frac{\partial v}{\partial t} = 0 \quad \text{in } \Omega \times \mathbb{R} \\ \rho \operatorname{div} v + \frac{1}{\widetilde{c}^2} \frac{\partial p}{\partial t} = 0 \quad \text{in } \Omega \times \mathbb{R} \end{aligned} \tag{1.1}$$

In the above equations, $\widetilde{c}$ signifies the speed of sound in the considered medium. Also,

[3] As a digression, we mention that systems described by two 'dual' observables admit a common graph-theoretical treatment. An example of a nonphysical system of the above variety is any economical system, comprising the observables 'price' and 'quantity'. It was duality consideration that led Maxwell to his celebrated augmentation of the existing equations of electromagnetism to what is now known as the Maxwell equations.

[4] We choose the homogeneous acoustic system.

$\Omega \subseteq \mathbb{R}^3$ is the spatial domain of validity of the equation system, while the time variable is defined on the whole real axis. Taking the divergence of the first equation in (1.1) and substituting for div$v$ using the second one yields the scalar *acoustic wave equation*, which is given by

$$\Delta p - \frac{1}{\tilde{c}^2}\frac{\partial^2 p}{\partial t^2} = 0 \quad \text{in } \Omega \times \mathbb{R} \tag{1.2}$$

If the case is considered where the field quantities p, v are time-harmonic, considerable simplification is achieved by applying the time-domain Fourier transform to (1.1).[5] The resultant equations are given by[6]

$$\nabla p - i\omega\rho v = 0 \quad \text{in } \Omega \tag{1.3}$$

$$-\text{div} v + \frac{i\omega}{\rho\tilde{c}^2} p = 0 \quad \text{in } \Omega \tag{1.4}$$

where ω is the so-called *pulsation*, or angular frequency, of the time-harmonic field quantities p, v. In terms of the *wavenumber* $k = \omega/\tilde{c}$, the system (1.3) may be written

$$\nabla p - ikc\rho v = 0 \quad \text{in } \Omega \tag{1.5}$$

$$-\text{div} v + \frac{ik}{\rho c} p = 0 \quad \text{in } \Omega \tag{1.6}$$

If the fields p, v are time-harmonic, an application of the time-domain Fourier transform to the acoustic wave equation (1.2) (or, equivalently, a simple manipulation of the system (1.5)–(1.6)) yields the time-harmonic version of the scalar wave equation:

$$(\Delta + k^2 I)p = 0 \quad \text{in } \Omega \tag{1.7}$$

Hence, the wave equation is transformed into a homogeneous Helmholtz equation on $\Omega \subset \mathbb{R}^3$.

The electromagnetic phenomena are described by the set of *Maxwell's equations*[7],

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[5] The Fourier transform, as used in this text, will be described in Chapter 2. Here, we only note that the chosen time dependence factor is $e^{-i\omega t}$.

[6] For notational simplicity, we use here the same symbols for the original and the Fourier transformed field quantities.

[7] Maxwell's equations constitute the modern mathematical description of macroscopic electric, magnetic and electromagnetic phenomena. Based on a substantial empirical and theoretical background developed by, among others, Gauss, Ampere and Faraday, the revolutionary work of James Clerk Maxwell, published in 1873, utilised purely theoretical (symmetry) considerations to construct a model. The major novel aspect of the work of Maxwell consisted in the prediction of the existence of electromagnetic waves. With the predicted speed of propagation in vacuum of such waves being that of light, it was apparent that the new model provided for an elegant unified description of already observed optical, low-frequency electromagnetic, as well as electrostatic and magnetostatic phenomena. Furthermore, the discussion on the existence of a medium for propagation of electromagnetic waves (aether) resulted in the introduction of Lorentz transformations and ultimately led Einstein in 1905 to formulate his special theory of relativity. After the publication of Maxwell's work, two crucial developments took place which were to elevate the theory to its modern state. On the purely theoretical side, during the years of 1885–1887, the original formulation of Maxwell was simplified and cast to the modern form (including the use of vector notation) by Oliver Heaviside.[32] On the experimental side, Heinrich Hertz conducted a series of experiments in the period 1887–1891, confirming the existence of electromagnetic waves.

given by

$$\begin{aligned} \mathrm{curl} E + \mu \frac{\partial H}{\partial t} &= 0 \\ \mathrm{curl} H - \varepsilon \frac{\partial E}{\partial t} &= j \\ \mathrm{div} H &= 0 \\ \mathrm{div} E &= \frac{\rho_e}{\varepsilon} \end{aligned} \tag{1.8}$$

for lossless simple media and valid on some space–time domain $\Omega \times \mathbb{R}$. The vector field quantity j is the *electric volume current density* and the scalar field ρ_e represents the *electric volume charge density*. The *source-free* equations of Maxwell are characterised by the vanishing of the fields j, ρ_e. By the well-known operator identity

$$\mathrm{div}\,\mathrm{curl} = 0 \tag{1.9}$$

the first two equations of (1.8) (also referred to as the *rotation equations*) imply, in the source-free case,

$$\frac{\partial}{\partial t} \mathrm{div} E = \frac{\partial}{\partial t} \mathrm{div} H = 0 \quad \text{in } \Omega \times \mathbb{R} \tag{1.10}$$

Hence, for $j \equiv 0, \rho_e \equiv 0$ in $\Omega \times \mathbb{R}$ and under the assumptions

$$\lim_{t \to -\infty} \{\mathrm{div} E(x,t)\} = 0 \quad \forall x \in \Omega \tag{1.11}$$

$$\lim_{t \to -\infty} \{\mathrm{div} H(x,t)\} = 0 \quad \forall x \in \Omega \tag{1.12}$$

the rotation equations imply the last pair of equations in (1.8) (also referred to as the *divergence equations*.)

Taking the curl of the rotation equations, it readily seen from (1.8) and the operator identity

$$\mathrm{curl}\,\mathrm{curl} = -\Delta + \nabla \mathrm{div} \tag{1.13}$$

that the electric field satisfies the vectorial wave equation

$$\Delta E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{1}{\varepsilon} \nabla \rho_e + \mu \frac{\partial j}{\partial t} \quad \text{in } \Omega \times \mathbb{R} \tag{1.14}$$

whereas the magnetic field satisfies the vectorial wave equation

$$\Delta H - \frac{1}{c^2} \frac{\partial^2 H}{\partial t^2} = -\mathrm{curl} j \quad \text{in } \Omega \times \mathbb{R} \tag{1.15}$$

In the above expressions, $c = 1/\sqrt{\mu\varepsilon}$ is the speed of light in the ambient medium specified by the permittivity ε and the permeability μ. Note that the validity of (1.14) and (1.15) does not constitute a sufficient condition for the validity of (1.8).

If time-harmonic fields are considered, the Maxwell system (1.8) may be transformed into

$$\mathrm{curl} E - i\omega\mu H = 0 \quad \text{in } \Omega \tag{1.16}$$

$$\mathrm{curl} H + i\omega\varepsilon E = j \quad \text{in } \Omega \tag{1.17}$$

$$\mathrm{div} H = 0, \quad \mathrm{div} E = \frac{\rho_e}{\varepsilon} \quad \text{in } \Omega \tag{1.18}$$

or, in terms of the *wavenumber* $k = \omega\sqrt{\mu\varepsilon}$ and the *intrinsic impedance* $\eta \equiv \sqrt{\mu/\varepsilon}$ of the ambient medium,

$$\text{curl}E - ik\eta H = 0 \quad \text{in } \Omega \tag{1.19}$$

$$\text{curl}H + \frac{ik}{\eta}E = j \quad \text{in } \Omega \tag{1.20}$$

$$\text{div}H = 0, \quad \text{div}E = \frac{\rho_e}{\varepsilon} \quad \text{in } \Omega \tag{1.21}$$

Furthermore, application of the time-domain Fourier transform on equations (1.14) and (1.15) (or simple manipulation of the system (1.19)–(1.21)) yields the time-harmonic versions of the vectorial wave equations (1.14) and (1.15):

$$(\Delta + k^2 I)E = \frac{1}{\varepsilon}\nabla\rho_e - i\omega\mu j \quad \text{in } \Omega \tag{1.22}$$

$$(\Delta + k^2 I)H = -\text{curl}j \quad \text{in } \Omega \tag{1.23}$$

The wave equations are thus transformed into inhomogeneous Helmholtz equations on $\Omega \subseteq \mathbb{R}^3$.

At least when dealing with scattering problems, considerable simplification is achieved by augmenting (and thereby generalising) the Maxwell equations (1.8) as follows:

$$\text{curl}E + \mu\frac{\partial H}{\partial t} = -m \quad \text{in } \Omega \times \mathbb{R} \tag{1.24}$$

$$\text{curl}H - \varepsilon\frac{\partial E}{\partial t} = j \quad \text{in } \Omega \times \mathbb{R} \tag{1.25}$$

$$\text{div}E = \frac{\rho_e}{\varepsilon} \quad \text{in } \Omega \times \mathbb{R} \tag{1.26}$$

$$\text{div}H = \frac{\rho_m}{\mu} \quad \text{in } \Omega \times \mathbb{R} \tag{1.27}$$

The vector quantity m is the *magnetic volume current density*, while the scalar field ρ_m represents the *magnetic volume charge density*. Note that the system (1.24)–(1.27) is the result of a symmetrisation of the original equations of Maxwell.

It is also useful to introduce, primarily as a computational device[8], the electric and magnetic scalar and vector potentials Φ_e, Φ_m and A_e, A_m, respectively. In the time-harmonic case, these potentials are defined by the postulates

$$\text{curl}A_e = \mu H,\ E = i\omega A_e - \nabla\Phi_e + \frac{1}{\mu}\text{curl}A_m,\ \text{div}A_e = i\omega\mu\varepsilon\Phi_e \tag{1.28}$$

$$\text{curl}A_m = \varepsilon E,\ H = i\omega A_m - \nabla\Phi_m - \frac{1}{\varepsilon}\text{curl}A_e,\ \text{div}A_m = i\omega\mu\varepsilon\Phi_m \tag{1.29}$$

chosen to be valid in the domain Ω. The last expression in (1.28) is commonly referred to as the *Lorentz gauge* [4], and the last expression in (1.29) is the corresponding dual.[9]

[8] The above potentials are usually introduced and used as purely mathematical, auxiliary quantities, with no immediate physical interpretation. It is, however, (perhaps not surprisingly) possible to construct an electromagnetic model comprising only e.g. the electric scalar and vector potentials, and omitting the electric and the magnetic fields altogether.

[9] The universally accepted designation *Lorentz gauge* is, in fact, historically incorrect, since the Danish scientist Lorenz, and not the Dutchman Lorentz, was the first to formulate the relation in (1.28) between the vectorial and scalar electric potentials. It is possible that the similarity in the pronounciation of the names of the two scientists has contributed to the presently prevailing erroneous reference to Lorentz.

Given, e.g., the time-harmonic case, the electromagnetic field is now found by solving one of the following, relatively simple, Helmholtz equations:

$$(\Delta + k^2 I)A_e = -\mu j \quad \text{in } \Omega \tag{1.30}$$

or

$$(\Delta + k^2 I)\Phi_e = -\frac{\rho_e}{\varepsilon} \quad \text{in } \Omega \tag{1.31}$$

It is convenient for our purposes to perform a *reduction* of the time-harmonic acoustic and electromagnetic systems of equations (1.5)–(1.6) and (1.19)–(1.21), respectively. To this end, several approaches are possible, and we here present a selection.

In the acoustic case, the standard normalisation of the medium density, $\rho \equiv 1$, and of the speed of propagation of the wave front, $\widetilde{c} \equiv 1$, clearly transforms the equations (1.5)–(1.6) to the reduced system

$$\nabla p - ikv = 0 \quad \text{in } \Omega \tag{1.32}$$

$$-\text{div} v + ikp = 0 \quad \text{in } \Omega \tag{1.33}$$

For the electromagnetic case, firstly observe that, by setting e.g. $\widetilde{E} \equiv -iE$, $\widetilde{H} \equiv \eta H$, it is possible to cast the original time-harmonic Maxwell system to a system of PDEs given on Ω as follows:

$$\text{curl}\widetilde{E} - k\widetilde{H} = 0 \tag{1.34}$$

$$\text{curl}\widetilde{H} - k\widetilde{E} = 0 \tag{1.35}$$

$$\text{div}\widetilde{E} = \frac{\rho_e}{\varepsilon}, \quad \text{div}\widetilde{H} = 0 \tag{1.36}$$

In this case both unknowns have, of course, the dimension of the electric field, which is V/m. In addition, it is seen that in the absence of free electric current and charge, the system (1.34)–(1.36) becomes symmetric with respect to the field quantities E, H.

Alternatively, setting

$$\widetilde{E} = \sqrt{\varepsilon}E, \ \widetilde{H} = \sqrt{\mu}H \tag{1.37}$$

we obtain the system

$$\text{curl}\widetilde{E} - ik\widetilde{H} = 0 \quad \text{in } \Omega \tag{1.38}$$

$$\text{curl}\widetilde{H} + ik\widetilde{E} = \sqrt{\mu}j \quad \text{in } \Omega \tag{1.39}$$

$$\text{div}\widetilde{E} = \frac{\rho_e}{\sqrt{\varepsilon}}, \quad \text{div}\widetilde{H} = 0 \quad \text{in } \Omega \tag{1.40}$$

The unit of $\widetilde{E}$, $\widetilde{H}$ is $(J/m^3)^{1/2}$, i.e. the normalised fields are related to electromagnetic energy density.

Finally, if the intrinsic impedance η is normalised to $\eta = 1$, the following set of reduced time-harmonic Maxwell's equations, valid on Ω, is obtained from the system (1.19)–(1.21):

$$\text{curl}E - ikH = 0 \quad \text{in } \Omega \tag{1.41}$$

$$\text{curl}H + ikE = j \quad \text{in } \Omega \tag{1.42}$$

$$\text{div}E = \frac{\rho_e}{\varepsilon}, \quad \text{div}H = 0 \quad \text{in } \Omega \tag{1.43}$$

It is seen that the reduction of the acoustic and electromagnetic systems brings the equations constituting these systems to the same general form, where the material parameters characteristic of the particular physical context are 'hidden' in the wavenumber k. The analysis of the boundary value problems conducted in this text disregards the physical dimension of the involved field quantities, and solely relies on the form of the reduced equation systems (1.32)–(1.33) and (1.41)–(1.43). Consequently, it may be assumed that any of the two normalisation procedures associated with the systems (1.38)–(1.40) and (1.41)–(1.43) has been performed.

It is possible to reformulate the acoustic system (1.32)–(1.33) and the homogeneous version of the electromagnetic system (1.41)–(1.43) in terms of the equivalent Helmholtz systems of equations. Using the operator identities

$$\mathrm{div}\nabla = \Delta, \quad \mathrm{curl}\,\mathrm{curl} = -\Delta + \nabla\mathrm{div} \tag{1.44}$$

it is easily verified that if the pair (p, v) satisfies

$$\begin{aligned} (\Delta + k^2 I)p = 0 \quad &\text{in } \Omega \\ -ikv = \nabla p \quad &\text{in } \Omega \end{aligned} \tag{1.45}$$

then it also solves (1.32)–(1.33). Conversely, we have already seen that (1.45) is implied by the system (1.32)–(1.33). Simlarly, if it holds true that

$$\begin{aligned} (\Delta + k^2 I)E = 0 \quad &\text{in } \Omega \\ \mathrm{div}E = 0 \quad &\text{in } \Omega \\ ikH = \mathrm{curl}E \quad &\text{in } \Omega \end{aligned} \tag{1.46}$$

or if we have

$$\begin{aligned} (\Delta + k^2 I)H = 0 \quad &\text{in } \Omega \\ \mathrm{div}H = 0 \quad &\text{in } \Omega \\ ikE = -\mathrm{curl}H \quad &\text{in } \Omega \end{aligned} \tag{1.47}$$

then the pair (E, H) satisfies the Maxwell system (1.41)–(1.43) for $j \equiv 0$ in Ω. Conversely, an earlier discussion shows that the system (1.41)–(1.43) implies (1.46) and (1.47).

1.2 The Generic Scattering Problem

The purpose of this section is to briefly present basic concepts common to all acoustic and electromagnetic scattering problems.[10] Some notational conventions used in the remainder of this text are also defined.

Figure 1.1 on the facing page shows the geometry of a general scattering problem in $\mathbb{R}^2$ and $\mathbb{R}^3$. In the two-dimensional case, the cross-section of the surface of the object is shown in the plane perpendicular to the axis of invariance. The object, immersed in the ambient medium, is referred to as the *scatterer*. The ambient medium is characterised by the pertaining constitutive parameters. The *interior domain* of the scatterer is denoted Ω_+, and the corresponding *exterior domain* is denoted Ω_-. The *scatterer surface*, which is the boundary $\partial\Omega$, is denoted Γ for notational simplicity. Clearly, we have $\Omega_+ \cup \Omega_- = \mathbb{R}^3 \setminus \Gamma$. The vector ν, also shown in Figure 1.1 on the next page, is

[10]Note that a scattering problem is an exterior boundary value problem. Our general treatment in later chapters includes both interior and exterior boundary value problems.

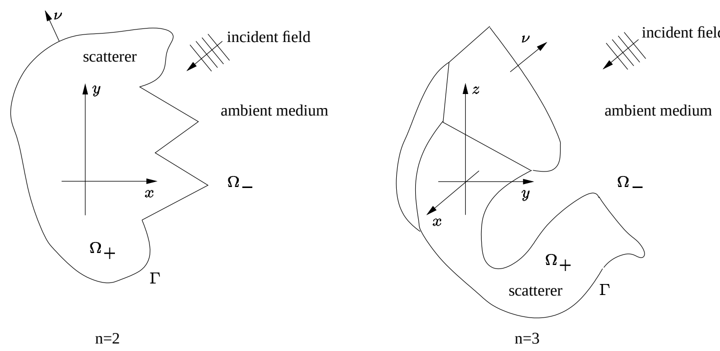

Figure 1.1: Instances of the geometry of the general scattering problem under consideration in the cases $n = 2$ and $n = 3$.

the unit outward normal associated with the interior domain Ω_+ of the scatterer (i.e. ν points into the domain Ω_-.) In fact, ν is a vector field well-defined also outside the scatterer surface. Reference [28] provides the following definition:

$$\nu(x) \equiv \pm\nabla|x - \mathcal{P}(x)| \quad \forall x \in \Omega_\mp \tag{1.48}$$

In the above expression, the quantity $\mathcal{P}(x)$ is the orthogonal projection of x upon Γ. It is convenient to operate with such a continuous field of unit outward normals, since limiting processes where the observation point approaches the surface Γ become easily expressible. The vector field ν is not defined at the singularities, i.e. edges, vertices and cusps, of the scatterer surface.

The general boundary value problem considered throughout this exposition comprises a scatterer represented by a compact domain $\Omega_+ \subset \mathbb{R}^n$ (in the case $n = 2$, only the scatterer cross section is assumed to be compact) with a Lipschitz-continuous surface $\Gamma = \partial\Omega_+$. The scatterer is illuminated by the *incident field*. The incident field is emitted by a source (or sources) placed in the exterior domain Ω_-. The difference between the total field constituting the solution of the boundary value problem at hand and the incident field is referred to as the *scattered field*. It is the objective of a scattering analysis to calculate, either analytically, asymptotically or numerically, the scattered field associated with a given scattering problem. Note that if the total field is found for a particular scattering problem, then solution (i.e. the scattered field) is known for all scattering problems differing from the original one only in the incident field. Expressed differently, any boundary value problem considered in this text can be recast as a scattering problem, upon the choice of the incident field. A classical scattering problem involves time-harmonic fields, since many aspects of analysis and numerical computation are simplified in this case.

In addition to its shape, a scatterer is characterised by the boundary condition valid on its surface. A general form of boundary condition is the *Standard Impedance Boundary Condition* (or SIBC for short.) In terms of the normalised electromagnetic field quantities, the Standard Impedance Boundary Condition is given by

$$\nu \times (E \times \nu) = \frac{\zeta}{\eta}\nu \times H \tag{1.49}$$

It is seen that the original surface impedance ζ is now normalised with respect to the intrinsic impedance η of the ambient medium. We choose in the following to denote by

ζ this normalised quantity. A *perfect electrical conductor* (also referred to as a PEC) is an idealisation of a highly conductive physical material, i.e. a material with the conductivity parameter σ satisfying, in the time-harmonic case, the relation $\sigma/\omega \gg 1$. The PEC approximation is the result of a limiting process with $\sigma \to \infty$. Reference [28] provides a detailed derivation justifying the PEC model for highly conductive materials. Perfect electric conductors are characterised by having zero surface impedance, i.e. the total tangential electric field vanishes identically on the surface of a PEC. Hence, scattering problems involving PECs include the following boundary condition:

$$\nu \times E = 0 \quad \text{on } \Gamma \tag{1.50}$$

where ν is the unit outward normal to Ω_+. The concept dual to that of a perfect electric conductor is a *perfect magnetic conductor* (or simply PMC). Although obviously nonphysical (the classical electromagnetic model does not include magnetic charges or currents), this model proves extremely useful in practical applications, in that duality relations can be used in certain cases to transform 'difficult' problems involving PECs into 'simple' problems with PMCs. Perfect magnetic conductors have infinite surface impedance, implying that the total tangential magnetic field vanishes on their surfaces. Electromagnetic scattering problems involving PMCs comprise the boundary condition

$$\nu \times H = 0 \quad \text{on } \Gamma \tag{1.51}$$

As will become apparent, the general exterior boundary value problem treated in this text corresponds to an electromagnetic scattering problem involving a PEC object.

The scattered field is always subject to a *radiation condition*. In terms of the normalised field quantities $\sqrt{\varepsilon}E, \sqrt{\mu}H$, the *Silver-Müller radiation conditions* are given by [28][11]

$$\begin{aligned} |\nu \times E - H| &= \mathcal{O}(r^{-2}) \quad \text{for } r \to \infty \\ |E + \nu \times H| &= \mathcal{O}(r^{-2}) \quad \text{for } r \to \infty \end{aligned} \tag{1.54}$$

In physical terms, a radiation condition ensures finiteness of energy (or power if $n = 2$) emitted by the scattered field. Mathematically, it ensures the uniqueness of solution to the boundary value problem.[28] Radiation conditions are also referred to as a *decay conditions*.[14]

An electromagnetic scattering problem may hence be written as

$$\begin{aligned} \text{curl} E^s - ikH^s = 0 &\quad \text{in } \Omega \\ \text{curl} H^s + ikE^s = 0 &\quad \text{in } \Omega \\ \nu \times E^s = \Psi &\quad \text{on } \Gamma \\ |\nu \times E^s - H^s| = \mathcal{O}(r^{-2}) &\quad \text{for } r \to \infty \end{aligned} \tag{1.55}$$

where the superscripts s signify the scattered field.

[11] Consider $f, g : I \subseteq \mathbb{R} \to \mathbb{C}$. We write $f = \mathcal{O}(g)$ iff

$$\exists x_0 \in I, c \geq 0 : \quad x > x_0, x \in I \Rightarrow f(x) < cg(x) \tag{1.52}$$

Consider $f, g : I \subseteq \mathbb{R} \to \mathbb{C}$. We write $f = o(g)$ iff

$$\forall c \in \mathbb{R}_+ \, \exists x_0 \in I : \quad x > x_0, x \in I \Rightarrow f(x) < cg(x) \tag{1.53}$$

1.3 Remarks

The literature on acoustic and electromagnetic scattering shows that the overwhelming majority of the problems considered (both classically and in contemporary works) for theoretical and practical analysis involves lossless simple media. For example, the problem of electromagnetic scattering by a perfectly electrically conducting body immersed in vacuum has historically received an extraordinary amount of attention, due to its ability to model a vast range of problems occurring in practice. Consequently, the restrictions imposed in this section on the constitutive parameters ρ, μ, ε are by no means specialising our subsequent analysis to 'uninteresting' or 'impractical' cases.

The acoustic and electromagnetic boundary conditions (as is the case for many other boundary conditions) involve limit values of field quantities as the observation point approaches the surface Γ (from the interior domain Ω_+ or the exterior domain Ω_-.) It is, however, customary to omit the appropriate notation emphasising this fact, and the form (1.50) of the boundary condition is typically used. Upon performing a generalisation of the considered boundary value problems to a system involving differential form-valued functions, we choose (for clarity) to maintain the explicit notation discriminating between limit values and other quantities. Nevertheless, in the acoustic and electromagnetic cases, we use the form (1.50).

Throughout the text, the electromagnetic case receives (marginally) more attention than the acoustic one, albeit this is of no significance whatsoever for the exposition. This assymetry is entirely due to the preferences of the author, which for the time being is involved in the field of electromagnetic scattering.

Chapter 2

The Mathematical Framework

In this chapter, a number of tools are developed which facilitate an analysis of the existence, uniqueness and regularity properties of solutions to acoustic and electromagnetic boundary value problems. After some basic definitions in Section 2.1, a brief account is made in Section 2.2 of the geometrical aspects of the boundary value problems considered in this text. In particular, the concept of a Lipschitz domain in $\mathbb{R}^n$ is introduced. Thereafter, we introduce in Section 2.4 the formalism of Clifford algebras, which constitutes an augmentation of the concept of a complex number. As will be shown in Chapter 3, the Clifford algebra setting may be used to perform a rather elegant generalisation of the considered boundary value problems. In Section 2.3, we present the framework of Sobolev spaces. An important generalisation of the notion of a differential operator is carried out in Section 2.5, where the subject of pseudodifferential operators is considered. Another significant element of the apparatus used in our subsequent analysis of boundary value problems is the theory of Fredholm operators. This matter is taken up in Section 2.6. The important Hodge decomposition result is derived in Section 2.7, and finally, in Section 2.8, we present the basic properties of fundamental solutions to the Helmholtz equation. References used in this chapter are [35, 36, 18, 28].

2.1 Preliminaries

In this section, we present some notational conventions and mathematical concepts used throughout the text.

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, we have $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$. For $\alpha \in \mathbb{N}_0^n$, we set $D^\alpha = \prod_{j=1}^n D_j^{\alpha_j}$ with $D_j = -i\partial/\partial x_j$ for all $j \in \{1, \ldots, n\}$. Furthermore, for $\alpha \in \mathbb{N}_0^n$, we define $|\alpha| = \sum_{j=1}^n \alpha_j$.

DEFINITION 2.1.1
The *Schwartz space of rapidly decreasing functions* is defined by

$$\mathcal{S}(\mathbb{R}^n) \equiv \{u \in C^\infty(\mathbb{R}^n) | x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n)\ \forall \alpha, \beta \in \mathbb{N}_0^n\} \tag{2.1}$$

□

Note that if $u \in \mathcal{S}(\mathbb{R}^n)$ then, for all $\alpha,\ \beta \in \mathbb{N}_0^n,\ x \in \mathbb{R}^n$,

$$|D^\alpha x^\beta u(x)| = \left|\sum_{a+b=\alpha} \binom{\alpha}{a} D^a x^\beta D^b u(x)\right| = \left|\sum_{\substack{a+b=\alpha \\ a\leq\beta}} \binom{\alpha}{a} C_{a\beta} x^{\beta-a} D^b u(x)\right| \leq \sum_{\substack{a+b=\alpha \\ a\leq\beta}} \binom{\alpha}{a} C_{a\beta} \left|x^{\beta-a} D^b u(x)\right| < \infty \tag{2.2}$$

Clearly, if $u \in \mathcal{S}(\mathbb{R}^n)$, then $\lim_{|x|\to\infty} |x^\beta D^\alpha u(x)| = 0$ for all $\alpha, \beta \in \mathbb{N}_0^n$.[1]

A *tempered distribution on* $\mathbb{R}^n$ is a continuous linear functional on the Schwarz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$. Hence, by construction, a tempered distribution is a bounded operator. The space of tempered distributions on $\mathbb{R}^n$ is denoted $\mathcal{S}'(\mathbb{R}^n)$. This space has the $weak^*$ topology, for which the following holds true, for all $y \in \mathcal{S}'(\mathbb{R}^n)$:

$$\begin{gathered} \lim_{n\to\infty} x_n = y \quad \text{in the } weak^* \text{ topology} \\ \Updownarrow \\ \lim_{n\to\infty} x_n u = yu \quad \forall u \in \mathcal{S}(\mathbb{R}^n) \end{gathered} \tag{2.3}$$

By the Riesz representation theorem, there exists an injective operator $T : L^p(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ for all $p \in \mathbb{N}$. We have, of course,

$$Tfg \equiv \int_{\mathbb{R}^n} g(x)\overline{f(x)}dx \quad \forall g \in \mathcal{S}(\mathbb{R}^n), \forall f \in L^p(\mathbb{R}^n) \tag{2.4}$$

Operators may now be conveniently defined on $S'(\mathbb{R}^n)$ as follows:

$$\tilde{A}wx \equiv wAx \quad \forall x \in S(\mathbb{R}^n), w \in S'(\mathbb{R}^n) \tag{2.5}$$

Next, we state the definitions of the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^*$:

$$\begin{aligned} \mathcal{F}u(\xi) &\equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x)e^{-ix\xi}dx \quad \forall \xi \in \mathbb{R}^n, u \in \mathcal{S}(\mathbb{R}^n), n \in \mathbb{N} \\ \mathcal{F}^*u(x) &\equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(\xi)e^{ix\xi}d\xi \quad \forall x \in \mathbb{R}^n, u \in \mathcal{S}(\mathbb{R}^n), n \in \mathbb{N} \end{aligned} \tag{2.6}$$

2.2 Smooth and Lipschitz Domains

As announced in Chapter 1, the boundary value problems treated in this text generally involve Lipschitz domains in $\mathbb{R}^n, n \in \mathbb{N}$. The distinction between smooth and nonsmooth domains is based on the notion of a *corner*, which we now define (with reference to [35]):

DEFINITION 2.2.1
Given a Riemannian manifold M, $\dim\{M\} = n \in \mathbb{N}$, and a positive integer $m \leq n-1$, let

$$K_m = \left\{ x \in \mathbb{R}^n \,\middle|\, x_j \leq 0\ \forall j \in \{1,\dots,n-m\} \right\} \tag{2.7}$$

[1] This, of course, justifies the name of the function space under consideration.

We refer to a point $x \in \partial M$ as a *corner of dimension* m iff there exists a neighbourhood $\overline{U} \subset \overline{M}$ of x and a diffeomorphism $\varphi \in C^2(\overline{U})$ which maps $\overline{U}$ onto a neighbourhood of 0 in K_m.

If M is a C^2-manifold, and if all $p \in \partial M$ are corners, then $\overline{M}$ is a *manifold with corners*.

□

DEFINITION 2.2.2
A manifold with a boundary containing no corners is referred to as *smooth*.

□

In the literature, the terms 'smooth' and 'regular' are used interchangeably when referring to smooth manifolds.

The notion of a Lipschitz domain is, not surprisingly, based on the concept of Lipschitz continuity of functions.

DEFINITION 2.2.3
A function f is referred to as *Lipschitz continuous* (or simply *Lipschitz*) iff

$$\exists C < \infty : \ |f(x) - f(y)| \leq C|x - y| \quad \forall x, y \in D\{f\} \tag{2.8}$$

□

Clearly, any function admitting a bounded first derivative is Lipschitz.

Expressed alternatively, if $n \in \mathbb{N}$, then an open set $\Omega_+ \subset \mathbb{R}^n$ is referred to as *Lipschitz* iff its boundary $\partial\Omega_+$ is everywhere locally the graph of a Lipschitz function. Formally, we have the following definition (based on [37]):

DEFINITION 2.2.4
Given $n \in \mathbb{N}$, an open set $\Omega \subset \mathbb{R}^n$ is said to be *Lipschitz* iff for every set $\omega \subset \partial\Omega$ there exists a rectangular coordinate system $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$, a set $\gamma \subset \mathbb{R}^n$ and a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ such that the relation

$$\gamma \cap \Omega = \{(x, y) | y > \phi(x)\} \cap \gamma \tag{2.9}$$

is satisfied.

The boundary of a Lipschitz domain is in general referred to as a Lipschitz hypersurface.

□

Note that any interval $\Omega_+ \subset \mathbb{R}$ is Lipschitz.

Reference [28] contains a particularly intuitive definition of a smooth (regular) surface in $\mathbb{R}^3$, which we here generalise to the case of $\mathbb{R}^n$.

DEFINITION 2.2.5
Let Ω_+ be an open, bounded set with boundary $\Gamma = \partial\Omega_+$, and consider a covering of Ω_+ by a finite union $\cup_i \omega_i$ of open sets. Assume that $\Gamma \subset \cup_i \omega_i$ and $\omega_0 \cap \Gamma = \emptyset$. The hypersurface Γ is referred to as regular iff for every set ω_i there exists the associated diffeomorphism ϕ_i mapping ω_i onto the unit ball centered at the origin, such that $\Omega_+ \cap \omega_i$ is mapped below the equatorial plane ($x_n < 0$), $\complement\Omega_+ \cap \omega_i$ is mapped above the equatorial plane ($x_n > 0$), and $\Gamma \cap \omega_i$ is mapped onto the equatorial plane

($x_n = 0$) of the unit ball. If all the diffeomorphisms ϕ_i and their inverses are in C^k, the hypersurface Γ is said to be of class C^k. The covering $\cup_i \omega_i$, together with the corresponding diffeomorphisms ϕ_i, is called an atlas. The pairs (ω_i, ϕ_i) are referred to as charts.

□

On many occasions throughout this text, we employ the concept of transport and localisation described above. Also, we define the diffeomorphisms ϕ_i, ϕ_i^{-1} to map orthonormal bases (in $\mathbb{R}^n$) to orthonormal bases (in $\mathbb{R}^n$), and hence the operators ϕ_i, ϕ_i^{-1} are unitary isometries. Consequently, we have $\phi_i^{-1}\phi_i x.\phi_i^{-1}\phi_i y = \phi_i x.\phi_i y = x.y$ for all $x, y \in \omega_i$.

A Lipschitz hypersurface in $\mathbb{R}^n$ may incorporate corners, and Lipschitz domains are thus generally nonsmooth. In the dimensions $n \in \{2, 3\}$, the corners take the form of edges, vertices and cusps. Examples of arbitrary Lipschitz surfaces in $\mathbb{R}^2$ and in $\mathbb{R}^3$ were shown in Figure 1.1 on page 9. In Figure 2.1, we depict some Lipschitz surfaces of more practical interest in the context of scattering theory.

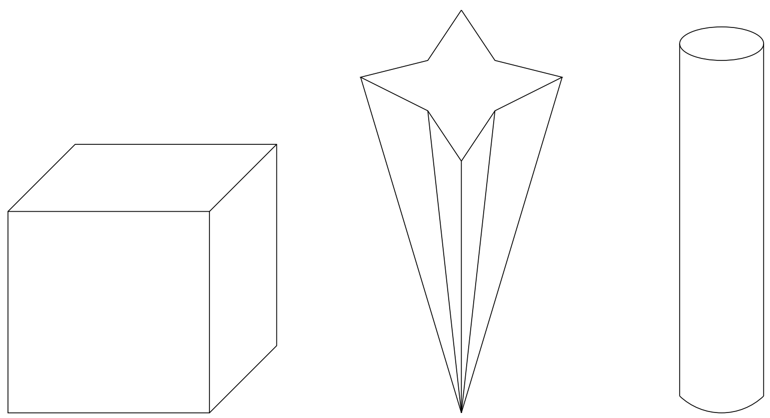

Figure 2.1: Examples of Lipschitz surfaces in $\mathbb{R}^3$ occurring in practical applications of acoustic and electromagnetic scattering theory.

2.3 Sobolev Spaces

In short, Sobolev spaces are spaces of functions with suitable regularity properties. By their very construction, these spaces constitute a useful tool in the analysis of regularity of solutions to boundary value problems. We present here only the rudimentary theory, and refer to [35] for an excellent account of the matter.

DEFINITION 2.3.1
Given $k \in \mathbb{Z}\backslash\mathbb{Z}_-, \alpha \in \mathbb{N}_0^n$, and an open set $\mathcal{D} \subseteq \mathbb{R}^n$, the Sobolev space $H^k(\mathcal{D})$ is defined by [35]

$$H^k(\mathcal{D}) \equiv \{u \in L^2(\mathcal{D}) : |\alpha| \leq k \Rightarrow D^\alpha u \in L^2(\mathcal{D})\} \tag{2.10}$$

The vector $D^\alpha u$ is written in the sense of a tempered distribution, i.e. $D^\alpha u \in L^2(\mathcal{D})$ actually means that the dual of $D^\alpha u$ is in $L^2(\mathcal{D})$. The Sobolev space is equipped with the norm

$$||u||^2_{H^k(\mathcal{D})} \equiv \sum_{\substack{\alpha\in\mathbb{R}^n \\ |\alpha|\leq k}} ||D^\alpha u||^2_{L^2(\mathcal{D})} \tag{2.11}$$

□

This definition implies

$$u \in H^k(\mathcal{D}), \alpha \in \mathbb{N}_0^n, |\alpha| \leq k \in \mathbb{Z}\backslash\mathbb{Z}_- \Rightarrow D^\alpha u \in H^{k-|\alpha|}(\mathcal{D}) \tag{2.12}$$

Obviously, $H^0(\mathcal{D}) = L^2(\mathcal{D})$.

THEOREM 2.3.2
The space $H^k(\mathcal{D}), k \in \mathbb{Z}\backslash\mathbb{Z}_-$, equipped with the norm (2.11), is a Hilbert space.
Proof: Assume $(u_n) \subset H^k(\mathcal{D})$ is a Cauchy sequence in the $H^k(\mathcal{D})$ norm. Then, since (2.11) is a sum of nonnegative terms, $(D^\alpha u_n), |\alpha| \leq k$ is a Cauchy sequence in $L^2(\mathcal{D})$. By construction, $L^2(\mathcal{D})$ is complete, so $\lim_{n\to\infty} D^\alpha u_n \in L^2(\mathcal{D})$ for $|\alpha| \leq k$. Due to continuity of D^α on the space of distributions on $C_0^\infty(\mathcal{D})$, it is true that $D^\alpha \lim_{n\to\infty} u_n \in L^2(\mathcal{D}), |\alpha| \leq k$, so $H^k(\mathcal{D})$ is a Hilbert space.

■

LEMMA 2.3.3
The following inclusions are valid:

$$\cdots \subset H^2(\mathcal{D}) \subset H^1(\mathcal{D}) \subset H^0(\mathcal{D}) = L^2(\mathcal{D}). \tag{2.13}$$

Proof: Since, for $a, b \in \mathbb{Z}\backslash\mathbb{Z}_-,\ a < b$, and $u \in H^b(\mathcal{D})$,

$$||u||^2_{H^a(\mathcal{D})} = \sum_{|\alpha|\leq a} ||D^\alpha u||^2_{L^2(\mathcal{D})} \leq \sum_{|\alpha|\leq b} ||D^\alpha u||^2_{L^2(\mathcal{D})} = ||u||^2_{H^b(\mathcal{D})}, \tag{2.14}$$

it is true that

$$a, b \in \mathbb{Z}\backslash\mathbb{Z}_-,\ a < b \Rightarrow H^b(\mathcal{D}) \subset H^a(\mathcal{D}) \tag{2.15}$$

■

DEFINITION 2.3.4
Given $s \in \mathbb{R}$, the *fractional order Sobolev space* $H^s(\mathbb{R}^n)$ is defined by [35]

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle\xi\rangle^s \mathcal{F}u \in L^2(\mathbb{R}^n)\} \tag{2.16}$$

The Sobolev space is equipped with the norm

$$||u||_{H^s(\mathbb{R}^n)} = ||\langle\xi\rangle^s \mathcal{F}u||_{L^2(\mathbb{R}^n)} \tag{2.17}$$

□

Note that, due to the continuity of the Fourier transform, a fractional order Sobolev space, with the above stated norm, is a Hilbert space.

DEFINITION 2.3.5
For any set $\Omega \subset \mathbb{R}^n$, the *restriction operator* r_Ω maps functions on $\mathbb{R}^n$ into their restrictions to Ω:

$$r_\Omega u = u|_\Omega \quad \forall u \text{ defined on } \mathbb{R}^n \tag{2.18}$$

The *extension by zero operator* e_Ω is defined as follows:

$$e_\Omega u(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases} \tag{2.19}$$

The *trace operator* γ_0, associated with a boundary Γ, restricts functions to Γ. Regularity results for the trace operator may be found in References [28, 35].

Finally, the *tangential trace operator* π_τ, associated with a boundary Γ, maps functions to tangential components of their traces on Γ.

□

2.4 Clifford Algebras and Differential Forms

As stated in the introduction, it is our aim in this text to present a unified treatment of acoustic and electromagnetic boundary value problems in arbitrary dimension. To this end, a generalised algebra is needed which incorporates scalar and vector quantities in a natural way, such that the acoustic and the electromagnetic fields can appear as functions with ranges in the same space. The framework of Clifford algebras proves suitable in this context.[2] In particular, the specialisation of the notion of a Clifford algebra-valued function to that of a differential form-valued function may be used for representation of the acoustic and the electromagnetic fields. In addition to providing a framework for a generalised description of boundary value problems, the theory of Clifford algebras facilitates a substantial simplification of the original systems of equations of acoustics and electromagnetism (the systems (1.5)–(1.6) and (1.19)–(1.21), respectively,) reducing them to special cases of a single homogeneous equation involving a perturbed Dirac operator.[3]

The material in this section is based on references [19, 20, 24, 22, 14, 35]. Clifford algebras and differential forms are, of course, research subjects of independent interest and with a vast number of applications, and only an introduction sufficient for our purposes is presented here. For further study, we refer to [19], which lists a number of relevant references.

DEFINITION 2.4.1
For any ordered sets $I, I' \in \mathbb{N}_0^n$, $n \in \mathbb{N}$, we define

$$\varepsilon_I^{I'} \equiv \begin{cases} 0 & \text{if } I \text{ and } J \text{ do not contain the same elements} \\ \operatorname{sgn}\{\pi_{I' \to I}\} & \text{otherwise} \end{cases} \tag{2.20}$$

where $\pi_{I' \to I}$ denotes the permutation taking I' into I.

□

[2] As will become clear shortly, Clifford algebras constitute generalisations of the algebra of quaternions.
[3] In fact, it was partly the wish to better understand the mathematical basis of the equations of Maxwell that motivated W. K. Clifford (who, incidentally, was J. C. Maxwell's student) to develop the theory of Clifford algebras in 1878. [19, 20]

DEFINITION 2.4.2
Let $\delta^K_{i,j}$ denote the usual Kronecker delta function with arguments $i, j \in \mathbb{N}_0$. Given $n \in \mathbb{N}_0$, the *complex* 2^n*-dimensional Clifford algebra* $\mathbb{C}_{(n)}$ is the 2^n-dimensional complex linear vector space containing a basis B which may be written in the manner

$$B = \left\{ e_I \middle| I \subseteq \{1, \ldots, n\},\, I \text{ strictly increasing} \right\} \tag{2.21}$$

and which is equipped with the following algebraic structure:

$$e_\emptyset = 1 \tag{2.22}$$

$$[e_{\{i\}}, e_{\{j\}}]_+ = -2\delta^K_{i,j} \quad \forall i, j \in \{1, \ldots, n\} \tag{2.23}$$

$$e_I = \prod_{j=1}^{|I|} e_{I_j} \quad \forall I \subseteq \{1, \ldots, n\} \tag{2.24}$$

where $|I| = \text{card}\{I\}$.

For notational convenience, we define

$$e_j \equiv e_{\{j\}} \quad \forall j \in \{1, \ldots, n\} \tag{2.25}$$

□

The basis vectors $e_I, |I| \in \{0, \ldots, n\}$ of a 2^n-dimensional Clifford algebra are also referred to as the *generators*.[14] For all $n \in \mathbb{N}_0$, we denote by $\mathbb{R}_{(n)}$ the obvious restriction of $\mathbb{C}_{(n)}$.

It is rewarding to write explicitly the two conditions imposed by (2.23), namely

$$e_i = -\, e_j \qquad \forall i, j \in \{1, \ldots, n\},\, i \neq j \tag{2.26}$$

$$e_i^2 = -\, 1 \qquad \forall i \in \{1, \ldots, n\} \tag{2.27}$$

DEFINITION 2.4.3
For any $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $l \leq n$, the operator $\Pi_l : \mathbb{C}_{(n)} \to \mathbb{C}_{(n)}$ is defined by

$$\Pi_l u = \sum_{|I|=l} u_I e_I \quad \forall u \in \mathbb{C}_{(n)} \tag{2.28}$$

Furthermore, we define

$$\Pi_{-1} u \equiv 0 \quad \forall u \in \mathbb{C}_{(n)},\, n \in \mathbb{N}_0 \tag{2.29}$$

□

DEFINITION 2.4.4
Given $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $l \leq n$, the linear space $\Lambda^{l,n}$ of n-dimensional complex differential l-forms (also referred to as n-dimensional complex l-vectors) is defined by

$$\Lambda^{l,n} \equiv \Pi_l \mathbb{C}_{(n)} \tag{2.30}$$

□

Hence, the space of n-dimensional complex differential l-forms, $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $l \leq n$, is but a restriction of the 2^n-dimensional complex Clifford algebra.

DEFINITION 2.4.5
The *Clifford conjugation* (denoted by $\bar{\cdot}$) on $\mathbb{C}_{(n)}$ is the unique linear complex involution (permutation operation consisting only of fixed points and transpositions) on $\mathbb{C}_{(n)}$ such that $e_I\overline{e_I} = \overline{e_I}e_I = 1$ for all multiindices I of cardinality $|I| \in \{0, \dots, n\}$. Consequently, given

$$u \in \mathbb{C}_{(n)}\,,\; u = \sum_{|I|\le n}{}' u_I e_I \tag{2.31}$$

we have

$$\overline{u} = \sum_{|I|\le n}{}' u_I \overline{e_I} \tag{2.32}$$

Also, the *complex conjugate* u^c of $u \in \mathbb{C}_{(n)}$ is defined as

$$u^c \equiv \sum_{|I|\le n}{}' \overline{u}_I e_I \tag{2.33}$$

□

For example, it is easily seen using (2.27) that

$$\overline{e_1} = -e_1,\; \overline{e_1e_2} = e_2e_1,\; \overline{e_1e_2e_3} = -e_3e_2e_1 \tag{2.34}$$

or, generally, that $\overline{e_I} = (-1)^{|I|}e_{I'}$, where the multiindex I' is the reversed I. Hence, by the property (2.26) of the basis vectors e_i, $i \in \{1, \dots, n\}$, it holds true that

$$\overline{u} = (-1)^l \varepsilon_I^{I'} u = (-1)^{l(l+1)/2} u \quad \forall u \in \Lambda^{l,n} \tag{2.35}$$

For an element u of the 2^n-dimensional complex Clifford algebra $\mathbb{C}_{(n)}$, the coefficient $u_0 \equiv u_\emptyset$ is referred to as the *scalar part*. We endow the algebra $\mathbb{C}_{(n)}$ with the natural inner product

$$\langle u, v\rangle_{\mathbb{C}_{(n)}} \equiv \sum_{|I|\le n}{}' u_I\overline{v_I} = (u\overline{v}^c)_0 \quad \forall u, v \in \mathbb{C}_{(n)} \tag{2.36}$$

It is now seen immediately from (2.35) and the definitions 2.4.2 and 2.4.5 that

$$\langle e_I, e_J\rangle_{\mathbb{C}_{(n)}} = (e_I\overline{e_J}^c)_0 = (e_I\overline{e_J})_0 = \delta^K_{I,J} \tag{2.37}$$

for all strictly increasing multiindices I, J. Hence, $\{e_I\}_{|I|\le n}$, I strictly increasing, constitutes an orthonormal basis in $\mathbb{C}_{(n)}$. It is also clear that the induced norm in $\mathbb{C}_{(n)}$ induces the natural Euclidean metric on $\mathbb{C}_{(n)}$:

$$\mathrm{dist}^2(u, v) = ||u - v||^2_{\mathbb{C}_{(n)}} = ((u - v)\overline{(u - v)}^c)_0 = \sum_{|I|\le n}{}' |(u - v)_I|^2 \tag{2.38}$$

where the last equality is due to the multiplication property (2.23) of the basis vectors e_I. Hence, the spaces $\mathbb{C}_{(n)}, \mathbb{C}^{2^n}$ are isometric isomorphic for all $n \in \mathbb{N}_0$. This, however, is not the only isomorphy in the present context. It is readily seen that $\mathbb{R}_{(0)} = \mathbb{R}$, $\mathbb{R}_{(1)} = \mathbb{C}$ (identifying $e_1 \in \mathbb{R}_{(1)}$ with the imaginary unit $i \in \mathbb{C}$,) while the algebra $\mathbb{R}_{(2)}$ can be identified with the quaternions (with $e_1 = i, e_2 = j, e_{12} = k$.) Due to the isomorphy between $\mathbb{C}$ and $\mathbb{R}_{(1)}$, it is observed that $\mathbb{C}_{(n)} \subseteq \mathbb{R}_{(n)}$ for all $n \in \mathbb{N}$. Since trivially $\mathbb{R}_{(n)} \subseteq \mathbb{C}_{(n)}$ for all $n \in \mathbb{N}_0$, we have $\mathbb{C}_{(n)} = \mathbb{R}_{(n)}$ for all $n \in \mathbb{N}$. More generally, we see that $\{\mathbb{R}_{(m)}\}_{(n)} = \mathbb{R}_{(n)}$ for all $m, n \in \mathbb{N}_0, m \le n$. Finally, the spaces $\mathbb{C}^n$, $\Lambda^{1,n} \subset \mathbb{C}_{(n)}$, $n \in \mathbb{N}$, as well as the spaces $\mathbb{C}$, $\Lambda^{0,n}$, $n \in \mathbb{N}_0$, are isometric isomorphic. In particular, there exists the embedding $\mathbb{C}^n \hookrightarrow \Lambda^{1,n}$, since any $u \in \mathbb{C}^n$ may be written in the representation

$$u = \sum_{i=1}^{n} u_i e_i \tag{2.39}$$

REMARK 2.4.6
The complex 2^n-dimensional Clifford algebra endowed with the Euclidean metric may equivalently be defined [20] as the minimal enlargement of the space $\mathbb{R}^n$ to a unitary complex algebra which is not generated by any proper subspace of $\mathbb{R}^n$, and which is such that $x^2 = -||x||^2_{\mathbb{R}^n}$ for all $x \in \mathbb{R}^n$.

□

Note that $\mathbb{R} \oplus \mathbb{R}^n \subset \mathbb{C}_{(n)}$ for all $n \in \mathbb{N}$. Every nonzero element of $\mathbb{R}_{(2)}$ has an inverse. Moreover, since the spaces $\Lambda^{1,n}$, $\mathbb{C}^n$, $n \in \mathbb{N}$ are isometric isomorphic, it holds true that $||u||_{\Lambda^{1,n}} = ||u||_{\mathbb{C}^n}$ for all $n \in \mathbb{N}$. Therefore, for all $u \in \Lambda^{1,n}$, $n \in \mathbb{N}$, we have by (2.35)

$$||u||^2_{\Lambda^{1,n}} \equiv (u,u)_{\Lambda^{1,n}} = (-uu^c)_0 = -uu^c \tag{2.40}$$

Thus, for any n-dimensional nonzero complex 1-form u, we have the result

$$u^{-1} = -||u||^{-2}_{\mathbb{C}^n} u^c \tag{2.41}$$

or, equivalently, $uu^c = -||u||^2_{\mathbb{C}^n}$. There is no similar result for l-forms with $l > 1$. [19]

It is clear that the space of n-dimensional differential 1-forms can be identified with the dual space of vector fields in $\mathbb{R}^n$.

A number of monadic and dyadic operators are now defined on the spaces of differential forms and differential form-valued functions.

DEFINITION 2.4.7
Given $n \in \mathbb{N}, l \in \{0, \ldots, n\}$ and the differential forms $a \in \Lambda^{1,n}$, $u \in \Lambda^{l,n}$, the *exterior product* $a \wedge u$ and the *interior product* $a \vee u$ are defined as follows:

$$a \wedge u = \Pi_{l+1}(au), \;\; a \vee u = -\Pi_{l-1}(au) \tag{2.42}$$

□

REMARK 2.4.8
If ν is the unit outward normal defined a.e. on a closed hypersurface $\Gamma \subset \mathbb{R}^n$,

□

The operator $\Pi_l : \mathbb{C}_{(n)} \to \Lambda^{l,n}$, $l \in \mathbb{N}_0$, $l \leq n$ is obviously linear. Since any element of $\mathbb{C}_{(n)}$, $n \in \mathbb{N}$ is a linear combination of differential forms in $\Lambda^{l,n}$, $l \in \{0, \ldots, n\}$, Definition 2.4.7 is readily extended to arbitrary $u \in \mathbb{C}_{(n)}$.

In order to proceed with the definition of the various monadic and dyadic operators, we need the notion of a differential form-valued function.

DEFINITION 2.4.9
Let $n \in \mathbb{N}, l \in \mathbb{N}_0, l \leq n$. Given an algebra $\mathcal{A}$ of distributions mapping $\Omega \subset \mathbb{R}^n$ into $\mathbb{C}$, we denote by $\mathcal{A}(\Omega, \Lambda^l)$ the vector space of differential l-forms with coefficients in $\mathcal{A}$. An element $u \in \mathcal{A}(\Omega, \Lambda^l)$ of this space is formally written

$$u = \sum_{|I|=l}{}' u_I dx_1 \wedge \cdots \wedge dx_{I_n} = \sum_{|I|=l}{}' u_I e_I \tag{2.43}$$

where $u_I \in \mathcal{A}$ for all multiindices I of cardinality l.

□

If the algebra $\mathcal{A}$ is normed, we equip the vector space $\mathcal{A}(\Omega, \Lambda^l)$ with the inner product

$$\langle u, v\rangle_{\mathcal{A}(\Omega,\Lambda^l)} \equiv \int_\Omega \langle u(x), v(x)\rangle_{\Lambda^{l,n}} dx = \int_\Omega (u(x)\overline{v(x)}^c)_0 dx \quad \forall u, v \in \mathcal{A}(\Omega, \Lambda^l) \tag{2.44}$$

and with the natural induced norm.

DEFINITION 2.4.10
Let $n \in \mathbb{N}, l \in \mathbb{N}_0, l \leq n$. Given an algebra $\mathcal{A}$ of complex functions defined on an open domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, the *Dirac operator* D_n on $\mathcal{A}(\Omega, \Lambda^l)$ is defined by

$$D_n \equiv \sum_{j=1}^{n} e_j \partial_j \tag{2.45}$$

The *exterior derivative* operator $d_{l,n}$ and the *interior derivative* operator $\delta_{l,n}$ on differential form-valued functions are defined as follows:

$$\begin{aligned} d_{l,n}u &\equiv \Pi_{l+1}(D_n u) \quad \forall u \in \mathcal{A}(\Omega, \Lambda^l) \\ \delta_{l,n}u &\equiv \Pi_{l-1}(D_n u) \quad \forall u \in \mathcal{A}(\Omega, \Lambda^l) \end{aligned} \tag{2.46}$$

The operator $\delta_{l,n}$ is also referred to as the *co-differential operator*.

□

Note that $\overline{D_n} = \sum_{j=1}^n \overline{e}_j \partial_j = -D_n$. Also, we have

$$\begin{aligned} d_{l,n} &: \mathcal{A}(\Omega, \Lambda^l) \to \mathcal{A}(\Omega, \Lambda^{l+1}) \quad \forall l \in \mathbb{N}_0,\ n \in \mathbb{N},\ l \leq n, \\ \delta_{l,n} &: \mathcal{A}(\Omega, \Lambda^l) \to \mathcal{A}(\Omega, \Lambda^{l-1}) \quad \forall l \in \mathbb{N},\ n \in \mathbb{N},\ l \leq n \end{aligned} \tag{2.47}$$

with $\mathcal{A}$, Ω as defined above. Furthermore, it is true that $\delta_{0,n}u = 0$ for all $n \in \mathbb{N}$ and $u \in \mathcal{A}(\Omega \subseteq \mathbb{R}^n, \Lambda^0)$. By the linearity of the operator Π_l, the above definitions of the exterior and the interior derivatives are readily extended to the appropriate spaces of Clifford algebra-valued functions.

Unless explicitly specified otherwise, we assume that operators acting on differential forms or differential form-valued functions are chosen in accordance to the dimension n and order l of the argument (arguments.) We simplify the notation for these operators and omit the subscripts l, n.

We now obtain the following results relating differential forms $dx_i, i \in \{1, \ldots, n\}$ and basis vectors $e_i, i \in \{1, \ldots, n\}$ of $\Lambda^{1,n}$:

$$dx_i = \Pi_1(\nabla x_i) = \Pi_1 e_i = e_i \quad \forall i \in \{1, \ldots, n\} \tag{2.48}$$

$$dx_i \wedge dx_j = \Pi_2(dx_i dx_j) = \Pi_2(e_i e_j) = (1 - \delta^K_{i,j})e_i e_j \quad \forall i, j \in \{1, \ldots, n\} \tag{2.49}$$

This justifies the formal expression (2.43) of a differential form-valued function.

DEFINITION 2.4.11
For all $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $l \leq n$, the *Hodge star operator*

$$* : \Lambda^{l,n} \to \Lambda^{n-l,n} \tag{2.50}$$

is defined by the relation

$$u \wedge *u = ||u||^2_{\Lambda^{l,n}} \omega \quad \forall u \in \Lambda^{l,n} \tag{2.51}$$

where the n-form $\omega = dx_1 \wedge \cdots \wedge dx_n$ is a volume element on the n-dimensional manifold $\mathbb{R}^n$.

□

An alternative definition of the Hodge star operator is given as follows:

DEFINITION 2.4.12
For all $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $l \leq n$. the Hodge star operator $*$ is defined as the unique linear mapping from $\Lambda^{l,n}$ into $\Lambda^{n-l,n}$ such that $e_I * e_I = e_1 \dots e_n$ for all multiindices I of length $|I| = l$.

□

The following remark proves useful in calculations involving operators on differential forms and differential form-valued functions.

REMARK 2.4.13
Let $a \in \Lambda^{1,n}, u \in \Lambda^{l,n}$ and $v \in C^\infty(\mathbb{R}^n, \Lambda^l)$, with

$$\begin{aligned} u &= \sum_{|I|=l}{}' u_I e_I \\ v &= \sum_{|I|=l}{}' v_I e_I \end{aligned} \tag{2.52}$$

Expressing (2.42) explicitly, we arrive at the following equivalent definitions of the exterior and interior products of differential forms:

$$\begin{aligned} a \wedge u &= \sum_{|J|=l+1}{}' \sum_{|I|=l}{}' \sum_{i=1}^{n} \varepsilon^{iI}_{J} a_i u_I e_J \\ a \vee u &= \sum_{|I|=l-1}{}' \sum_{i=1}^{n} a_i u_{iI} e_I \end{aligned} \tag{2.53}$$

Similarly, the action of the Hodge star operator on a differential form can be written explicitly as

$$*u = \sum_{|I|=l}{}' \varepsilon^{I \complement I}_{\{1,\dots,n\}} u_I e_{\complement I} \tag{2.54}$$

where $\complement I \equiv \{1, \dots, n\} \backslash I$. Finally, (2.46) may be transformed into the following expressions for the exterior and interior derivatives of differential form-valued functions:

$$\begin{aligned} dv &= \sum_{|I|=l}{}' \sum_{i=1}^{n} \partial_i v_I e_i e_I \\ \delta v &= -\sum_{|I|=l-1}{}' \sum_{i=1}^{n} \partial_i v_{iI} e_I \end{aligned} \tag{2.55}$$

□

A simple calculation shows that, for any $n \in \mathbb{N}$, we have

$$D^2 = \sum_{j,j'=1}^{n} e_j e_{j'} \partial^2_{jj'} = -\sum_{j=1}^{n} \partial_j^2 = -\Delta \tag{2.56}$$

where the operator

$$-\Delta : \mathcal{A}(\Omega, \Lambda^l) \to \mathcal{A}(\Omega, \Lambda^l), \quad l \in \mathbb{N}_0, l \leq n \tag{2.57}$$

is referred to as the *Hodge Laplacian*. Any differential form u satisfying $\Delta u = 0$ (possibly in the sense of distributions) is called a *harmonic* differential form. Note that the following relation is valid:

$$(d + \delta)u = \Pi_{l+1}(Du) + \Pi_{l-1}(Du) = Du \quad \forall u \in \mathcal{A}(\mathbb{R}^n, \Lambda^l),\, l \in \{0, \ldots, n\} \tag{2.58}$$

or simply $D = d + \delta$. Thus, since obviously $[D^2, \Pi_l] = 0$ for all $l \in \mathbb{N}_0$, it is true that

$$d^2 = 0,\ \delta^2 = 0 \tag{2.59}$$

Furthermore, in light of (2.56), we may deduce the following important expression for the Hodge Laplacian:

$$-\Delta = d\delta + \delta d = [d, \delta]_+ \tag{2.60}$$

In words, the Hodge Laplacian is the negative of the anticommutator of the exterior and the interior derivative operators.

We now state a number of technical results which will prove useful in the subsequent analysis.

LEMMA 2.4.14
Assume that $n \in \mathbb{N}, l \in \{0, \ldots, n\}$. Given the differential form-valued functions $a, b \in \mathcal{A}(\mathbb{R}^n, \Lambda^1), u \in \mathcal{A}(\mathbb{R}^n, \Lambda^l), c \in \mathcal{A}(\mathbb{R}^n, \Lambda^{n-l}), f \in \mathcal{A}(\mathbb{R}^n, \Lambda^0)$, the following holds true:

$$a \wedge (a \wedge u) = 0,\ a \vee (a \vee u) = 0 \tag{2.61}$$

$$*(a \wedge u) = (-1)^l a \vee *u,\ *(a \vee u) = (-1)^{l+1} a \wedge *u \tag{2.62}$$

$$au = a \wedge u - a \vee u \tag{2.63}$$

$$a \wedge (b \vee u) + b \vee (a \wedge u) = \langle a, b \rangle u \tag{2.64}$$

$$a \wedge (b \vee u) + a \vee (b \wedge u) = \langle a, b \rangle u \tag{2.65}$$

$$\langle a \vee u, c \rangle = \langle u, a \wedge c \rangle \tag{2.66}$$

$$**u = (-1)^{l(n-l)} u \tag{2.67}$$

$$\delta u = (-1)^{n(l+1)+1} *d*u \tag{2.68}$$

$$*\delta u = (-1)^l d*u,\ \delta *u = (-1)^{l+1} *du \tag{2.69}$$

$$\langle u, *c \rangle = (-1)^{l(n-l)} \langle *u, c \rangle,\ \langle *u, *c \rangle = \langle u, c \rangle \tag{2.70}$$

If the function u is constant, we have

$$d(fu) = (df) \wedge u, \quad \delta(fu) = -(df) \vee u \tag{2.71}$$

Proof: All the above relations may be readily deduced using the expressions for the interior and exterior products and derivatives on differential forms, as well as the expression for the Hodge star operator, presented in Remark 2.4.13. These calculations, while trivial, are bulky, and we choose to omit them. Less tedious proofs do exist for some of the relations, where the definitions (2.42) and (2.46) of the interior and the exterior products and derivatives, respectively, are used. Specifically, relation (2.63) is easily seen to hold, since we have that

$$a \wedge u - a \vee u = \Pi_{l+1} au + \Pi_{l-1} au = au \tag{2.72}$$

Ad (2.64), we write

$$\begin{aligned} a \wedge (b \vee u) + b \vee (a \wedge u) = &- a \wedge \Pi_{l-1} bu + b \vee \Pi_{l+1} au = \\ &- \Pi_l \left\{ a\Pi_{l-1} bu + b\Pi_{l+1} au \right\} = \langle a, b\rangle u \end{aligned} \tag{2.73}$$

and similarly for the proof of relation (2.65):

$$\begin{aligned} a \wedge (b \vee u) + a \vee (b \wedge u) = &- a \wedge \Pi_{l-1} bu + a \vee \Pi_{l+1} bu = \\ &- \Pi_l a \left\{ \Pi_{l-1} bu + \Pi_{l+1} bu \right\} = -\Pi_l abu = \langle a, b\rangle u \end{aligned} \tag{2.74}$$

Also, if u is a constant differential form-valued function, we have

$$d(fu) = \Pi_{l+1} D(fu) = \Pi_{l+1} (\nabla f) u = (\nabla f) \wedge u = (df) \wedge u, \tag{2.75}$$

$$\delta(fu) = \Pi_{l-1} D(fu) = \Pi_{l-1} (\nabla f) u = -(df) \vee u \tag{2.76}$$

■

DEFINITION 2.4.15
Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega = \Gamma \subset \mathbb{R}^n$, let ν be the pertaining unit outward normal (defined a.e. on Γ.) A differential form u defined a.e. on the hypersurface Γ is referred to as *tangential* iff $\nu \vee u = 0$ a.e. on Γ. Furthermore, u is referred to as *normal* iff $\nu \wedge u = 0$ a.e. on Γ.

□

Using the notation of the above definition, note that relation (2.64) implies

$$\nu \wedge (\nu \vee u) + \nu \vee (\nu \wedge u) = \langle \nu, \nu\rangle u = u \tag{2.77}$$

for all differential forms u defined a.e. on the hypersurface Γ. Consequently, we have

$$u = \nu \wedge (\nu \vee u) \tag{2.78}$$

for any normal form u on Γ, and

$$u = \nu \vee (\nu \wedge u) \tag{2.79}$$

for any tangential form u on Γ. We may introduce the notions of the *tangential component* u_t and the *normal component* u_n of a differential form on Γ as follows:

$$u_t \equiv \nu \vee (\nu \wedge u), \quad u_n \equiv \nu \wedge (\nu \vee u) \tag{2.80}$$

Note that, according to the relations in (2.61) and (2.66), we have

$$\langle u_t, u_n \rangle = \langle \nu \vee (\nu \wedge u), \nu \wedge (\nu \vee u) \rangle = \langle \nu \wedge u, \nu \wedge \nu \wedge (\nu \vee u) \rangle = 0 \tag{2.81}$$

and, by (2.77), it holds true that

$$u_t + u_n = u \tag{2.82}$$

These results are consistent with, and hence a generalisation of, the usual case involving surface normal and tangential vectors in $\mathbb{R}^3$.

COROLLARY 2.4.16
The Hodge star operator $* : \Lambda^{l,n} \to \Lambda^{n-l,n}$ is an isomorphism.
Proof: By (2.67), the linear operator $*$ is surjective on $\Lambda^{n-l,n}$. Furthermore, it is seen immediately from the explicit expression (2.54) that it is an injective isometry.

■

Some of the identities presented in Lemma 2.4.14 constitute generalisations to differential forms of certain well-known results involving geometrical vector fields in $\mathbb{R}^n$. It is useful to express explicitly the action of the presented operators in the special cases $(l,n) = (0,3)$ and $(l,n) = (1,3)$, using the pertaining conventional notation.

REMARK 2.4.17
Assume $f \in \Lambda^{0,n}$, $a, u \in \Lambda^{1,n}$ for some $n \in \mathbb{N}$. It is readily seen from the relevant definitions and from e.g. Remark 2.4.13 that the following holds true:

$$\begin{gathered} a \vee f = 0,\ a \wedge f = af \\ *(a \wedge u) = a \times u,\ \ a \vee u = a.u \\ df = Df = \nabla f,\ \ \delta u = -\mathrm{div} u,\ \ \delta * u = *du = \mathrm{curl} u \\ \delta * \delta * u = \delta du = \mathrm{curl}\,\mathrm{curl} u = (-\Delta + \nabla \mathrm{div}) u \\ *d*du = *d\delta * u = \mathrm{curl}\,\mathrm{curl} u = (-\Delta + \nabla \mathrm{div}) u \end{gathered} \tag{2.83}$$

□

The interior differential operator δ is actually the transpose of the exterior differential operator d in certain special cases, as follows from the next lemma.

LEMMA 2.4.18
General Green-Stokes Identities
Given $n \in \mathbb{N}, l \in \{0, \ldots, n\}$, let Ω be a smooth bounded domain in $\mathbb{R}^n$, with boundary Γ and unit outward normal ν. Also, let $\mathcal{A} = C_0^\infty(\mathbb{R}^n)$. Then, for all $u \in \mathcal{A}(\overline{\Omega}, \Lambda^l)$, $v \in \mathcal{A}(\overline{\Omega}, \Lambda^{l+1})$, it holds true that

$$\int_\Omega \left\{ \langle du, v \rangle - \langle u, \delta v \rangle \right\} d\Omega = \int_\Gamma \langle \nu \wedge u, v \rangle d\Gamma \tag{2.84}$$

Also, for all $u \in \mathcal{A}(\overline{\Omega}, \Lambda^{l+1})$, $v \in \mathcal{A}(\overline{\Omega}, \Lambda^l)$, we have

$$\int_\Omega \left\{ \langle \delta u, v \rangle - \langle u, dv \rangle \right\} d\Omega = -\int_\Gamma \langle \nu \vee u, v \rangle d\Gamma \tag{2.85}$$

Proof: We refer to Reference [35, pp. 158–163] for a proof of this lemma.

■

The validity of the above Green-Stokes identities is extended by means of a standard density argument.

REMARK 2.4.19
Using the results of Proposition 2.4.17, we see that for any differential form-valued functions u, v defined in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with boundary Γ, it holds true that

$$\int\limits_{\Omega} \Big\{ \langle du, *v \rangle - \langle u, \delta * v \rangle \Big\} d\Omega = \int\limits_{\Omega} \Big\{ (-1)^{l(n-l)} \langle *du, v \rangle + (-1)^l \langle u, *dv \rangle \Big\} d\Omega \tag{2.86}$$

and also

$$\int\limits_{\Gamma} \langle \nu \wedge u, *v \rangle d\Gamma = \int\limits_{\Gamma} \langle u, \nu \vee *v \rangle d\Gamma = (-1)^l \int\limits_{\Gamma} \langle u, *\nu \wedge v \rangle d\Gamma \tag{2.87}$$

Consequently, in the special case of three-dimensional geometrical vector fields, specified by the parameter pair $(l, n) = (1, 3)$, expression (2.84) reduces to the classical Stokes formula:

$$\int\limits_{\Omega} \Big\{ u.\mathrm{curl} v - v.\mathrm{curl} u \Big\} d\Omega = \int\limits_{\Gamma} u.\nu \times v d\Gamma \tag{2.88}$$

LEMMA 2.4.20
Given $n \in \mathbb{N}, l \in \mathbb{N}_0, l \leq n$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. If a differential form-valued function $u \in C^\infty(\Gamma, \Lambda^l)$ satisfies $(\Delta + k^2 I)u = 0$ in Ω, then it holds true that

$$\begin{aligned} ||du||^2_{L^2(\Omega,\Lambda^{l+1})} + ||\delta u||^2_{L^2(\Omega,\Lambda^{l-1})} - k^2 ||u||^2_{L^2(\Omega,\Lambda^l)} = \\ \int\limits_{\Gamma} \Big\{ \langle du, \nu \wedge u \rangle - \langle \delta u, \nu \vee u \rangle \Big\} d\Gamma \end{aligned} \tag{2.89}$$

Proof: The general Green-Stokes identities of Lemma 2.4.18 imply the following:

$$\begin{aligned} ||du||^2_{L^2(\Omega,\Lambda^{l+1})} &= \int\limits_{\Omega} \langle du, du \rangle d\Omega = \int\limits_{\Omega} \langle \delta du, u \rangle d\Omega + \int\limits_{\Gamma} \langle du, \nu \wedge u \rangle d\Gamma \\ ||\delta u||^2_{L^2(\Omega,\Lambda^{l-1})} &= \int\limits_{\Omega} \langle \delta u, \delta u \rangle d\Omega = \int\limits_{\Omega} \langle d\delta u, u \rangle d\Omega - \int\limits_{\Gamma} \langle \delta u, \nu \vee u \rangle d\Gamma \end{aligned} \tag{2.90}$$

Adding the above equalities yields

$$\begin{aligned} ||du||^2_{L^2(\Omega,\Lambda^{l+1})} + ||\delta u||^2_{L^2(\Omega,\Lambda^{l-1})} = \\ \int\limits_{\Omega} \langle -\Delta u, u \rangle d\Omega + \int\limits_{\Gamma} \langle du, \nu \wedge u \rangle d\Gamma - \int\limits_{\Gamma} \langle \delta u, \nu \vee u \rangle d\Gamma = \\ k^2 ||u||_{L^2(\Gamma,\Lambda^l)} + \int\limits_{\Gamma} \langle du, \nu \wedge u \rangle d\Gamma - \int\limits_{\Gamma} \langle \delta u, \nu \vee u \rangle d\Gamma \end{aligned} \tag{2.91}$$

■

LEMMA 2.4.21
Green Identities for Differential Forms
Given $n \in \mathbb{N}, l \in \{0, \ldots, n\}$, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary. Given $u, v \in C_0^\infty(\overline{\Omega}, \Lambda^l)$, we have the following generalisations of the classical identities of Green:

$$\int\limits_{\Omega} \Big\{ \langle u, \Delta v \rangle + \langle du, dv \rangle + \langle \delta u, \delta v \rangle \Big\} d\Omega = \int\limits_{\Gamma} \Big\{ \langle u, \nu \vee dv \rangle - \langle u, \nu \wedge \delta v \rangle \Big\} d\Gamma \quad \text{(Green's first identity)} \tag{2.92}$$

and

$$\int\limits_{\Omega} \Big\{ \langle u, (\Delta + k^2 I) v \rangle - \langle (\Delta + k^2 I) u, v \rangle \Big\} d\Omega = \int\limits_{\Gamma} \Big\{ \langle u, \nu \vee dv \rangle - \langle u, \nu \wedge \delta v \rangle + \langle \nu \wedge \delta u, v \rangle - \langle \nu \vee du, v \rangle \Big\} d\Gamma$$

$$\text{(Green's second identity)} \tag{2.93}$$

Proof: Using the already derived relation $\Delta = -\delta d - d\delta$ for the Hodge Laplacian, as well as Lemma 2.4.18, we have

$$\int\limits_{\Omega_+} \langle u, \Delta v \rangle dx = - \int\limits_{\Omega_+} \Big\{ \langle u, \delta dv \rangle + \langle u, d\delta v \rangle \Big\} dx = - \int\limits_{\Omega_+} \Big\{ \langle du, dv \rangle + \langle \delta u, \delta v \rangle \Big\} dx + \int\limits_{\partial\Omega} \Big\{ \langle u, \nu \vee dv \rangle - \langle u, \nu \wedge \delta v \rangle \Big\} d\Gamma \tag{2.94}$$

Furthermore, due to symmetry, it also holds true that

$$\int\limits_{\Omega_+} \langle \Delta u, v \rangle dx = - \int\limits_{\Omega_+} \Big\{ \langle du, dv \rangle + \langle \delta u, \delta v \rangle \Big\} dx + \int\limits_{\partial\Omega} \Big\{ \langle \nu \vee du, v \rangle - \langle \nu \wedge \delta u, v \rangle \Big\} d\Gamma \tag{2.95}$$

Clearly, we have

$$\int\limits_{\Omega_+} \Big\{ \langle u, (\Delta + k^2 I) v \rangle - \langle (\Delta + k^2 I) u, v \rangle \Big\} dx = \int\limits_{\Omega_+} \Big\{ \langle u, \Delta v \rangle - \langle \Delta u, v \rangle \Big\} dx \tag{2.96}$$

and the result follows.

■

The validity of the generalised Green identities presented above is extended using a standard density argument.

It is seen from the definitions of the exterior derivative operator d, its formal adjoint δ, and the Hodge star operator $*$ that the following holds true for all $k \in \mathbb{N}$:

$$dC^k(\Omega, \Lambda^l) \subseteq C^{k-1}(\Omega, \Lambda^{l+1}) \tag{2.97}$$

$$\delta C^k(\Omega, \Lambda^l) \subseteq C^{k-1}(\Omega, \Lambda^{l-1}) \tag{2.98}$$

$$*C^k(\Omega, \Lambda^l) \subseteq C^k(\Omega, \Lambda^{n-l}) \tag{2.99}$$

DEFINITION 2.4.22
The following two closed subspaces of $L^p(\Gamma, \Lambda^l)$ are defined for $p \in]1, \infty[$:

$$\begin{aligned} L^p_t(\Gamma, \Lambda^l) &= \left\{ u \in L^p(\Gamma, \Lambda^l) \middle| \nu \vee u = 0 \text{ a.e. on } \Gamma \right\} \\ L^p_n(\Gamma, \Lambda^l) &= \left\{ u \in L^p(\Gamma, \Lambda^l) \middle| \nu \wedge u = 0 \text{ a.e. on } \Gamma \right\} \end{aligned} \tag{2.100}$$

□

DEFINITION 2.4.23
Given $n \in \mathbb{N}, p \in]1, \infty[$, the operators

$$\delta_\partial : L^p_t(\Gamma, \Lambda^l) \to C^1(\mathbb{R}^n, \Lambda^{l-1}), \quad l \in \{1, \ldots, n\} \tag{2.101}$$

$$d_\partial : L^p_n(\Gamma, \Lambda^l) \to C^1(\mathbb{R}^n, \Lambda^{l+1}), \quad l \in \{0, \ldots, n\} \tag{2.102}$$

are defined by

$$\begin{aligned} &\int_\Gamma \langle du, v \rangle d\Gamma \equiv \int_\Gamma \langle u, \delta_\partial v \rangle d\Gamma \quad \forall u \in C^1(\mathbb{R}^n, \Lambda^{l-1}), v \in L^p_{\text{tan}}(\Gamma, \Lambda^l), l \in \{1, \ldots, n\} \\ &\int_\Gamma \langle \delta u, v \rangle d\Gamma \equiv \int_\Gamma \langle u, d_\partial v \rangle d\Gamma \quad \forall u \in C^1(\mathbb{R}^n, \Lambda^{l+1}), v \in L^p_{\text{nor}}(\Gamma, \Lambda^l), l \in \{0, \ldots, n\} \end{aligned} \tag{2.103}$$

We set $L^p_t(\Gamma, \Lambda^0) \subset \ker\{\delta_\partial\}$. Furthermore, the vector $\delta_\partial u$ is referred to as the *boundary co-derivative* of u in L^p.

□

We now have the following definitions [14]:

DEFINITION 2.4.24
Given $n \in \mathbb{N}, l \in \{0, \ldots, n\}$, we set

$$\begin{aligned} L^{p,\delta}_t(\Gamma, \Lambda^l) &\equiv \{u \in L^p_t(\Gamma, \Lambda^l) | \delta_\partial u \in L^p_t(\Gamma, \Lambda^{l-1})\} \\ L^{p,d}_n(\Gamma, \Lambda^l) &\equiv \{u \in L^p_n(\Gamma, \Lambda^l) | d_\partial u \in L^p_n(\Gamma, \Lambda^{l+1})\} \end{aligned} \tag{2.104}$$

□

We note that

$$*L^{p,\delta}_t(\Gamma, \Lambda^{n-l}) = L^{p,d}_n(\Gamma, \Lambda^l) \tag{2.105}$$

and

$$*L^p_t(\Gamma, \Lambda^l) = L^p_n(\Gamma, \Lambda^{n-l}) \tag{2.106}$$

2.5 Pseudodifferential Operators

The notion of a pseudodifferential operator constitutes a generalisation of that of a differential operator. In this section, the theory of pseudodifferential operators is developed to the extent needed for an investigation of the existence, uniqueness and regularity properties of solutions to acoustic and electromagnetic boundary value problems. As will become apparent, the theory is applicable only to problems comprising smooth boundaries. The general case involving Lipschitz scatterers will be treated by other methods in Chapter 3. Consequently, all surfaces considered here in conjunction with pseudodifferential operators are assumed to be smooth. In the context of the basic theory of pseudodifferential operators, we have primarily used the references [36, 29, 11] and [10].

In the treatment of the pseudodifferential operators, we shall adopt the following convention on the expression of the Fourier transform[4]:

$$\hat{u}(\xi) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} u(x) e^{-ix.\xi} dx \quad \forall \xi \in \mathbb{R}^n, u \in C_0^\infty(\mathbb{R}^n) \tag{2.107}$$

As will become apparent in the subsequent development of Chapter 3, this choice impacts the constants appearing in the jump relations for the layer potential operators, and hence has influence on the corresponding constant terms in certain boundary integral operators. This, however, is of insignificant effect.

DEFINITION 2.5.1
Given a domain $\Omega \subseteq \mathbb{R}^n, n \in \mathbb{N}$, the space $O_M(\Omega)$ of *slowly increasing functions* is defined as follows:

$$\begin{aligned} O_M(\Omega) \equiv \{ u \in C^\infty(\Omega) \,|\, \forall \alpha \in \mathbb{N}_0^n \ \exists C_\alpha \geq 0, a_\alpha \in \mathbb{R} : \\ |D^\alpha u(x)| \leq C_\alpha \langle x \rangle^{a_\alpha} \ \forall x \in \Omega \} \end{aligned} \tag{2.108}$$

□

Given $\Omega \subseteq \mathbb{R}^n$, let P be a differential operator of order less than or equal to $m \in \mathbb{N}_0$ and of the form

$$P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \tag{2.109}$$

with

$$a_\alpha \in C^\infty(\Omega) \quad \forall \alpha : |\alpha| \leq m \tag{2.110}$$

It is readily seen that the following holds true for all $x \in \mathbb{R}^n$, $u \in C_0^\infty(\Omega)$:

$$Pu(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \int\limits_{\mathbb{R}^n} \hat{u}(\xi) e^{ix.\xi} d\xi = \int\limits_{\mathbb{R}^n} \hat{u}(\xi) \Big\{ \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \Big\} e^{ix.\xi} d\xi \tag{2.111}$$

This last expression is commonly referred to as the *Fourier integral representation* of the operator P. [36] The function

$$p(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n \tag{2.112}$$

[4]This convention is also utilised in references [36, 11] in the present context.

is called the *symbol* of P. Since the integral in (2.111) is defined for a larger set of functions than those of the form (2.112), it is possible to extend the class of operators specified in (2.111). Through the next few definitions, the notion of a *pseudodifferential operator* is introduced as a generalisation of that of a differential operator.

DEFINITION 2.5.2
Given $n \in \mathbb{N}$, $\rho, \delta \in [0,1]$, $m \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$ open, the space (class) $S^m_{\rho,\delta}(\Omega, \mathbb{R}^n)$ of *symbols* of degree m and type ρ, δ is defined as follows:

$$S^m_{\rho,\delta}(\Omega, \mathbb{R}^n) \equiv \Big\{ p \in C^\infty(\Omega, \mathbb{R}^n) \Big| \forall \alpha, \beta \in \mathbb{N}_0^n \;\; \exists C_{\alpha,\beta} < \infty : \\ |D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|} \; \forall (x,\xi) \in \Omega \times \mathbb{R}^n \Big\} \tag{2.113}$$

A function p of one variable only, satisfying

$$|D_\xi^\alpha p(\xi)| \le C_\alpha \langle \xi \rangle^{m - \rho|\alpha|} \quad \forall \xi \in \mathbb{R}^n \tag{2.114}$$

is said to belong to the space S^m_ρ.

□

According to the above definition, thus, the elements of a symbol space and all associated derivatives are slowly increasing functions in each of the variables $x \in \Omega \subseteq \mathbb{R}^n$, $\xi \in \mathbb{R}^n$. Also seen from Definition 2.5.2 is that the adjective 'pseudodifferential' emphasises the fact that the operators in question may be written in the same form as differential operators, albeit with the use of more general symbol functions.

DEFINITION 2.5.3
Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $m \in \mathbb{R}$, $\rho, \delta \in [0,1]$. If, for some $p \in S^m_{1,0}(\Omega, \mathbb{R}^n)$, there exists a sequence $(p_{m-j})_{j\in\mathbb{N}_0} \subset C^\infty(\Omega \times \mathbb{R}^n)$ of homogeneous functions satisfying

$$p_{m-j}(x, t\xi) = t^{m-j} p_{m-j}(x,\xi) \quad \forall j \in \mathbb{N}_0, t \ge 1, x \in \Omega, |\xi| \ge 1 \tag{2.115}$$

such that

$$p - \sum_{j=0}^{N} p_{m-j} \in S^{m-N}_{1,0}(\Omega, \mathbb{R}^n) \quad \forall N \in \mathbb{N}_0 \tag{2.116}$$

holds true, we denote this by $p \sim \sum_{j\in\mathbb{N}_0} p_{m-j}$. The space $S^m(\Omega, \mathbb{R}^n)$ of *polyhomogeneous symbols* of degree m is now defined as follows:

$$S^m(\Omega, \mathbb{R}^n) \equiv \Big\{ p \in C^\infty(\Omega \times \mathbb{R}^n) \Big| \; p \sim \sum_{j\in\mathbb{N}_0} p_{m-j} \Big\} \tag{2.117}$$

Furthermore, we define

$$S^\infty(\Omega, \mathbb{R}^n) = \bigcup_{m\in\mathbb{R}} S^m(\Omega, \mathbb{R}^n), \quad S^{-\infty}(\Omega, \mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} S^m(\Omega, \mathbb{R}^n) \tag{2.118}$$

□

Note that in Definition 2.5.3 it is not implied that the asymptotic series $\sum_{j\in\mathbb{N}_0} p_{m-j}$ is convergent. The polyhomogeneous symbols are also referred to as the *classical* symbols, and accordingly the space S^m is sometimes denoted S^m_{cl} [36].

REMARK 2.5.4
Let $m \in \mathbb{R}$. From the above definitions, it clearly holds true that $S^m(\Omega,\mathbb{R}^n) \subset S^m_{1,0}(\Omega,\mathbb{R}^n)$. Furthermore, the space $\bigcap_{m\in\mathbb{R}} S^m_{\rho,\delta}(\Omega,\mathbb{R}^n)$ contains the same elements as the space $S^{-\infty}(\mathbb{R}^n)$. Also, for $\rho,\rho',\delta,\delta' \in [0,1]$, $\rho \geq \rho'$, $\delta \leq \delta'$, we have $S^m_{\rho,\delta}(\Omega,\mathbb{R}^n) \subset S^m_{\rho',\delta'}(\Omega,\mathbb{R}^n)$.

Finally, for any $p \in S^m$ we have $p_{m-j} \in S^{m-j}_{1,0}$ for all $j \in \mathbb{N}_0$. Indeed, by definition, it is true that $p - \sum_{j=0}^{N} p_{m-j} \in S^{m-N}_{1,0} \subset S^{m-N-1}$ for all $N \in \mathbb{N}_0$, so necessarily

$$p - \sum_{j=0}^{N-1} p_{m-j} - \left\{ p - \sum_{j=0}^{N} p_{m-j} \right\} = p_{m-N} \in S^{m-N}_{1,0} \quad \forall N \in \mathbb{N}_0 \tag{2.119}$$

where the fact is used that $S^{m-N}_{1,0}$, $m \in \mathbb{R}, N \in \mathbb{N}_0$, is a space.

□

REMARK 2.5.5
The restriction in (2.113) on symbol functions in $S^m_{\rho,\delta}$ may not seem severe at all for $m > 0$. Nevertheless, at least the maximum increase rate of elements of $S^m_{\rho,\delta}$ is specified in this case.

□

DEFINITION 2.5.6
For $n \in \mathbb{N}$, $\rho,\delta \in [0,1]$, $m \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$, the space $OPS^m_{\rho,\delta}(\Omega)$ of *pseudodifferential operators* of order m and type ρ,δ is defined as follows:

$$\begin{aligned} OPS^m_{\rho,\delta}(\Omega) \equiv \Bigg\{ P \in \mathcal{L}(C_0^\infty(\mathbb{R}^n)) \Bigg| \exists p \in S^m_{\rho,\delta}(\Omega,\mathbb{R}^n) : \\ Pu(x) = \int_{\mathbb{R}^n} \hat{u}(\xi) p(x,\xi) e^{ix\cdot\xi} d\xi \ \forall x \in \Omega, u \in C_0^\infty(\mathbb{R}^n) \Bigg\} \end{aligned} \tag{2.120}$$

The function p is called the *symbol* of the operator P, and this is stressed by writing $P = p(x,D)$. The alternative notation $P = OP(p)$ is also used in the literature.

An entirely similar definition is made for the spaces $OPS^m(\Omega)$ (also denoted $OPS^m_{\mathrm{cl}}(\Omega)$ [36]) of pseudodifferential operators.

□

Let $p(x,D)$ be a pseudodifferential operator. Making use of the theorem of Fubini, as well as of the definition of the Fourier transform, we write

$$
\begin{aligned}
p(x, D)u(x) = &\int_{\mathbb{R}^n} \hat{u}(\xi) e_{\mathbb{R}^{2n}} p(x, \xi) e^{i\xi . x} d\xi = \\
&(2\pi)^{-n} \int_{\mathbb{R}^{2n}} u(y) e_{\mathbb{R}^{2n}} p(x, \xi) e^{i\xi .(x-y)} dy d\xi = \\
&(2\pi)^{-n} \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} e_{\mathbb{R}^{2n}} p(x, \xi) e^{i\xi .(x-y)} d\xi dy \quad \forall x \in \mathbb{R}^n,\, u \in C_0^\infty(\mathbb{R}^n)
\end{aligned}
\tag{2.121}
$$

Hence, any pseudodifferential operator with symbol p is an integral operator with the pertaining kernel

$$
\begin{aligned}
K(x, y) \equiv &(2\pi)^{-n} \int_{\mathbb{R}^n} e_{\mathbb{R}^{2n}} p(x, \xi) e^{i\xi .(x-y)} d\xi = \\
&(2\pi)^{-n} \mathcal{F}^{-1}_{\xi \mapsto y} e_{\mathbb{R}^{2n}} p(x, x-y) \quad \forall x, y \in \mathbb{R}^n
\end{aligned}
\tag{2.122}
$$

In this context, the kernel K is commonly referred to as the *Schwartz kernel* of the pseudodifferential operator $p(x, D)$ [36]. Note that the kernel function K is expressed in the form of an oscillatory integral.

PROPOSITION 2.5.7
Let the operator $\kappa : \mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$ be defined by

$$
\begin{aligned}
\kappa p(x, y) \equiv &(2\pi)^{-n} \int_{\mathbb{R}^n} e_{\mathbb{R}^{2n}} p(x, \xi) e^{i\xi .(x-y)} d\xi = \\
&(2\pi)^{-n} \mathcal{F}^{-1}_{\xi \mapsto y} e_{\mathbb{R}^{2n}} p(x, x-y) \quad \forall x, y \in \mathbb{R}^n
\end{aligned}
\tag{2.123}
$$

Alternatively expressed, let the operator κ map symbols of pseudodifferential operators into pertaining Schwartz kernels. This operator is isomorphic and isometric up to a constant.
Proof: The Fourier transform is a bijective linear operator on $\mathcal{S}'(\mathbb{R}^n)$. Furthermore, by the theorem of Plancherel, we have $||\kappa p|| \propto ||p||$ for all $p \in \mathcal{S}'(\mathbb{R}^{2n})$.

■

It is now clear that a pseudodifferential operator is of the Hilbert-Schmidt type iff its symbol belongs to $L^2(\mathbb{R}^{2n})$.

DEFINITION 2.5.8
A pseudodifferential operator in $OPS^{-\infty}$ is referred to as *negligible*.[11]

□

REMARK 2.5.9
Given a symbol $p \in S^m_{\rho,\delta}$, straightforward integration by parts yields

$$
(I - \Delta_z) \mathcal{F}^{-1}_{\xi \mapsto z} p = \mathcal{F}^{-1}_{\xi \mapsto z} \left\{ (1 + |\xi|^2) p \right\} = \mathcal{F}^{-1}_{\xi \mapsto z} (\langle \xi \rangle^2 p)
\tag{2.124}
$$

Hence, for all $N \in \mathbb{N}_0$, we have

$$(I - \Delta_z)^N \mathcal{F}^{-1}_{\xi \mapsto z} p = \mathcal{F}^{-1}_{\xi \mapsto z}(\langle \xi \rangle^{2N} p) \tag{2.125}$$

Furthermore, again integrating by parts, we see that

$$\mathcal{F}^{-1}_{\xi \mapsto z}(-\Delta_\xi) p = |z|^2 \mathcal{F}^{-1}_{\xi \mapsto z} p \tag{2.126}$$

Thus, for any $M \in \mathbb{N}_0$, it is true that

$$|z|^{2M} \mathcal{F}^{-1}_{\xi \mapsto z} p = \mathcal{F}^{-1}_{\xi \mapsto z}(-\Delta_\xi)^M p \tag{2.127}$$

□

In view of the relation (2.122) between the kernel and the symbol of a pseudodifferential operator, we have the following proposition (our proof differs significantly from the corresponding one in [36]):

PROPOSITION 2.5.10
Let $p(x, D) \in OPS^m_{\rho,\delta}$. The kernel K associated with the operator $p(x, D)$ is in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \backslash \text{diag}\{\mathbb{R}^n \times \mathbb{R}^n\})$. Furthermore, it is true that

$$|K(x, y)| = \mathcal{O}(|x - y|^{-2c}) \tag{2.128}$$

for any nonnegative integer $c > n + m$.
Proof: Indeed, for all $x, y \in \mathbb{R}^n$ and all $M, N \in \mathbb{N}_0$ such that $2(M - N) > n + m$, it holds true that (see Remark 2.5.9)

$$\begin{aligned} |x - y|^{2M} (I - \Delta_y)^N K(x, y) =& (2\pi)^{-n} |x - y|^{2M} (I - \Delta_y)^N \mathcal{F}^{-1}_{\xi \mapsto y} p(x, x - y) = \\ & (2\pi)^{-n} \mathcal{F}^{-1}_{\xi \mapsto y}(-\Delta_\xi)^M (\langle \xi \rangle^{2N} p)(x, x - y) \end{aligned} \tag{2.129}$$

Subject to the above restriction on M, N, it is true that $(-\Delta_\xi)^M (\langle \xi \rangle^{2N} p) \in L^1(\mathbb{R}^n)$ as a function of ξ for all $x \in \mathbb{R}^n$. Indeed, we have the following:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} (-\Delta_\xi)^M \langle \xi \rangle^{2N} p(x, \xi) d\xi \right| \leq& C \left| \int_{\mathbb{R}^n} \sum_i D^{\alpha_i}_\xi (\langle \xi \rangle^{2N} p)(x, \xi) d\xi \right| = \\ & C \left| \int_{\mathbb{R}^n} \sum_i \sum_{a+b=\alpha_i} \binom{\alpha_i}{a} D^a_\xi \langle \xi \rangle^{2N} D^b_\xi p(x, \xi) d\xi \right| \leq \\ & \sum_i \sum_{a+b=\alpha_i} C_b \binom{\alpha_i}{a} \int_{\mathbb{R}^n} |D^a_\xi \langle \xi \rangle^{2N}| \langle \xi \rangle^{m - |b|} d\xi \leq \\ & \sum_i \sum_{a+b=\alpha_i} C_b \binom{\alpha_i}{a} \int_{\mathbb{R}^n} \langle \xi \rangle^{2N + m - (|a| + |b|)} d\xi = \\ & \sum_i C_i \binom{\alpha_i}{a} \int_{\mathbb{R}^n} \langle \xi \rangle^{2N + m - |\alpha_i|} d\xi \end{aligned} \tag{2.130}$$

Now since the operator $(-\Delta_\xi)^M$ involves differentiation of order greater than or equal to $2M$, it is true that

$$\left|\int\limits_{\mathbb{R}^n}(-\Delta_\xi)^M\langle\xi\rangle^{2N}p(x,\xi)d\xi\right| \le \sum_i C_i\binom{\alpha_i}{a}\int\limits_{\mathbb{R}^n}\langle\xi\rangle^{2N-2M+m}d\xi \le C\int\limits_{\mathbb{R}^n}\langle\xi\rangle^{2(N-M)+m}d\xi \tag{2.131}$$

By assumption, we have $2(N-M)+m<-n$, and thus the above integral is bounded. It is hence true that $\mathcal{F}^{-1}_{\xi\mapsto y}(-\Delta_\xi)^M(\langle\xi\rangle^{2N}p) \in C^\infty(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Therefore, for $2(M-N)>n+m$, we have

$$(I-\Delta_y)^N K(x,y) = |x-y|^{-2M} f_{M,N}(x,x-y) \tag{2.132}$$

where $f_{M,N}\in C^\infty(\mathbb{R}^n)$ for all relevant M,N, and the first part of the result follows. Setting $N=0,\ M>n+m$ in equation (2.132) now yields

$$|K(x,y)| \le C|x-y|^{-2M} = \mathcal{O}(|x-y|^{-2M}) \quad \forall x,y\in\mathbb{R}^n \tag{2.133}$$

This completes the proof.

■

REMARK 2.5.11
Let K be the kernel of a pseudodifferential operator in $OPS^m_{\rho,\delta}$. Based on the result of Proposition 2.5.10, we conclude that K is rapidly decreasing for $|x-y|\to\infty$. In particular, we have $|K(x,y)|=\mathcal{O}(|x-y|^{-2(n+m)})$. Now if $n+m\ge 0$, then clearly $|K(x,y)| = \mathcal{O}(|x-y|^{-n-m})$. Alternatively, we have $(n+m)/2 > n+m$, and by Proposition 2.5.10, we again obtain $|K(x,y)| = \mathcal{O}(|x-y|^{-n-m})$. This, in turn, implies

$$|D^\alpha_{x,y}K(x,y)| = \mathcal{O}(|x-y|^{-n-m-|\alpha|}) \quad \forall\alpha\in\mathbb{N}_0^n \tag{2.134}$$

□

Reference [36] provides a different approach to obtaining the result of Remark 2.5.11.

PROPOSITION 2.5.12
Let $m\in\mathbb{Z}, n\in\mathbb{N}$ be such that $-n<m<0$. Given any $q\in\mathcal{S}'(\mathbb{R}^n)$, smooth in $\mathbb{R}^n\backslash\{0\}$ and rapidly decreasing for $|x|\to\infty$, the following holds true:

$$\begin{gathered} q\in L^1_{\text{loc}}(\mathbb{R}^n), \\ \forall\alpha\in\mathbb{N}_0^n\ \ \exists C_\alpha<\infty:\ |D^\alpha_x q(x)| \le C_\alpha|x|^{-n-m-|\alpha|} \quad \forall x\in\mathbb{R}^n\backslash\{0\} \\ \Downarrow \\ \exists p\in S^m_1(\mathbb{R}^n):\ q=\mathcal{F}^{-1}p \end{gathered} \tag{2.135}$$

Proof: Let $\{\psi_j\}_{j\in\mathbb{N}_0}$ be a family of functions on $\mathbb{R}^n$, satisfying the following conditions:

$$\begin{gathered} \psi_0\in C_0^\infty(\mathbb{R}^n), \quad \text{supp}\{\psi_0\} = \left\{x\in\mathbb{R}^n \,\middle|\, |x|\in\,]1/2,2[\right\}, \\ \psi_j(x) = \psi_0(2^j x) \quad \forall x\in\mathbb{R}^n,\ j\in\mathbb{N} \\ \sum_{j\in\mathbb{N}_0}\psi_j(x) = 1 \quad \forall x\in\overline{B_{\mathbb{R}^n}(0,1)} \end{gathered} \tag{2.136}$$

We note in passing that

$$\psi_j \in C_0^\infty(\mathbb{R}^n), \quad \operatorname{supp}\{\psi_j\} = \left\{ x \in \mathbb{R}^n \,\middle|\, |x| \in]2^{-j-1}, 2^{-j+1}[\right\} \quad \forall j \in \mathbb{N}_0 \tag{2.137}$$

necessarily holds true. Clearly, we may write

$$q \equiv q_0 + \left\{ \sum_{j \in \mathbb{N}_0} \psi_j \right\} q \tag{2.138}$$

Now, due to the assumption on q, as well as the fact that $m < 0$, it is true that

$$|| \sum_{j \in \mathbb{N}_0} \psi_j q ||_{L^1(\mathbb{R}^n)} \le \sum_{j \in \mathbb{N}_0} \int_{\mathbb{R}^n} |\psi_0(2^j x)| \, |q(x)| dx \le \sum_{j \in \mathbb{N}_0} C_j \int_{\mathbb{R}^n} |x|^{-n-m} dx < \infty \tag{2.139}$$

Hence, $q_0 \in \mathcal{S}(\mathbb{R}^n)$. By the assumption on q, we have furthermore

$$2^{-j(n+m)} \psi_j(2^{-j}x) q(2^{-j}x) = 2^{-j(n+m)} \psi_0(x) q(2^{-j}x) \in \mathcal{S}(\mathbb{R}^n) \tag{2.140}$$

It is thus true that

$$\mathcal{F}^{-1} q_0 + \sum_{j \in \mathbb{N}_0} \mathcal{F}^{-1}(\psi_j q) \in S_1^m(\mathbb{R}^n) \tag{2.141}$$

■

Given the connection between the symbol and the kernel of a pseudodifferential operator, it is hardly surprising that a given space of pseudodifferential operators can be characterised by the behaviour of the associated kernels. The next proposition, which we take from [36] without proof (but which is based on Proposition 2.5.10 and Proposition 2.5.12,) expresses this characterisation.

PROPOSITION 2.5.13
Let $-n < m < 0$, and assume that $L \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ is a smooth function of x with values in $\mathcal{S}_0'(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then the function K given by

$$K(x, y) = L(x, x - y) \quad \forall x, y \in \mathbb{R}^n$$

defines the Schwartz kernel of an operator in $OPS_{1,0}^m$ iff it holds true that

$$|D_x^\beta D_z^\gamma L(x, z)| \le C_{\beta\gamma} |z|^{-n-m-|\gamma|} \quad \forall (x, z) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$$

Furthermore, for any $j \in \mathbb{N}$, the function K defines the Schwartz kernel of an operator in OPS^{-j} iff

$$L(x, z) \sim \sum_{l \ge 0} (q_l(x, z) + p_l(x, z) \log |z|) \quad \forall x, z \in \mathbb{R}^n$$

where each $D_x^\beta q_l(x, \cdot)$ is a bounded continuous function of x with values in the space $\mathcal{H}_{j+l-n}^\#(\mathbb{R}^n)$, and $p_l(x, z)$ is a polynomial homogeneous of degree $j + l - n$ in z, with coefficients that are bounded, together with all their x-derivatives. (We denote by $\mathcal{H}_\mu^\#(\mathbb{R}^n)$ the space of distributions on $\mathbb{R}^n$, homogeneous of degree μ, which are smooth on $\mathbb{R}^n \setminus \{0\}$.

■

The following elementary result will prove useful in establishing the regularity properties of pseudodifferential operators.

PROPOSITION 2.5.14
Let X be a space equipped with a measure μ, and assume a function $k : X^2 \to \mathbb{C}$ is measurable on X^2, i.e. let

$$\iint_{X^2} |k(x,y)| d\mu(x) d\mu(y) < \infty \tag{2.142}$$

hold true. Furthermore, assume that there exist constants C_1, C_2 such that

$$\begin{aligned} ||k_y||_{L^1(X,\mu)} &\equiv \int_X |k(x,y)| d\mu(x) \le C_1 \quad \forall y \in X \\ ||k_x||_{L^1(X,\mu)} &\equiv \int_X |k(x,y)| d\mu(y) \le C_2 \quad \forall x \in X \end{aligned} \tag{2.143}$$

Then the integral operator K with kernel k, defined by

$$Kf(x) \equiv \int_X k(x,y) f(y) d\mu(y) \quad \forall x \in X,\ f \in D\{K\} \tag{2.144}$$

is bounded on $L^p(X,\mu)$, $p \in [1,\infty[$, in that the following holds true for any pair of dual exponents (q,q'):

$$||K|| \le C_1^{1/q} C_2^{1/q'} \tag{2.145}$$

Proof: By the well-known inequality of Young (see, e.g., Reference [31] for proof), given by

$$ab \le \frac{a^q}{q} + \frac{b^{q'}}{q'} \quad \forall a,b \ge 0,\ q,q' > 0,\ \frac{1}{q} + \frac{1}{q'} = 1 \tag{2.146}$$

we have, for all $f \in L^q(X,\mu)$, $g \in L^{q'}(X,\mu)$,

$$\begin{aligned} |(Kf,g)_X| = \left| \iint_{X^2} k(x,y) f(y) g(x) d\mu(x) d\mu(y) \right| \le \\ \int_X \int_X |k(x,y)| \left\{ \frac{|f(y)|^q}{q} + \frac{|g(x)|^{q'}}{q'} \right\} d\mu(x) d\mu(y) = \\ \frac{1}{q} ||k_x f^q||_{L^1(X,\mu)} + \frac{1}{q'} ||k_y g^{q'}||_{L^1(X,\mu)} \end{aligned} \tag{2.147}$$

Employing the Cauchy-Schwartz inequality, as well as the assumptions (2.143), now yields

$$|(Kf,g)_X| \le \frac{C_1}{q} ||f||^q_{L^q(X,\mu)} + \frac{C_2}{q'} ||g||^{q'}_{L^{q'}(X,\mu)} \quad \forall f \in L^q(X,\mu),\ g \in L^{q'}(X,\mu) \tag{2.148}$$

But for any $t \in]0, \infty[$, we have $(Ktf, t^{-1}g)_X = (Kf, g)_X$, so it holds that

$$|(Kf,g)_X| = |(Ktf, t^{-1}g)_X| \leq \frac{C_1 t^q}{q}||f||^q_{L^q(X,\mu)} + \frac{C_2}{t^{q'}q'}||g||^{q'}_{L^{q'}(X,\mu)} \tag{2.149}$$

for all $f \in L^q(X,\mu)$, $g \in L^{q'}(X,\mu)$. Clearly, a better estimate may be obtained by minimising the right-hand side of the above equation with respect to the parameter t. The optimum value t_0 of t is easily found to be

$$t_0 = \left\{ \frac{C_2||g||^{q'}_{L^{q'}(X,\mu)}}{C_1||f||^q_{L^q(X,\mu)}} \right\}^{1/qq'} = \frac{C_2^{1/qq'}||g||^{1/q}_{L^{q'}(X,\mu)}}{C_1^{1/qq'}||f||^{1/q'}_{L^q(X,\mu)}} \tag{2.150}$$

Using the fact that q, q' are dual exponents, it is now straightforward to derive that

$$|(Kf,g)_X| \leq 2C_1^{1/q}C_2^{1/q'}||f||_{L^q(X,\mu)}||g||_{L^{q'}(X,\mu)} \quad \forall f \in L^q(X,\mu),\, g \in L^{q'}(X,\mu) \tag{2.151}$$

■

REMARK 2.5.15
Assume $m \in \mathbb{R}$, and let $\rho, \delta \in \mathbb{R}$ satisfy $0 \leq \delta < \rho \leq 1$. Due to the obvious general commutator relations

$$[\Re\{\cdot\}, D^\alpha] = 0 \quad \forall \alpha \in \mathbb{N}_0^n \tag{2.152}$$
$$[\Im\{\cdot\}, D^\alpha] = 0 \quad \forall \alpha \in \mathbb{N}_0^n \tag{2.153}$$

we have the following:

$$\begin{gathered} p(x,D) \in OPS^m_{\rho,\delta}(\mathbb{R}^n) \\ \Downarrow \\ |D_x^\beta D_\xi^\alpha \Re\{p(x,\xi)\}| = |\Re\{D_x^\beta D_\xi^\alpha p(x,\xi)\}| \leq \\ |D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \quad \forall x,\xi \in \mathbb{R}^n \\ \Updownarrow \\ \Re\{p(x,D)\} \in OPS^m_{\rho,\delta}(\mathbb{R}^n) \end{gathered} \tag{2.154}$$

where $\Re\{p(x,D)\}$ is the pseudodifferential operator whose symbol is the real part of the symbol of $p(x,D)$. An entirely similar argument can be made for the operator $\Im\{p(x,D)\} \in OPS^m_{\rho,\delta}(\mathbb{R}^n)$.

□

The following proposition, taken from Reference [11], contains some important elements of the calculus of pseudodifferential operators.

PROPOSITION 2.5.16
For all $m, m' \in \mathbb{R}$ and $\Omega \subseteq \mathbb{R}^n$ open, the following holds true:

$$p \in S^m_{1,0}(\Omega^2) \Rightarrow p(x,y,D) \sim p_1(x,D) \sim p_2(x,D) \tag{2.155}$$

where $p_1 \in S^m_{1,0}(\Omega)$, $p_2 \in S^m_{1,0}(\Omega)$ satisfy

$$\begin{aligned} p_1(x,\xi) &= \sum_{\alpha\geq 0} \frac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha p(x,x,\xi) \mod S^{-\infty} \quad \forall (x,\xi)\in\Omega\times\mathbb{R}^n \\ p_2(y,\xi) &= \sum_{\alpha\geq 0} \frac{1}{\alpha!}\partial_x^\alpha \overline{D}_\xi^\alpha p(y,y,\xi) \mod S^{-\infty} \quad \forall (y,\xi)\in\Omega\times\mathbb{R}^n \end{aligned} \tag{2.156}$$

Also, for $p \in S^m_{1,0}(\Omega)$, it holds true that

$$p(x,D)^* = p_3(x,D) \mod S^{-\infty} \tag{2.157}$$

where $p_3 \in S^m_{1,0}(\Omega)$ satisfies

$$p_3 = \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_x^\alpha D_\xi^\alpha p^* \tag{2.158}$$

Hence, the adjoint of $OP(p(x,\xi))$ is the operator $OP(\overline{p}(x,\xi))$.

As for the product (composition) of pseudodifferential operators, assume $p_j(x,D) \in OPS^{m_j}_{\rho_j,\delta_j}(\Omega)$ for some $m_j \in \mathbb{R}$, $\rho_j, \delta_j \in [0,1]$ with $j \in \{0,1\}$ such that $0 \leq \delta_2 < \rho_1 \leq 1$. It then holds true that

$$p_1(x,D)p_2(x,D) \in OPS^{m_1+m_2}_{\min\{\rho_1,\rho_2\},\max\{\delta_1,\delta_2\}}(\Omega) \tag{2.159}$$

Furthermore, the symbol $p_1 \circ p_2$ of the operator product $p_1(x,D)p_2(x,D)$ is given by

$$p_1 \circ p_2(x,\xi) \sim \sum_{\alpha\geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi) \quad \forall (x,\xi)\in\Omega\times\mathbb{R}^n \tag{2.160}$$

Clearly, equation (2.160) specifies a generalisation of the well-known Leibnitz formula.
Proof: We refer to [36] and [11] for two different proofs of the assertions made in this proposition.

■

REMARK 2.5.17
Given $m_1, m_2, \rho, \delta \in \mathbb{R}$, $0 \leq \delta < \rho \leq 1$, let $p_1(x,D) \in OPS^{m_1}_{\rho,\delta}, p_2(x,D) \in OPS^{m_2}_{\rho,\delta}$ be pseudodifferential operators. We see from the expansion (2.160) of the symbol of a product of two pseudodifferential operators that the terms corresponding to $\alpha = 0$ are identical for the operators $p_1(x,D)p_2(x,D)$ and $p_2(x,D)p_1(x,D)$. Consequently, the commutator $[p_1(x,D), p_2(x,D)]$ is a pseudodifferential operator of order lower than $m_1 + m_2$. In fact, according to Reference [36], we have

$$[p_1(x,D), p_2(x,D)] \in OPS^{m_1+m_2-(\rho-\delta)}_{\rho,\delta} \tag{2.161}$$

□

DEFINITION 2.5.18
Given $m \in \mathbb{R}$, $\rho, \delta \in [0,1]$, a pseudodifferential operator $p(x,D) \in OPS^m_{\rho,\delta}$ is *elliptic* iff there exists $r < \infty$ such that

$$|p^{-1}(x,\xi)| \leq C\langle\xi\rangle^{-m} \quad \forall (x,\xi)\in\Omega\times\mathbb{R}^n : |\xi| \geq r \tag{2.162}$$

□

We have the following useful result regarding ellipticity of pseudodifferential operators:

PROPOSITION 2.5.19
Let $m \in \mathbb{Z}, n \in \mathbb{N}$ be such that $-n < m < 0$, and let $p(x,D) \in OPS^m$ be a pseudodifferential operator with pertaining Schwartz kernel $\kappa p(x,D) = K$. Assume that, at some $x_0 \in \mathbb{R}^n$, we have

$$K(x_0,y) = a|x_0-y|^{-m-n} + \cdots \quad \text{for } |x_0-y| \to 0,\, a \neq 0 \tag{2.163}$$

with remainder terms of increasing regularity. It then holds true for the principal symbol p_m that

$$p_m(x_0,\xi) \propto |\xi|^m \quad \forall \xi \in \mathbb{R}^n \tag{2.164}$$

and thus $p(x,D)$ is elliptic at x_0.

Proof: By assumption, there exists a set of functions $\{p_{m-j}\}_{j\in\mathbb{N}_0}$, satisfying $p_{m-j} \in S^{m-j}_{1,0}$ and $p_{m-j}(x,t\xi) = t^{m-j}p_{m-j}(x,\xi)$ for all $x,\xi \in \mathbb{R}^n$, $j \in \mathbb{N}_0$, and furthermore such that

$$p - \sum_{j=0}^{N} p_{m-j} \in S^{m-N}_{1,0} \quad \forall N \in \mathbb{N}_0 \tag{2.165}$$

For any $j \in \mathbb{N}_0$, it holds true that

$$\begin{aligned}\mathcal{F}^{-1}_{\xi\mapsto y}p_{m-j}(x_0,x_0-y) = \int_{\mathbb{R}^n} p_{m-j}(x_0,\xi)e^{i(x_0-y)\cdot\xi}d\xi = \\ |x_0-y|^{-n-m-j}\int_{\mathbb{R}^n} p_{m-j}(x_0,\psi)e^{i\psi\cdot(x_0-y)/|x_0-y|}d\psi \quad \forall y \in \mathbb{R}^n\end{aligned} \tag{2.166}$$

In view of this last result, as well as referring to the relation (2.123), we see that the following is implied by the assumption in (2.163):

$$\int_{\mathbb{R}^n} p_m(x_0,\psi)e^{i\psi\cdot(x_0-y)/|x_0-y|}d\psi \propto 1 \quad \forall \psi \in \mathbb{R}^n \tag{2.167}$$

or equivalently, after a transformation of the variables,

$$\int_{\mathbb{R}^n} |x_0-y|^{m+n}p_m(x_0,\xi)e^{i\xi\cdot(x_0-y)}d\xi \propto 1 \quad \forall y \in \mathbb{R}^n \tag{2.168}$$

Using the already derived relation (2.127), we infer from (2.168) that

$$(-\Delta_\xi)^{(m+n)/2}p_m(x_0,\cdot) \propto \delta_D \tag{2.169}$$

The result follows.

■

The ellipticity property of a pseudodifferential operator will prove to be a notion of fundamental significance in an analysis of existence and uniqueness of solutions to boundary value problems with smooth geometry.

DEFINITION 2.5.20
Let $p(x,D),\ q(x,D)$ be pseudodifferential operators on $\Omega \subseteq \mathbb{R}^n$. If

$$p(x,D)q(x,D) \sim I, \quad q(x,D)p(x,D) \sim I \tag{2.170}$$

then $q(x,D)$ is called a *two-sided parametrix* for $p(x,D)$. Accordingly, the function q is referred to as a *two-sided parametrix symbol* for p.

□

Clearly, if $q(x,D)$ is a two-sided parametrix for $p(x,D)$, then necessarily $p \circ q \sim 1$ and $q \circ p \sim 1$. We now show an important property of elliptic pseudodifferential operators.

PROPOSITION 2.5.21
Given $m \in \mathbb{R}$, $\rho, \delta \in [0,1]$, $0 \leq \delta < \rho \leq 1$, let $p(x,D) \in OPS^m_{\rho,\delta}(\Omega)$ be a pseudodifferential operator. There exists a two-sided parametrix for $p(x,D)$ iff the latter operator is elliptic.
Proof: Let $\psi_r \in C^\infty(\mathbb{R}^n)$ be such that

$$\psi_r(\xi) = \begin{cases} 0, & |\xi| \leq r \\ 1, & |\xi| \geq 2r \end{cases} \tag{2.171}$$

where the constant $r \in \mathbb{R}$ is asssociated with $p(x,D)$ as described in Definition 2.5.18. Define the product function $q_0 \equiv \psi_r p^{-1}$. By Proposition 2.5.16, it holds true that

$$p \circ q_0 \sim \sum_{|\alpha| \geq 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi p D^\alpha_x q_0 = \psi + r_0 \tag{2.172}$$

with

$$r_0 \equiv \sum_{|\alpha| > 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi p D^\alpha_x q_0 \tag{2.173}$$

But since $\psi - 1 \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\psi + r_0 = 1 + r_0 + (\psi - 1) \sim 1 + r_0 \tag{2.174}$$

and hence $p \circ q_0 \sim 1 + r_0$. Using the formula of Leibnitz, as well as the expression from (2.173) for the function r_0, we write

$$
\begin{aligned}
|D_x^\beta D_\xi^\alpha r_0(x,\xi)| = &\left| \sum_{\substack{|\alpha'|>0 \\ a+b=\alpha \\ c+d=\beta}} \frac{i^{|\alpha'|}}{\alpha'!} \binom{\alpha}{a} \binom{\beta}{c} D_x^c D_\xi^{\alpha'+a} p(x,\xi) D_x^{\alpha'+d} D_\xi^b q_0(x,\xi) \right| \leq \\
&\sum_{\substack{|\alpha'|>0 \\ a+b=\alpha \\ c+d=\beta}} \frac{1}{\alpha'!} \binom{\alpha}{a} \binom{\beta}{c} |D_x^c D_\xi^{\alpha'+a} p(x,\xi)| |D_x^{\alpha'+d} D_\xi^b q_0(x,\xi)| \leq \\
&\sum_{\substack{|\alpha'|>0 \\ a+b=\alpha \\ c+d=\beta}} \frac{1}{\alpha'!} \binom{\alpha}{a} \binom{\beta}{c} C_{\alpha',a,b,c,d} \langle\xi\rangle^{-\rho(|\alpha'+a|+|b|)+\delta(|\alpha'+d|+|c|)} = \\
&\sum_{\substack{|\alpha'|>0 \\ a+b=\alpha \\ c+d=\beta}} \frac{1}{\alpha'!} \binom{\alpha}{a} \binom{\beta}{c} C_{\alpha',a,b,c,d} \langle\xi\rangle^{(\delta-\rho)|\alpha'|-\rho|\alpha|+\delta|\beta|} \leq \\
&\left\{ \sum_{\substack{|\alpha'|>0 \\ a+b=\alpha \\ c+d=\beta}} \frac{1}{\alpha'!} \binom{\alpha}{a} \binom{\beta}{c} C_{\alpha',a,b,c,d} \right\} \langle\xi\rangle^{\delta-\rho-\rho|\alpha|+\delta|\beta|}
\end{aligned}
\tag{2.175}
$$

Hence, $r_0 \in S^{\delta-\rho}_{\rho,\delta}$. An entirely similar analysis yields $q_0 \circ p \sim 1 + \widetilde{r}_0$ with

$$
\widetilde{r}_0 = \sum_{|\alpha|>0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha q_0 D_x^\alpha p \in S^{\delta-\rho}_{\rho,\delta} \tag{2.176}
$$

We have thus arrived at

$$
p(x,D)q_0(x,D) \sim I + r_0(x,D), \quad q_0(x,D)p(x,D) \sim I + \widetilde{r}_0(x,D) \tag{2.177}
$$

Let us now formally define the operators $s(x,D)$, $q(x,D)$ by

$$
s(x,D) \equiv \sum_{j\in\mathbb{N}} (-1)^j \widetilde{r}_0^j(x,D), \; q(x,D) \equiv (I + s(x,D)) q_0(x,D) \tag{2.178}
$$

The operators $\widetilde{s}(x,D)$, $\widetilde{q}(x,D)$ are defined in a similar manner:

$$
\widetilde{s}(x,D) \equiv \sum_{j\in\mathbb{N}} (-1)^j r_0^j(x,D), \; \widetilde{q}(x,D) \equiv q_0(x,D)(I + \widetilde{s}(x,D)) \tag{2.179}
$$

It holds true that

$$\begin{aligned}
q(x,D)p(x,D) =&(I+s(x,D))q_0(x,D)p(x,D) = I+\widetilde{r}_0(x,D)+\\
&\sum_{j\in\mathbb{N}}(-1)^j\widetilde{r}_0^j(x,D)+\sum_{j=2}^{\infty}(-1)^{j-1}\widetilde{r}_0^j(x,D) \mod OPS^{-\infty} =\\
&I \mod OPS^{-\infty}\\
p(x,D)\widetilde{q}(x,D) =&p(x,D)q_0(x,D)(I+\widetilde{s}(x,D)) = I+\sum_{j\in\mathbb{N}}(-1)^j r_0^j(x,D)+\\
&r_0(x,D)+\sum_{j=2}^{\infty}(-1)^{j-1}r_0^j(x,D) \mod OPS^{-\infty} =\\
&I \mod OPS^{-\infty}
\end{aligned} \tag{2.180}$$

The above relations imply that

$$\begin{aligned}
(q(x,D)p(x,D))\widetilde{q}(x,D) \sim \widetilde{q}(x,D)\\
q(x,D)(p(x,D)\widetilde{q}(x,D)) \sim q(x,D)
\end{aligned} \tag{2.181}$$

The associativity property for operators now yields $q(x,D) \equiv \widetilde{q}(x,D)$. By virtue of the results in (2.180), the operator $q(x,D)$ is the desired two-sided parametrix for $p(x,D)$. The proof is complete.

■

EXAMPLE 2.5.22
We are now in a position to demonstrate an application of the theory of pseudodifferential operators on equations of the type

$$p(x,D)u = f, \quad u,f\in\mathcal{S}'(\mathbb{R}^n),\ p \text{ elliptic in } OPS^m_{\rho,\delta},\ 0\le\delta<\rho\le 1 \tag{2.182}$$

If $q(x,D)$ is a two-sided parametrix for $p(x,D)$, then necessarily

$$q(x,D)f = q(x,D)p(x,D)u = u + r(x,D)u, \quad r(x,D)\in OPS^{-\infty} \tag{2.183}$$

Since $r(x,D)\in OPS^{-\infty}$ implies $r(x,D):\mathcal{S}'(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)$, we have

$$u = q(x,D)f \mod C^\infty(\mathbb{R}^n) \tag{2.184}$$

□

LEMMA 2.5.23
Given $m,\rho,\delta\in\mathbb{R}$ satisfying $\rho>0$, $m<-n+\rho(n-1)$, any pseudodifferential operator $p(x,D)\in OPS^m_{\rho,\delta}$ maps $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $p\in[1,\infty[$. In particular, if $\rho=1$, then the above statement holds for any $m<0$.
Proof: Under the first set of assumptions, the kernel K of the operator $p(x,D)$ satisfies the estimates

$$\begin{aligned}
|K(x,y)| \le C_N|x-y|^N \quad \forall x,y\in\mathbb{R}^n,\ |x-y|\ge 1,\\
|K(x,y)| \le C|x-y|^{1-n} \quad \forall x,y\in\mathbb{R}^n,\ |x-y|\le 1
\end{aligned} \tag{2.185}$$

Furthermore, if $\rho = 1$, we know from Remark 2.5.11 that the following estimate holds:

$$|K(x,y)| \leq C|x-y|^{-n-m} \quad \forall x, y \in \mathbb{R}^n \tag{2.186}$$

Application of Proposition 2.5.14 with $X = \mathbb{R}^n$ immediately yields the asserted results.

■

LEMMA 2.5.24
Given $\rho, \delta, m \in \mathbb{R}$ such that $m > 0$ and $0 \leq \delta < \rho \leq 1$, a pseudodifferential operator $p(x,D) \in OPS^m_{\rho,\delta}$ maps $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.
Proof: It is sufficient to establish the boundedness of $(p(x,D)^* p(x,D))^N$ on $L^2(\mathbb{R}^n)$ for some $N \in \mathbb{N}$, since of course

$$\begin{aligned} ||p(x,D)^N u||^2_{L^2(\mathbb{R}^n)} =& (p(x,D)^N u, p(x,D)^N u) = |((p(x,D)^* p(x,D))^N u, u)| \leq \\ & ||(p(x,D)^* p(x,D))^N u||_{L^2(\mathbb{R}^n)} ||u||_{L^2(\mathbb{R}^n)} \end{aligned} \tag{2.187}$$

for all $u \in L^2(\mathbb{R}^n)$, $N \in \mathbb{N}$, and $||p(x,D)|| \leq ||p(x,D)^N||^{1/N}$ for any $N \in \mathbb{N}$. As mentioned in Proposition 2.5.16, it is true that

$$(p(x,D)^* p(x,D))^N \in OPS^{-2Nm}_{\rho,\delta} \tag{2.188}$$

for any $N \in \mathbb{N}$. By Lemma 2.5.23, if N is chosen such that

$$N > \frac{1}{2m}\left\{n - \rho(n-1)\right\} \tag{2.189}$$

then the operator $(p(x,D)^* p(x,D))^N$ maps $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

■

THEOREM 2.5.25
Given ρ, $\delta \in \mathbb{R}$ such that $0 \leq \delta < \rho \leq 1$, an operator in $OPS^0_{\rho,\delta}$ maps $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.
Proof: By assumption and Proposition 2.5.16, the product operator

$$q(x,D) \equiv p(x,D)^* p(x,D) \tag{2.190}$$

belongs to the space $OPS^0_{\rho,\delta}$, and therefore the pertaining symbol $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is in $L^\infty(\mathbb{R}^{2n})$. Define $M, b \in \mathbb{R}$ such that $b > 0$ and $|q(x,\xi)| \leq M - b$ for all $x, \xi \in \mathbb{R}^n$. Then it holds true that $M - \Re\{q(x,\xi)\} \geq M - |q(x,\xi)| \geq b$ for all $x, \xi \in \mathbb{R}^n$. Hence, the symbol A defined by

$$A(x,\xi) \equiv (M - \Re\{q(x,\xi)\})^{1/2} \quad \forall x, \xi \in \mathbb{R}^n \tag{2.191}$$

belongs to $S^0_{\rho,\delta}$. Moreover,

$$A(x,D)^* A(x,D) = M - q(x,D) + r(x,D) \tag{2.192}$$

where $r(x,D) \in OPS^{\delta-\rho}_{\rho,\delta}$. The following holds true:

$$\begin{aligned} M||u||^2_{L^2(\mathbb{R}^n)} - ||p(x,D)u||^2_{L^2(x,D)} = ((M - p(x,D)^* p(x,D))u, u) = \\ ((A(x,D)^* A(x,D) - r(x,D))u, u) \geq -|(r(x,D)u, u)| \geq -C||u||^2_{L^2(\mathbb{R}^n)} \end{aligned} \tag{2.193}$$

Hence, it is true that

$$||p(x,D)u||_{L^2(\mathbb{R}^n)} \leq (M+C)^{1/2}||u||_{L^2(\mathbb{R}^n)} \quad \forall u \in L^2(\mathbb{R}^n) \tag{2.194}$$

and the proof is complete.

■

The next proposition is crucial for regularity analysis of solutions to PDEs (hereunder solutions to scattering problems) using the theory of pseudodifferential operators.

PROPOSITION 2.5.26
For $m,\, s \in \mathbb{R}$, $\rho, \delta \in [0,1]$, $0 \leq \delta < \rho \leq 1$, a pseudodifferential operator in $OPS^m_{\rho,\delta}(\mathbb{R}^n)$ maps the space $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$.[5]
Proof: Given $u \in H^s(\mathbb{R}^n)$, there exists $w \in L^2(\mathbb{R}^n)$ such that $u = \Lambda^{-s}w$. Hence, it holds true that

$$\Lambda^{s-m}p(x,D)u = \Lambda^{s-m}p(x,D)\Lambda^{-s}w \tag{2.195}$$

But since $\Lambda^{s'} \in OPS^{s'}$ for all $s' \in \mathbb{R}$, we have from the result (2.160) on products of pseudodifferential operators that

$$\Lambda^{s-m}p(x,D)\Lambda^{-s} \in OPS^0 \tag{2.196}$$

By Theorem 2.5.25, it follows that $\Lambda^{s-m}p(x,D)u = \Lambda^{s-m}p(x,D)\Lambda^{-s}w \in L^2(\mathbb{R}^n)$. This completes the proof.

■

In light of the above result, Proposition 2.5.13 specifies a method for determining the mapping properties of a pseudodifferential operator from the properties of the associated kernel.

REMARK 2.5.27
Note that if $P = OP(p) \in OPS^{-\infty}(\Omega, \mathbb{R}^n)$ then $p \in \mathcal{S}(\mathbb{R}^n)$ as a function of ξ.
Proof: By assumption, it holds true that for any α there exists a constant $C_\alpha < \infty$ such that

$$|\xi^\beta D^\alpha_\xi p(x,\xi)| \leq |\xi|^\beta C_\alpha \langle\xi\rangle^N \tag{2.197}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}^n$, $N \in \mathbb{R}$. We find the following asymptotic equivalence for all $(x,\xi) \in \Omega \times \mathbb{R}^n$, $N \in \mathbb{R}$ and all α, β:

$$|\xi|^\beta C_\alpha \langle\xi\rangle^N \underset{|\xi|\gg 1}{\sim} C_\alpha \sqrt{n}(\max_j |\xi_j|)^{N+n\max_j \beta_j} \tag{2.198}$$

Hence, $\xi^\beta D^\alpha_\xi p \in L^\infty(\mathbb{R}^n)$ with respect to the variable ξ, as asserted.

■

REMARK 2.5.28
In view of Remark 2.5.27 and Proposition 2.5.7, we infer that the kernel of a negligible operator belongs to the Schwartz space $\mathcal{S}$. Consequently, every negligible operator is a compact integral operator.

□

[5] Note the error in the statement of this proposition in Reference [36, p. 18].

2.6 Fredholm Operators

In this section, the fundamentals of the classical theory of Fredholm operators are presented. Together with the apparatus of the pseudodifferential operator theory, this constitutes a powerful tool in the analysis of the existence and uniqueness of solutions to boundary value problems on domains with smooth boundary. As it turns out, the concept of a Fredholm operator of index zero is of particular utility, in that it greatly simplifies the establishing of the existence and uniqueness of solution of boundary value problems considered in this text. The principal reference used in this section is [35].

A fairly recent development [14, 22] is the realisation that the Fredholm operator theory may furthermore be employed in an analysis of problems involving Lipschitz domains, if used in conjunction with the so-called Rellich energy estimates (see Chapter 3.)

We begin by stating a number of preliminary results.

LEMMA 2.6.1
For any Banach space U, the space of compact operators $\mathcal{K}(U)$ is a closed two-sided ideal in the ring of bounded linear operators $\mathcal{L}(U)$.
Proof: Let $(u_n)_{n\in\mathbb{N}} \subset U$ be a convergent sequence, and consider $S \in \mathcal{L}(U)$. Since S is bounded, there exists a constant $C > 0$ such that

$$||Su_n - Su_m||_U \leq C||u_n - u_m||_U \quad \forall m, n \in \mathbb{N} \tag{2.199}$$

The sequence $(Su_n)_{n\in\mathbb{N}} \subset U$ is Cauchy and thus convergent in U. Furthermore, consider some bounded set $Q \subset U$. Again due to the boundedness of the operator S, it holds true that

$$||Su||_U \leq C||u||_U \quad \forall u \in Q \tag{2.200}$$

and thus $S(Q)$ is a bounded set.

■

LEMMA 2.6.2
All eigenspaces of a compact operator, which correspond to nonzero eigenvalues, are finite-dimensional.
Proof: Given a compact operator A, let us assume that there exists an infinite-dimensional eigenspace EV_λ of that operator (associated with the eigenvalue $\lambda \neq 0$.) The set of basis vectors of EV_λ constitutes a bounded subset of the domain of A, but it is not relatively compact, i.e. it does not have compact closure.

■

PROPOSITION 2.6.3
For Banach spaces U, V, W, let $T \in \mathcal{L}(U, V)$, $K \in \mathcal{K}(U, W)$, and assume that there exist constants C_1, C_2 such that

$$||u||_U \leq C_1||Tu||_V + C_2||Ku||_W \quad \forall u \in U \tag{2.201}$$

Then the operator T has closed range.
Proof: Consider a converging sequence $(Tu_n)_{n\in\mathbb{N}} \subset V$, $\lim_{n\to\infty}\{Tu_n\} = f \in V$. If dist$(u_n, \ker\{T\})$ is bounded for all $n \in \mathbb{N}$ by some constant C, we consider a sequence $(v_n)_{n\in\mathbb{N}} \subset V$ characterised by $v_n \equiv u_n \bmod \ker\{T\}$, $||v_n|| \leq 2C$, $n \in \mathbb{N}$. It holds

true that $Tv_n = Tu_n$ for all $n \in \mathbb{N}$, so $\lim_{n\to\infty}\{Tv_n\} = f$. Also, $\lim_{n\to\infty}\{Kv_n\} = g \in W$. From the assumption (2.201), we obtain

$$||v_n - v_m||_U \leq C_1||Tv_n - Tv_m||_V + C_2||Kv_n - Kv_m||_W \quad \forall m, n \in \mathbb{N} \quad (2.202)$$

Hence, the sequence $(v_n)_{n\in\mathbb{N}}$ is obviously Cauchy, and there exists $v \in V$ satisfying $\lim_{n\to\infty}\{v_n\} = v$ and consequently $Tv = f$.

If, on the other hand, the sequence $\text{dist}(u_n, \ker\{T\})$, $n \in \mathbb{N}$, is unbounded, we may assume that $\text{dist}(u_n, \ker\{T\}) \geq 2$ for all $n \in \mathbb{N}$. Consider a sequence $(v_n)_{n\in\mathbb{N}} \subset V$ characterised by

$$v_n = u_n \mod \ker\{T\},\ \text{dist}(u_n, \ker\{T\}) \leq ||v_n|| \leq \text{dist}(u_n, \ker\{T\}) + 1 \quad \forall n \in \mathbb{N} \quad (2.203)$$

Also, define $w_n = v_n/||v_n||$ for all $n \in \mathbb{N}$. It holds true that $\text{dist}(w_n, \ker\{T\}) \geq 1/2$ for all $n \in \mathbb{N}$. The operator K is compact, and the set $\{w_n\}_{n\in\mathbb{N}} \subset V$ is bounded, hence the set $\{Kw_n\}_{n\in\mathbb{N}} \subset W$ is relatively compact. Consequently, there exists a subsequence $(\widetilde{w}_n)_{n\in\mathbb{N}} \subset (w_n)_{n\in\mathbb{N}}$ such that $\lim_{n\in\mathbb{N}}\{T\widetilde{w}_n\} \in W$. Furthermore, we have that $\lim_{n\to\infty}\{Tw_n\} = 0$. An application of the assumption (2.201) similar to what was done earlier thus implies that $\lim_{n\to\mathbb{N}}\{w_n\} \in V$. But then $T\lim_{n\to\infty}\{Tw_n\} = 0$ and $\text{dist}(\lim_{n\to\infty}\{Tw_n\}, \ker\{T\}) \geq 1/2$, which is a contradiction.

■

DEFINITION 2.6.4
A set X is referred to as *nowhere dense* iff

$$\overline{X}^\circ = \emptyset \quad (2.204)$$

□

We shall need the following classical topological result.

PROPOSITION 2.6.5
The Baire Category Theorem
Let U be a complete metric space or a locally compact Hausdorff space. We assume U is nonempty. Furthermore, let $(U_j)_{j\in\mathbb{N}}$ be a sequence of nowhere dense subsets of U. It holds true that

$$\bigcup_{j\in\mathbb{N}} U_j \neq U \quad (2.205)$$

Proof: For any $p \in U$, $r > 0$ denote by $B(p, r)$ the closed ball in U with centre at $p \in U$ and of radius r. Since, by assumption, $\overline{U}_1^\circ = \emptyset$, we have $\overline{U}_1 \neq U$. Accordingly, the set $U\backslash\overline{U}_1$ is open, so there exist $p_1 \in U$ and $r_1 > 0$ such that $B(p_1, r_1)$ is a subset of $U\backslash U_1$. Consequently, there are $p_2 \in U$ and $r_2 \in]0, r_1/2]$ such that $B(p_2, r_2) \subset B(p_1, r_1)\backslash U_2$. In general, for every $j \in \mathbb{N}\backslash\{1\}$ there exist $p_j \in U$, $r_j \in]0, 2^{1-j}r_1]$ such that

$$B(p_j, r_j) \subset B(p_{j-1}, r_{j-1})\backslash U_j \quad (2.206)$$

The sequence $(p_j)_{j\in\mathbb{N}} \subset U$ is Cauchy and hence converges to some $p \in U$. Since $p \in B(p_j, r_j)$ for all $j \in \mathbb{N}$, it is true that $p \notin \cup_{j\in\mathbb{N}} U_j$.

■

PROPOSITION 2.6.6
Open Mapping Theorem
If U, V are Banach spaces and the operator $T \in \mathcal{L}(U, V)$ is surjective, then T maps any neighbourhood of 0 in U onto a neighbourhood of 0 in V.
Proof: See, e.g., Reference [35].

■

There is a useful corollary to the open mapping theorem:

COROLLARY 2.6.7
If U, V are Banach spaces and $T \in \mathcal{L}(U, V)$ is bijective, then $T^{-1} : V \to U$ is continuous, so T is an isomorphism.
Proof: Since T^{-1} is continuous, there exists a nonzero constant C such that $||T^{-1}v||_U \leq C||v||_V$ for all $v \in V$. Furthermore, T is bijective, so $||u||_U \leq C||Tu||_V$ for all $u \in U$. Finally, the operator T is bounded.

■

The notion of a Fredholm operator is now introduced.

DEFINITION 2.6.8
Let U, V be Banach spaces and let $T \in \mathcal{L}(U, V)$. Then T is referred to as a *Fredholm operator* (denoted $T \in \text{Fred}(U, V)$) iff it holds true that

$$\dim\{\ker\{T\}\} < \infty,\ T(V) = \overline{T(V)},\ \text{Codim}\{T(V)\} < \infty \tag{2.207}$$

The *index* $\text{Ind}\{T\}$ of the operator T is defined by

$$\text{Ind}\{T\} \equiv \dim\{\ker\{T\}\} - \text{Codim}\{T(V)\} \tag{2.208}$$

□

As a simple example, the operator zI defined on a Banach space is a Fredholm operator with index zero for all $z \in \mathbb{C}\backslash\{0\}$. Indeed, zI is bijective for $z \in \mathbb{C}\backslash\{0\}$ and, the domain of definition being a Banach space, it is true that $R\{zI\} = \overline{R\{zI\}}$.

REMARK 2.6.9
Let U, V be Hilbert spaces. It is an elementary result that, for any $T \in \mathcal{L}(U, V)$, we have

$$\ker\{T^*\} = R\{T\}^{\perp}, \quad \ker\{T\} = R\{T^*\}^{\perp} \tag{2.209}$$

Hence, the following holds true:

$$\text{Ind}\{T\} = \dim\{\ker\{T\}\} - \dim\{\ker\{T^*\}\} \quad \forall T \in \text{Fred}(U, V) \tag{2.210}$$

and

$$T \in \text{Fred}(U, V) \Rightarrow T^* \in \text{Fred}(V, U),\ \text{Ind}\{T^*\} = -\text{Ind}\{T\} \tag{2.211}$$

□

COROLLARY 2.6.10
Let $p(x, D)$ be a pseudodifferential Fredholm operator with a real-valued symbol p, i.e. with $p(x, \xi) = \overline{p(x, \xi)}$ for all $x, \xi \in \mathbb{R}^n$. Then $\text{Ind}\{p(x, D)\} = 0$.
Proof: The result follows from Proposition 3.4.1 and Remark 2.6.9. We see that, by assumption, $p(x, D)$ is self-adjoint, and hence it holds true that

$$\text{Ind}\{p(x, D)\} = -\text{Ind}\{p(x, D)\} = 0 \tag{2.212}$$

■

It is clear that a Fredholm operator with index zero is injective iff it is surjective. This fact significantly simplifies the analysis of the existence and uniqueness properties of solutions to a particular class of boundary value problems. Indeed, if a boundary value problem can be reduced to an equation of the type

$$Tu = f, \quad T \text{ Fredholm of index zero} \tag{2.213}$$

then existence and uniqueness of solution to (2.213) are equivalent properties, and it is sufficient only to prove one of them.

The following result on Fredholm operators finds direct application in the classical analysis of boundary value problems, where it plays a significant role.

PROPOSITION 2.6.11
Given Banach spaces U, V and an operator $T \in \mathcal{L}(U, V)$, it holds true that

$$T \in \text{Fred}(U, V) \tag{2.214}$$

$$\Updownarrow$$

$$\exists S_1,\ S_2 \in \mathcal{L}(V, U) :\ S_1 T = I + K_1,\ TS_2 = I + K_2, \quad K_1, K_2 \text{ compact} \tag{2.215}$$

Proof: Assume firstly that (2.215) is true. Clearly, we then have $\ker\{T\} \subset \ker\{I + K_1\}$, and therefore $\dim\{\ker\{T\}\} < \infty$ by Lemma 2.6.2. By the first assumption in (2.215), as well as the triangle inequality and the definition of operator norm, it is true that

$$||u||_U = ||S_1 Tu - K_1 u||_U \leq ||S_1||_{\mathcal{L}(V,U)} ||Tu||_U + ||K_1 u||_U \quad \forall u \in U \tag{2.216}$$

Consequently, by Proposition 2.6.3, we have $R\{T\} = \overline{R\{T\}}$. Also, the second assumption in (2.215) implies $R\{I + K_2\} \subset T(V)$. It holds true that Codim$\{R\{I + K_2\}\}$.[35]

Conversely, assume that (2.214) is true. In this case, the following decomposition of the spaces U, V is in order:

$$U = U_1 \oplus \ker\{T\},\ V = R\{T\} \oplus V_0 \tag{2.217}$$

where $R\{T\} = \overline{R\{T\}}$ and the spaces $\ker\{T\}, V_0$ are finite-dimensional. Hence, if we define the orthogonal projections $K_1 : U \to \ker\{T\}$, $K_2 : V \to V_0$, these operators are of finite rank and therefore compact. Furthermore, it is clear that the restriction $\widetilde{T} : U_1 \to R\{T\}$ of T to the domain U_1 is bijective, so there exist the pertaining left and right inverse operators $S_1,\ \hat{S}_2 \in \mathcal{L}(R\{T\}, U_1)$ defined by

$$S_1 \widetilde{T} u = u \ \ \forall u \in U_1, \quad \widetilde{T} \hat{S}_2 v = v \ \ \forall v \in R\{T\} \tag{2.218}$$

We now introduce an extension $S_2 : V \to U_1$ of the operator $\hat{S}_2$:

$$S_2 v = \begin{cases} \hat{S}_2 v & v \in R\{T\} \\ 0 & \text{otherwise} \end{cases} \tag{2.219}$$

The following is seen to hold true:

$$\begin{aligned} S_1 Tu &= u - K_1 u \quad \forall u \in U \\ TS_2 v &= v - K_2 v \quad \forall v \in V \end{aligned} \tag{2.220}$$

This completes the proof.

■

If $T \in \mathrm{Fred}(U, V)$, the associated operators S_1, S_2 are called the *Fredholm inverses* of T. Note that, provided (2.214) and (2.215) hold true, we have

$$S_1(I + K_2) = S_1 T S_2,\ (I + K_1)S_2 = S_1 T S_2 \tag{2.221}$$

which implies

$$S_1 - S_2 = K_1 S_2 - S_1 K_2 \tag{2.222}$$

Thus, by Lemma 2.6.1, the operators $S_1,\ S_2$ differ by a compact operator.

For any Banach space U, there exists a natural algebra homomorphism $\pi : \mathcal{L}(U) \mapsto \mathcal{L}(U)/\mathcal{K}(U)$. The quotient space $\mathcal{L}(U)/\mathcal{K}(U)$ is called the *Calkin algebra*.

COROLLARY 2.6.12
If U, V are Banach spaces, an operator $T \in \mathcal{L}(U, V)$ is an element of $\mathrm{Fred}(U, V)$ iff T has a left inverse in $\mathcal{L}(U)/\mathcal{K}(U)$ and a right inverse in $\mathcal{L}(V)/\mathcal{K}(V)$. In particular, an operator $T \in \mathcal{L}(U)$, where U is a Banach space, is in $\mathrm{Fred}(U)$ iff it is invertible in $\mathcal{L}(U)/\mathcal{K}(U)$.

□

COROLLARY 2.6.13
Let U, V, W be Banach spaces. If operators T_1, T_2, K satisfy $T_1 \in \mathrm{Fred}(U, V)$, $T_2 \in \mathrm{Fred}(V, W)$, $K \in \mathcal{K}(U, V)$, then it holds true that

$$T_1 + K \in \mathrm{Fred}(U, V),\ T_2 T_1 \in \mathrm{Fred}(U, W) \tag{2.223}$$

Proof: By Proposition 2.6.11, there exist operators $S_1^{(1)}, S_2^{(1)} \in \mathcal{L}(V, U),\ S_1^{(2)}, S_2^{(2)} \in \mathcal{L}(W, V)$ such that

$$\begin{aligned} S_1^{(1)} T_1 = I + K_1^{(1)}, \quad T_1 S_2^{(1)} = I + K_2^{(1)} \\ S_1^{(2)} T_2 = I + K_1^{(2)}, \quad T_2 S_2^{(2)} = I + K_2^{(2)} \end{aligned} \tag{2.224}$$

where $K_i^{(j)}$, $i, j \in \{1, 2\}$, are compact operators. Hence we may write

$$\begin{aligned} S_1^{(1)}(T_1 + K) = I + K_1^{(1)} + S_1^{(1)} K, \\ (T_1 + K)S_2^{(1)} = I + K_2^{(1)} + K S_2^{(1)} \end{aligned} \tag{2.225}$$

as well as

$$\begin{aligned} S_1^{(1)} S_1^{(2)} T_2 T_1 = S_1^{(1)}(I + K_1^{(2)})T_1 = I + K_1^{(1)} + S_1^{(1)} K_1^{(2)} T_1, \\ T_2 T_1 S_2^{(1)} S_2^{(2)} = T_2(I + K_2^{(1)})S_2^{(2)} = I + K_2^{(2)} + T_2 K_2^{(1)} S_2^{(2)} \end{aligned} \tag{2.226}$$

The result now follows from Lemma 2.6.1 and Proposition 2.6.11.

■

PROPOSITION 2.6.14
The set of invertible elements of a Banach algebra $\mathcal{A}$ with unit operator is open.
Proof: Let T, S be operators on $\mathcal{A}$ such that T is invertible and $||T - S|| < ||T^{-1}||^{-1}$ in operator norm. By assumption, $||T^{-1}(T - S)|| < 1$. Also, it holds true that $S = S + T - T = T(I - T^{-1}(T - S))$, and hence the operator $S^{-1} = (I - T^{-1}(T - S))^{-1} T^{-1}$ is well-defined on $\mathcal{A}$.

■

COROLLARY 2.6.15
If U, V denote Banach spaces, the space Fred(U, V) is open.
Proof: The Calkin algebra $\mathcal{L}(U, V)/\mathcal{K}(U, V)$ is a Banach algebra with unit operator.

■

PROPOSITION 2.6.16
A Stability Theorem for the Index of Fredholm Operators
Let U, V be Banach spaces. The index mapping

$$\text{Ind}\{T\} = \dim\{\ker\{T\}\} - \text{Codim}\{R\{T\}\} \quad \forall T \in \text{Fred}(U, V) \tag{2.227}$$

is constant on every connected component of the set Fred(U, V) of Fredholm operators.
Proof: Consider some $T \in$ Fred(U, V). By definition 2.6.8 of Fredholm operators, we may write

$$U = U_1 \oplus \ker\{T\}, \; V = R\{T\} \oplus V_0 \tag{2.228}$$

where the spaces $\ker\{T\}, V_0$ are finite-dimensional. Next, for any $S \in \mathcal{L}(U, V)$ introduce the bilinear form $\tau_S : U_1 \otimes V_0 \to V$ by

$$\tau_S(u, v) \equiv Su + v \quad \forall (u, v) \in U_1 \otimes V_0 \tag{2.229}$$

By construction, the form τ_T is bijective, so by Corollary 2.6.7, it is an isomorphism. Furthermore, by Proposition 2.6.15, there exists $\varepsilon > 0$ satisfying

$$||T - S||_{\mathcal{L}(U,V)} < \varepsilon \Rightarrow S \in \text{Fred}(U, V) \quad \forall S \in \mathcal{L}(U, V) \tag{2.230}$$

In the rest of this proof, we assume the operator $S \in \mathcal{L}(U, V)$ is such that $||T - S||_{\mathcal{L}(U,V)} < \varepsilon$. Then, in a similar manner as above, it is seen that τ_S is an isomorphism. It is true that

$$\tau_S(U_1) = S(U_1) = \overline{S(U_1)}, \quad \text{Codim}\{S(U_1)\} = \dim\{V_0\} \tag{2.231}$$

Also, the following expression of semicontinuity is valid [17]:

$$\text{Codim}\{S(U)\} \leq \text{Codim}\{T(U)\} \tag{2.232}$$

Now since

$$\ker\{S\} \cap U_1 = \emptyset \tag{2.233}$$

it holds true that

$$U = U_1 \oplus Z \oplus \ker\{S\} \tag{2.234}$$

for some finite-dimensional space $Z \subset U$. Obviously, we have

$$\dim\{\ker\{S\}\} + \dim\{Z\} = \dim\{\ker\{T\}\} \tag{2.235}$$

By construction, S is injective on $U_1 \oplus Z$. Also, since S is Fredholm, we have $S(U) = \overline{S(U)}$, Codim$\{S(U)\} < \infty$. Hence, it is true that

$$\text{Codim}\{S(U)\} = \text{Codim}\{T(U)\} - \dim\{S(Z)\} \tag{2.236}$$

But since $\dim\{S(Z)\} = \dim\{Z\}$, we may combine equations (2.235) and (2.236) to arrive at

$$\dim\{\ker\{S\}\} - \text{Codim}\{S(U)\} = \dim\{\ker\{T\}\} - \text{Codim}\{T(U)\} \tag{2.237}$$

or simply Ind$\{S\} =$ Ind$\{T\}$.

■

A specialisation of Proposition 2.6.16 yields the following useful result:

COROLLARY 2.6.17
If U, V are Banach spaces and given $T \in \text{Fred}(U, V)$, $K \in \mathcal{K}(U, V)$, it holds true that

$$\text{Ind}\{T\} = \text{Ind}\{T + K\} \tag{2.238}$$

Proof: The operators T and $T+K$ are in the same connected component of $\text{Fred}(U, V)$, since, by Proposition 2.6.13, we have

$$T + tK \in \text{Fred}(U, V) \quad \forall t \in [0, 1] \tag{2.239}$$

■

We now have the following fundamental result as a corollary:

COROLLARY 2.6.18
Fredholm's alternative
Let V be a Banach space, and assume $z \neq 0$, $K \in \mathcal{K}(V)$. Then the operator $zI - K$ is surjective iff it is injective.

■

PROPOSITION 2.6.19
Let U, V, W be Banach spaces. Then, for any $T \in \text{Fred}(U, V)$, $S \in \text{Fred}(V, W)$ it holds true that

$$\text{Ind}\{ST\} = \text{Ind}\{S\} + \text{Ind}\{T\} \tag{2.240}$$

Proof: Let us define the following family of operators:

$$P(t) \equiv \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \quad \forall t \in \mathbb{R} \tag{2.241}$$

Clearly, it is true that $P(t) \in \mathcal{L}(U \otimes V, V \otimes W)$ for all $t \in \mathbb{R}$. The operator

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \tag{2.242}$$

is Fredholm with index zero. Furthermore, we have $P(t) \in \text{Fred}(U \otimes V, V \otimes W)$ for all $t \in \mathbb{R}$. Consequently, the operators

$$P(0) = \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}, \quad P(-\frac{\pi}{2}) = \begin{bmatrix} 0 & -I \\ ST & 0 \end{bmatrix} \tag{2.243}$$

are in the same connected component of the set $\text{Fred}(U \otimes V, V \otimes W)$. It holds true that

$$\text{Ind}\{P(0)\} = \text{Ind}\{S\} + \text{Ind}\{T\}, \quad \text{Ind}\{P(-\frac{\pi}{2})\} = \text{Ind}\{ST\} \tag{2.244}$$

The result now follows by Proposition 2.6.16 on stability of the index of Fredholm operators.

■

Based on the material presented in this section, an appropriate qualitative characterisation of Fredholm operators seems to be '*almost bijective*'. While equations involving truly bijective operators exhibit the desirable properties of existence and uniqueness of solution, those comprising Fredholm operators allow for finite-dimensional spaces of homogeneous solutions, and in addition finite-dimensional spaces of data for which no solution exists. In particular, in view of the concept of the Fredholm inverse, we could loosely assert that Fredholm operators exhibit 'bijectivity modulo C^∞.'

2.7 The Hodge Decomposition

We need some preliminary work in order to derive the Hodge decomposition result.

PROPOSITION 2.7.1
The sesquilinear form $s_\Delta : (H^1(M,\Lambda^1))^2 \to \mathbb{C}$, defined by

$$s_\Delta(u,v) = -(\Delta u, v) \quad \forall u,v \in H^1(M,\Lambda^1) \tag{2.245}$$

is $H^1(M,\Lambda^1)$-coercive[6], i.e. there exist positive constants C_1, C_2 such that

$$s_\Delta(u,u) = -(\Delta u, u) \geq C_1 ||u||^2_{H^1(M,\Lambda^1)} - C_2 ||u||^2_{L^2(M,\Lambda^1)} \tag{2.246}$$

is true for all $u \in H^1(M,\Lambda^1)$.
Proof: We refer to [35].

■

The estimate

$$\exists C > 0 : \; ||(-\Delta + V)u||_{H^{-1}(M,\Lambda^l)} \geq C||u||_{H^1(M,\Lambda^l)} \; \forall u \in H^1(M,\Lambda^l) \tag{2.247}$$

holds.[35] The operator $\Delta + V : H^1(M,\Lambda^l) \to H^{-1}(M,\Lambda^l)$ is injective with closed range. Since the operator is self-adjoint, it is also surjective. Indeed, the following holds true:

$$\begin{gathered} (\varphi, (-\Delta + V)u) = 0 \quad \forall u \in H^1(M,\Lambda^l) \\ \Updownarrow \\ ((-\Delta + V)\varphi, u) = 0 \quad \forall u \in H^1(M,\Lambda^l) \\ \Updownarrow \\ \varphi \in \ker\{-\Delta + V\} \\ \Updownarrow \\ \varphi = 0 \end{gathered} \tag{2.248}$$

Hence, $-\Delta + V$ is bijective, and there exists a two-sided inverse $T : H^{-1}(M,\Lambda^l) \to H^1(M,\Lambda^l)$. Define $\varphi = (-\Delta + V)u$, $\psi = (-\Delta + V)v$ for some $u,v \in H^1_0(M,\Lambda^l)$. It holds true that

$$\begin{gathered} (T\varphi, \psi)_{L^2(M,\Lambda^l)} = (u, (-\Delta + V)v)_{L^2(M,\Lambda^l)} = \\ (u, d\delta v + \delta d v + Vv)_{L^2(M,\Lambda^l)} = ((-\Delta + V)u, v)_{L^2(M,\Lambda^l)} = \end{gathered} \tag{2.249}$$

In the above derivation, the fact has been used that the interior derivative operator δ is the formal adjoint of the exterior derivative operator d. Consequently, the operator T is self-adjoint. By Rellich's theorem, it is a compact self-adjoint operator on $L^2(M,\Lambda^l)$.[35] Hence, there exists in $L^2(M,\Lambda^l)$ an orthonormal basis of eigenvectors of T, specified by

$$Tu_j^{(V)} = \mu_j^{(V)} u_j^{(V)}, \quad u_j^{(V)} \in H^1(M,\Lambda^l) \tag{2.250}$$

$$0 < (Tu,u) \leq V^{-1} ||u||^2_{L^2(M,\Lambda^l)} \tag{2.251}$$

[6] In Chapter 4, we treat in more detail the concept of coercivity of sesquilinear forms.

Inserting $u = u_j^{(V)}$ into (2.251), it follows easily that

$$0 < \mu_j^{(V)} \leq V^{-1} \tag{2.252}$$

Since, by definition, $(-\Delta + V) = T^{-1}$, we see from the expression (2.250) that

$$-\Delta u_j^{(V)} = \left\{ \frac{1}{\mu_j^{(V)}} - V \right\} u_j^{(V)} \quad \forall j \tag{2.253}$$

Due to the relation (2.252), the following holds true for the eigenvalues $\lambda_j^{(V)} \equiv (\mu_j^{(V)})^{-1} - V$ of the Hodge Laplacian:

$$\lambda_j^{(V)} \geq 0 \tag{2.254}$$

$$u_j^{(V)} \in C^\infty(M, \Lambda^l) \tag{2.255}$$

DEFINITION 2.7.2
The kernel of the Hodge Laplacian operator on l-forms is referred to as the space of *harmonic l-forms*, and it is denoted by $\mathcal{H}_l$.[35]

□

LEMMA 2.7.3
The following holds true for all $l \in \mathbb{N}_0$:

$$u \in \mathcal{H}_l \Leftrightarrow du = 0,\, \delta u = 0 \text{ on M} \tag{2.256}$$

Proof: Assume that $u \in \mathcal{H}_l$. Then, necessarily, we have

$$0 = -(\Delta u, v) = (du, dv) + (\delta u, \delta v) \quad \forall u, v \in H^1(M, \Lambda^l) \tag{2.257}$$

Hence $||du||^2_{L^2(M,\Lambda^l)} + ||\delta u||^2_{L^2(M,\Lambda^l)} = 0$. The converse result is easily seen to be true.

■

We are now in a position to state the celebrated *Hodge decomposition* result.

PROPOSITION 2.7.4
The Hodge Decomposition on Smooth Boundaryless Riemannian Manifolds
Let M be a smooth boundaryless Riemannian manifold. If $u \in H^s(M, \Lambda^l),\ s \in \mathbb{R}, l \in \mathbb{N}$, then there exist differential forms

$$\alpha \in H^s(M, \Lambda^{l-1}),\, \beta \in H^s(M, \Lambda^{l+1}),\, \gamma \in \mathcal{H}_l \tag{2.258}$$

such that the decomposition

$$u = d\alpha + \delta\beta + \gamma \tag{2.259}$$

is valid. The terms in the right-hand side of equation (2.259) are mutually orthogonal in $L^2(M, \Lambda^l)$.

If $u \in H^s(M, \Lambda^0),\ s \in \mathbb{R}$, then there exist the differential forms

$$\beta \in H^s(M, \Lambda^1),\, \gamma \in \mathcal{H}_0 \tag{2.260}$$

such that $u = \delta\beta + \gamma$. The terms in the right-hand side of equation (2.260) are mutually orthogonal in $L^2(M, \Lambda^0)$.

Proof: Given $l \in \mathbb{N}_0$, denote by P_l the orthogonal projection operator mapping $L^2(M, \Lambda^l)$ onto the kernel $\mathcal{H}_l$ of the Hodge Laplacian on l-forms. Furthermore, let $G : L^2(M, \Lambda^l) \to L^2(M, \Lambda^l)$ be the continuous linear operator defined by

$$Gu_j^{(V)} \equiv \begin{cases} 0, & \lambda_j^{(V)} = 0 \\ \frac{1}{\lambda_j^{(V)}} u_j^{(V)}, & \lambda_j^{(V)} > 0 \end{cases} \tag{2.261}$$

where $(-\Delta + V)u_j^{(V)} = \lambda_j^{(V)}$ for all j, V, in accordance with the above discussion.

The operator $\Delta : L^2(M, \Lambda^l) \to H^{-2}(M, \Lambda^l)$ is continuous. It is hence true that, for all $V > 0,\ u \in L^2(M, \Lambda^l)$,

$$-\Delta G u = \sum_j (u, u_j^{(V)})(-\Delta G)u_j^{(V)} = \sum_{j:\,\lambda_j \neq 0} \frac{(u, u_j^{(V)})}{\lambda_j^{(V)}}(-\Delta)u_j^{(V)} = (I - P_l)u \tag{2.262}$$

In a similar manner, we obtain $G(-\Delta)u = (I - P_l)u$ for all $V > 0,\ u \in L^2(M, \Lambda^l)$. The operator G is, hence, the two-sided inverse of $-\Delta :\ L^2(M, \Lambda^l)\backslash\mathcal{H}_l \to H^{-2}(M, \Lambda^l)$. It is customary to call G the *Green operator*. Employing now the identity $-\Delta = d\delta + \delta d$ together with equation (2.262) yields

$$u = d\delta G u + \delta d G u + P_l u \quad \forall u \in L^2(M, \Lambda^l) \tag{2.263}$$

Since $H^j(M, \Lambda^l) \subset H^0(M, \Lambda^l) = L^2(M, \Lambda^l)$ for all $j \in \mathbb{N}_0$, the expression (2.262) is valid for every $u \in H^j(M, \Lambda)$, $j \in \mathbb{N}_0$. Therefore, given $j \in \mathbb{N}_0$, we have $-\Delta G u \in H^j(M, \Lambda^l)$ for all $u \in H^j(M, \Lambda^l)$. It is hence true that the operator G maps $H^j(M, \Lambda^l)$ onto $H^{j+2}(M, \Lambda^l)$ for all $j \in \mathbb{N}_0$.

The operator δ is the formal adjoint of d, and hence it holds true that

$$(d\delta G u, \delta d G u)_{L^2(M,\Lambda^l)} = (d^2\delta G u, dGu)_{L^2(M,\Lambda^l)} = 0 \quad \forall u \in L^2(M, \Lambda^l) \tag{2.264}$$

Furthermore, by Lemma 2.7.3, it is easily established that, for all $u \in L^2(M, \Lambda^l)$,

$$\begin{gathered} v \in \mathcal{H}_l \\ \Downarrow \\ (dGu, v)_{L^2(M,\Lambda^l)} = (Gu, \delta v)_{L^2(M,\Lambda^l)} = 0 \\ (\delta G u, v)_{L^2(M,\Lambda^l)} = (Gu, dv)_{L^2(M,\Lambda^l)} = 0 \end{gathered} \tag{2.265}$$

This completes the proof.

■

The significance of the Hodge decomposition in analysis is paramount, and it exceeds the field of potential theory.[22] From our perspective, the result resolves positively the question of existence of electromagnetic scalar and vector potentials introduced in Chapter 1, and furthermore, as we shall see in Chapter 3, it puts into perspective the integral representation formula for differential form-valued solutions of the Helmholtz equation in $\mathbb{R}^n$.

2.8 Fundamental Solutions of the Helmholtz Equation

The analysis presented in Chapter 3 relies heavily on fundamental solutions of the Helmholtz equation in $\mathbb{R}^n$. This is hardly surprising, having in mind that the general boundary value problem considered there comprises the n-dimensional Helmholtz operator. In particular, the fundamental solutions and their directional derivatives define the kernels of the layer potential and boundary integral operators.

It is therefore of interest to provide a section containing a precise description and a presentation of the key properties of the mentioned family of functions. In order to avoid a lengthy digression from the main line of reasoning in this text, we do not embark on proving the assertions made. Consult References [20, 14, 24, 28] for the relevant proofs and/or further literature on the subject.

The unique (up to a proportionality factor) solution u (in the sense of distributions) of the inhomogeneous Helmholtz equation

$$(\Delta + k^2 I)u = \delta_D \quad \text{in } \mathbb{R}^n \tag{2.266}$$

which satisfies the radiation condition is referred to as the *fundamental solution* of the Helmholtz operator. [20, 28][7] It is necessary to impose the radiation condition in order to obtain a unique form of solution to (2.266), since clearly any uniform plane wave in $\mathbb{R}^n$ is also a solution. The solution satisfying the outgoing radiation condition is referred to as the *radial outgoing fundamental solution*, while the solution satisfying the ingoing radiation condition is called the *radial ingoing fundamental solution*.[28] The type of radiation condition, as well as of the fundamental solution, is changed by complex conjugation, as shown in [28]. In scattering problems, the source of the scattered field is located at the surface of the scatterer, which is bounded. Hence, the scattered field is 'outgoing' (i.e. it carries energy away from the scatterer towards infinity,) and we are therefore here interested only in the outgoing fundamental solution of the Helmholtz equation (from now on referred to simply as 'the fundamental solution.')

By definition, the fundamental solution is a Green function pertaining to the Helmholtz equation which satisfies the radiation condition.

A general expression for the fundamental solution $\Phi_{k,n}$ of the Helmholtz operator $\Delta + k^2 I$ on $\mathbb{R}^n$ is given by [20][8]

$$\Phi_{k,n}(x,y) = \begin{cases} (-2\pi|x-y|)^{\frac{1-n}{2}} \frac{\partial}{\partial|x-y|}^{\frac{n-1}{2}} \left(\frac{1}{2ik}\right) e^{ik|x-y|}, & \text{n odd} \\ (-2\pi|x-y|)^{\frac{2-n}{2}} \frac{\partial}{\partial|x-y|}^{\frac{n-2}{2}} \left(\frac{1}{4i}\right) H_0^{(1)}(k|x-y|), & \text{n even} \end{cases} \tag{2.267}$$

for $x, y \in \mathbb{R}^n, x \neq y$, $k \in \mathbb{C}$, $\Im k \geq 0$. For $n = 2$ and $n = 3$, the above formula reduces to the well-known cases

$$\Phi_{k,2}(x,y) = \frac{1}{4i} H_0^{(1)}(k|x-y|), \quad x, y \in \mathbb{R}^2, x \neq y, k \in \mathbb{C}, \Im k \geq 0 \tag{2.268}$$

$$\Phi_{k,3}(x,y) = -\frac{e^{ik|x-y|}}{4\pi|x-y|} = -\frac{k}{4\pi} h_0^{(1)}(k|x-y|), \quad x, y \in \mathbb{R}^3, x \neq y, k \in \mathbb{C}, \Im k \geq 0 \tag{2.269}$$

[7] There seems to be a lack of concensus regarding the sign of the inhomogeneous term of (2.266). Thus, Reference [20] is in correspondence with (2.266), while Reference [28] adopts the convention $(\Delta + k^2 I)E = -\delta_D$.

[8] Note that we have altered slightly the original expressions, such that the fundamental solutions in our case are not radial, but are functions of two variables.

involving the Hankel function $H_0^{(1)}$ and the spherical Hankel function $h_0^{(1)}$, respectively, of zero order and first kind.

REMARK 2.8.1
The fundamental solution $\Phi_{k,n}$ of the Helmholtz equation in $\mathbb{R}^n$ satisfies

$$|\Phi_{k,n}(x,y)| = \mathcal{O}(|x-y|^{2-n}) \quad \text{for } x,y \in \mathbb{R}^n,\ |x-y| \in]0,1[\tag{2.270}$$

In particular, it holds true that [36]

$$\Phi_{k,n}(x,y) \sim -\frac{1}{(n-2)\sigma_{n-1}}|x-y|^{2-n} + \dots \quad \text{for } |x-y| \to 0 \text{ and } n \geq 3 \tag{2.271}$$

$$\Phi_{k,2}(x,y) \sim \frac{1}{2\pi}\log|x-y| + \dots \quad \text{for } |x-y| \to 0 \tag{2.272}$$

where the remainder terms are of increasing regularity. Furthermore, we have

$$\left|\frac{\partial \Phi_{k,n}}{\partial \nu_y}(x,y)\right| = \mathcal{O}(|x-y|^{1-n}) \quad \text{for } x,y \in \mathbb{R}^n,\ |x-y| \in]0,1[\tag{2.273}$$

and in particular [36]

$$\frac{\partial \Phi_{k,n}}{\partial \nu_y}(x,y) \sim \frac{1}{\sigma_{n-1}}|x-y|^{1-n}\nu_y.\frac{x-y}{|x-y|} + \dots \quad \text{for } |x-y| \to 0,\ n \geq 3 \tag{2.274}$$

$$\tag{2.275}$$

with remainder terms of increasing regularity.
For all $k \in \mathbb{C}, n \in \mathbb{N}, n \geq 3$, it holds true that [14]

$$\Phi_{k,n}(x,y) - \Phi_{0,n}(x,y) = \mathcal{O}(|x-y|^{3-n}) \quad \text{for } |x-y| \to 0 \tag{2.276}$$

□

Let us expand the fundamental solution $\Phi_{k,3}$:

$$\begin{aligned} -4\pi\Phi_{k,3}(x,y) = & \frac{e^{ik|x-y|}}{|x-y|} = |x-y|^{-1}\sum_{j\in\mathbb{N}_0}\frac{(ik|x-y|)^j}{j!} = \\ & \sum_{j\in\mathbb{N}_0}\frac{(ik)^j}{j!}|x-y|^{j-1} = \frac{1}{|x-y|} + ik - \frac{k^2}{2}|x-y| + \dots \end{aligned} \tag{2.277}$$

Obviously, the principal singularity of $\Phi_{k,3}$ is in the term corresponding to $k=0$. This is the case for all fundamental solutions $\Phi_{k,n}$.

In the course of analysis in Chapter 3, we make multiple references to an antisymmetry inherent in the derivatives of the fundamental solutions $\Phi_{k,n}$. The antisymmetry which we have in mind is the one with respect to the arguments x and y, and it originates from the fact that the fundamental solutions are functions of the argument $|x-y|$ only, and not of x and/or y separately. We have, for example,

$$d_x\Phi_{k,n} = -d_y\Phi_{k,n} \tag{2.278}$$

Fundamental solutions of the Helmholtz equation which are described above, and which we employ in the analysis of Chapter 3, belong to the class of $(0,0)$-biform-valued functions, i.e. functions scalar in both variables. Consequently, the so-called *layer* *potentials*, which are introduced in Chapter 3 and which essentially are integral operators with kernels derived from $\Phi_{k,n}$, are of the scalar variety. However, a generalisation of the notion of a fundamental solution to (l,l)-biform-valued functions, where $l \in \mathbb{N}$, is possible. References [22, 23] rely on this generalisation to present a more powerful analysis of boundary value problems by means of generalised (nonscalar) layer potential operators. For our purposes, however, the scalar layer potential operators are adequate.

Chapter 3

Potential Theory on Smooth and Lipschitz Domains

The theory of layer potentials constitutes a major tool in the analysis of boundary value problems, and it has been used extensively in this context for decades. In particular, while the three-dimensional Maxwell boundary value problem of (1.55), as well as the corresponding generalisation to Euclidean spaces of arbitrary dimension, have been solved in the 1950's using the layer potential approach [14], the method is at present still a subject of intense research[1]. The techniques associated with potential theory provide an alternative to the calculus of variations in the analysis of boundary value problems. The theoretical fundament upon which these techniques are built is more elaborate, requiring an introduction of the notions of pseudodifferential and Fredholm operators, and indeed the following sections show extensive use of the theory presented in Chapter 2. The gain obtained is the adaptability to nonsmooth boundary value problems, as illustrated in Section 3.4.2 of this chapter. It can furthermore be argued that the layer potential approach allows for better physical interpretation of the results, be it abstract regularity theorems or specific numerical data. The description involving surface densities on the scatterer boundary as sources of radiation of the scattered field constitutes a natural setting for an 'interpretation-oriented' treatment of acoustic and electromagnetic scattering problems. It is the general experience of the author that analytical, asymptotic and numerical techniques based on potential theory today are by far prevalent in the field of electromagnetic scattering.

Prior to beginning the development of this chapter, we state the overall plan. Firstly, a generalisation is presented of the governing equations and boundary conditions of acoustics and electromagnetism, rendering the systems in question special cases of a single boundary value problem. This is achieved using the formalism of Clifford algebras and operators on differential forms. Thereafter, we introduce layer potentials and boundary integral operators. For the smooth case, these operators are shown to be of the pseudodifferential variety, and we establish their properties in the context of the Fredholm theory. In particular, a number of boundary integral operators are found to be Fredholm of index zero in this case. Next, a link is established by the integral

[1]Due to the computational power of the digital computer, the interest in layer potential techniques is perhaps higher than ever. An example of modern theoretical research in the layer potentials may be found in Reference [22], while practical examples abound in e.g. the IEEE Antennas and Propagation Magazine, and even in the author's own work [15, 16].

representation theorem between the layer potentials and the solutions of the general boundary value problem comprising the acoustic and the electromagnetic cases. More precisely, it is shown that the solutions of the n-dimensional Helmholtz equation on l-forms, $n \in \mathbb{N}, l \in \mathbb{N}_0, l \leq n$, may be expressed in terms of the layer potential operators. The stage is now set for the analysis of the existence, uniqueness and regularity of solution to the general boundary value problem in the smooth setting, in that it is sufficient to prove e.g. the injectivity of a certain boundary integral operator. Upon finishing the analysis of the smooth case, we introduce the Rellich identities and treat the more general Lipschitz class of problems. Since the apparatus of pseudodifferential operators in general does not apply in this case, the proof of the fredholmness of the relevant boundary integral operators is more difficult. It is, however, possible to extend the result found in the smooth setting and find appropriate boundary integral operators of the Fredholm type and index zero on Lipschitz surfaces. Once this is achieved, the existence and uniqeness analysis proceeds as in the smooth case.

As is apparent, analysis is conducted separately for the smooth case and the Lipschitz case, but in both instances the Fredholm properties of the involved operators are exploited towards a simplification of the proofs of existence and uniqueness. Furthermore, the regularity part of the analysis is, for the smooth setting, simplified considerably as soon as we establish the identity of the spaces of pseudodifferential operators in which the considered boundary operators lie.

Clearly, there is a redundancy in our exposition, in that the existence, uniqueness and regularity analysis of boundary value problems involving smooth domains is implied in the corresponding analysis of the more general class of problems comprising Lipschitz domains. We have decided, however, to treat the smooth case separately, presenting thereby an application the important and beautiful theory of pseudodifferential operators, and moreover following the 'historical' line of reasoning.[2]

3.1 The General Helmholtz Boundary Value Problem

This section is devoted to showing that the governing equations of acoustics and electromagnetism, discussed in Section 1.1, are but special cases of a generalised system of equations involving differential form-valued functions. Furthermore, under a suitable transformation of the boundary data, boundary value problems comprising this latter system are equivalent to Helmholtz boundary value problems. Thus, not only is a unified analysis of the classical acoustic and electromagnetic boundary value problems possible, but we may also generalise matters considerably.

Consider a time-harmonic, source-free electromagnetic boundary value problem in $\Omega_3 \subseteq \mathbb{R}^3$, comprising a lossless simple medium (i.e. constant real scalar constitutive parameters μ, ε.) As shown in Chapter 1, the pertaining reduced system of equations of Maxwell is given by

$$\begin{aligned} \mathrm{curl} E - ikH = 0 \quad &\text{in } \Omega_3 \\ \mathrm{curl} H + ikE = 0 \quad &\text{in } \Omega_3 \end{aligned} \tag{3.1}$$

with $k = \omega/c = \omega\sqrt{\mu\varepsilon} \in \mathbb{R}$. Identifying scalar functions with 0-forms and geometrical vector fields in $\mathbb{R}^3$ with 1-forms[3], we see using Remark 2.4.17 and the second

[2] The approach utilising the theory of pseudodifferential operators in conjunction with the Fredholm theory precedes the use of the Rellich inequalities in this context.[14, 22]

[3] These canonical identifications are natural, as may be seen from our presentation of Clifford algebras in Section 2.4.

identity of (2.69) that the system (3.1) may be written as

$$\begin{aligned} *dE - ikH &= 0 \quad \text{in } \Omega_3 \\ \delta{*}H + ikE &= 0 \quad \text{in } \Omega_3 \end{aligned} \tag{3.2}$$

or, with

$$\mathscr{H} \equiv *H \tag{3.3}$$

and utilising (2.67), as

$$\begin{aligned} dE - ik\mathscr{H} &= 0 \quad \text{in } \Omega_3 \\ \delta\mathscr{H} + ikE &= 0 \quad \text{in } \Omega_3 \end{aligned} \tag{3.4}$$

Since in this case the fields (E, H) are characterised as differential form-valued functions by the parameter values $(l, n) = (1, 3)$, the quantity $\mathscr{H}$ is a 2-form-valued function defined on Ω_3 (see Definition 2.4.11.) Denoting by $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n$ suitable algebras of functions on some set $\Omega \subseteq \mathbb{R}^n$, we are now motivated to consider the general operator

$$\begin{gathered} \mathscr{M}_{l,n}(z) : \mathcal{A}_n(\Omega, \Lambda^l) \times \mathcal{B}_n(\Omega, \Lambda^{l+1}) \to \mathcal{C}_n(\Omega, \Lambda^{l+1}) \times \mathcal{D}_n(\Omega, \Lambda^l), \\ n \in \mathbb{N},\ l \in \{0, \dots, n\}, z \in \mathbb{C} \end{gathered} \tag{3.5}$$

defined by

$$\mathscr{M}_{l,n}(z) \equiv \begin{bmatrix} d_{l,n} & \overline{z}I \\ zI & \delta_{l,n} \end{bmatrix} \quad \forall n \in \mathbb{N}, l \in \{0, \dots, n\}, z \in \mathbb{C} \tag{3.6}$$

where we use the subscripts l, n to emphasise that we are now dealing with general interior and exterior derivative operators. It is clear from the above discussion that the system

$$\mathscr{M}_{l,n}(ik) \begin{bmatrix} E \\ \mathscr{H} \end{bmatrix} = 0, \quad k \in \mathbb{R} \tag{3.7}$$

comprising an l-form-valued function E and an $l+1$-form-valued function $\mathscr{H}$ for general values of the parameter pair (l, n), $n \in \mathbb{N}$, $l \in \{1, \dots, n\}$, constitutes a generalisation of the three-dimensional Maxwell equations given in (3.1). In addition, it is seen using the results of Remark 2.4.17 that

$$\mathscr{M}_{0,n}(ik) \begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} \nabla_n & -ikI \\ ikI & -div_n \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} \nabla p - ikv \\ -\text{div} v + ikp \end{bmatrix} \tag{3.8}$$

holds for all wavenumbers $k \in \mathbb{R}$ and all suitable 0-form-valued (i.e. scalar) functions p and 1-form-valued functions v. Thus, the reduced three-dimensional reduced acoustic system of equations (1.32)–(1.33) is also a special case of (3.7)!

It may be argued that a new notation is in order for the general field quantities $(E, \mathscr{H})$ appearing in the system (3.7), a notation which would not necessarily bear association to the special cases of acoustic or electromagnetic fields. For simplicity, however, we use the symbols $E, \mathscr{H}$ in the remainder of this text.[4]

REMARK 3.1.1
In light of the above discussion, the system of equations (3.7) may be said to constitute the basis of a general model, which comprises the governing equations of acoustics

[4] A similar convention is also followed e.g. in references [14, 22].

and of electromagnetism as special cases characterised by the value of the parameter l. Clearly, each special case is furthermore distinguished by the associated constitutive parameters of the various media, but this is also merely a matter of the choice of parameter values. The general model involves one n-dimensional l-form-valued field quantity and one n-dimensional $l+1$-form-valued field quantity, both defined in a subset of $\mathbb{R}^n$, with $n \in \mathbb{N}, l \in \{0, \ldots, n\}$. Note that while the original acoustic system of equations follows this pattern (with $l = 0$,) the equations of Maxwell involve, in their standard form, a pair of 1-form-valued functions, and the extra step (3.3) is needed in order to satisfy the mentioned pattern (with $l = 1$.) Geometrical vector fields (with complex coefficients) are perhaps more suitable for representing physical force fields than complex 2-form-valued functions, but, having the above discussion in mind, the traditional Maxwellian system may in our context be characterised as less convenient than the corresponding system involving the quantities $(E, *H)$.

Loosely put, the general system (3.7) 'transcends' the physical details[5] pertaining to each particular value of the parameter l, and simply describes general *wave phenomena* involving pairs of complementary physical quantities, such as pressure and displacement velocity, electric and magnetic field, etc. The generalised wave equations carry, in a sense (modulo physical details,) the essence of the wave phenomenon. Consequently, a model based on this system could be referred to in general terms as a '*wave model*'. The wave model describes wave phenomena in an arbitrary number of dimensions $n \in \mathbb{N}$, and is thus general also in this respect.

It is highly improbable that the possibility of a unified treatment of acoustic and electromagnetic wave phenomena is unknown. We note, however, that none of the references studied mentions, let alone elaborates on, this possibility. In fact, a clear distinction is maintained between the acoustic and the electromagnetc cases, even in the modern literature. References [28, 14, 22] serve to illustrate this point.[6]

□

REMARK 3.1.2
We note that the time-harmonic Maxwell system (3.1) may alternatively be recast to the following system of equations involving a 2-form-valued function $\mathscr{E} \equiv *E$ and a 1-form-valued function H on Ω_3:

$$\begin{aligned} \delta\mathscr{E} - ikH &= 0 \quad \text{in } \Omega_3 \\ dH + ik\mathscr{E} &= 0 \quad \text{in } \Omega_3 \end{aligned} \tag{3.9}$$

In contrast to the case of representation (3.4) of the original Maxwellian system, we do not have a special notation for the differential operators occuring in (3.9).

Since (3.9) can be derived by simple application of the Hodge star operator to the equations of (3.4), we refer to the system (3.9) as the *Hodge dual* of (3.4). Generalising the parameters l, n as above, we express the connection between the systems (3.9) and (3.7) by

$$\mathscr{M}_{l,n}(-ik) \begin{bmatrix} H \\ \mathscr{E} \end{bmatrix} = \begin{bmatrix} dH + ik\mathscr{E} \\ \delta\mathscr{E} - ikH \end{bmatrix} \quad \forall n \in \mathbb{N}, l \in \{0, \ldots, n\}, k \in \mathbb{R} \tag{3.10}$$

[5]Details such as different speed of propagation, different interaction mechanisms between the fields and matter, different physical manifestations

[6]While Reference [14] develops the (in principle) same theory firstly for the acoustic and then for the electromagnetic case, references [14, 22] use exclusively the electromagnetic nomenclature when dealing with the generalised systems of equations.

where $\mathscr{E}$ is an $l+1$-form-valued function and H is an l-form-valued function on $\Omega \subseteq \mathbb{R}^n$. We choose in the following to employ the formulation presented in (3.7).

□

Given an open bounded Lipschitz domain $\Omega_3 \subset \mathbb{R}^3$, consider the boundary $\Gamma_3 = \partial\Omega_3$. Using the expression (2.80) for the tangential component of a differential form, as well as a result from Remark 2.4.17 and the second relation of (2.62), the Standard Impedance Boundary Condition of electromagnetics,

$$\nu \times (E \times \nu) = \zeta \nu \times H \quad \text{a.e. on } \Gamma_3,\ \zeta \in \mathbb{C} \tag{3.11}$$

associated with the surface Γ_3 may be expressed as follows in terms of 1-form-valued functions:

$$\nu \vee (\nu \wedge \gamma_0 E) = \zeta * (\nu \wedge \gamma_0 H) = -\zeta \nu \vee \gamma_0(*H) = -\zeta \nu \vee \gamma_0 \mathscr{H} \quad \text{a.e. on } \Gamma_3, \zeta \in \mathbb{C} \tag{3.12}$$

The last equality above is obtained upon defining $\mathscr{H} \equiv *H$. Since in the present case the fields (E, H) are characterised by the parameter pair $(l, n) = (1, 3)$, the quantity $\mathscr{H}$ is a 2-form-valued function on Ω_3. Now generalise the expression (3.12) to include an l-form-valued function E and an $l+1$-form-valued function $\mathscr{H}$ defined on $\Omega \subseteq \mathbb{R}^n, n \in \mathbb{N}, l \in \{0, \ldots, n\}$, with $\Gamma = \partial\Omega$. The condition

$$\nu \vee (\nu \wedge \gamma_0 E) = -\zeta \nu \vee \gamma_0 \mathscr{H} \quad \text{a.e. on } \Gamma,\ \zeta \in \mathbb{C} \tag{3.13}$$

could in this case be referred to as the *generalised Standard Impedance Boundary Condition*. If the field quantities $(E, \mathscr{H})$ are assumed to satisfy the equation system (3.7), and if the dynamic case is considered (i.e. if $k \neq 0$,) the relation (3.13) may be written as follows:

$$\nu \vee (\nu \wedge \gamma_0 E) = i\zeta k^{-1} \nu \vee \gamma_0 dE \quad \text{a.e. on } \Gamma,\ \zeta \in \mathbb{C} \tag{3.14}$$

If furthermore $l = 0$, we apply to (3.14) the results from the calculus of differential forms presented in Chapter 2 and find that the generalised SIBC reads

$$\nu.\nu E = \zeta i k^{-1} \nu.\nabla E \quad \text{a.e. on } \Gamma,\ \zeta \in \mathbb{C} \tag{3.15}$$

This is simply the classical homogeneous Robin boundary condition, written alternatively as

$$ik^{-1}\zeta \frac{\partial E}{\partial \nu} - E = 0 \quad \text{a.e. on } \Gamma,\ \zeta \in \mathbb{C} \tag{3.16}$$

Thus, in the framework of differential forms, a direct link is established between the SIBC of electromagnetics and the standard Robin boundary condition, in that a common generalisation is found of these boundary conditions. Note that while this correspondence is easily derived (without use of the calculus of differential forms) when considering two-dimensional electromagnetic scattering problems ($l = 1, n = 2$), we here present the general result valid for all values of the parameter pair (l, n) satisfying the condition $n \in \mathbb{N}, l \in \{0, \ldots, n\}$.

It remains to express the Silver-Müller radiation conditions in terms of differential form-valued functions. We have [20]

$$\begin{aligned} \frac{x}{|x|} \wedge E(x) - \mathscr{H}(x) &= o(|x|^{-(n-1)/2}) \quad \text{for } |x| \to \infty \\ \frac{x}{|x|} \vee \mathscr{H}(x) - E(x) &= o(|x|^{-(n-1)/2}) \quad \text{for } |x| \to \infty \end{aligned} \tag{3.17}$$

Before the formal introduction of a general boundary value problem comprising the electromagnetic and the acoustic case, the following definition is in order:

DEFINITION 3.1.3
Given an l-form-valued function u defined on a domain $\Omega \subset \mathbb{R}^n$ with compact boundary $\partial\Omega = \Gamma$, as well as a constant $C > 1$, we define the *nontangential maximal function* $\mathcal{N}_C(u)$ pertaining to u as

$$\mathcal{N}_C(u)(x) \equiv \sup\left\{ |u(y)| \,\middle|\, y \in \Omega,\ |x-y| \leq C\mathrm{dist}(y,\Gamma) \right\} \quad \forall x \in \Gamma \tag{3.18}$$

Accordingly, $\mathcal{N}_C$ denotes the *nontangential maximal operator* associated with Γ. We omit the subscript C in the following, and interpret the symbol $\mathcal{N}$ as the nontangential maximal operator for *some* constant $C > 1$.

□

Given any $x_0 \in \Gamma$ and $\varepsilon > 0$, consider the open ball $B_{\mathbb{R}^n}(x_0, \varepsilon)$. For any constant $C > 1$ and sufficiently small ε, we see that the set

$$\{y \in \Omega |\, |x_0 - y| \leq C\mathrm{dist}(y,\Gamma)\} \cap B_{\mathbb{R}^n}(x_0,\varepsilon) \tag{3.19}$$

is a truncated circular cone in $B_{\mathbb{R}^n}(x_0,\varepsilon)$ with the vertex at x_0. The limiting case $C = 1$ corresponds to a line segment in $B_{\mathbb{R}^n}(x_0,\varepsilon)$ orthogonal to Γ and containing the point x_0. Thus the origin of the name of the *nontangential* maximal function.

DEFINITION 3.1.4
The Interior and Exterior General Maxwell Boundary Value Problems
Let $n \in \mathbb{N}, l \in \{0,\ldots,n\}, k \in \mathbb{C}, p \in \mathbb{R}$, and consider an open bounded domain $\Omega_+ \subset \mathbb{R}^n$ with $\Gamma = \partial\Omega_+$ and $\Omega_- = \complement\overline{\Omega}_+$. The unit outward normal to the hypersurface Γ, defined a.e. on Γ, is denoted by ν.

The *general* (interior or exterior) *Maxwell boundary value problem* is given by[7][14]

$$\begin{gathered}
E_\pm(x) \in \Lambda^{l,n},\ \mathscr{H}_\pm(x) \in \Lambda^{l+1,n} \quad \forall x \in \Omega_\pm \\
\mathscr{M}_{l,n}(ik)\begin{bmatrix} E_\pm \\ \mathscr{H}_\pm \end{bmatrix} = 0 \quad \text{in } \Omega_\pm \\
\mathcal{N}E_\pm,\ \mathcal{N}\mathscr{H}_\pm \in L^p(\Gamma) \\
\nu \wedge \gamma_0 E_\pm = \Psi \in L^{p,d}_n(\Gamma, \Lambda^{l+1}) \\
(\text{or, equivalently, } \nu \wedge \gamma_0 \delta\mathscr{H}_\pm = -ik\Psi \in L^{p,d}_n(\Gamma,\Lambda^{l+1}))
\end{gathered} \tag{3.20}$$

The exterior problem is supplied by the requirement that the forms $E_-, \mathscr{H}_-$ satisfy the Silver-Müller radiation condition (decay condition) (3.17), namely

$$\frac{x}{|x|} \wedge E_- - \mathscr{H}_- = o(|x|^{-(n-1)/2}) \quad \text{for } |x| \to \infty \tag{3.21}$$

□

[7]We use the script letter $\mathscr{H}$ for the 'generalised magnetic field', as opposed to the plain E for the 'generalised electric field', in order to emphasise that $\mathscr{H}$ is an $l+1$-form-valued function, and hence is of higher order than E.

REMARK 3.1.5
The generalised formulation elegantly 'covers' both acoustic and electromagnetic phenomena. At the same time, the well-known conception in classical physics[8] that the acoustic and the electromagnetic fields are, in general, 'decoupled' (that is, that the two phenomena are unrelated in nature,) is expressed through the orthogonality of the scalar and 1-form elements of Clifford algebras.

□

We note that the general Maxwell boundary value problems defined above correspond to electromagnetic scattering problems involving PEC structures, in that the surface impedance ζ is set to zero, and the generalised electric field must 'counter' an imposed field Ψ. Indeed, for $l = 1, n = 3$, we may write the boundary condition on E given above as

$$*\nu \wedge \gamma_0 E_\pm = \nu \times E_\pm = *\Psi \in L_t^{p,\delta}(\Gamma, \Lambda^1) \tag{3.22}$$

Although the boundary value problems of Definition 3.1.4 perhaps deserve a terminology of their own (e.g. the name 'generalised wave problems' could be considered,) we choose to follow Reference [14] and call them 'general Maxwell boundary value problems.' In Section 1.1, we have seen that the source-free (homogeneous) governing equations of acoustics and electromagnetism are equivalent to appropriate Helmholtz systems of equations. The next result constitutes a generalisation of this fact and hence provides motivation for considering certain types of Helmholtz boundary value problems in the context of the analysis of the general Maxwell boundary value problem introduced in Definition 3.1.4.

PROPOSITION 3.1.6
Given a domain $\Omega \subseteq \mathbb{R}^n$, the system of equations

$$\begin{aligned} dE - ik\mathscr{H} = 0 \quad \text{in } \Omega \\ \delta\mathscr{H} + ikE = 0 \quad \text{in } \Omega \end{aligned} \tag{3.23}$$

is equivalent to each of the systems

$$\begin{aligned} (\Delta + k^2 I)E = 0 \quad \text{in } \Omega \\ \delta E = 0 \quad \text{in } \Omega \\ \mathscr{H} = \frac{1}{ik} dE \quad \text{in } \Omega \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} (\Delta + k^2 I)\mathscr{H} = 0 \quad \text{in } \Omega \\ d\mathscr{H} = 0 \quad \text{in } \Omega \\ E = -\frac{1}{ik} \delta\mathscr{H} \quad \text{in } \Omega \end{aligned} \tag{3.25}$$

for any pair of functions $(E, \mathscr{H}) \in C^\infty(\Omega, \Lambda^l) \times C^\infty(\Omega, \Lambda^{l+1})$.
Proof: The proof is straightforward. The identities $\delta E = 0$, $d\mathscr{H} = 0$ follow immediately from (3.23) and the relations involving the exterior and interior derivative

[8] In the modern picture, elastic (acoustic) waves exist fundamentally due to the electromagnetic interactions between the charges carried by the subatomic particles of neighbouring atoms in a medium. The term 'classical physics' above implies the understanding of any medium as a continuum.

operators given in (2.59). Hence, if the system (3.23) is satisfied by some $(E, \mathscr{H})$, we may generalise the derivations performed in Section 1.1 and write

$$-\Delta E = \delta dE = ik\delta\mathscr{H} = ik(-ikE) = k^2 E \tag{3.26}$$

In a dual manner, the same result may be deduced for the field $\mathscr{H}$. Starting from (3.23), we see that

$$-\Delta\mathscr{H} = d\delta\mathscr{H} = -ikdE = -ik(ik\mathscr{H}) = k^2\mathscr{H} \tag{3.27}$$

If, conversely, the system (3.24) is satisfied, then necessarily

$$\delta\mathscr{H} = \frac{1}{ik}\delta dE = \frac{1}{ik}k^2 E = -ikE \tag{3.28}$$

and from (3.25) we obtain

$$dE = -\frac{1}{ik}d\delta\mathscr{H} = -\frac{1}{ik}k^2\mathscr{H} = ik\mathscr{H} \tag{3.29}$$

■

Let us formally introduce the Helmholtz boundary value problems considered in this text. (The terminology again follows Reference [14].)

DEFINITION 3.1.7
The Interior and Exterior Electric and Magnetic Boundary Value Problems
Let $n \in \mathbb{N}, l \in \{0, \ldots, n\}, k \in \mathbb{C}, p \in \mathbb{R}$, and consider an open bounded domain $\Omega_+ \subset \mathbb{R}^n$ with $\Gamma = \partial\Omega_+$ and $\Omega_- = \complement\overline{\Omega}_+$. Denote by ν the unit outward normal to Γ, defined a.e. on Γ.

The (interior or exterior) *electric boundary value problem* is given by

$$\begin{gathered} E_\pm(x) \in \Lambda^{l,n} \quad \forall x \in \Omega_\pm \\ (\Delta + k^2 I)E_\pm = 0 \quad \text{in } \Omega_\pm \\ \mathcal{N}E_\pm, \mathcal{N}dE_\pm, \mathcal{N}\delta E_\pm, \mathcal{N}\delta dE_\pm \in L^p(\Gamma) \\ \nu \wedge \gamma_0 E_\pm = \Psi_1 \in L^{p,d}_n(\Gamma, \Lambda^{l+1}) \\ \nu \wedge \gamma_0\delta E_\pm = \Psi_2 \in L^{p,d}_n(\Gamma, \Lambda^l) \quad (\text{if } l \in \{1, \ldots, n\}) \end{gathered} \tag{3.30}$$

Again, the exterior problem is supplied by the Silver-Müller radiation condition on E_-.

The (interior or exterior) *magnetic boundary value problem* is defined in a dual manner:

$$\begin{gathered} \mathscr{H}_\pm(x) \in \Lambda^{l,n} \quad \forall x \in \Omega_\pm \\ (\Delta + k^2 I)\mathscr{H}_\pm = 0 \quad \text{in } \Omega_\pm \\ \mathcal{N}\mathscr{H}_\pm, \mathcal{N}d\mathscr{H}_\pm, \mathcal{N}\delta\mathscr{H}_\pm, \mathcal{N}\delta d\mathscr{H}_\pm \in L^p(\Gamma) \\ \nu \vee \gamma_0\mathscr{H}_\pm = \Psi_1 \in L^{p,\delta}_t(\Gamma, \Lambda^{l-1}) \quad (\text{if } l \in \{1, \ldots, n\}) \\ \nu \vee \gamma_0 d\mathscr{H}_\pm = \Psi_2 \in L^{p,\delta}_t(\Gamma, \Lambda^l) \end{gathered} \tag{3.31}$$

with the Silver-Müller radiation condition imposed for the exterior problem.

□

REMARK 3.1.8
Note that, in the special case of $l = 0$, results of Remark 2.4.17 imply that the electric and the magnetic boundary value problems are Helmholtz problems with a (generally inhomogeneous) Dirichlet and Neumann boundary condition, respectively. In particular, the Dirichlet boundary condition reads

$$\nu\gamma_0 E_\pm = \Psi_1 \in L_n^{p,d}(\Gamma, \Lambda^1) \tag{3.32}$$

while the Neumann boundary condition is given by

$$\gamma_0 \frac{\partial \mathscr{H}_\pm}{\partial \nu} = \Psi_2 \in L_t^{p,\delta}(\Gamma, \Lambda^0) \tag{3.33}$$

in this case.

□

The (interior or exterior) electric and magnetic boundary value problems are more general than the corresponding Maxwell problem of Definition 3.1.4. In particular, we will prove in Section 3.4.3 that solution of one of the presented Helmholtz systems in the special case $\Psi_2 = 0$ is sufficient for solution of the associated general Maxwell boundary value problem. We refer to the systems (3.30) and (3.31) as *general Helmholtz boundary value problems*, for obvious reasons. It is the analysis of a general Helmholtz boundary value problem, or more precisely the analysis of the (interior and exterior) electric boundary value problem, that constitutes the essence of the present work.

Before we continue with the principal development of this chapter, let us show an important application of the Clifford algebra framework on the general Maxwell system (see, e.g., Reference [20]). In classical complex function theory, a function $u : \mathbb{C} \to \mathbb{C}$ is referred to as *monogenic* (as opposed to polygenic) on some domain $\Omega \subseteq \mathbb{C}$ iff $u \in D^1(\Omega)\backslash D^2(\Omega)$. The following generalisation is in order:

DEFINITION 3.1.9
Given an open set $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, a smooth, Clifford algebra-valued function

$$u : \Omega \to \mathbb{R}_{(n)} \tag{3.34}$$

is referred to as *k-monogenic* iff the following is true:

$$\mathbb{D}_k u = 0 \quad \text{in } \Omega \tag{3.35}$$

where the operator $\mathbb{D}_k$ is the perturbed Dirac operator, namely

$$\mathbb{D}_k = D + k e_{n+1} \tag{3.36}$$

A 0-monogenic function is also referred to as *Clifford holomorphic* or *Clifford regular*. [19]

□

A great deal of motivation for considering the formalism of Clifford algebras arises from the following simple result:

PROPOSITION 3.1.10
Given an open set $\Omega \subset \mathbb{R}^n$, let $E, \mathscr{H}$ be smooth differential forms of degree l and $l+1$, respectively, where $l \in \{0, \dots, n\}$, which are furthermore defined on Ω. Define $k \in \mathbb{C}$, as well as the function

$$u = \mathscr{H} - ie_{n+1}E \tag{3.37}$$

The following holds true:

$$\begin{gathered} \mathbb{D}_k u = 0 \quad \text{on } \Omega \\ \Updownarrow \\ dE - ik\mathscr{H} = 0,\, \delta\mathscr{H} + ikE = 0 \\ \delta E = 0,\, d\mathscr{H} = 0 \quad \text{on } \Omega \end{gathered} \tag{3.38}$$

That is, the k-monogenicity of u on Ω is equivalent with the differential forms $E, \mathscr{H}$ constituting a solution to the generalised Maxwell system on Ω.
Proof: Using the definition of the perturbed Dirac operator $\mathbb{D}_k$, as well as the fact that $D = d + \delta$, we clearly have the following:

$$\begin{gathered} \mathbb{D}_k u = (D + ke_{n+1})(\mathscr{H} - ie_{n+1}E) = \\ d\mathscr{H} + \delta\mathscr{H} - ide_{n+1}E - i\delta e_{n+1}E + ke_{n+1}\mathscr{H} + ikE = \\ (d\mathscr{H} + \delta\mathscr{H} + ikE) + i(-1)^{l+1}(dE + \delta E - ik\mathscr{H})e_{n+1} \end{gathered} \tag{3.39}$$

Clearly, the two terms in the right-hand side of the above equation are mutually orthogonal, which leads to the conclusion that the equation $\mathbb{D}_k u = 0$ is equivalent to

$$d\mathscr{H} + \delta\mathscr{H} + ikE = 0,\, dE + \delta E - ik\mathscr{H} = 0 \tag{3.40}$$

Simple degree considerations now reveal that this is equivalent to

$$d\mathscr{H} = 0,\, \delta\mathscr{H} + ikE = 0,\, \delta E = 0,\, dE - ik\mathscr{H} = 0 \tag{3.41}$$

■

3.2 Layer Potentials and Boundary Integral Operators

The general Helmholtz boundary value problems presented in Section 3.1 admit solutions expressible in terms of the *layer potential operators*. This enables us to reduce each of the problems (3.30) and (3.31) to an integral equation involving boundary integral operators on the hypersurface Γ. Knowledge of the properties of the relevant boundary integral operators facilitates an analysis of the existence, uniqueness and regularity of solutions to the general Helmholtz problems. In particular, it turns out that the boundary integral operators are Fredholm of index zero in case the hypersurface Γ is smooth. For Lipschitz boundaries Γ, the boundary integral operators are also invertible, albeit under certain additional conditions.

In Section 3.2.1, we take up the definition of the layer potentials and of the associated boundary integral operators. Next, the important jump relations and regularity results associated with these operators are presented in Section 3.2.2 for the smooth case. Section 3.2.3 elaborates on the ellipticity and fredholmness properties of the boundary integral operators, again for the case of smooth boundaries.

3.2.1 Definition of Layer Potentials and Boundary Integral Operators

DEFINITION 3.2.1
Let $k \in \mathbb{C}$, $n \in \mathbb{N}$, and let $\Gamma \subset \mathbb{R}^n$ be a bounded[9] $n-1$-dimensional Lipschitz hypersurface. Consider an l-form-valued function $u : \Gamma \to \Lambda^{l,n}, l \in \{0,\dots,n\}$. The *single layer acoustic potential operator*[10] $\mathcal{S}_{k,n}$ on Γ is formally defined by

$$\mathcal{S}_{k,n}u(x) \equiv \int_\Gamma \Phi_{k,n}(x,y)u(y)d\Gamma(y) \quad \forall x \in \mathbb{R}^n \backslash \Gamma \tag{3.42}$$

The *double layer acoustic potential operator* $\mathcal{D}_{k,n}$ on Γ is formally defined by

$$\mathcal{D}_{k,n}u(x) \equiv \int_\Gamma \frac{\partial \Phi_{k,n}}{\partial \nu_y}(x,y)u(y)d\Gamma(y) \quad \forall x \in \mathbb{R}^n \backslash \Gamma \tag{3.43}$$

We also define formally the normal derivatives of the single and double layer potentials:

$$\mathcal{D}^*_{k,n}u(x) \equiv \int_\Gamma \frac{\partial \Phi_{k,n}}{\partial \nu_x}(x,y)u(y)d\Gamma(y) \quad \forall x \in \mathbb{R}^n \backslash \Gamma \tag{3.44}$$

$$\mathcal{N}_{k,n}u(x) \equiv \int_\Gamma \frac{\partial^2 \Phi_{k,n}}{\partial \nu_x \partial \nu_y}(x,y)u(y)d\Gamma(y) \quad \forall x \in \mathbb{R}^n \backslash \Gamma \tag{3.45}$$

In general, the directional derivative $\partial \Phi_{k,n}/\partial \nu_x$ is defined as $\nu(x).\nabla_x \Phi_{k,n}$, where $\nu(x)$ is the unit outward normal to Γ at $x \in \Gamma$. In the following, the subscripts k,n signifying the value of the wavenumber and the domain dimension associated with the relevant operator will be suppressed for simplicity.

□

REMARK 3.2.2
Assume u is a differential form-valued function on Γ such that the first three integrals appearing in Definition 3.2.1 are convergent.[11] Now note that, due to the properties of the functions $\Phi_{k,n}$, as well as the obvious commutator relations

$$\left[\Delta_x + k^2 I, \frac{\partial}{\partial \nu_y}\right] = 0, \quad \left[\Delta_x + k^2 I, \frac{\partial}{\partial \nu_x}\right] = 0, \quad \left[\frac{\partial}{\partial \nu_x}, \frac{\partial}{\partial \nu_y}\right] = 0 \tag{3.46}$$

it holds true that

$$(\Delta + k^2 I)\mathcal{S}u(x) = (\Delta + k^2 I)\mathcal{D}u(x) = (\Delta + k^2 I)\mathcal{D}^*u(x) = 0 \tag{3.47}$$

for all $x \in \mathbb{R}^n \backslash \Gamma$. Also, (again due to the properties of the fundamental solutions $\Phi_{k,n}$,) the functions $\mathcal{S}u, \mathcal{D}u, \mathcal{D}^*u$ satisfy the Silver-Müller radiation conditions (3.17).

□

[9] In the two-dimensional case ($n = 2$), only the cross-section of the surface Γ with the plane perpendicular to the axis of invariance is assumed to be bounded.

[10] Some references use the term 'simple' instead of 'single' (see, e.g., [7].)

[11] The kernel of the normal derivative $\mathcal{N}_{k,n}$ of the double layer potential is strongly singular and therefore not integrable [28], and the integral in (3.45) is actually improper.

We now define the boundary integral operators pertaining to the layer potentials.

DEFINITION 3.2.3
Let $\Omega_+ \subset \mathbb{R}^n$ be an open bounded Lipschitz domain with $\Gamma = \partial\Omega_+$ and $\Omega_- = \complement\overline{\Omega}_+$. Also, let $u : \Gamma \to \Lambda^{l,n}$ be a differential form-valued function. The (inner or outer) *acoustic single layer potential boundary integral operator* $\mathfrak{S}^{\pm}_{k,n}$ is defined by

$$\mathfrak{S}^{\pm}_{k,n} \equiv \gamma_0 r_{\Omega_\pm} \mathcal{S}_{k,n} \tag{3.48}$$

We furthermore formally introduce the principal value boundary integral operator $S_{k,n}$ as follows:

$$S_{k,n}u(x) \equiv p.v. \int_\Gamma \Phi_{k,n}(x,y)u(y)d\Gamma(y) \quad \forall x \in \Gamma \tag{3.49}$$

where $p.v.$ signifies that the integral in question is the usual principal value integral. The (inner or outer) *acoustic double layer potential boundary integral operator* $\mathfrak{D}^{\pm}_{k,n}$ is defined by

$$\mathfrak{D}^{\pm}_{k,n} \equiv \gamma_0 r_{\Omega_\pm} \mathcal{D}_{k,n} \tag{3.50}$$

Similarly, we define

$$\mathfrak{D}^{*\,\pm}_{k,n} \equiv \gamma_0 r_{\Omega_\pm} \mathcal{D}^*_{k,n} \tag{3.51}$$

Next, we formally introduce the principal value boundary integral operators $D_{k,n}, D^*_{k,n}$ as follows:

$$\begin{aligned} D_{k,n}u(x) &\equiv p.v. \int_\Gamma \frac{\partial\Phi_{k,n}}{\partial\nu_y}(x,y)u(y)d\Gamma(y) \quad \forall x \in \Gamma \\ D^*_{k,n}u(x) &\equiv p.v. \int_\Gamma \frac{\partial\Phi_{k,n}}{\partial\nu_x}(x,y)u(y)d\Gamma(y) \quad \forall x \in \Gamma \end{aligned} \tag{3.52}$$

We introduce the boundary integral operators $\mathfrak{N}^{\pm}_{k,n}$ by

$$\mathfrak{N}^{\pm}_{k,n} \equiv \gamma_0 r_{\Omega_\pm} \mathcal{N}_{k,n} \tag{3.53}$$

The principal value boundary integral operator $\mathscr{N}_{k,n}$ is formally defined in the following manner:

$$\mathscr{N}_{k,n}u(x) \equiv p.v. \int_\Gamma \frac{\partial^2\Phi_{k,n}}{\partial\nu_x\partial\nu_y}(x,y)u(y)d\Gamma(y) \quad \forall x \in \Gamma \tag{3.54}$$

The *magnetostatic operator* $M_{k,n}$ is formally defined by [22, 14]

$$M_{k,n}u \equiv p.v. \int_\Gamma \nu(x) \vee d_x\Phi_{k,n}(x,y)u(y)d\Gamma(y) \quad \forall x \in \Gamma \tag{3.55}$$

The *electrostatic operator* $N_{k,n}$ is formally defined by [22, 14]

$$N_{k,n}u(x) \equiv p.v. \int_\Gamma \nu(x) \wedge \delta_x\Phi_{k,n}(x,y)u(y)d\Gamma(y) \quad \forall x \in \Gamma \tag{3.56}$$

For notational convenience, the subscripts k, n will be suppressed in the following.

□

As may be seen from the above definition, the boundary integral operators on a hypersurface Γ are restrictions to Γ of the associated layer potential operators.

The next two remarks are made using the calculus of differential forms presented in Section 2.4.

REMARK 3.2.4
Assume u is a differential form-valued function on a Lipschitz hypersurface Γ. By the relations in (2.75), as well as by the antisymmetry in the fundamental solution $\Phi_{k,n}$, we may write formally

$$\begin{aligned} Mu(x) =& p.v. \int_\Gamma \nu(x) \vee \{(\nabla_x \Phi_{k,n}) \wedge u\}(x,y) d\Gamma(y) = \\ &- p.v. \int_\Gamma \nu(x) \vee \{(\nabla_y \Phi_{k,n}) \wedge u\}(x,y) d\Gamma(y) \quad \forall x \in \Gamma \end{aligned} \tag{3.57}$$

$$\begin{aligned} Nu(x) =& - p.v. \int_\Gamma \nu(x) \wedge \{(\nabla_x \Phi_{k,n}) \vee u\}(x,y) d\Gamma(y) = \\ & p.v. \int_\Gamma \nu(x) \wedge \{(\nabla_y \Phi_{k,n}) \vee u\}(x,y) d\Gamma(y) \quad \forall x \in \Gamma \end{aligned} \tag{3.58}$$

□

REMARK 3.2.5
Given any bounded Lipschitz hypersurface $\Gamma \in \mathbb{R}^n$, the pertaining magnetostatic operator M and electrostatic operator N satisfy the relation

$$M* = -*N \tag{3.59}$$

Furthermore, these boundary layer operators are related to the operator D in the following manner:

$$D^* = M - N \tag{3.60}$$

Proof: If u is a differential form-valued function defined on Γ, we have formally

$$-*Nu = -*\nu \wedge p.v.\, \delta \mathcal{S}u = (-1)^l \nu \vee *p.v.\, \delta \mathcal{S}u = \nu \vee d*p.v.\, \mathcal{S}u = M*u \tag{3.61}$$

and the first result follows. Next, use Remark 3.2.4 and the relation (2.65) to write

$$\begin{aligned} (M-N)u(x) =& p.v. \int_\Gamma \Big\{ \nu(x) \vee (\nabla_x \Phi_{k,n} \wedge u)(x,y) + \\ & \nu(x) \wedge (\nabla_x \Phi_{k,n} \vee u)(x,y) \Big\} d\Gamma(y) = \\ & p.v. \int_\Gamma \langle \nu_x, \nabla_x \Phi_{k,n} \rangle (x,y) u(y) d\Gamma(y) = \\ & p.v. \int_\Gamma \frac{\partial \Phi_{k,n}}{\partial \nu_x}(x,y) u(y) d\Gamma(y) \end{aligned} \tag{3.62}$$

for all $x \in \Gamma$.

■

As a final remark in this section, we note that additional types of layer potentials and associated boundary integral operators are used in the context of the analysis of boundary value problems (see, e.g., Reference [30].) However, the set of operators presented here is sufficient when dealing with boundary value problems based on the Helmholtz equation.

3.2.2 Regularity Results and Jump Relations - The Smooth Case

Assuming that the hypersurface Γ is *smooth*, we now perform an analysis of the associated layer potential operators $\mathcal{S}, \mathcal{D}$. This analysis yields, on one hand, the *jump relations* establishing the connection between the operators $\mathcal{S}, S$ and $\mathcal{D}, D$, respectively, and, on the other, the regularity results for the operators $\mathfrak{S}^{\pm}, \mathfrak{D}^{\pm}$.

By definition, any regular hypersurface Γ has an atlas associated with it. Since the Sobolev spaces $W^{s,p}(\Gamma, \Lambda^l)$, $s \in \mathbb{R}, p \in]1, \infty[, l \in \{0, \ldots, n\}$, are defined by transport and localisation using local coordinate charts on Γ, a regularity analysis of boundary operators on Γ involving these spaces may also be performed utilising transport and localisation.

Let $\widetilde{u}$ be a differential form-valued distribution defined on $\mathbb{R}^n$, and u a surface differential form-valued function on Γ, with $\widetilde{u} \equiv u\delta_D^\Gamma$, where δ_D^Γ is the suitable Dirac delta function. Furthermore, let (ω_i, ϕ_i) be a local coordinate chart pertaining to an atlas covering Γ, and let $\alpha_i \in C_0^\infty(\omega_i)$ be a function from a partition of unity associated with that atlas. By definition, the diffeomorphism $\phi_i : \omega_i \to \mathbb{R}^n$ effects a local 'straightening out' of the hypersurface Γ, mapping the patch $\Gamma_i \equiv \omega_i \cap \Gamma$ to a subset of the hyperplane $\{x \in \mathbb{R}^n | x_n = 0\}$ containing the origin. In this context, for each i, the usual coordinate system of $\mathbb{R}^n$ is referred to as the *local coordinate system*, and the set $\phi_i\Gamma_i$ as the *local coordinate patch*.[35] Hence, in the local coordinate system, the x_n-axis is perpendicular to the image $\phi_i\Gamma_i$ of the hypersurface element Γ_i, while the axes $x_1, \ldots, x_{n-1}$ can be chosen to run along the $n-1$ orthonormal basis vectors spanning the set $\{x \in \mathbb{R}^n | x_n = 0\}$.

Given a pseudodifferential operator $p(x, D)$ on $\mathbb{R}^{2n}$, with Schwartz kernel K, we define its localisation $p_i(x, D)$ to ω_i by

$$p_i(x, \xi) \equiv \alpha_i(\xi)p(x, \xi) \quad \forall x, \xi \in \mathbb{R}^n \tag{3.63}$$

Clearly, it holds true that

$$\left\{\sum_i \alpha_i p\right\}(x, D) = p(x, D) \tag{3.64}$$

in a tubular neighbourhood of Γ. Also, the vector $p_i(x, D)\widetilde{u}$, given by

$$p_i(x, D)\widetilde{u}(x) = \int_{\mathbb{R}^n} p_i(x, \xi)\widehat{\widetilde{u}}(\xi)e^{ix\cdot\xi}d\xi \quad \forall x \in \mathbb{R}^n \tag{3.65}$$

constitutes a localisation of $p(x, D)\widetilde{u}$ to Γ_i. Now let

$$\delta_D^t(y) = \delta_D(y - t) \quad \forall y, t \in \mathbb{R} \tag{3.66}$$

and furthermore define, for all i,

$$P_i(x, \xi) \equiv p_i(\phi_i^{-1}(x), \phi_i^{-1}(\xi)) \quad \forall x, \xi \in \mathbb{R}^n \tag{3.67}$$

Clearly, we have supp$\{P_i(x,\cdot)\} = \phi_i\omega_i$. Finally, let $U_i = u \circ \phi_i^{-1}$. In the local coordinate system associated with the unitary mapping ϕ_i, it holds true that

$$\begin{aligned} p_i(x,D)\widetilde{u}(x) = \int\limits_{\phi_i\omega_i} & P_i(x,\xi)\mathcal{F}_{y\mapsto\xi}(\delta_D^0 U_i)(\xi)e^{i\phi_i^{-1}x.\phi_i^{-1}\xi}d\xi = \\ & \int\limits_{\mathbb{R}^n} P_i(x,\xi)\mathcal{F}_{y'\mapsto\xi'}U_i(\xi')e^{ix.\xi}d\xi = \\ & \int\limits_{\mathbb{R}^{n-1}}\left\{\int\limits_{\mathbb{R}} P_i(x,\xi',\xi_n)e^{ix_n\xi_n}d\xi_n\right\}\widehat{U}_i(\xi')e^{ix'.\xi'}d\xi' = \\ & \int\limits_{\mathbb{R}^{n-1}} q_i(x',x_n,\xi')\widehat{U}_i(\xi')e^{ix'.\xi'}d\xi' \quad \forall x \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \end{aligned} \tag{3.68}$$

where the symbol q_i is given by

$$q_i(x',x_n,\xi') = \int\limits_{\mathbb{R}} P_i(x,\xi',\xi_n)e^{ix_n\xi_n}d\xi_n \quad \forall (x,\xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \tag{3.69}$$

By construction, it is true that supp$\{q_i(x,\cdot)\} = \phi_i\omega_i$. The exact local representation of the operator $p(x,D)$ is thus specified by the symbol given in local coordinates by (3.69). It furthermore holds true that

$$\begin{aligned} p(x,D)\widetilde{u}(x) = & \int\limits_{\mathbb{R}^{n-1}} q_i(x',x_n,\xi')\widehat{U}_i(\xi')e^{ix'.\xi'}d\xi' + \\ & \int\limits_{\Gamma\setminus\Gamma_i} K(\phi_i^{-1}x,y)u(y)d\Gamma(y) \quad \forall x \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \end{aligned} \tag{3.70}$$

in the local coordinate system associated with the chart (ω_i, ϕ_i).

We shall follow the references [36, 29] and write the local representation q of the principal symbol p_0 of a pseudodifferential operator in the form

$$\begin{aligned} q(x',x_n,\xi') = & \int\limits_{\mathbb{R}}\left\{\sum_i P_i(x,\xi',\xi_n)\right\}e^{ix_n\xi_n}d\xi_n = \\ & \int\limits_{\mathbb{R}} p(\phi_i^{-1}(x),\phi_i^{-1}(\xi))e^{ix_n\xi_n}d\xi_n \quad \forall (x,\xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \end{aligned} \tag{3.71}$$

where we use the local coordinates (x,ξ).

REMARK 3.2.6
By Proposition 2.5.13, the order $m \in \mathbb{R}$ of a pseudodifferential operator may be characterised in terms of the boundedness of the corresponding Schwartz kernel by terms proportional to $|x-y|^{-m-n}$. Consequently, the order m is invariant under coordinate transformations by diffeomorphisms ϕ_i.

□

We now present a general result concerning the regularity and jump relations of boundary operators associated with pseudodifferential operators on $\mathbb{R}^n$. Firstly, we need the following lemma:

LEMMA 3.2.7
Let $p \in S^m_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^n)$, $m \in \mathbb{R}$, be the symbol of a pseudodifferential operator. If $m < -1$, it holds true that the symbol $\tilde{p}$ defined by

$$\tilde{p}(x', \xi') \equiv \int_{\mathbb{R}} p(x', \xi', \xi_n) d\xi_n \quad \forall x', \xi' \in \mathbb{R}^{n-1} \tag{3.72}$$

is in the space $S^{m+1}_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$.
Proof: Clearly, we have for all $\alpha, \beta \in \mathbb{N}_0^{n-1}$ and all $x', \xi' \in \mathbb{R}^{n-1}$ that

$$|D^\beta_{x'} D^\alpha_{\xi'} \tilde{p}(x', \xi')| \leq \int_{\mathbb{R}} |D^\beta_{x'} D^\alpha_{\xi'} p(x', \xi', \xi_n)| d\xi_n \leq C_0 \int_{\mathbb{R}} \langle \xi \rangle^{m-|\alpha|} d\xi_n \tag{3.73}$$

For any $j < -1$, the integral

$$\int_{\mathbb{R}} \langle \xi \rangle^j d\xi_n \tag{3.74}$$

is convergent, and a calculation yields

$$\int_{\mathbb{R}} \langle \xi \rangle^j d\xi_n = \sqrt{\pi} \frac{\Gamma(-\frac{j+1}{2})}{\Gamma(-\frac{j}{2})} \langle \xi' \rangle^{j+1} \quad \forall j < -1,\, \xi' \in \mathbb{R}^{n-1} \tag{3.75}$$

Consequently, it is true that

$$|D^\beta_{x'} D^\alpha_{\xi'} \tilde{p}(x', \xi')| \leq C \langle \xi' \rangle^{m+1-|\alpha|} \quad \forall x', \xi' \in \mathbb{R}^{n-1},\, \alpha, \beta \in \mathbb{N}_0^{n-1} \tag{3.76}$$

■

PROPOSITION 3.2.8
Given a bounded smooth hypersurface $\Gamma \subset \mathbb{R}^n$ and $m \in \mathbb{Z}_-$, let $p(x, D)$ be an associated pseudodifferential operator in $OPS^m(\mathbb{R}^n \backslash \Gamma)$. We assume that the pertaining symbol is a rational function of (x, ξ), exhibiting only simple poles in ξ.

If $m < -1$, the boundary operators $\gamma_0 r_{\Omega_\pm} p(x, D)$ corresponding to $p(x, D)$ satisfy

$$\gamma_0 r_{\Omega_+} p(x, D) = \gamma_0 r_{\Omega_-} p(x, D) \in OPS^{m+1}(\Gamma) \tag{3.77}$$

If $m = -1$, and if the symbol p satisfies

$$p(x, \xi \pm \tau \nu_x) = \pm C(x, \xi) \tau^{-1} + \mathcal{O}(\tau^{-2}) \quad \forall x \in \Gamma \tag{3.78}$$

with ν_x denoting the unit outward normal vector to Γ at x, it holds true that

$$\gamma_0 r_{\Omega_\pm} p(x, D) \in OPS^0(\Gamma) \tag{3.79}$$

or, more precisely,

$$\gamma_0 r_{\Omega_\pm} p(x, D) \propto \mp i\pi I + h(x', D) \tag{3.80}$$

for some pseudodifferential operator $h(x', D) \in OPS^{-1}(\Gamma)$.
Proof: Retaining the notation of the above discussion, we rewrite for convenience the local representation (3.69) of the symbol of the operator $p(x, D)$:

$$q_i(x', x_n, \xi') = \int_{\mathbb{R}} P_i(x, \xi', \xi_n) e^{ix_n \xi_n} d\xi_n \quad \forall (x, \xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \tag{3.81}$$

For $m < -1$, the integral in (3.81) is absolutely convergent for all $x_n \neq 0$, in that we arrive at the following estimate:

$$\begin{aligned} |q_i(x, \xi')| \leq \int_{\mathbb{R}} |P_i(x, \xi', \xi_n)| d\xi_n \leq & C_0 \int_{\mathbb{R}} \langle \xi \rangle^m d\xi_n \leq C \langle \xi' \rangle^{m+1} \\ & \forall (x, \xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \end{aligned} \tag{3.82}$$

The inequality $|P_i(x, \xi)| \leq C_0 \langle \xi \rangle^m$ employed in the above derivation is deduced from Definition 2.5.3 of polyhomogeneous symbols, having in mind Remark 3.2.6 on invariance of symbol order under coordinate transformations. Also, the result (3.75) from the proof of Lemma 3.2.7 is used in (3.82). In light of (3.82), the function q_i is continuous in (x, ξ'). It therefore holds true that the symbol $\gamma_0 r_{\mathbb{R}^n_\pm} q_i$ is well-defined and may be written as

$$\gamma_0 r_{\mathbb{R}^n_\pm} q_i(x', \xi') = \int_{\mathbb{R}} P_i(x', 0, \xi', \xi_n) d\xi_n \quad \forall x', \xi' \in \mathbb{R}^{n-1} \tag{3.83}$$

Now, by Lemma 3.2.7, we obviously have

$$\gamma_0 r_{\mathbb{R}^n_\pm} q_i \in S^{m+1}_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}) \tag{3.84}$$

In fact, by Definition 2.5.3 and remarks 3.2.6 and 2.5.4, we know that there exists a set $\{f_{i,m-j}\}_{j \in \mathbb{N}_0}$ of ξ-homogeneous functions, each of degree of homogeneity $m - j$ and furthermore satisfying $f_{i,m-j} \in S^{m-j}_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^n)$, such that

$$P_i|_{x_n=0} - \sum_{j=0}^{N} f_{i,m-j} \in S^{m-N}_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^n) \quad \forall N \in \mathbb{N}_0 \tag{3.85}$$

Consequently, there exists by Lemma 3.2.7 a set of functions $\{g_{i,m-j}\}_{j \in \mathbb{N}_0}$, each homogeneous in ξ with degree $m-j$ and furthermore satisfying $g_{i,m-j} \in S^{m-j}_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^n)$ such that the following holds true:

$$\gamma_0 r_{\mathbb{R}^n_\pm} q_i - \sum_{j=0}^{N} g_{i,m-j} \in S^{m+1-N}_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^n) \quad \forall N \in \mathbb{N}_0 \tag{3.86}$$

Therefore, we may refine (3.84) to $\gamma_0 r_{\mathbb{R}^n_\pm} q_i \in S^{m+1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$.

For $m = -1$, it is true that

$$|P_i(x, \xi)| \leq C \langle \xi \rangle^{-1} \quad \forall x, \xi \in \mathbb{R}^n \tag{3.87}$$

and therefore, regarded as a function of ξ_n, the symbol P_i satisfies

$$P_i(x, \xi', \xi_n) = \mathcal{O}(|\xi_n|^{-1}) \quad \text{for } |\xi_n| \to \infty \tag{3.88}$$

Consequently, the local representation q_i of p, which is the inverse Fourier transform (3.69) of P_i with respect to the variable ξ_n, is well-defined and smooth for all $x_n \neq 0$. [36] Now according to Reference [36], if the symbol p satisfies (3.78), it then holds true that

$$\gamma_0 r_{\Omega_\pm} p(x,D) \in OPS^0(\Gamma) \tag{3.89}$$

Let us now evaluate q_i in (3.81) for $m \leq -1$. Apart from a confirmation of the continuity of q_i in the case $m < -1$, we also obtain in this way the jump relation (3.80). We extend the integrand in (3.81) to the whole complex plane, obtaining

$$I(x,\xi') = \int_{\mathbb{R}} P_i(x,\xi',z)e^{ix_n z}dz \quad \forall (x,\xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \tag{3.90}$$

Clearly, for $x_n > 0$ ($x_n < 0$), the integrand in (3.90) is bounded and decaying in the upper (lower) complex half-plane. For every $z \in \mathbb{C}\backslash\{0\}$, we denote by $\gamma_{|z|}$ the oriented curve (upper half circle) in $\mathbb{C}$ parameterised by $\gamma_{|z|}(\varphi) = |z|e^{i\varphi}, \varphi \in [0,\pi]$. Similarly, the symbol $-\gamma_{|z|}$ denotes, for $z \in \mathbb{C}\backslash\{0\}$, the oriented curve (lower half circle) parameterised by $-\gamma_{|z|}(\varphi) = |z|e^{i\varphi}, \varphi \in [0,-\pi]$. By assumption, the function P_i is rational (in that the rectangular, 'global' coordinate system in $\mathbb{R}^n$ is mapped by the diffeomorphism ϕ_i to a rectangular local coordinate system,) and only exhibits simple poles in the second variable (since the mapping ϕ_i is a diffeomorphism and hence is injective,) so straightforward application of the residue theory yields

$$I(x,\xi') = \pm\, 2\pi i \sum_j^{\pm} \mathrm{Res}(P_i e^{ix_n z}, y_j) - \lim_{|z|\to\infty} \int_{\pm\gamma_{|z|}} P_i(x,\xi',z)e^{ix_n z}dz \tag{3.91}$$

for all $(x,\xi') \in \mathbb{R}^n_\pm \times \mathbb{R}^{n-1}$, where the summation is performed of the residuals of $P_i e^{ix_n z}$ associated with the poles y_j in the upper and lower half of the complex plane, respectively. The properties of the function P_i imply

$$\sum_j^{+} \mathrm{Res}(P_i e^{ix_n z}, y_j) = -\sum_j^{-} \mathrm{Res}(P_i e^{ix_n z}, y_j) \tag{3.92}$$

Furthermore, for all $p(x,D) \in OPS^m(\mathbb{R}^n)$, $m \in \mathbb{Z}_-$, we have

$$\begin{aligned}
\lim_{|z|\to\infty} \int_{\pm\gamma_{|z|}} P_i(x,\xi',z)e^{ix_n z}dz \propto \lim_{|z|\to\infty} \int_{\pm\gamma_{|z|}} z^m e^{ix_n z}dz &= \\
\frac{1}{2}\Big\{ \pm 2\pi i \mathrm{Res}(z^m e^{ix_n z}, 0) \Big\} &= \\
\pm\, i\pi \frac{1}{(|m|-1)!} \lim_{z\to 0}\Big\{ \frac{d^{|m|-1}}{dz^{|m|-1}} e^{ix_n z} \Big\} &= \\
\pm\, i\pi \frac{(ix_n)^{|m|-1}}{(|m|-1)!} \quad \forall (x,\xi') \in \mathbb{R}^n_\pm \cup \mathbb{R}^{n-1} &
\end{aligned} \tag{3.93}$$

and hence we arrive at

$$q_i(x',x_n,\xi') = 2\pi i \sum_j^{+} \mathrm{Res}(P_i e^{ix_n z}, y_j) \mp Ci\pi \frac{(ix_n)^{|m|-1}}{(|m|-1)!} \quad \forall (x,\xi') \in \mathbb{R}^n_\pm \cup \mathbb{R}^{n-1} \tag{3.94}$$

for all $m \in \mathbb{Z}_-$, with C denoting a constant. Now for $|m| > 1$ it is true that

$$\lim_{x_n \searrow 0} \{q_i(x', x_n, \xi')\} = \lim_{x_n \nearrow 0} \{q_i(x', x_n, \xi')\} = 2\pi i \sum_j^+ \operatorname{Res}(P_i|_{x_n=0}, y_j) \quad (3.95)$$

for all $x', \xi' \in \mathbb{R}^{n-1}$, and we recover the already found continuity of q_i with respect to x_n. For $m = -1$, we have

$$\begin{aligned} \lim_{x_n \searrow 0} \{q_i(x', x_n, \xi')\} &= 2\pi i \sum_j^+ \operatorname{Res}(P_i|_{x_n=0}, y_j) - Ci\pi \quad \forall x', \xi' \in \mathbb{R}^{n-1} \\ \lim_{x_n \nearrow 0} \{q_i(x', x_n, \xi')\} &= 2\pi i \sum_j^+ \operatorname{Res}(P_i|_{x_n=0}, y_j) + Ci\pi \quad \forall x', \xi' \in \mathbb{R}^{n-1} \end{aligned} \quad (3.96)$$

The result follows.

■

Note that our proof of the limiting expressions in (3.77) and (3.80) of Proposition 3.2.8 differs considerably from the approach to the matter presented in Reference [36]. In particular, we obtain (general) expressions for the localisations q_i of the symbols of the considered pseudodifferential operators.

The following holds true:

PROPOSITION 3.2.9
The layer potential operators $\mathcal{S}, \mathcal{D}, \mathcal{D}^*$ associated with a smooth hypersurface $\Gamma \subset \mathbb{R}^n$ satisfy

$$\begin{aligned} \mathcal{S} &\in OPS^{-2}(\mathbb{R}^n \backslash \Gamma) \\ \mathcal{D}, \mathcal{D}^* &\in OPS^{-1}(\mathbb{R}^n \backslash \Gamma) \end{aligned} \quad (3.97)$$

Proof: Clearly, the Schwartz kernels of the operators $\mathcal{S}_{k,n}, \mathcal{D}_{k,n}, \mathcal{D}^*_{k,n}$ are the fundamental solution $\Phi_{k,n}$ and its directional derivatives $\partial\Phi_{k,n}/\partial\nu_y, \partial\Phi_{k,n}/\partial\nu_x$, respectively. The result now follows immediately from Proposition 2.5.13 and Remark 2.8.1.

■

In view of Proposition 3.2.9, the results of Proposition 3.2.8 establish the regularity of some boundary integral operators of Section 3.2. Also, we obtain a form of the associated jump relations:

PROPOSITION 3.2.10
Given any smooth hypersurface $\Gamma \subset \mathbb{R}^n$, the pertaining layer potential (interior and exterior) boundary integral operators $\mathfrak{S}^\pm, \mathfrak{D}^\pm$ satisfy

$$\begin{aligned} \mathfrak{S}^\pm &\in OPS^{-1}(\Gamma) \\ \mathfrak{D}^\pm, \mathfrak{D}^{*\,\pm} &\in OPS^{0}(\Gamma) \end{aligned} \quad (3.98)$$

Furthermore, it is true that

$$\mathfrak{S}^+ = \mathfrak{S}^- \quad (3.99)$$

and the difference operators $(\mathfrak{D}^+ - \mathfrak{D}^-)$, $(\mathfrak{D}^{*\,+} - \mathfrak{D}^{*\,-})$ are nonvanishing.

■

Recalling Proposition 2.5.26, as well as the fact that $OPS^m \subset OPS^m_{1,0}$, we see that this last result has the following immediate consequence for the boundary integral operators $\mathfrak{S}^\pm, \mathfrak{D}^\pm, \mathfrak{D}^{*\,\pm}$:

PROPOSITION 3.2.11

Let $\Gamma \subset \mathbb{R}^n$ be a smooth hypersurface. For any $s \in \mathbb{R}$ and $l \in \mathbb{N}_0, n \in \mathbb{N}$ such that $l \leq n$, the operators

$$\begin{aligned} \mathfrak{S}^\pm &: H^s(\Gamma) \to H^{s+1}(\Gamma, \Lambda^l) \\ \mathfrak{D}^\pm &: H^s(\Gamma) \to H^s(\Gamma, \Lambda^l) \\ \mathfrak{D}^{*\,\pm} &: H^s(\Gamma) \to H^s(\Gamma, \Lambda^l) \end{aligned} \tag{3.100}$$

are continuous.

■

In order to apply the general results of Proposition 3.2.8 to the layer potentials and boundary integral operators of interest to us, we may calculate explicitly the expressions of the symbols of these operators. Clearly, once these expressions are found, the regularity of the operators in question can be determined directly using the definition of the symbol spaces $S^m_{\rho,\delta}$. However, it turns out that the calculation of the local representations of the boundary integral operators associated with the layer potentials allows for derivation of more detailed jump relations than those presented in Proposition 3.2.8. Also, we justify the explicit calculations by the desire for completeness in our exposition.

In Section 2.5 on pseudodifferential operators, we have found a simple relation between the symbol p and the Schwartz kernel K of an operator $p(x, D) \in OPS^m_{\rho,\delta}(\Omega)$, namely

$$K(x, y) = (2\pi)^{-n} \mathcal{F}^{-1}_{\xi \mapsto y} e_{\mathbb{R}^{2n}} p(x, x - y) \quad \forall x, y \in \mathbb{R}^n \tag{3.101}$$

In the following, we shall for notational convenience assume that $p = e_{\mathbb{R}^{2n}} p$.

Let us firstly turn attention to the acoustic single layer potential operator $\mathcal{S}$, and hence to the special case where the Schwartz kernel $K_\mathcal{S}$ is the fundamental solution $\Phi_{k,n}$ (see Section 2.8) of the Helmholtz equation in $\mathbb{R}^n$, i.e. $(\Delta + k^2 I) K_\mathcal{S}(x, y) = \delta_D(x - y)$ for all $x, y \in \mathbb{R}^n$. Application of the Fourier transform operator $\mathcal{F}_{y \mapsto \xi}$ on both sides of equation (3.101) yields

$$p^{k,n}_\mathcal{S}(x, \xi) = (2\pi)^n e^{-ix.\xi} \mathcal{F}_{y \mapsto \xi} \Phi_{k,n}(x, -\xi) \quad \forall x, \xi \in \mathbb{R}^n \tag{3.102}$$

where $p^{k,n}_\mathcal{S}$ is the symbol associated with the Schwartz kernel $\Phi_{k,n}$. By construction, it is true that

$$\mathcal{F}_{y \mapsto \xi} \Phi_{k,n}(x, \xi) = \frac{e^{-ix.\xi}}{|\xi|^2 + k^2} \quad \forall x, \xi \in \mathbb{R}^n \tag{3.103}$$

and thus we have

$$p^{k,n}_\mathcal{S}(x, \xi) = (2\pi)^n \frac{1}{|\xi|^2 + k^2} \quad \forall x, \xi \in \mathbb{R}^n \tag{3.104}$$

In view of Proposition 3.2.8 and Proposition 3.2.9, we now see that

$$\mathfrak{S}^\pm \in OPS^{-1}(\Gamma) \tag{3.105}$$

Note that the symbol $p_{\mathcal{S}}^{k,n}$ of the single layer potential operator is a function of $|\xi|$ only, and thus it is invariant to transformations by the local diffeomorphisms ϕ_i. Furthermore, it is seen that this symbol is locally given by

$$q_{\mathcal{S}}^{k,n}(x', x_n, \xi') = (2\pi)^n \int_{\mathbb{R}} \frac{e^{ix_n\xi_n}}{\xi_n^2 + |\xi'|^2 + k^2} d\xi_n \quad \forall (x, \xi') \in \mathbb{R}_+^n \cup \mathbb{R}_-^n \times \mathbb{R}^{n-1} \tag{3.106}$$

Based on an elementary discussion found in Section A.1, we have

$$\int_{\mathbb{R}} \frac{e^{ix_n\xi_n}}{\xi_n^2 + |\xi'|^2 + k^2} d\xi_n = \begin{cases} -\int_{\mathbb{R}} \frac{e^{ix_n\xi_n}}{2i\sqrt{|\xi'|^2+k^2}} \left\{ \frac{1}{\xi_n + i\sqrt{|\xi'|^2+k^2}} \right\} d\xi_n, & x_n < 0 \\ \int_{\mathbb{R}} \frac{e^{ix_n\xi_n}}{2i\sqrt{|\xi'|^2+k^2}} \left\{ \frac{1}{\xi_n - i\sqrt{|\xi'|^2+k^2}} \right\} d\xi_n, & x_n > 0 \end{cases} \tag{3.107}$$

Consequently, we may write the following for the local representations $q_{\mathcal{S}^\pm}^{k,n}$ of the symbols of operators $r_{\Omega_\pm}\mathcal{S}$:

$$q_{\mathcal{S}^+}^{k,n} = -(2\pi)^n \int_{\mathbb{R}} \frac{e^{ix_n\xi_n}}{2i\sqrt{|\xi'|^2+k^2}} \left\{ \frac{1}{\xi_n + i\sqrt{|\xi'|^2+k^2}} \right\} d\xi_n \quad \forall (x, \xi') \in \mathbb{R}_-^n \times \mathbb{R}^{n-1} \tag{3.108}$$

$$q_{\mathcal{S}^-}^{k,n} = (2\pi)^n \int_{\mathbb{R}} \frac{e^{ix_n\xi_n}}{2i\sqrt{|\xi'|^2+k^2}} \left\{ \frac{1}{\xi_n - i\sqrt{|\xi'|^2+k^2}} \right\} d\xi_n \quad \forall (x, \xi') \in \mathbb{R}_+^n \times \mathbb{R}^{n-1} \tag{3.109}$$

In Section A.1, an evaluation is performed of the integrals appearing in (3.108) and (3.109), and it is found that

$$\int_{\mathbb{R}} \frac{e^{ix_n\xi_n}}{\xi_n^2 + |\xi'|^2 + k^2} d\xi_n = \frac{\pi e^{-|x_n|\sqrt{|\xi'|^2+k^2}}}{\sqrt{|\xi'|^2+k^2}} \quad \forall x_n \in \mathbb{R}\backslash\{0\} \tag{3.110}$$

As a result, we have the following explicit local expression for the symbol $p_{\mathcal{S}}^{k,n}$ of the single layer potential operator:

$$q_{\mathcal{S}}^{k,n}(x', x_n, \xi') = (2\pi)^n \pi \frac{e^{-|x_n|\sqrt{|\xi'|^2+k^2}}}{\sqrt{|\xi'|^2+k^2}} \quad \forall (x, \xi') \in \mathbb{R}_+^n \cup \mathbb{R}_-^n \times \mathbb{R}^{n-1} \tag{3.111}$$

Furthermore, since x_n is the normal coordinate with respect to the hypersurface Γ in the current local representation, the symbols $q_{\mathfrak{S}^\pm}^{k,n}$ of the boundary layer operators $\mathfrak{S}^\pm = \gamma_0 r_{\Omega_\pm}\mathcal{S}$ in the local coordinates are obtained by taking the limit $x_n \nearrow 0$ and $x_n \searrow 0$ in the expressions (3.108) and (3.109), respectively. From (3.111), we have

$$\lim_{x_n \searrow 0} \{q_{\mathcal{S}}^{k,n}(x', x_n, \xi')\} = \lim_{x_n \nearrow 0} \{q_{\mathcal{S}}^{k,n}(x', x_n, \xi')\} = \frac{(2\pi)^n \pi}{\sqrt{|\xi'|^2+k^2}} \quad \forall x', \xi' \in \mathbb{R}^{n-1} \tag{3.112}$$

Thus, it holds true that

$$q^{k,n}_{\mathfrak{S}\pm}(x',\xi') = \frac{(2\pi)^n \pi}{\sqrt{|\xi'|^2+k^2}} \quad \forall x',\xi' \in \mathbb{R}^{n-1} \tag{3.113}$$

and we may write $q^{k,n}_{\mathfrak{S}} \equiv q^{k,n}_{\mathfrak{S}+} = q^{k,n}_{\mathfrak{S}-}$.

Note that the essence of the discussion presented above can actually be expressed by the sole result (3.110). This, however, is a characteristic of the single layer potential operator, and we decide to develop a complete argument on the grounds of clarity and consistency with the following analogous discussion of the double layer potential operator $\mathcal{D}$ and of the operators $\mathcal{D}^*,\mathcal{N}$.

In the introduced local coordinates, the kernels $K_{\mathcal{D}},K_{\mathcal{D}^*},K_{\mathcal{N}}$ of the operators $\mathcal{D},\mathcal{D}^*,\mathcal{N}$ are clearly given by

$$\begin{aligned} K_{\mathcal{D}} &= \frac{\partial}{\partial \nu_y} K_{\mathcal{S}} = \frac{\partial}{\partial x_n} K_{\mathcal{S}} \\ K_{\mathcal{D}^*} &= \frac{\partial}{\partial \nu_x} K_{\mathcal{S}} = \frac{\partial}{\partial x_n} K_{\mathcal{S}} \\ K_{\mathcal{N}} &= \frac{\partial^2}{\partial \nu_x \partial \nu_y} K_{\mathcal{S}} = \frac{\partial^2}{\partial x_n^2} K_{\mathcal{S}} \end{aligned} \tag{3.114}$$

Hence, using the localised version of (3.101), we have

$$\begin{aligned} K_{\mathcal{D}} = K_{\mathcal{D}^*} &= i\xi_n K_{\mathcal{S}} \\ K_{\mathcal{N}} &= -\xi_n^2 K_{\mathcal{S}} \end{aligned} \tag{3.115}$$

and consequently, in view of (3.106), the local expressions of the symbols of $\mathcal{D},\mathcal{D}^*,\mathcal{N}$ take the form

$$\begin{aligned} q^{k,n}_{\mathcal{D}}(x',x_n,\xi') =& q^{k,n}_{\mathcal{D}^*}(x',x_n,\xi') = \\ & i(2\pi)^n \int_{\mathbb{R}} \frac{\xi_n e^{ix_n\xi_n}}{\xi_n^2+|\xi'|^2+k^2} d\xi_n \quad \forall (x,\xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \\ q^{k,n}_{\mathcal{N}}(x',x_n,\xi') =& -(2\pi)^n \int_{\mathbb{R}} \frac{\xi_n^2 e^{ix_n\xi_n}}{\xi_n^2+|\xi'|^2+k^2} d\xi_n \\ & \forall (x,\xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \end{aligned} \tag{3.116}$$

Referring to Section A.1, we may write the following for the local representations $q^{k,n}_{\mathcal{D}\pm}, q^{k,n}_{\mathcal{D}^*\pm}, q^{k,n}_{\mathcal{N}\pm}$,:

$$\begin{aligned} q^{k,n}_{\mathcal{D}+}(x',x_n,\xi') =& q^{k,n}_{\mathcal{D}^*+}(x',x_n,\xi') = \\ & i(2\pi)^n \frac{1}{2} \int_{\mathbb{R}} \frac{e^{ix_n\xi_n}}{\xi_n + i\sqrt{|\xi'|^2+k^2}} d\xi_n \quad \forall (x,\xi') \in \mathbb{R}^n_- \times \mathbb{R}^{n-1} \end{aligned} \tag{3.117}$$

$$\begin{aligned} q^{k,n}_{\mathcal{D}-}(x',x_n,\xi') =& q^{k,n}_{\mathcal{D}^*-}(x',x_n,\xi') = i(2\pi)^n \frac{1}{2} \int_{\mathbb{R}} \frac{e^{ix_n\xi_n}}{\xi_n - i\sqrt{|\xi'|^2+k^2}} d\xi_n \\ & \forall (x,\xi') \in \mathbb{R}^n_+ \times \mathbb{R}^{n-1} \end{aligned} \tag{3.118}$$

$$q_{\mathcal{N}^+}^{k,n}(x', x_n, \xi') = -(2\pi)^n \frac{1}{2} \int\limits_{\mathbb{R}} \frac{\xi_n e^{i x_n \xi_n}}{\xi_n^2 + i\sqrt{|\xi'|^2 + k^2}} d\xi_n \quad \forall (x, \xi') \in \mathbb{R}^n_- \times \mathbb{R}^{n-1} \tag{3.119}$$

$$q_{\mathcal{N}^-}^{k,n}(x', x_n, \xi') = -(2\pi)^n \frac{1}{2} \int\limits_{\mathbb{R}} \frac{\xi_n e^{i x_n \xi_n}}{\xi_n^2 - i\sqrt{|\xi'|^2 + k^2}} d\xi_n \quad \forall (x, \xi') \in \mathbb{R}^n_+ \times \mathbb{R}^{n-1} \tag{3.120}$$

The integrals occurring in (3.117)-(3.120) are computed in Section A.1, with the result

$$\int\limits_{\mathbb{R}} \frac{\xi_n^j e^{i x_n \xi_n}}{\xi_n^2 + \sqrt{|\xi'|^2 + k^2}} d\xi_n = (\pm i)^j \pi (|\xi'|^2 + k^2)^{(j-1)/2} e^{-|x_n|\sqrt{|\xi'|^2+k^2}} \qquad x_n \gtrless 0, j \in \{1, 2\} \tag{3.121}$$

Hence, we obtain the following explicit local expressions for the symbols $p_{\mathcal{D}}^{k,n}, p_{\mathcal{D}^*}^{k,n}, p_{\mathcal{N}}^{k,n}$ of the considered layer potential operators:

$$q_{\mathcal{D}}^{k,n}(x', x_n, \xi') = q_{\mathcal{D}^*}^{k,n}(x', x_n, \xi') = \begin{cases} -(2\pi)^n \pi e^{-x_n\sqrt{|\xi'|^2+k^2}} & \forall (x, \xi') \in \mathbb{R}^n_+ \times \mathbb{R}^{n-1} \\ (2\pi)^n \pi e^{x_n\sqrt{|\xi'|^2+k^2}} & \forall (x, \xi') \in \mathbb{R}^n_- \times \mathbb{R}^{n-1} \end{cases}$$

$$q_{\mathcal{N}}^{k,n}(x', x_n, \xi') = (2\pi)^n \pi \sqrt{|\xi'|^2 + k^2} e^{-|x_n|\sqrt{|\xi'|^2+k^2}} \quad \forall (x, \xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \tag{3.122}$$

This clearly implies

$$\begin{aligned} \lim_{x_n \searrow 0} \{q_{\mathcal{D}}^{k,n}(x', x_n, \xi')\} &= \lim_{x_n \searrow 0} \{q_{\mathcal{D}^*}^{k,n}(x', x_n, \xi')\} = -(2\pi)^n \pi \quad \forall x', \xi' \in \mathbb{R}^{n-1} \\ \lim_{x_n \nearrow 0} \{q_{\mathcal{D}}^{k,n}(x', x_n, \xi')\} &= \lim_{x_n \searrow 0} \{q_{\mathcal{D}^*}^{k,n}(x', x_n, \xi')\} = (2\pi)^n \pi \quad \forall x', \xi' \in \mathbb{R}^{n-1} \\ \lim_{x_n \searrow 0} \{q_{\mathcal{N}}^{k,n}(x', x_n, \xi')\} &= \lim_{x_n \nearrow 0} \{q_{\mathcal{D}}^{k,n}(x', x_n, \xi')\} = (2\pi)^n \pi \sqrt{|\xi'|^2 + k^2} \quad \forall x', \xi' \in \mathbb{R}^{n-1} \end{aligned} \tag{3.123}$$

Now, since by definition

$$\begin{aligned} q_{\mathfrak{D}^\pm}^{k,n} &= \gamma_0 r_{\Omega_\pm} q_{\mathcal{D}}^{k,n} \\ q_{\mathfrak{D}^*\pm}^{k,n} &= \gamma_0 r_{\Omega_\pm} q_{\mathcal{D}^*}^{k,n} \\ q_{\mathfrak{N}^\pm}^{k,n} &= \gamma_0 r_{\Omega_\pm} q_{\mathcal{N}}^{k,n} \end{aligned} \tag{3.124}$$

it holds true that the local expressions for the symbols of the (inner or outer) boundary integral operators $\mathcal{D}, \mathcal{D}^*, \mathcal{N}$ are given by

$$\begin{aligned} q_{\mathfrak{D}^\pm}^{k,n}(x', \xi') &= \pm\pi(2\pi)^n \quad \forall (x', \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \\ q_{\mathfrak{D}^*\pm}^{k,n}(x', \xi') &= \pm\pi(2\pi)^n \quad \forall (x', \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \\ q_{\mathfrak{N}^\pm}^{k,n}(x', \xi') &= \pi(2\pi)^n \pi \sqrt{|\xi'|^2 + k^2} \quad \forall (x', \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \end{aligned} \tag{3.125}$$

Note that the functions $q_{\mathfrak{D}^\pm}^{k,n}, q_{\mathfrak{D}^*\pm}^{k,n}$ are independent of the wavenumber k.

REMARK 3.2.12

For $k = 0$, the above analysis specialises to the case of pseudodifferential operators whose kernels are fundamental solutions of the Laplace equation. It is seen from the expressions (3.111), (3.122) that the symbols of the layer potential operators $\mathcal{S}, \mathcal{D}, \mathcal{D}^*, \mathcal{N}$

are in this case given by (in local representation)

$$\begin{aligned} q_{\mathcal{S}}^{0,n}(x',\xi') &= (2\pi)^n \pi e^{-|x_n||\xi'|}|\xi'|^{-1} \quad \forall (x,\xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \\ q_{\mathcal{D}}^{0,n}(x',\xi') = q_{\mathcal{D}^*}^{0,n}(x',\xi') &= \mp (2\pi)^n \pi e^{-|x_n||\xi'|} \quad \forall (x,\xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \\ q_{\mathcal{N}}^{0,n}(x',\xi') &= (2\pi)^n \pi |\xi'| e^{-|x_n||\xi'|} \quad \forall (x,\xi') \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- \times \mathbb{R}^{n-1} \end{aligned} \tag{3.126}$$

□

In the above discussion, the local representations of the symbols of the boundary integral operators $\mathfrak{S}^{\pm}, \mathfrak{D}^{\pm}, \mathfrak{D}^{*\,\pm}, \mathfrak{N}^{\pm}$ were found to satisfy

$$\begin{aligned} q_{\mathfrak{S}^{\pm}}^{k,n}(x',\xi') &\propto (k^2+|\xi'|^2)^{-1/2} \quad \forall x',\xi' \in \mathbb{R}^{n-1} \\ q_{\mathfrak{D}^{\pm}}^{k,n}(x',\xi') &\propto 1 \quad \forall x',\xi' \in \mathbb{R}^{n-1} \\ q_{\mathfrak{D}^{*\,\pm}}^{k,n}(x',\xi') &\propto 1 \quad \forall x',\xi' \in \mathbb{R}^{n-1} \\ q_{\mathfrak{N}^{\pm}}^{k,n}(x',\xi') &\propto (k^2+|\xi'|^2)^{1/2} \quad \forall x',\xi' \in \mathbb{R}^{n-1} \end{aligned} \tag{3.127}$$

Obviously, the function $q_{\mathfrak{S}^+}^{k,n} = q_{\mathfrak{S}^-}^{k,n}$ is bounded by $\langle\xi'\rangle^{-1}$, while $q_{\mathfrak{D}^{\pm}}^{k,n}, q_{\mathfrak{D}^{*\,\pm}}^{k,n}$ are bounded by $\langle\xi'\rangle^{0}$, on the whole domain $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. Also, the local symbol $q_{\mathfrak{N}^+}^{k,n} = q_{\mathfrak{N}_-}^{k,n}$ is bounded by $\langle\xi'\rangle$ on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. This, in turn, implies that

$$\begin{aligned} |D_{x'}^{\beta} D_{\xi'}^{\alpha} q_{\mathfrak{S}^{\pm}}^{k,n}(x',\xi')| &\le C\langle\xi'\rangle^{-1-|\alpha|} \quad \forall x',\xi' \in \mathbb{R}^{n-1} \\ |D_{x'}^{\beta} D_{\xi'}^{\alpha} q_{\mathfrak{D}^{\pm}}^{k,n}(x',\xi')| &\le C\langle\xi'\rangle^{-|\alpha|} \quad \forall x',\xi' \in \mathbb{R}^{n-1} \\ |D_{x'}^{\beta} D_{\xi'}^{\alpha} q_{\mathfrak{D}^{*\,\pm}}^{k,n}(x',\xi')| &\le C\langle\xi'\rangle^{-|\alpha|} \quad \forall x',\xi' \in \mathbb{R}^{n-1} \\ |D_{x'}^{\beta} D_{\xi'}^{\alpha} q_{\mathfrak{N}^{\pm}}^{k,n}(x',\xi')| &\le C\langle\xi'\rangle^{1-|\alpha|} \quad \forall x',\xi' \in \mathbb{R}^{n-1} \end{aligned} \tag{3.128}$$

The first three estimates above are, of course, implied by Proposition 3.2.8 and Proposition 3.2.9, but the last inequality cannot be inferred from these results.

Let us summarise the obtained regularity results:

PROPOSITION 3.2.13
Given a smooth hypersurface $\Gamma \subset \mathbb{R}^n$, the following holds true for the associated boundary value operators:

$$\begin{aligned} \mathfrak{S}^{\pm} &\in OPS^{-1}(\Gamma) \\ \mathfrak{D}^{\pm} &\in OPS^{0}(\Gamma) \\ \mathfrak{D}^{*\,\pm} &\in OPS^{0}(\Gamma) \\ \mathfrak{N}^{\pm} &\in OPS^{1}(\Gamma) \end{aligned} \tag{3.129}$$

■

We are now also in a position to establish the precise jump relations involving the operator pairs $(\mathcal{S}, S)$, $(\mathcal{D}, D)$, $(\mathcal{D}^*, D^*)$ and $(\mathcal{N}, \mathscr{N})$.

PROPOSITION 3.2.14
Jump Relations for the Boundary Integral Operators $\mathfrak{S}, \mathfrak{D}, \mathfrak{D}^*, \mathfrak{N}$
Let $\Gamma \subset \mathbb{R}^n$ be an $n-1$-dimensional, smooth hypersurface. The following holds true

for any $u \in L^p(\Gamma, \Lambda^l)$, $p \in]1, \infty[$, $l \in \{0, \dots, n\}$:

$$\lim_{x \to x_0 \pm} \{\mathcal{S}u(x)\} = \mathfrak{S}^{\pm} u(x_0) = p.v. \int_\Gamma \Phi_{k,n}(x_0, y) u(y) d\Gamma(y) = Su(x_0) \quad \forall x_0 \in \Gamma \tag{3.130}$$

$$\lim_{x \to x_0 \pm} \{\mathcal{D}u(x)\} = \mathfrak{D}^{\pm} u(x_0) = \pm\, (2\pi)^n \pi u(x_0) + p.v. \int_\Gamma \frac{\partial \Phi_{k,n}}{\partial \nu_y}(x_0, y) u(y) d\Gamma(y) = \\ \pm\, (2\pi)^n \pi u(x_0) + Du(x_0) \quad \forall x_0 \in \Gamma \tag{3.131}$$

$$\lim_{x \to x_0 \pm} \{\mathcal{D}^* u(x)\} = \mathfrak{D}^{\pm} u(x_0) = \pm\, (2\pi)^n \pi u(x_0) + p.v. \int_\Gamma \frac{\partial \Phi_{k,n}}{\partial \nu_x}(x_0, x) u(y) d\Gamma(y) = \\ \pm\, (2\pi)^n \pi u(x_0) + D^* u(x_0) \quad \forall x_0 \in \Gamma \tag{3.132}$$

$$\lim_{x \to x_0 \pm} \{\mathcal{N}u(x)\} = \mathfrak{N}^{\pm} u(x_0) = p.v. \int_\Gamma \frac{\partial^2 \Phi_{k,n}}{\partial \nu_x \partial \nu_y}(x_0, y) u(y) d\Gamma(y) = \mathscr{N} u(x_0) \quad \forall x_0 \in \Gamma \tag{3.133}$$

Proof: Clearly, the result $\mathfrak{S}^+ = \mathfrak{S}^-$ follows immediately from Proposition 3.2.8. Assume $u \in L^p(\Gamma, \Lambda^l)$ for some $p \in]1, \infty[$, $l \in \{0, \dots, n\}$. For any $x_0 \in \Gamma$, we consider the open ball $B_{\mathbb{R}^n}(x_0, \varepsilon)$ with centre at x_0 and of radius $\varepsilon > 0$. Since the hypersurface Γ is regular, there exists a local chart $(B_{\mathbb{R}^n}(x_0, \varepsilon), \phi_\varepsilon)$ with the diffeomorphism ϕ_ε mapping the ball $B_{\mathbb{R}^n}(x_0, \varepsilon)$ into a neighbourhood of the origin in $\mathbb{R}^n$ and satisfying $\phi_\varepsilon x_0 = 0$. For notational convenience, we set $\Gamma_{x_0,\varepsilon} \equiv \Gamma \backslash B_{\mathbb{R}^n}(x_0, \varepsilon)$. By the expressions in (3.113) for the local representations of the symbols of $\mathfrak{S}^{\pm}$, it holds true that, for all $x_0 \in \Gamma$,

$$\lim_{x \to x_0 \pm} \{\mathcal{S}u(x)\} = (2\pi)^n \pi \int_{\phi_\varepsilon B_{\mathbb{R}^n}(x_0, \varepsilon)} \frac{\hat{u}(\xi')}{\sqrt{|\xi'|^2 + k^2}} d\xi' + \int_{\Gamma_{x_0,\varepsilon}} \Phi_{k,n}(x_0, y) u(y) d\Gamma(y) \tag{3.134}$$

Due to the regularity of the function $\mathcal{F}_{y' \mapsto \xi'} u$, the first term in the above equation admits the following (writing the integration variable ξ' in the polar coordinates (r, θ) in $\mathbb{R}^{n-1}$):

$$\lim_{\varepsilon \to 0} \int_{\phi_\varepsilon B_{\mathbb{R}^n}(x_0, \varepsilon)} \frac{\hat{u}(\xi')}{\sqrt{|\xi'|^2 + k^2}} d\xi' = \lim_{\varepsilon \to 0} \int_0^\varepsilon \int_\theta \frac{\hat{u}(\xi')}{\sqrt{r^2 + k^2}} r d\theta dr = \\ \hat{u}(0) \sigma_{n-1} \lim_{\varepsilon \to 0} \int_0^\varepsilon \frac{r}{\sqrt{r^2 + k^2}} dr = 0 \tag{3.135}$$

By definition of a principal value integral, we may write the following for the second term in (3.134):

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{x_0,\varepsilon}} \Phi_{k,n}(x_0, y) u(y) d\Gamma(y) = p.v. \int_\Gamma \Phi_{k,n}(x_0, y) u(y) d\Gamma(y) = Su(x_0) \tag{3.136}$$

Hence, the first result follows. Next, by the expressions in (3.125) for the local representations of the symbols of $\mathfrak{D}^{\pm}$, we have

$$\lim_{x \to x_0 \pm} \{\mathcal{D}u(x)\} = \mp(2\pi)^n \pi \int\limits_{\mathbb{R}^{n-1}} \hat{u}(\xi')d\xi' + \int\limits_{\Gamma_{x_0,\varepsilon}} \frac{\partial \Phi_{k,n}}{\partial \nu_y}(x_0, y)u(y)d\Gamma(y) \quad (3.137)$$

For the first term in the above expression, it clearly holds true that

$$\lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^{n-1}} \hat{u}(\xi')d\xi' = \lim_{\varepsilon \to 0} \{\mathcal{F}^{-1}_{\xi' \mapsto y'} \mathcal{F}_{y' \mapsto \xi'} u|_{y_n = 0}(0)\} = u(0) \quad (3.138)$$

We recall that $x_0 = 0$ in the local coordinates. By definition of a principal value integral, it is true that

$$\lim_{\varepsilon \to 0} \int\limits_{\Gamma_{x_0,\varepsilon}} \frac{\partial \Phi_{k,n}}{\partial \nu_y}(x_0, y)u(y)d\Gamma(y) = p.v. \int\limits_{\Gamma} \frac{\partial \Phi_{k,n}}{\partial \nu_y}(x_0, y)d\Gamma(y) = Du(x_0) \quad (3.139)$$

The analysis for the operator $\partial \mathcal{S}/\partial \nu_x$ is identical to that for $\mathcal{D}$, modulo the sign of the first term in (3.137). Finally, we refer to [28] and [36] for proofs of (3.133). This completes the proof. It should be noted that standard texts (see, e.g., [36, 28]) present the relations (3.131) and (3.132) in the form

$$\mathfrak{D}^{\pm} = \pm\frac{1}{2}I + D \quad (3.140)$$

i.e. with other constants than those shown here. We readily see that this corresponds to the choice of the fundamental solution given by $(2\pi)^{-(n+1)}\Phi_{k,n}$.

■

The following is a result on regularity properties of the principal value integral operators S, D, D^*.

PROPOSITION 3.2.15
For any smooth hypersurface $\Gamma \subset \mathbb{R}^n$, the principal value integral operators S, D are pseudodifferential operators in the space $OPS^{-1}(\Gamma)$.
Proof: The result on S follows e.g. from the identity $\mathfrak{S}^+ = \mathfrak{S}^- = S$ derived in Proposition (3.2.14) and the already established relation $\mathfrak{S}^{\pm} \in OPS^{-1}(\Gamma)$. We may also note that the functions $\Phi_{k,n}$ and $\partial\Phi_{k,n}/\partial\nu_y$ are the Schwartz kernels of the operators $S_{k,n}$ and $D_{k,n}$ on domains Γ and $\Gamma \backslash \text{diag}\{\Gamma \times \Gamma\}$, respectively. The result concerning the operators S and D now follows [36] by (2.271) and (2.274) of Remark 2.8.1, used in conjunction with Proposition 2.5.13 and the fact that

$$\langle \nu_y - \nu_x, \nu_y \rangle \quad (3.141)$$

vanishes on the diagonal $x = y$. (As before, the quantities ν_x, ν_y are the unit outward normals on the hypersurface Γ at x and y, respectively.)

■

Remembering Proposition 2.5.26, as well as the relation $OPS^m \subset OPS^m_{1,0}$, we now have

COROLLARY 3.2.16
Given an open bounded set $\Omega_+ \subset \mathbb{R}^n$ with a smooth boundary $\Gamma = \partial\Omega_+$, as well as $k \in \mathbb{R}$, $s \in \mathbb{R}$, the following holds true:

$$S: H^s(\Gamma) \to H^{s+1}(\Gamma) \quad \text{continuously} \tag{3.142}$$

$$D: H^s(\Gamma) \to H^{s+1}(\Gamma) \quad \text{continuously} \tag{3.143}$$

$$D^*: H^s(\Gamma) \to H^{s+1}(\Gamma) \quad \text{continuously} \tag{3.144}$$

■

3.2.3 Considerations on Ellipticity and Fredholmness

The result presented next renders relevant the theory of Fredholm operators from Section 2.6 in the context of the analysis of the general Helmholtz problem (3.30) (with *smooth boundary*) using boundary integral operators. It represents the final preparatory step prior to the existence, uniqueness and regularity analysis of the smooth case in Section 3.4.

PROPOSITION 3.2.17
Given a smooth hypersurface $\Gamma \subset \mathbb{R}^n$, the pertaining principal value boundary integral operators

$$S: H^s(\Gamma) \to H^{s+1}(\Gamma) \tag{3.145}$$

$$\pm(2\pi)^n\pi I + D: H^s(\Gamma) \to H^s(\Gamma) \tag{3.146}$$

$$\pm(2\pi)^n\pi I + D^*: H^s(\Gamma) \to H^s(\Gamma) \tag{3.147}$$

are Fredholm of index zero on $L^p(\Gamma, \Lambda^l)$ for all $p \in]1, \infty[$.
Proof: Our proof regarding the operator S is valid for $n \in \mathbb{N}, n > 2$. With reference to Proposition 2.5.19, every point x_0 is in $\mathbb{R}^{n-1}$ in the local coordinates. Now the Schwartz kernel of the operator S is the fundamental solution $\Phi_{k,n}$, and according to Remark 2.8.1, it is true that the associated function L satisfies

$$L(x_0, z) \sim a|z|^{2-n} + \ldots \quad \text{for } z \to 0 \tag{3.148}$$

But since $S \in OPS^{-1}$, we have, still in the notation of Proposition 2.5.19, that

$$-m - (n-1) = 1 - n + 1 = 2 - n \tag{3.149}$$

Hence, by Proposition 2.5.19, S is elliptic at every $x_0 \in \Gamma$. Then, by Proposition 2.5.21, there exists a two-sided parametrix associated with S, and finally, by Proposition 2.6.11, S is a Fredholm operator. Expression (3.113) shows that the local representation of the symbol of the boundary integral operator $\mathfrak{S}^+ = \mathfrak{S}^- = S$ is real-valued, i.e.

$$q^{k;n}_{\mathfrak{S}^\pm}(x', \xi') = \overline{q^{k,n}_{\mathfrak{S}^\pm}(x', \xi')} \quad \forall x', \xi' \in \mathbb{R}^{n-1} \tag{3.150}$$

Now by Corollary 2.6.10, it holds true that $\mathrm{Ind}\{S\} = 0$. In view of the result in Remark (2.8.1) concerning the normal derivative of the fundamental solution $\Phi_{k,n}$, we conclude that the operators $D, D^* \in OPS^{-1}(\Gamma)$ are compact. Hence, utilising the index stability theorem 2.6.16, it is clear that $\pm(2\pi)^n\pi I + D$, $\pm(2\pi)^n\pi I + D^*$ are Fredholm operators of index zero.

■

REMARK 3.2.18
In the standard literature, the notation for the (principal value) boundary integral operators is more simple than the one presented in this text. In this instance, we have chosen our notation for clarity, not compactness.

□

3.3 Integral Representation Formulae for Differential Forms

Integral representations of solutions of the general Helmholtz systems in Definition (3.1.7) enable a reduction of these systems to integral equations involving boundary integral operators introduced in Section 3.2. In this way, contact is made with the theory of pseudodifferential operators, as well as the Fredholm theory. The general representation theorem 3.3.3 applying to differential form-valued solutions to the general Helmholtz equation may be specialised to the solutions $(E, \mathscr{H})$ of the general Maxwell system. As a result, a number of representation formulas is obtained for the generalised electric and magnetic field. These expressions have a straightforward physical interpretation in terms of electric and magnetic *surface current densities*.

DEFINITION 3.3.1
Let $\Omega_+ \subset \mathbb{R}^n$ be an open bounded domain with boundary $\Gamma \subset \mathbb{R}^n$, and set $\Omega_- = \complement\overline{\Omega}_+$. Given any differential form-valued function (or, more generally, distribution) u defined in $\mathbb{R}^n \setminus \Gamma$ and any $x_0 \in \Gamma$, we write

$$[u](x_0) \equiv \gamma_0 r_{\Omega_+} u(x_0) - \gamma_0 r_{\Omega_-} u(x_0) \tag{3.151}$$

In particular, if v is a differential 1-form (i.e. a geometrical vector field) defined on the hypersurface Γ, we make the following definitions:

$$[\nu \wedge u] \equiv \gamma_0(e_{\Omega_+}\nu) \wedge r_{\Omega_+} u - \gamma_0(e_{\Omega_-}\nu) \wedge r_{\Omega_-} u \tag{3.152}$$
$$[\nu \vee u] \equiv \gamma_0(e_{\Omega_+}\nu) \vee r_{\Omega_+} u - \gamma_0(e_{\Omega_-}\nu) \vee r_{\Omega_-} u \tag{3.153}$$

where ν is the standard unit outward normal field on Γ, and where we furthermore assume that nontangential approach paths are taken from the interior and the exterior of Γ.

□

REMARK 3.3.2
Let $\Gamma \subset \mathbb{R}^n$ be a closed hypersurface with unit outward normal ν, and let u be a differential form-valued function defined on Γ. Using Definition 3.2.1 regarding the single and double layer potential operators, as well as the relations (2.71), (2.65) and

the antisymmetry in the fundamental solution $\Phi_{k,n}$, we write

$$\begin{aligned}-dS\nu \vee u + \delta S\nu \wedge u = -\int_\Gamma d_x\left\{\Phi_{k,n}\nu \vee u\right\}d\Gamma + \int_\Gamma \delta_x\left\{\Phi_{k,n}\nu \wedge u\right\}d\Gamma = \\ -\int_\Gamma (d_x\Phi_{k,n}) \wedge (\nu \vee u)d\Gamma - \int_\Gamma (d_x\Phi_{k,n}) \vee (\nu \wedge u)d\Gamma = \\ -\int_\Gamma \langle d_x\Phi_{k,n}, \nu\rangle u d\Gamma = \int_\Gamma \langle d_y\Phi_{k,n}, \nu\rangle u d\Gamma = \mathcal{D}u\end{aligned} \tag{3.154}$$

on $\mathbb{R}^n \setminus \Gamma$.

In particular, we have the operator equality $\mathcal{D} = -d\mathcal{S}\nu\vee$ on normal forms, as well as $\mathcal{D} = \delta\mathcal{S}\nu\wedge$ on tangential forms.

In light of the above result, we have, using the jump relation (3.131) for the double layer potential operator,

$$\gamma_0 r_{\Omega_\pm}\mathcal{D}\nu \vee u = \pm(2\pi)^n\pi\nu \vee u + p.v.\delta\mathcal{S}\nu \wedge (\nu \vee u) \tag{3.155}$$

$$\gamma_0 r_{\Omega_\pm}\mathcal{D}\nu \wedge u = \pm(2\pi)^n\pi\nu \wedge u - p.v.d\mathcal{S}\nu \vee (\nu \wedge u) \tag{3.156}$$

and hence, by Definition 3.2.3,

$$\nu \wedge \gamma_0 r_{\Omega_\pm}\mathcal{D}\nu \vee u = \left\{\pm(2\pi)^n\pi I + N\right\}\nu \wedge (\nu \vee u) \tag{3.157}$$

$$\nu \vee \gamma_0 r_{\Omega_\pm}\mathcal{D}\nu \wedge u = -\left\{\mp(2\pi)^n\pi I + M\right\}\nu \vee (\nu \wedge u) \tag{3.158}$$

In particular, if u is a normal differential form-valued function on Γ, it holds true that

$$\nu \wedge \gamma_0 r_{\Omega_\pm}\delta\mathcal{S}u = \nu \wedge \gamma_0 r_{\Omega_\pm}\mathcal{D}\nu \vee u = \left\{\pm(2\pi)^n\pi I + N\right\}u \tag{3.159}$$

and similarly, if u is a tangential function on Γ, we have

$$\nu \vee \gamma_0 r_{\Omega_\pm}d\mathcal{S}u = \nu \vee \gamma_0 r_{\Omega_\pm}\mathcal{D}\nu \wedge u = \left\{\pm(2\pi)^n\pi I + M\right\}u \tag{3.160}$$

□

The following general result provides the basis for integral representations of acoustic and electromagnetic fields. It is the author's experience that electromagnetics textbooks rely heavily in their exposition on integral representations of the electromagnetic field, yet provide insufficient, if any, proof of their existence.[12]

THEOREM 3.3.3
Integral Representation Theorem for Differential Form-Valued Functions
Let, as usual, $\Phi_{k,n}$ denote the fundamental solution of the Helmholtz equation in $\mathbb{R}^n$. Assume the differential form-valued functions $u_\pm \in C^\infty(\Omega_\pm, \Lambda^l)$ satisfy

$$(\Delta + k^2 I)u_\pm = 0 \quad \text{in } \Omega_\pm \tag{3.161}$$

[12] See, for instance, references [32, 6].

where $\Omega_+ \subset \mathbb{R}^n$ is an open bounded Lipschitz domain with boundary $\Gamma = \partial\Omega_+$ and complement $\Omega_- = \complement\Omega_+$. Also, assume that u_- satisfies the Silver-Müller radiation condition. Let $\tilde{u}_\pm$ be the extensions of $u_\pm$ by zero into the respective complementary domains, and set $u = \tilde{u}_+ + \tilde{u}_-$. Then, for all $x \in \mathbb{R}^n\backslash\Gamma$, it holds true that

$$\begin{aligned}
u(x) = &\int_\Gamma \Phi_{k,n}(x,y)\Big\{[\nu\wedge\delta u - \nu\vee du](y)\Big\}d\Gamma(y) + \int_\Gamma \frac{\partial\Phi_{k,n}}{\partial\nu(y)}[u]d\Gamma(y) = \\
&\int_\Gamma \Phi_{k,n}(x,y)\Big\{[\nu\wedge\delta u - \nu\vee du](y)\Big\}d\Gamma(y) - d\int_\Gamma \Phi_{k,n}(x,y)[\nu\vee u](y)d\Gamma(y) + \\
&\delta\int_\Gamma \Phi_{k,n}(x,y)[\nu\wedge u](y)d\Gamma(y)
\end{aligned} \tag{3.162}$$

or, equivalently,

$$u = \mathcal{S}[\nu\wedge\delta u - \nu\vee du] + \mathcal{D}[u] = \mathcal{S}[\nu\wedge\delta u - \nu\vee du] - d\mathcal{S}[\nu\vee u] + \delta\mathcal{S}[\nu\wedge u] \tag{3.163}$$

in $\mathbb{R}^n\backslash\Gamma$.

Proof: Denoting by $K(\mathbb{R}^n)$ the space of constant functions on $\mathbb{R}^n$, let $M \in K(\mathbb{R}^n, \Lambda^l)$, and set $v_x(y) = \Phi_{k,n}(x,y)M$ for all $x, y \in \mathbb{R}^n$. Clearly, application of the Green identity (2.93) on the differential forms $u_\pm$ and v_x yields, on $\mathbb{R}^n\backslash\Gamma$,

$$\begin{aligned}
\int_{\Omega_\pm} \langle u_\pm, \delta_D M\rangle d\Omega = &\int_\Gamma \Big\{\langle u_\pm, \pm\nu\vee dv_x\rangle - \langle u_\pm, \pm\nu\wedge\delta v_x\rangle + \\
&\langle\pm\nu\wedge\delta u_\pm, v_x\rangle - \langle\pm\nu\vee du_\pm, v_x\rangle\Big\}d\Gamma = \\
&\int_\Gamma \Big\{\langle\pm\nu\wedge u_\pm, dv_x\rangle - \langle\pm\nu\vee u_\pm, \delta v_x\rangle + \\
&\Phi_{k,n}\langle\pm\nu\wedge\delta u_\pm, M\rangle - \Phi_{k,n}\langle\pm\nu\vee du_\pm, M\rangle\Big\}d\Gamma
\end{aligned} \tag{3.164}$$

The last equality is due to the identity (2.66). Now, using the relations (2.75) and (2.66), as well as the antisymmetry in the fundamental solution $\Phi_{k,n}$, we see that the following holds true:

$$\begin{aligned}
\langle\pm\nu\wedge u_\pm, dv_x\rangle = &\langle\pm\nu\wedge u_\pm, (d_y\Phi_{k,n})\wedge M\rangle = \langle(d_y\Phi_{k,n})\vee(\pm\nu\wedge u_\pm), M\rangle = \\
&-\langle(d_x\Phi_{k,n})\vee(\pm\nu\wedge u_\pm), M\rangle = \langle\delta_x(\Phi_{k,n}(\pm\nu)\wedge u_\pm), M\rangle
\end{aligned} \tag{3.165}$$

Also, we have

$$\begin{aligned}
\langle\pm\nu\vee u_\pm, \delta v_x\rangle = &\langle\mp\nu\vee u_\pm, (d_y\Phi_{k,n})\vee M\rangle = \langle(d_y\Phi_{k,n})\wedge(\mp\nu\vee u_\pm), M\rangle = \\
&-\langle(d_x\Phi_{k,n})\wedge(\mp\nu\vee u_\pm), M\rangle = -\langle d_x(\Phi_{k,n}(\mp\nu)\vee u_\pm), M\rangle
\end{aligned} \tag{3.166}$$

By (2.65), it thus holds true that

$$\langle\pm\nu\wedge u_\pm, dv_x\rangle - \langle\pm\nu\vee u_\pm, \delta v_x\rangle = \langle\langle d_y\Phi_{k,n}, \pm\nu\rangle u_\pm, M\rangle = \pm\left\langle\frac{\partial\Phi_{k,n}}{\partial\nu_y}u_\pm, M\right\rangle \tag{3.167}$$

and, equivalently,

$$\langle \pm\nu \wedge u_\pm, dv_x\rangle - \langle \pm\nu \vee u_\pm, \delta v_x\rangle = \langle \delta_x(\Phi_{k,n}(\pm\nu) \wedge u_\pm) - d_x(\Phi_{k,n}(\pm\nu) \vee u_\pm), M\rangle \tag{3.168}$$

Consequently, by equations (3.164), (3.165), (3.166) and (3.167), the following holds true on $\mathbb{R}^n \setminus \Gamma$:

$$\begin{aligned}\langle u_\pm, M\rangle = \pm \left\langle \int_\Gamma \left\{\delta_x(\Phi_{k,n}\nu \wedge u_\pm)\right\} d\Gamma, M\right\rangle \mp \left\langle \int_\Gamma \left\{d_x(\Phi_{k,n}\nu \vee u_\pm)\right\} d\Gamma, M\right\rangle \pm \\ \left\langle \int_\Gamma \Phi_{k,n}\left\{\nu \wedge \delta u_\pm - \nu \vee du_\pm\right\} d\Gamma, M\right\rangle = \\ \pm \left\langle \int_\Gamma \frac{\partial \Phi_{k,n}}{\partial \nu_y} u_\pm d\Gamma, M\right\rangle \pm \left\langle \int_\Gamma \Phi_{k,n}\left\{\nu \wedge \delta u_\pm - \nu \vee du_\pm\right\} d\Gamma, M\right\rangle\end{aligned} \tag{3.169}$$

Since M is an arbitrary constant-coefficient l-form on $\mathbb{R}^n$, the above identity clearly implies

$$\begin{aligned}u_\pm = \mp\, d \int_\Gamma \Phi_{k,n}\nu \vee u_\pm d\Gamma \pm \delta \int_\Gamma \Phi_{k,n}\nu \wedge u_\pm d\Gamma \pm \\ \int_\Gamma \Phi_{k,n}\left\{\nu \wedge \delta u_\pm - \nu \vee du_\pm\right\} d\Gamma = \\ \pm \int_\Gamma \frac{\partial \Phi_{k,n}}{\partial \nu_y} u d\Gamma \pm \int_\Gamma \Phi_{k,n}\left\{\nu \wedge \delta u_\pm - \nu \vee du_\pm\right\} d\Gamma\end{aligned} \tag{3.170}$$

in $\Omega_\pm$. The proof is complete.

■

A traditional argument facilitating the derivation of the acoustic integral representation theorem is merely a special case of the development presented in the proof of Theorem 3.3.3.[13] However, derivations of variants of the electromagnetic integral representation theorem are typically less rigorous, and no connection with the acoustic case is demonstrated or mentioned.[14] Clearly, equation (3.162) is of the form $u = d\alpha + \delta\beta + \gamma$, where $\gamma \in \mathcal{H}_l(\mathbb{R}^n \setminus \Gamma)$. Hence, we observe that the integral representation formula (3.162) constitutes a specialisation and refinement of the general Hodge decomposition result presented in Proposition 2.7.4.

We note that the integral representation formula can be expressed as

$$u = \mathcal{S}[\Pi_l \nu D u] + \mathcal{D}[u] \tag{3.171}$$

Moreover, fields not satisfying the radiation condition cannot be represented using the above integral representation formula. In the context of scattering theory, an incident plane wave will not be representable by (3.171); only the scattered field will admit the integral representation shown above.

[13] See, for example, Reference [28].
[14] See, e.g., references [6, 28].

COROLLARY 3.3.4
The field components $(E, \mathscr{H})$ satisfying the generalised Maxwell system (3.7) admit the following representations (decompositions):

$$\begin{aligned} E =& -\mathcal{S}[\nu \vee dE] + \mathcal{D}[E] = -ik\mathcal{S}[\nu \vee \mathscr{H}] + \mathcal{D}[E] = \\ & -\mathcal{S}[\nu \vee dE] - d\mathcal{S}[\nu \vee E] + \delta\mathcal{S}[\nu \wedge E] = \\ & -ik\mathcal{S}[\nu \vee \mathscr{H}] - d\mathcal{S}[\nu \vee E] + \delta\mathcal{S}[\nu \wedge E] \quad \text{in } \mathbb{R}^n \backslash \Gamma \end{aligned} \tag{3.172}$$

$$\begin{aligned} \mathscr{H} =& \mathcal{S}[\nu \wedge \delta\mathscr{H}] + \mathcal{D}[\mathscr{H}] = -ik\mathcal{S}[\nu \wedge E] + \mathcal{D}[\mathscr{H}] = \\ & \mathcal{S}[\nu \wedge \delta\mathscr{H}] - d\mathcal{S}[\nu \vee \mathscr{H}] + \delta\mathcal{S}[\nu \wedge \mathscr{H}] = \\ & -ik\mathcal{S}[\nu \wedge E] - d\mathcal{S}[\nu \vee \mathscr{H}] + \delta\mathcal{S}[\nu \wedge \mathscr{H}] \quad \text{in } \mathbb{R}^n \backslash \Gamma \end{aligned} \tag{3.173}$$

Proof: The result is straightforward and follows upon combining the general representation formula in (3.163) with the equations constituting the system (3.7).

■

REMARK 3.3.5
The integral representation theorem 3.3.3 on page 87 is well-suited as a basis for a unified treatment of existence, uniqueness and regularity issues associated with acoustic and electromagnetic problems. Moreover, it facilitates the expression of solutions of the generalised Maxwell boundary value problem in a form which is desirable from a physical point of view, in that only quantities with a clear (direct) physical interpretation are involved. In particular, in the n-dimensional acoustic ($l = 0$) and electromagnetic ($l = 1$) cases, the integral representation formula (3.163) can be interpreted as a linear combination of *radiation integrals* involving *surface densities* as sources. It is indeed in these terms that the representation formula is traditionally introduced and referred to in the electromagnetics literature. [6]

□

In the n-dimensional acoustic case specified by the parameter pair ($l = 0, n \in \mathbb{N}$), the integral representation formula directly yields a convenient expression, involving only quantities with a clear physical interpretation. Indeed, setting $l = 0$ and using Remark 2.4.17, the general integral representation formula (3.163) can be specialised to

$$p = -\mathcal{S}\left[\frac{\partial p}{\partial \nu}\right] + \mathcal{D}[p] = -\mathcal{S}\left[\frac{\partial p}{\partial \nu}\right] - \operatorname{div}\mathcal{S}[\nu p] \quad \text{in } \mathbb{R}^n \backslash \Gamma \tag{3.174}$$

In the electromagnetic setting, the direct expressions arising from the integral representation are, however, not physically convenient. Consequently, what is typically considered in theoretical as well as applied electromagnetic texts are derived representations involving densities with direct physical interpretations. In order to express the representations of Corollary 3.3.4 in the standard form used in electromagnetics, the so-called *Stratton-Chu* formulae [6], further manipulation of the right-hand side terms in (3.172) and (3.173) is required. Setting $H \equiv *\mathscr{H}$ and using the theory of Chapter 2, we write

$$d\mathcal{S}[\nu \vee E] = -\frac{1}{ik} d\mathcal{S}[\nu \vee \delta \mathscr{H}] = -\frac{1}{ik} d\mathcal{S}[\delta_\partial \nu \vee \mathscr{H}] = \tag{3.175}$$

$$-\frac{1}{ik} d\delta\mathcal{S}[\nu \vee \mathscr{H}] = \frac{(-1)^{l+1}}{ik} d\delta\mathcal{S}[*(\nu \wedge H)] \tag{3.176}$$

$$\delta\mathcal{S}[\nu \wedge E] = (-1)^{(l+1)(n-l-1)} \delta{*}{*}\mathcal{S}[\nu \wedge E] = \tag{3.177}$$

$$(-1)^{l(n-l+1)+1} * d\mathcal{S}[*(\nu \wedge E)] \tag{3.178}$$

$$ik\mathcal{S}[\nu \vee \mathscr{H}] = (-1)^l ik\mathcal{S}[*(\nu \wedge H)] \tag{3.179}$$

Furthermore, it holds true that

$$*d\mathcal{S}[\nu \vee \mathscr{H}] = (-1)^l * d\mathcal{S}[*(\nu \wedge H)] \tag{3.180}$$

$$*\delta\mathcal{S}[\nu \wedge \mathscr{H}] = \frac{1}{ik} * \delta\mathcal{S}[\nu \wedge dE] = \tag{3.181}$$

$$\frac{1}{ik} * \delta\mathcal{S}[d_\partial \nu \wedge E] = \tag{3.182}$$

$$\frac{1}{ik} * \delta d\mathcal{S}[\nu \wedge E] = \frac{(-1)^l}{ik} d * d\mathcal{S}[\nu \wedge E] = \frac{1}{ik} d\delta\mathcal{S}[*(\nu \wedge E)] \tag{3.183}$$

$$ik * \mathcal{S}[\nu \wedge E] = ik\mathcal{S}[*(\nu \wedge E)] \tag{3.184}$$

Motivated by the classical, three-dimensional case, we define for the general values of the parameter pair (l, n), $n \in \mathbb{N}$, $l \in \{0, \ldots, n\}$, the *electric surface current density*,

$$j \equiv [*(\nu \wedge H)] \tag{3.185}$$

and the *magnetic surface current density*,

$$m \equiv [*(\nu \wedge E)] \tag{3.186}$$

The following result is now readily obtained.

PROPOSITION 3.3.6
The Stratton-Chu Representation Formulae for Solutions of the General Helmholtz System
The field quantities $(E, H = *\mathscr{H})$ related to the 'generalised electromagnetic field' $(E, \mathscr{H})$ may be written as follows:

$$\begin{aligned} E = &\frac{(-1)^l}{ik} d\delta\mathcal{S}[*(\nu \wedge H)] + \\ &(-1)^{l(n-l+1)+1} * d\mathcal{S}[*(\nu \wedge E)] + (-1)^{l+1} ik\mathcal{S}[*(\nu \wedge H)] = \\ &\frac{(-1)^{l+1}}{ik} d\delta\mathcal{S}j + (-1)^{l(n-l+1)} * d\mathcal{S}m + (-1)^l ik\mathcal{S}j \quad \text{in } \mathbb{R}^n \backslash \Gamma \end{aligned} \tag{3.187}$$

$$\begin{aligned} H = &\frac{(-1)^{l(n-l)+1}}{ik} d\delta\mathcal{S}[*(\nu \wedge E)] + \\ &(-1)^{l(n-l+1)+1} * d\mathcal{S}[*(\nu \wedge H)] + (-1)^{l(n-l)+1} ik\mathcal{S}[*(\nu \wedge E)] = \\ &\frac{(-1)^{l(n-l)+1}}{ik} d\delta\mathcal{S}m + (-1)^{l(n-l+1)} * d\mathcal{S}j + (-1)^{l(n-l)+1} ik\mathcal{S}m \quad \text{in } \mathbb{R}^n \backslash \Gamma \end{aligned} \tag{3.188}$$

■

Setting $(l,n)=(1,3)$ yields the standard integral representation formulae for the (normalised) electromagnetic field [28, 6] in $\mathbb{R}^3$:

$$\begin{aligned} E &= -ik\mathcal{S}j - \frac{1}{ik}\nabla\text{div}\mathcal{S}j - \text{curl}\mathcal{S}m = -ikA_e - \frac{1}{ik}\nabla\text{div}A_e - \text{curl}A_m, \\ H &= -ik\mathcal{S}m + \frac{1}{ik}\nabla\text{div}\mathcal{S}m - \text{curl}\mathcal{S}j = -ikA_m + \frac{1}{ik}\nabla\text{div}A_m - \text{curl}A_e \end{aligned} \tag{3.189}$$

where $A_e \equiv \mathcal{S}j, A_m \equiv \mathcal{S}m$ are the (normalised) vector electric and vector magnetic potential, respectively. We have hereby shown the existence of the vector electromagnetic potentials. Since the corresponding scalar potentials are determined through the Lorenz gauge and its dual, the existence of the scalar electromagnetic potentials has also been demonstrated.

Using the general Maxwell equations, as well as (3.187) and (3.188), a straightforward calculation shows that

$$\begin{aligned} E &= -\frac{1}{ik}\delta\mathscr{H} = \frac{(-1)^l}{ik}\delta d\mathcal{S}[*(\nu\wedge H)] + (-1)^{l(n-l-1)} * d\mathcal{S}[*(\nu\wedge E)] \\ H &= \frac{(-1)^{l(n-l)}}{ik} * dE = (-1)^{l(n+1)+1}\delta d\mathcal{S}[*(\nu\wedge E)] + (-1)^{l(n-l+1)} * d\mathcal{S}[*(\nu\wedge H)] \end{aligned} \tag{3.190}$$

This can be specialised to an alternative representation of the normalised electromagnetic field in $\mathbb{R}^3$, given by

$$\begin{aligned} -E &= \text{curl}\mathcal{S}m + \frac{1}{ik}\text{curlcurl}\mathcal{S}j = \text{curl}A_m + \frac{1}{ik}\text{curl}\,\text{curl}A_e \\ -H &= \text{curl}\mathcal{S}j + \text{curlcurl}\mathcal{S}m = \text{curl}A_e + \text{curl}\,\text{curl}A_m \end{aligned} \tag{3.191}$$

3.4 Existence, Uniqueness and Regularity Analysis

In this section, the actual analysis is performed of the existence, uniqueness and regularity properties of solutions to the general Helmholtz boundary value problem of Section 3.1. The analysis is divided into two distinct cases, according to the assumed smoothness of the hypersurface Γ. Thus, in Section 3.4.1, we consider the case where Γ is smooth, while Section 3.4.2 is concerned with boundary value problems with Lipschitz continuous boundaries. The analysis presented in this section is based on recasting the general Helmholtz boundary value problem presented in Definition (3.1.7) to an integral equation involving boundary integral operators introduced in Section 3.2.

Let us, for convenience, repeat here the (main parts of) general electric and magnetic interior and exterior boundary value problems defined in Section 3.1:

$$\begin{aligned} (\Delta + k^2 I)E_\pm &= 0 \quad \text{in } \Omega_\pm \\ \nu\wedge\gamma_0 E_\pm &= \Psi_1 \in L_n^{p,d}(\Gamma,\Lambda^{l+1}) \\ \nu\wedge\gamma_0\delta E_\pm &= \Psi_2 \in L_n^{p,d}(\Gamma,\Lambda^{l}) \end{aligned} \tag{3.192}$$

$$\begin{aligned} (\Delta + k^2 I)\mathscr{H}_\pm &= 0 \quad \text{in } \Omega_\pm \\ \nu\vee\gamma_0\mathscr{H}_\pm &= \Psi_1 \in L_t^{p,\delta}(\Gamma,\Lambda^{l}) \\ \nu\vee\gamma_0 d\mathscr{H}_\pm &= \Psi_2 \in L_t^{p,\delta}(\Gamma,\Lambda^{l+1}) \end{aligned} \tag{3.193}$$

3.4.1 Smooth Domains

We include here without proof the following auxiliary result from Reference [14]:

LEMMA 3.4.1
Let $\Omega_+ \subset \mathbb{R}^n$ be an open bounded C^1 domain with boundary Γ, and assume $p \in]1,\infty[$. Furthermore, let the following hold true:

$$\begin{gathered} E \in C^\infty(\Omega_+, \Lambda^l) \\ (\Delta + k^2 I)E = 0 \quad \text{in } \Omega_+ \\ \mathcal{N}E, \mathcal{N}(dE), \mathcal{N}(\delta E) \in L^p(\Gamma) \\ \gamma_0 dE = 0, \nu \vee \gamma_0 E = 0 \quad \text{a.e. on } \Gamma \quad (\text{or } \gamma_0 \delta E = 0, \nu \wedge \gamma_0 E = 0 \quad \text{a.e. on } \Gamma) \end{gathered} \tag{3.194}$$

Then $E \equiv 0$ in Ω_+. A similar result holds in the complement $\Omega_- = \complement\overline{\Omega}_+$ if E also satisfies the Silver-Müller radiation condition in Ω_-.

□

We assume in the following that k^2 is not an eigenvalue of the Hodge Laplacian on Ω_+[15]. Consequently, if $k^2 = 0$, we have $E = 0$, and we are thus only interested in nonzero values of the wavenumber. We now have the following result, also presented in Reference [14]:

LEMMA 3.4.2
Assume $k \in \mathbb{C}\backslash\{0\}$, and let $\Omega_+ \subset \mathbb{R}^n$ be an open bounded smooth domain with boundary Γ and complement Ω_-. The operators $\pm(2\pi)^n\pi I + M$ are invertible on $L_t^{p,\delta}(\Gamma, \Lambda^l)$ for all $p \in]1,\infty[$ and all $l \in \{0,\dots,n\}$.
Proof: Since smooth domains are also in the more general class C^1, Lemma 3.4.1 is relevant for the present result. Assume that $(-(2\pi)^n\pi I + M)u = 0$. Then, by Lemma 3.4.1, we have

$$d\mathcal{S}u \equiv 0 \quad \text{in } \Omega_+ \tag{3.195}$$

so

$$\delta E \equiv 0 \quad \text{in } \Omega_+ \tag{3.196}$$

Now since $\nu \wedge \gamma_0 r_{\Omega_+} \delta E = \nu \wedge \gamma_0 r_{\Omega_-} \delta E$, it is true that

$$\nu \wedge \gamma_0 r_{\Omega_-} \delta E \equiv 0 \quad \text{on } \Gamma \tag{3.197}$$

Applying Lemma 3.4.1 on the quantity δE yields

$$\delta E \equiv 0 \quad \text{in } \Omega_- \tag{3.198}$$

Hence, we have $E = -k^{-2}\Delta E = k^{-2} d\delta E = 0$ in the exterior domain Ω_-, such that, by construction,

$$u = \nu \vee \gamma_0 r_{\Omega_-} E - \nu \vee \gamma_0 r_{\Omega_+} E = 0 \tag{3.199}$$

This completes the proof.

■

[15]The values $k \in \mathbb{R}\backslash\{0\}$ for which the interior Maxwell boundary value problem $\mathcal{M}_+$ has more than one solution are referred to as *Maxwell eigenvalues*. [14]

We can now prove the invertibility of the operators $\pm(2\pi)^n \pi I + M$ on suitable function spaces.

PROPOSITION 3.4.3
Let $\Omega_+ \subset \mathbb{R}^n$ be an open bounded smooth domain with boundary Γ, and assume $k \in \mathbb{C}\backslash\{0\}$. The operators $\pm(2\pi)^n \pi I + M$ are invertible on $L_t^p(\Gamma, \Lambda^l)$ for all $p \in]1, \infty[$ and all $l \in \{0, \dots, n\}$.
Proof: The space $L_t^{p,\delta}(\Gamma, \Lambda^l), p \in]1, \infty[$, is dense in $L_t^p(\Gamma, \Lambda^l)$.[14] By Lemma 3.4.2, the ranges of the operators $\pm(2\pi)^n \pi I + M$ are dense in $L_t^p(\Gamma, \Lambda^l)$. In other words, the deficiency of these operators is of dimension zero in $L_t^p(\Gamma, \Lambda^l)$. Since $\pm(2\pi)^n \pi I + M$ are Fredholm of index zero, the kernel dimension of $\pm(2\pi)^n \pi I + M$ on $L_t^p(\Gamma, \Lambda^l)$ is also zero, and the result follows.

■

REMARK 3.4.4
Note that, by Hodge duality, a similar discussion as above can be used to infer that, in the smooth case and with $k \neq 0$, the operators $\pm(2\pi)^n \pi I + N$ are invertible on $L_n^p(\Gamma, \Lambda^l)$ for all $p \in]1, \infty[$ and all $l \in \{0, \dots, n\}$.

□

REMARK 3.4.5
For $p \in]1, \infty[$ we have $H^p \subset L^p$. Now, in view of Proposition 3.2.17 and Remark 3.3.2, the following operators are isomorphisms for $n \geq 3$ (and under same restrictions on k as above):

$$\begin{aligned} \pm(2\pi)^n \pi I + M &: H_t^p(\Gamma, \Lambda^l) \to H^p(\Gamma, \Lambda^l), \quad p \in]1, \infty[\\ \pm(2\pi)^n \pi I + N &: H_n^p(\Gamma, \Lambda^l) \to H^p(\Gamma, \Lambda^l), \quad p \in]1, \infty[\end{aligned} \tag{3.200}$$

□

We now present the main result of this section.

PROPOSITION 3.4.6
Existence and Uniqueness of Solution to the General Helmholtz Boundary Value Problem - The Smooth Case
Assume $k \in \mathbb{C}\backslash\{0\}$. The (interior and exterior) electric and magnetic boundary value problems (3.192) and (3.193) (stated in full in (3.30) and (3.31)) each have unique solution, given any boundary data

$$\Psi_1 \in L_n^{p,d}(\Gamma, \Lambda^{l+1}), \Psi_2 \in L_n^{p,d}(\Gamma, \Lambda^l), \, p \in]1, \infty[\tag{3.201}$$

and

$$\Psi_1 \in L_t^{p,\delta}(\Gamma, \Lambda^l), \Psi_2 \in L_t^{p,\delta}(\Gamma, \Lambda^{l+1}), \, p \in]1, \infty[\tag{3.202}$$

respectively.
Proof: By Hodge duality, it suffices to prove existence and uniqueness for, say, only the interior and exterior electric boundary value problems.

Inspired by the general integral representation theorem, which clearly applies to solutions of the interior and exterior electric boundary value problems, we define $E_\pm = \mathcal{S}u_1 + \mathcal{D}\nu \vee u_2$ in $\Omega_\pm$ with $u_1 \in L_n^{p,d}(\Gamma, \Lambda^l), u_2 \in L_n^{p,d}(\Gamma, \Lambda^{l+1})$. By Remark 3.2.2, it is true that $(\Delta + k^2 I)E_\pm = 0$ in $\Omega_\pm$, the function E_- satisfying the radiation condition. Also, it is seen from Remark 3.3.2 that $E_\pm = r_{\Omega_\pm}\mathcal{S}u_1 + r_{\Omega_\pm}\delta\mathcal{S}u_2$, and hence $\delta E_\pm = r_{\Omega_\pm}\delta\mathcal{S}u_1$. Since the functions u_1, u_2 yield, by definition, normal forms on the hypersurface Γ, the result in (3.159) implies

$$\nu \wedge \gamma_0 \delta E_\pm = \nu \wedge \gamma_0 r_{\Omega_\pm}\delta\mathcal{S}u_1 = \Big\{ \pm(2\pi)^n \pi I + N \Big\} u_1 \tag{3.203}$$

and

$$\nu \wedge \gamma_0 E_\pm = \nu \wedge \gamma_0 r_{\Omega_\pm}\mathcal{S}u_1 + \nu \wedge \gamma_0 r_{\Omega_\pm}\delta\mathcal{S}u_2 = \nu \wedge Su_1 + \Big\{ \pm(2\pi)^n \pi I + N \Big\} u_2 \tag{3.204}$$

on Γ. It now follows from the already established bijectivity of the operators $\pm(2\pi)^n\pi I + N$ on $L^{p,d}_n(\Gamma, \Lambda^l), L^{p,d}_n(\Gamma, \Lambda^{l+1})$ that $E_\pm$ is a solution of the (interior or exterior) electric boundary value problem iff

$$u_1 = \Big\{ \pm(2\pi)^n \pi I + N \Big\}^{-1} \Psi_2 \quad \text{on } \Gamma \tag{3.205}$$

$$u_2 = \Big\{ \pm(2\pi)^n \pi I + N \Big\}^{-1} (\Psi_1 - \nu \wedge Su_1) \quad \text{on } \Gamma \tag{3.206}$$

Having established the existence part, we now turn to proving the uniqueness of the solution of the system (3.192).

If a differential form-valued function $E_\pm \in L^p(\Omega_\pm, \Lambda^l)$ satisfies the homogeneous system

$$\begin{aligned} (\Delta + k^2 I)E_\pm &= 0 \quad \text{in } \Omega_\pm \\ \nu \wedge \gamma_0 E_\pm &= 0 \quad \text{on } \Gamma \\ \nu \wedge \gamma_0 \delta E_\pm &= 0 \quad \text{on } \Gamma \end{aligned} \tag{3.207}$$

then, by the integral representation formula (3.163), it holds true that

$$\pm E_\pm = -r_{\Omega_\pm}\mathcal{S}\nu \vee \gamma_0 dE_\pm - r_{\Omega_\pm} d\mathcal{S}\nu \vee \gamma_0 E_\pm \quad \text{in } \Omega_\pm \tag{3.208}$$

and thus

$$dE_\pm = \mp r_{\Omega_\pm} d\mathcal{S}\nu \vee \gamma_0 dE_\pm \quad \text{in } \Omega_\pm \tag{3.209}$$

Now, using the result in 3.160, we see that

$$\nu \vee \gamma_0 dE_\pm = \mp\nu \vee \gamma_0 r_{\Omega_\pm} d\mathcal{S}\nu \vee \gamma_0 dE_\pm = \Big\{ (2\pi)^n \pi I \mp M \Big\} \nu \vee \gamma_0 dE_\pm \quad \text{on } \Gamma \tag{3.210}$$

or, equivalently,

$$\Big\{ \pm(2\pi)^n \pi I + M \Big\} \nu \vee \gamma_0 dE_\pm = 0 \quad \text{on } \Gamma \tag{3.211}$$

The injectivity of the operators $\pm(2\pi)^n\pi I + M$ implies

$$\nu \vee \gamma_0 dE_\pm = 0 \quad \text{on } \Gamma \tag{3.212}$$

This identity may, in turn, be combined with (3.208) to yield

$$E_\pm = \mp r_{\Omega_\pm} d\mathcal{S}\nu \vee \gamma_0 E_\pm \quad \text{in } \Omega_\pm \tag{3.213}$$

But then it holds true that

$$\nu \vee \gamma_0 E_\pm = \mp \nu \vee \gamma_0 r_{\Omega_\pm} d\mathcal{S} \nu \vee \gamma_0 E_\pm = \Big\{ (2\pi)^n \pi I \mp M \Big\} \nu \vee \gamma_0 E_\pm \quad \text{on } \Gamma \tag{3.214}$$

or, equivalently,

$$\Big\{ \pm (2\pi)^n \pi I + M \Big\} \nu \vee \gamma_0 E_\pm = 0 \quad \text{on } \Gamma \tag{3.215}$$

Since the operators $\pm(2\pi)^n \pi I + M$ are injective, we necessarily have

$$\nu \vee \gamma_0 E_\pm = 0 \quad \text{on } \Gamma \tag{3.216}$$

Expressions (3.208), (3.212) and (3.216) obviously imply that $E_\pm = 0$ in $\Omega_\pm$, and the proof is complete.

As already asserted, the Hodge duality now implies existence and uniqueness for the interior and exterior magnetic boundary value problems (3.193) (stated in full in (3.31).) More precisely, the proof of existence relies on the already established bijectivity of the operators $\pm(2\pi)^n \pi I + M$, while the uniqueness follows from the injectivity of $\pm(2\pi)^n \pi I + N$.

■

Let us summarise formally the results implied by the above proposition and the pertaining proof:

REMARK 3.4.7
For $k \in \mathbb{C} \backslash \{0\}$, and for any boundary datum $\Psi_1 \in L_n^{p,d}(\Gamma, \Lambda^{l+1})$, $\Psi_2 \in L_n^{p,d}(\Gamma, \Lambda^l)$, $p \in]1, \infty[$, the exact, unique solution of the (interior or exterior) electric boundary value problem (3.192) (stated fully in (3.30)) is given by

$$\begin{aligned} E_\pm =& r_{\Omega_\pm} \mathcal{S}(\pm(2\pi)^n \pi I + N)^{-1} \Psi_2 + \\ & r_{\Omega_\pm} \delta \mathcal{S}(\pm(2\pi)^n \pi I + N)^{-1}(\Psi_1 - \nu \wedge S(\pm(2\pi)^n \pi I + N)^{-1} \Psi_2) \quad \text{in } \Omega_\pm \end{aligned} \tag{3.217}$$

Furthermore, for any boundary datum $\Psi_1 \in L_t^{p,\delta}(\Gamma, \Lambda^l)$, $\Psi_2 \in L_t^{p,\delta}(\Gamma, \Lambda^{l+1})$, $p \in]1, \infty[$, the exact, unique solution of the (interior or exterior) magnetic boundary value problem (3.193) (stated fully in (3.31)) is given by

$$\begin{aligned} \mathscr{H}_\pm =& r_{\Omega_\pm} \mathcal{S}(\pm(2\pi)^n \pi I + M)^{-1} \Psi_2 + \\ & r_{\Omega_\pm} d\mathcal{S}(\pm(2\pi)^n \pi I + M)^{-1}(\Psi_1 - \nu \vee S(\pm(2\pi)^n \pi I + M)^{-1} \Psi_2) \quad \text{in } \Omega_\pm \end{aligned} \tag{3.218}$$

The results of Remark 3.4.5 may be used to find the regularity of solution in case the boundary data are known to be in $H^p(\Gamma, \Lambda), p \in]1, \infty[$, and $n \geq 3$.[16]

□

[16]Reference [36] asserts that the result of Proposition 3.2.17 is valid also for $n = 2$.

3.4.2 Lipschitz Domains

In the general setting of boundary value problems on Lipschitz domains, the theory of pseudodifferential operators falls short of describing the associated boundary integral operators. This is due to the fact that the localisation-and-transport process introduced earlier no longer applies in the neighbourhoods of singularities of the domain boundaries. It is, however, still the case that the boundary integral operators are Fredholm of index zero, and hence the generalisation of boundary geometry to the class of Lipschitz surfaces does not, in fact, considerably increase the complexity of existence and uniqueness considerations relative to the smooth case. The exposition of this section is based primarily on Reference [14]. Accordingly, we choose to replace our constant $\pm(2\pi)^n\pi$ with the standard constant $1/2$ in the expressions of the relevant boundary integral operators.

It is shown in Reference [14] that the relevant boundary integral operators are well-defined on Lipschitz hypersurfaces. Also, we note that the jump relations found earlier for the smooth case continue to hold in the Lipschitz case, by virtue of the results firstly presented in Reference [5]. In [14], limiting arguments are used in order to justify the validity of the Green-Stokes formulae (and some additional results) derived earlier for the smooth case. A straightforward proof [14] also shows that

$$\delta S u = S \delta_\partial u \quad \forall u \in L_t^{p,\delta}(\Gamma, \Lambda^l) \tag{3.219}$$

We start our analysis with the following lemma, which we present without proof (Reference [36, p. 188] provides proof of a similar result, while Reference [14] refers to [26] for proof.)

LEMMA 3.4.8
Given a Hilbert space H and a Hausdorff topological space X, let

$$\mathfrak{T} : X \to \mathcal{L}(H) \tag{3.220}$$

be a continuous mapping. Also, assume $\mathfrak{T}\lambda$ is injective with closed range for all $\lambda \in X$. If for every component X_i of X there exists $\lambda_i \in X_i$ such that $\mathfrak{T}\lambda_i$ is an isomorphism of H, then $\mathfrak{T}\lambda$ is an isomorphism of H for all $\lambda \in X$.

■

We now introduce the notation $a \approx b$ when there exist constants C_1, C_2 independent of a, b such that

$$C_1 b \leq a \leq C_2 b \tag{3.221}$$

As mentioned in the introduction of Chapter 1, the lack of the tools of the pseudodifferential calculus is compensated in the Lipschitz case by the *Rellich identities*. We present these identities, without proof, in the following theorem.

THEOREM 3.4.9
Assume $\Omega \subset \mathbb{R}^n$ is an open bounded Lipschitz domain. Let $\{e_j\}_{j\in\{1,\ldots,n\}}$ be the canonical basis in $\mathbb{R}^n$. For any $E \in C^\infty(\overline{\Omega}, \Lambda^l)$ and any vector field θ in $\mathbb{R}^n$ with

smooth, real-valued components, it holds true that

$$\Re \int_{\Omega} \left\{ \frac{1}{2}|E|^2 \mathrm{div}\theta + \langle \delta E, \theta \vee \overline{E} \rangle + \langle dE, \theta \wedge \overline{E} \rangle - \sum_i \langle \nabla_i \vee E, e_i \vee \overline{E} \rangle \right\} d\Omega =$$
$$\int_{\Gamma} \frac{1}{2}|E|^2 \langle \theta, \nu \rangle d\Gamma - \Re \int_{\Gamma} \langle \theta \vee \overline{E}, \nu \vee E \rangle d\Gamma =$$
$$- \int_{\Gamma} \frac{1}{2}|E|^2 \langle \theta, \nu \rangle d\Gamma + \Re \int_{\Gamma} \langle \theta \wedge E, \nu \wedge \overline{E} \rangle d\Gamma \tag{3.222}$$

■

We note that the above theorem, as well as the rather straightforward proof, are found in Reference [14], and also that other equivalent versions of the Rellich identities are commonly used (see Reference [20].) The following result is a direct consequence of the Rellich identities given above [14]:

PROPOSITION 3.4.10
Let $E \in C^\infty(\Omega, \Lambda^l)$, and assume $\mathcal{N}E, \mathcal{N}(\delta E), \mathcal{N}(d\delta E) \in L^2(\Gamma)$. Furthermore, assume $(\Delta + k^2 I)E = 0, dE = 0$ in Ω. Then it holds true that

$$||\nu \vee \mathscr{H}||_{L_t^{2,\delta}(\Gamma, \Lambda^l)} \approx ||\nu \wedge E||_{L_n^{2,d}(\Gamma, \Lambda^{l+1})} \tag{3.223}$$

■

We next establish the injectivity of the operators $\lambda I - M, \lambda \in \mathbb{R}, |\lambda| \geq 1/2$ on $L_t^{2,\delta}(\Gamma, \Lambda^l)$.

LEMMA 3.4.11
Let $k \in \mathbb{C} \backslash \{0\}, \Im\{k\} \geq |\Re\{k\}|$, and let $\Omega_+ \subset \mathbb{R}^n$ be an open bounded Lipschitz domain with complement Ω_- and boundary Γ. Given $l \in \{0, \ldots, n\}$, $\lambda \in \mathbb{C}, |\lambda| \geq 1/2$, the operator

$$\lambda I - M \tag{3.224}$$

is injective on the space $L_t^{2,\delta}(\Gamma, \Lambda^l)$.
Proof: Let $u \in L_t^{2,\delta}(\Gamma, \Lambda^l)$ be an eigenvector of the operator M with the associated eigenvalue λ. The Green formula (2.92) yields

$$\pm \int_{\Gamma} \langle \gamma_0 r_{\Omega_\pm} \delta d\mathcal{S}u, \nu \vee \gamma_0 r_{\Omega_\pm} d\mathcal{S}u \rangle d\Gamma = \int_{\Omega_\pm} \left\{ |\delta d\mathcal{S}u|^2 - k^2 |d\mathcal{S}u|^2 \right\} d\Omega \tag{3.225}$$

We define $\mu_\pm$ to be the right-hand side of the above equation. The integrand occurring in $\mu_\pm$ may be written

$$\left\{ |\delta d\mathcal{S}u|^2 + [(\Im\{k\})^2 - (\Re\{k\})^2] |d\mathcal{S}u|^2 \right\} + i \left\{ 2\Re\{k\}\Im\{k\} |d\mathcal{S}u|^2 \right\} \tag{3.226}$$

By assumption, we have

$$(\Im\{k\})^2 - (\Re\{k\})^2 \geq 0 \tag{3.227}$$

as well as

$$||\delta d\mathcal{S}u||_{L^2(\Omega_\pm,\Lambda^l)} > 0, ||d\mathcal{S}u||_{L^2(\Omega_\pm,\Lambda^{l+1})} > 0 \tag{3.228}$$

Inspection of the expression (3.226) hence shows that the angle between the complex numbers $\mu_\pm$ is less than $\pi/2$, i.e. that these numbers lie in the same quadrant of the complex plane.

It holds true that [14]

$$\gamma_0 r_{\Omega_+}\delta d\mathcal{S}u = \gamma_0 r_{\Omega_-}\delta d\mathcal{S}u \quad \text{on } \Gamma \tag{3.229}$$

Using this relation and referring to Remark 3.3.2, we write the expression (3.225) as follows:

$$\begin{aligned}\mu_\pm = \pm \int_\Gamma \left\langle \gamma_0 r_{\Omega_+}\delta d\mathcal{S}u, \left\{\mp\frac{1}{2} + M\right\}u\right\rangle d\Gamma = \\ \int_\Gamma \left\langle \gamma_0 r_{\Omega_+}\delta d\mathcal{S}u, \left\{-(2\pi)^n\pi \pm \lambda I\right\}u\right\rangle d\Gamma\end{aligned} \tag{3.230}$$

Straightforward calculation now yields

$$\mu_- - \mu_+ = -2\lambda \int_\Gamma \langle \gamma_0 r_{\Omega_+}\delta d\mathcal{S}u, u\rangle d\Gamma \tag{3.231}$$

as well as

$$\mu_- + \mu_+ = -\int_\Gamma \langle \gamma_0 r_{\Omega_+}\delta d\mathcal{S}u, u\rangle d\Gamma \tag{3.232}$$

Hence, we have

$$\frac{1}{2}\frac{|\mu_- - \mu_+|}{|\mu_- + \mu_+|} = |\lambda| \tag{3.233}$$

But since the angle between μ_+ and μ_- is less than $\pi/2$, we have

$$|\mu_- - \mu_+| < |\mu_- + \mu_+| \tag{3.234}$$

and consequently the eigenvalue λ of M must satisfy the condition

$$|\lambda| < \frac{1}{2} \tag{3.235}$$

■

LEMMA 3.4.12
Given $k \in \mathbb{C}$, $\Im\{k\} > 0$, and a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$, the following equivalence relations hold true:

$$\left|\left|\left\{\pm\frac{1}{2}I + M\right\}u\right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \approx ||u||_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \quad \forall u \in L_t^{2,\delta}(\Gamma,\Lambda^l) \tag{3.236}$$

$$\left|\left|\left\{\pm\frac{1}{2}I + N\right\}u\right|\right|_{L_n^{2,d}(\Gamma,\Lambda^l)} \approx ||u||_{L_n^{2,d}(\Gamma,\Lambda^l)} \quad \forall u \in L_n^{2,d}(\Gamma,\Lambda^l) \tag{3.237}$$

The constants involved in these equivalence relations are positive and depend only on the dimension n, the wavenumber k and the Lipschitz surface Γ.

Proof: For some $u \in L_t^{2,\delta}(\Gamma, \Lambda^l)$, define $E_\pm = r_{\Omega_\pm} d\mathcal{S}u$ in the domain $\Omega_\pm$. According to Remark 3.3.2, it holds true that

$$\nu \vee \gamma_0 E_\pm = \nu \vee \gamma_0 r_{\Omega_\pm} d\mathcal{S}u = \left\{ \pm \frac{1}{2} I + M \right\} u \quad \text{on } \Gamma \tag{3.238}$$

Furthermore, we have by definition

$$\delta E_\pm = (-\Delta - d\delta) r_{\Omega_\pm} \mathcal{S}u = k^2 r_{\Omega_\pm} \mathcal{S}u - r_{\Omega_\pm} d\mathcal{S}\delta_\partial u \quad \text{on } \Omega_\pm \tag{3.239}$$

and hence, by Remark 3.3.2,

$$\begin{aligned} \nu \wedge \gamma_0 \delta E_\pm = & k^2 \nu \wedge \gamma_0 r_{\Omega_\pm} \mathcal{S}u - \nu \wedge \gamma_0 r_{\Omega_\pm} d\mathcal{S}\delta_\partial u = \\ & k^2 \nu \wedge Su - \nu \wedge p.v. d\mathcal{S}\delta_\partial u \quad \text{on } \Gamma \end{aligned} \tag{3.240}$$

Consequently, it holds true that

$$\nu \wedge \gamma_0 \delta E_+ = \nu \wedge \gamma_0 \delta E_- \quad \text{on } \Gamma \tag{3.241}$$

Using the relations (3.238), (3.241), and invoking the results presented in Proposition 3.4.10, we arrive at

$$\begin{aligned} \left|\left| \left\{ -\frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} = ||\nu \vee \gamma_0 E_-||_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \approx ||\nu \wedge \gamma_0 \delta E_-||_{L_n^{2,d}(\Gamma,\Lambda^l)} \approx \\ ||\nu \vee \gamma_0 E_+||_{L_t^{2,\delta}(\Gamma,\Lambda^l)} = \left|\left| \left\{ \frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \end{aligned} \tag{3.242}$$

Employing the result of (3.242), we obtain

$$\begin{aligned} ||u||_{L_t^{2,\delta}(\Gamma,\Lambda^l)} = & \left|\left| \left\{ \frac{1}{2} I + M \right\} u - \left\{ -\frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \leq \\ & \left|\left| \left\{ \frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} + \left|\left| \left\{ -\frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \leq \\ & C_\pm \left|\left| \left\{ \pm\frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \end{aligned} \tag{3.243}$$

for some constants C_+, C_-. Another application of the triangle inequality yields

$$\begin{aligned} \left|\left| \left\{ \pm\frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} = & \left|\left| \left\{ \pm\frac{1}{2} I \mp \frac{1}{2} I \pm \frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \leq \\ & ||u||_{L_t^{2,\delta}(\Gamma,\Lambda^l)} + \left|\left| \left\{ \mp\frac{1}{2} + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \leq \\ & ||u||_{L_t^{2,\delta}(\Gamma,\Lambda^l)} + C_\pm \left|\left| \left\{ \pm\frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \end{aligned} \tag{3.244}$$

for some constants C_+, C_-, and we thus have

$$\left|\left| \left\{ \pm\frac{1}{2} I + M \right\} u \right|\right|_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \leq \frac{1}{1 - C_\pm} ||u||_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \tag{3.245}$$

The result (3.237) follows from (3.236) by the Hodge duality, or, more precisely, it may be obtained by substituting $*u$ for u in the above derivation.

■

We now establish the main result of this section.

THEOREM 3.4.13
Let $\Omega_+ \subset \mathbb{R}^n$ be an open bounded Lipschitz domain with complement Ω_- and boundary Γ, and let $l \in \{0, \dots, n\}$. For any $k \in \mathbb{C}, \lambda \in \mathbb{R}, |\lambda| \geq 1/2$, the operator $\lambda I - M$ on $L_t^{2,\delta}(\Gamma, \Lambda^l)$ is Fredholm with index zero. In particular, for any $k \in \mathbb{C}\backslash\{0\}, \Im\{k\} \geq |\Re\{k\}|$ $\lambda \in \mathbb{R}, |\lambda| \geq 1/2$, the mapping $\lambda I - M$ on $L_t^{2,\delta}(\Gamma, \Lambda^l)$ is an isomorphism.
Proof: Assume firstly that $k \in \mathbb{C}\backslash\{0\}, \Im\{k\} \geq |\Re\{k\}|$ holds true. In view of Lemma 3.4.11, which establishes the relation

$$\Big\{\sigma(M) \cap \mathbb{R}\Big\} \subset \Big]-\frac{1}{2}, \frac{1}{2}\Big[\tag{3.246}$$

for the spectrum of the magnetostatic operator M, the operator $\lambda I - M$ is invertible on $L_t^{2,\delta}(\Gamma, \Lambda^l)$ for the parameter λ large enough. Hence, by Lemma 3.4.8, we only need to show the closedness of the range of the operators $\lambda I - M, \lambda \in \mathbb{R}\backslash]-1/2, 1/2[$ on $L_t^{2,\delta}(\Gamma, \Lambda^l)$.

For any $u \in L_t^{2,\delta}(\Gamma, \Lambda^l)$, Theorem 3.4.9 (regarding the Rellich identities) and Lemma 2.4.20 imply[17]

$$\begin{aligned}\frac{1}{2}\int_\Gamma |\nu \vee \gamma_0 r_{\Omega_+} d\mathcal{S}u|^2 \langle \theta, \nu\rangle d\Gamma \leq &\Re \int_\Gamma \langle \theta \vee \gamma_0 r_{\Omega_+} d\mathcal{S}u, \nu \vee \gamma_0 r_{\Omega_+} d\mathcal{S}u\rangle d\Gamma + \\ &C\left|\int_\Gamma \langle \gamma_0 r_{\Omega_+} \delta d\mathcal{S}u, \nu \vee \gamma_0 r_{\Omega_+} d\mathcal{S}u\rangle d\Gamma\right|\end{aligned} \tag{3.247}$$

For convenience, we introduce the notation

$$I = \Re \int_\Gamma \langle \theta \vee \gamma_0 r_{\Omega_+} d\mathcal{S}u, \nu \vee \gamma_0 r_{\Omega_+} d\mathcal{S}u\rangle d\Gamma \tag{3.248}$$

$$II = C\left|\int_\Gamma \langle \gamma_0 r_{\Omega_+} \delta d\mathcal{S}u, \nu \vee \gamma_0 r_{\Omega_+} d\mathcal{S}u\rangle d\Gamma\right| \tag{3.249}$$

By Remark 3.3.2, we may write

$$\nu \vee \gamma_0 r_{\Omega_+} d\mathcal{S}u = \left\{-\frac{1}{2}I + M\right\}u = (\lambda - 1/2)u - (\lambda I - M)u \quad \text{on } \Gamma \tag{3.250}$$

as well as

$$\begin{aligned}\theta(x) \vee \gamma_0 r_{\Omega_+} d\mathcal{S}u(x) = &-\frac{1}{2}\theta(x) \vee (\nu(x) \wedge u(x)) + \\ &\theta(x) \vee p.v.d \int_\Gamma \Phi_{k,n}(x,y)u(y)d\Gamma(y) \quad \forall x \in \Gamma\end{aligned} \tag{3.251}$$

[17] We assume here that Lemma 2.4.20 generalises to Lipschitz domains.

Using the result in (3.250) in conjunction with the inequality of (3.247), we arrive at

$$\frac{1}{2}\left(\lambda-\frac{1}{2}\right)^2\int_\Gamma |u|^2\langle\theta,\nu\rangle d\Gamma \le C_1||u||^2_{L_t^{2,\delta}(\Gamma,\Lambda^l)} + C_2||(\lambda-M)u||^2_{L_t^{2,\delta}(\Gamma,\Lambda^l)} + I + II \tag{3.252}$$

Furthermore, we have from (3.250) and (3.251) that

$$I \le I' + I'' + C_1||u||^2_{L_t^{2,\delta}(\Gamma,\Lambda^l)} + C_2||(\lambda-M)u||^2_{L_t^{2,\delta}(\Gamma,\Lambda^l)} \tag{3.253}$$

with the definitions

$$\begin{aligned} I' &\equiv -\frac{1}{2}\left(\lambda-\frac{1}{2}\right)\Re\int_\Gamma \langle\theta\vee(\nu\wedge u),u\rangle d\Gamma \\ I'' &\equiv \left(\lambda-\frac{1}{2}\right)\Re\int_\Gamma\left\langle\theta(x)\vee p.v.d\int_\Gamma \Phi_{k,n}(x,y)u(y)d\Gamma(y),u(x)\right\rangle d\Gamma(x) \end{aligned} \tag{3.254}$$

Using the relations of differential form calculus presented in Chapter 2, we easily see that

$$\begin{aligned} \langle\theta\vee(\nu\wedge u),u\rangle =&\langle u\langle\theta,\nu\rangle,u\rangle - \langle\nu\wedge(\theta\vee u),u\rangle = \\ & |u|^2\langle\theta,\nu\rangle - \langle\theta\vee u,\nu\vee u\rangle = |u|^2\langle\theta,\nu\rangle \quad \text{a.e. on } \Gamma \end{aligned} \tag{3.255}$$

due to the fact that the differential form-valued function u is, by definition, tangential, i.e. that it satisfies $\nu\vee u=0$ a.e. on Γ. Application of (3.255) and (3.252) now yields the following estimate for the quantity I':

$$\begin{aligned} I' =&\frac{1}{2}\left(\lambda-\frac{1}{2}\right)\Re\int_\Gamma |u|^2\langle\theta,\nu\rangle d\Gamma \le \\ &\frac{1}{2}\left(\lambda-\frac{1}{2}\right)^{-1}\left\{C_1||u||^2_{L_t^{2,\delta}(\Gamma,\Lambda^l)} + C_2||(\lambda-M)u||^2_{L_t^{2,\delta}(\Gamma,\Lambda^l)} + I + II\right\} \end{aligned} \tag{3.256}$$

In order to arrive at a suitable estimate of I'', we write, for all $x\in\Gamma$,

$$\begin{aligned} \theta(x)\vee p.v.d\int_\Gamma \Phi_{k,n}(x,y)u(y)d\Gamma(y) =& \int_\Gamma \theta(x)\vee[(d_x\Phi_{k,n})(x,y)\wedge u(y)]d\Gamma(y) = \\ & \int_\Gamma (\theta(x)-\theta(y))\vee[(d_x\Phi_{k,n})(x,y)\wedge u(y)]d\Gamma(y)+ \\ & \int_\Gamma \theta(y)\vee[(d_x\Phi_{k,n})(x,y)\wedge u(y)]d\Gamma(y) = \\ & Ku(x)+I_1''(x)+I_2''(x) \end{aligned} \tag{3.257}$$

where K signifies a compact operator on $L^2(\Gamma,\Lambda^l)$, and where the following definitions are made:

$$I_1''(x) \equiv -\int_\Gamma (d_x\Phi_{k,n})(x,y) \wedge (\theta(y) \vee u(y))d\Gamma(y) \quad \forall x \in \Gamma \tag{3.258}$$

$$I_2''(x) \equiv \int_\Gamma \langle d_y\Phi_{k,n}(x,y), \theta(y)\rangle u(y)d\Gamma(y) \quad \forall x \in \Gamma \tag{3.259}$$

It is seen from Definition 2.4.23, introducing the boundary co-derivative δ_∂, and from the Cauchy-Schwartz inequality that the following holds true:

$$\begin{aligned}\Re\int_\Gamma \langle I_1'', u\rangle d\Gamma = -\Re\int_\Gamma \langle \gamma_0 r_{\Omega_+}\mathcal{S}\theta \vee u, u\rangle d\Gamma = \Re\int_\Gamma \langle S\theta \vee u, \delta_\partial u\rangle d\sigma \leq \\ C_1||\delta_\partial u||_{L_t^{2,\delta}(\Gamma,\Lambda^{l-1})} + C_2||S\theta \vee u||_{L_t^{2,\delta}(\Gamma,\Lambda^{l-1})}\end{aligned} \tag{3.260}$$

We also have, by the triangle inequality, that

$$\begin{aligned}&\left|\Re\int_\Gamma\left\langle\int_\Gamma \frac{\partial\Phi_{k,n}}{\partial\theta_y}(x,y)u(y)d\Gamma(y), u(x)\right\rangle d\Gamma(x)\right| \leq \\ &\quad\left|\int_\Gamma\left\langle\int_\Gamma \frac{\partial(\Phi_{k,n}-\Phi_{0,n})}{\partial\theta_y}(x,y)u(y)d\Gamma(y), u(x)\right\rangle d\Gamma(x)\right| + \\ &\quad\left|\Re\int_\Gamma\left\langle\int_\Gamma \frac{\partial\Phi_{0,n}}{\partial\theta_y}(x,y)u(y)d\Gamma(y), u(x)\right\rangle d\Gamma(x)\right| \equiv I_1'''(x) + I_2'''(x) \quad \forall x \in \Gamma\end{aligned} \tag{3.261}$$

As already noted in Section 2.8, the fundamental solution $\Phi_{0,n}$ of the n-dimensional Laplace equation contains the principal singularity of any fundamental solution $\Phi_{k,n}$, $k \in \mathbb{N}_0$. Therefore, the kernel $\partial(\Phi_{k,n} - \Phi_{0,n})/\partial\theta_y$ is only weakly singular, and we have

$$I_1'''(x) \leq C_1||Ku||_{L^2(\Gamma,\Lambda^l)} + C_2||u||_{L^2(\Gamma,\Lambda^l)} \quad \forall x \in \Gamma \tag{3.262}$$

By definition of the adjoint operator, it is true that

$$\Re\int_\Gamma\left\langle\frac{\partial S_{0,n}}{\partial\theta_y}u, u\right\rangle = \frac{1}{2}\int_\Gamma\left\langle\left(\frac{\partial S_{0,n}}{\partial\theta} + \left(\frac{\partial S_{0,n}}{\partial\theta}\right)^*\right)u, u\right\rangle d\Gamma \tag{3.263}$$

The kernel of $\partial S_{0,n}/\partial\theta + (\partial S_{0,n}/\partial\theta)^*$ equals

$$(\theta(x) - \theta(y)).\nabla\Phi_{0,n}(x,y) \quad \forall x, y \in \Gamma,\ x \neq y \tag{3.264}$$

and hence it is weakly singular. Consequently, we have

$$I_2'''(x) \leq C_1||Ku||_{L^2(\Gamma,\Lambda^l)} + C_2||u||_{L^2(\Gamma,\Lambda^l)} \quad \forall x \in \Gamma \tag{3.265}$$

Next, we see that

$$\begin{aligned}\gamma_0 r_{\Omega_+}\delta d\mathcal{S}u =& \gamma_0 r_{\Omega_+}(-\Delta - d\delta)\mathcal{S}u = \gamma_0 r_{\Omega_+}(k^2 I - d\delta)\mathcal{S}u = \\ & k^2\gamma_0 r_{\Omega_+}\mathcal{S}u - \gamma_0 r_{\Omega_+}d\mathcal{S}\delta_\partial u\end{aligned} \tag{3.266}$$

Now since $\nu \vee \gamma_0 r_{\Omega_+} d\mathcal{S} = -1/2I + M$, estimates of the quantity II may be deduced from the following estimates, obtained using the Cauchy-Schwartz inequality:

$$\left|\int_\Gamma \langle \gamma_0 r_{\Omega_+} \mathcal{S}u, (\lambda I - M)u\rangle d\Gamma\right| \le ||Su||_{L^2(\Gamma,\Lambda^l)} + ||(\lambda I - M)u||_{L^2(\Gamma,\Lambda^l)} \tag{3.267}$$

$$\left|\int_\Gamma \langle \gamma_0 r_{\Omega_+} \mathcal{S}u, u\rangle d\Gamma\right| \le ||Su||_{L^2(\Gamma,\Lambda^l)} + ||u||_{L^2(\Gamma,\Lambda^l)} \tag{3.268}$$

$$\left|\int_\Gamma \langle \gamma_0 r_{\Omega_+} d\mathcal{S}\delta_\partial u, (\lambda I - M)u\rangle d\Gamma\right| \le ||\gamma_0 r_{\Omega_+} d\mathcal{S}u||_{L^2(\Gamma,\Lambda^l)} + ||(\lambda I - M)u||_{L^2(\Gamma,\Lambda^l)} \tag{3.269}$$

$$\left|\int_\Gamma \langle \gamma_0 r_{\Omega_+} d\mathcal{S}\delta_\partial u, u\rangle d\Gamma\right| = \left|\int_\Gamma \langle S\delta_\partial u, \delta_\partial u\rangle d\Gamma\right| \le C_1||S\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} + C_2||\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} \tag{3.270}$$

Note that we have used Definition 2.4.22 in the derivation of the last inequality. By (3.252), we may now write

$$\begin{aligned} ||u||_{L^2(\Gamma,\Lambda^l)} \le & C_1||(\lambda - M)u||_{L^2(\Gamma,\Lambda^l)} + ||Ku||_{L^2(\Gamma,\Lambda^l)} + ||K\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} + \\ & C_2||u||_{L^2(\Gamma,\Lambda^l)} + C_3||\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} \quad \forall u \in L_t^{2,\delta}(\Gamma,\Lambda^l) \end{aligned} \tag{3.271}$$

It holds true that [14]

$$\delta_\partial M u = -k^2 \nu \vee Su + M\delta_\partial u \quad \forall u \in L_t^{2,\delta}(\Gamma,\Lambda^l) \tag{3.272}$$

on the hypersurface Γ, and we therefore have

$$\delta_\partial(\lambda I - M)u = k^2\nu \vee Su + (\lambda I - M)\delta_\partial u \quad \forall u \in L_t^{2,\delta}(\Gamma,\Lambda^l), \lambda \in \mathbb{C} \tag{3.273}$$

Hence, employing the estimate (3.271), we arrive at

$$\begin{aligned} ||\delta_\partial u||_{L^2(\Gamma,\Lambda^l)} \le & C_1||(\lambda I - M)\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} + ||K\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} + \\ & C_2||\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} \le \\ & C_1||\delta_\partial(\lambda I - M)u||_{L^2(\Gamma,\Lambda^{l-1})} + C_2||Ku||_{L^2(\Gamma,\Lambda^l)} + \\ & ||K\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} + C_3||\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} \quad \forall u \in L_t^{2,\delta}(\Gamma,\Lambda^l) \end{aligned} \tag{3.274}$$

or simply

$$\begin{aligned} ||\delta_\partial u||_{L^2(\Gamma,\Lambda^l)} \le & C_1||\delta_\partial(\lambda I - M)u||_{L^2(\Gamma,\Lambda^{l-1})} + C_2||Ku||_{L^2(\Gamma,\Lambda^l)} + \\ & ||K\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} \quad \forall u \in L_t^{2,\delta}(\Gamma,\Lambda^l) \end{aligned} \tag{3.275}$$

for $\lambda \in \mathbb{R}, |\lambda| \ge 1/2$. This result may be used with (3.271) to produce, for $\lambda \in \mathbb{R}, |\lambda| \ge 1/2$, the estimate

$$\begin{aligned} ||(\lambda I - M)u||_{L_t^{2,\delta}(\Gamma,\Lambda^l)} + ||Ku||_{L^2(\Gamma,\Lambda^l)} + & \\ ||K\delta_\partial u||_{L^2(\Gamma,\Lambda^{l-1})} \ge ||u||_{L_t^{2,\delta}(\Gamma,\Lambda^l)} & \quad \forall u \in L_t^{2,\delta}(\Gamma,\Lambda^l) \end{aligned} \tag{3.276}$$

We conclude that the operator $\lambda I - M$, modulo compact operators, is bounded from below for all $\lambda \in \mathbb{R} \backslash]-1/2, 1/2[$. Consequently, $\lambda I - M$ has closed range on $L_t^{2,\delta}(\Gamma, \Lambda^l)$.

The operator $\lambda I - M_{k,n}, n \in \mathbb{N}, k \in \mathbb{C}, \Im\{k\} \geq |\Re\{k\}|$, on $L_t^{2,\delta}(\Gamma, \Lambda^l)$ is readily seen to be of Fredholm type upon noting that

$$\lambda I - M_{k,n} = (\lambda I - M_{k_1,n}) + (M_{k_1,n} - M_{k,n}) \quad \forall n \in \mathbb{N}, k_1, k \in \mathbb{C} \tag{3.277}$$

By Lemma 3.4.11, the operator $\lambda I - M_{k_1,n}$ is invertible on $L_t^{2,\delta}(\Gamma, \Lambda^l)$ for $\lambda \in \mathbb{R}, |\lambda| \geq 1/2$. According to the property (2.276) of the fundamental solution $\Phi_{k,n}$, the kernel of $M_{k_1,n} - M_{k,n}$ is weakly singular, and hence, by the relation (3.272), the operator $M_{k_1,n} - M_{k,n}$ is compact on $L_t^{2,\delta}(\Gamma, \Lambda^l)$. Therefore, $\lambda I - M$ constitutes a compact perturbation of a bijection, and by the index stability theorem for Fredholm operators, the operator $\lambda I - M$ is Fredholm of index zero. For $\Im\{k\} > 0$, this last result, in conjunction with Lemma 3.4.12, yields $\pm 1/2 \notin \sigma(M)$.

■

The spectral radius of the magnetostatic operator M acting on $L_t^{2,\delta}(\Gamma, \Lambda^l)$, for some Lipschitz surface Γ, is thus less than $1/2$.

REMARK 3.4.14
We have, of course,

$$\pm \frac{1}{2} I + M = -\left\{ \mp \frac{1}{2} I - M \right\} \tag{3.278}$$

and hence the above discussion is relevant in the context of analysis of existence and uniqueness of solution to the general Maxwell boundary value problem (3.20) on Lipschitz domains.

□

An entirely similar derivation may be conducted for the exterior case, where $\Omega_- = \complement\Omega_+$ is considered, and where the 1-form θ is assumed to have compact support. Due to this last assumption, the behaviour of the differential form u at infinity is irrelevant in the present context.

We utilise the Hodge duality between the operators M, N to extend the above result to the operator N on $L_n^{2,d}(\Gamma, \Lambda^l)$.

3.4.3 Results for the General Maxwell Boundary Value Problem

It is finally possible to deduce the existence, uniqueness and regularity results for the solutions of the general Maxwell boundary value problem, in that the next proposition provides the last required element of the argumentation.

REMARK 3.4.15
Assume a differential form-valued function u is a solution of the homogeneous Helmholtz equation, i.e. $(\Delta + k^2 I)u = 0$, on some domain $\Omega \subseteq \mathbb{R}^n$. Then, using the already established relations

$$\begin{aligned} -\Delta &= d\delta + \delta d \\ d^2 &= 0, \delta^2 = 0 \end{aligned} \tag{3.279}$$

involving the Hodge Laplacian and the exterior and interior derivative operators, we see that the following holds true:

$$\begin{aligned} \Delta\delta u &= -\delta d\delta u = \delta(\delta du - k^2 u) = -k^2 u \\ \Delta du &= -d\delta du = d(d\delta u - k^2 u) = -k^2 u \end{aligned} \tag{3.280}$$

■

PROPOSITION 3.4.16
For any open bounded Lipschitz domain $\Omega_+ \subset \mathbb{R}^n$, solution of the (interior or exterior) electric boundary value problem (3.30) or of the (interior or exterior) magnetic boundary value problem (3.31) with the boundary datum $\Psi_2 \equiv 0$ on Γ is sufficent for the solution of the corresponding general Maxwell boundary value problem (3.20) specified by $\Psi \equiv \Psi_1$.
Proof: Assume that a differential form-valued function $E_\pm$ satisfies the (interior or exterior) electric boundary value problem (3.30) in the domain $\Omega_\pm$. The $(l-1)$ form-valued function $\delta E_\pm$ then necessarily solves the homogeneous Helmholtz system

$$\begin{aligned} (\Delta + k^2 I)\delta E_\pm = 0 \quad &\text{in } \Omega_\pm \\ \nu \wedge \gamma_0 \delta E_\pm = 0 \quad &\text{on } \Gamma \\ \nu \wedge \gamma_0 \delta(\delta E_\pm) = 0 \quad &\text{on } \Gamma \end{aligned} \tag{3.281}$$

The first identity in (3.281) follows from Remark 3.4.15. By the uniqueness part of Theorem 3.4.6, it is true that $\delta E_\pm$ identically vanishes in $\Omega_\pm$. The result now follows from Proposition 3.1.6.

In a dual manner, if $E_\pm$ satisfies the (interior or exterior) magnetic boundary value problem (3.31) in $\Omega_\pm$, we then have

$$\begin{aligned} (\Delta + k^2 I) dE_\pm = 0 \quad &\text{in } \Omega_\pm \\ \nu \vee \gamma_0 dE_\pm = 0 \quad &\text{on } \Gamma \\ \nu \vee \gamma_0 d(dE_\pm) = 0 \quad &\text{on } \Gamma \end{aligned} \tag{3.282}$$

Theorem 3.4.6 implies $dE_\pm \equiv 0$ in $\Omega_\pm$, and the result follows by Proposition 3.1.6.

■

Using the results presented in Remark 3.4.7, we may now specify the solutions of the interior and exterior generalised Maxwell boundary value problems.

COROLLARY 3.4.17
Given an open bounded Lipschitz domain Ω_+, the exact, unique solution $(E_\pm, \mathscr{H}_\pm)$ of the (interior or exterior) generalised Maxwell boundary value problem (3.20) is given by[18]

$$E_\pm = r_{\Omega_\pm} \delta \mathcal{S}(\pm(2\pi)^n \pi I + N)^{-1} \Psi \qquad \text{in } \Omega_\pm \tag{3.283}$$

$$\mathscr{H}_\pm = \frac{1}{ik} dE_\pm = \frac{1}{ik} r_{\Omega_\pm} d\delta \mathcal{S}(\pm(2\pi)^n \pi I + N)^{-1} \Psi \qquad \text{in } \Omega_\pm \tag{3.284}$$

for any boundary datum $\nu \wedge \gamma_0 E_\pm = \Psi \in L^{2,d}_n(\Gamma, \Lambda^{l+1})$, or, equivalently,

$$\mathscr{H}_\pm = r_{\Omega_\pm} d\mathcal{S}(\pm(2\pi)^n \pi I + M)^{-1} \Psi \qquad \text{in } \Omega_\pm \tag{3.285}$$

$$E_\pm = -\frac{1}{ik} \delta \mathscr{H}_\pm = -\frac{1}{ik} r_{\Omega_\pm} \delta d \mathcal{S}(\pm(2\pi)^n \pi I + M)^{-1} \Psi \qquad \text{in } \Omega_\pm \tag{3.286}$$

[18] We again use our convention for the constants in the relevant boundary integral operators.

for any boundary datum $\nu \vee \gamma_0 \mathscr{H}_\pm = \Psi \in L_t^{2,\delta}(\Gamma, \Lambda^l)$.

If the hypersurface Γ is smooth, the boundary data may be in $L_n^{p,d}, L_t^{p,\delta}$ with $p \in]1, \infty[$.

□

We note that Reference [14] asserts that there exists $\varepsilon > 0$, dependent only on n, k and the Lipschitz character of Γ, such that the above result is valid for Lipschitz domains and data in $L_t^{p,\delta}, p \in]2-\varepsilon, 2+\varepsilon[$. We furthermore present some regularity results on solutions of the generalised Maxwell boundary value problem.

THEOREM 3.4.18
Given a solution $(E, \mathscr{H})$ of the generalised Maxwell boundary value problem (3.20), the following holds true:

$$\begin{gathered} \mathcal{N}E, \mathcal{N}\mathscr{H} \in L^2(\Gamma) \\ \Updownarrow \\ E \in W^{1/2}(\Omega_+, \Lambda^l),\ \mathscr{H} \in W^{1/2}(\Omega_+, \Lambda^{l+1}) \end{gathered} \tag{3.287}$$

Furthermore, if any of the above conditions is satisfied, we have

$$||\mathcal{N}E||_{L^2(\Gamma)} + ||\mathcal{N}\mathscr{H}||_{L^2(\Gamma)} \approx ||E|| \tag{3.288}$$

Proof: A proof is found in Reference [14].[19]

■

REMARK 3.4.19
Of immediate physical interest are the three-dimensional acoustic and electromagnetic cases specified by $(l, n) = (0, 3)$ and $(l, n) = (1, 3)$, respectively. The expression (3.283)–(3.284) for the unique exact solution of the generalised Maxwell boundary value problem (3.20) specialises in these cases to

$$p_\pm = -\, r_{\Omega_\pm} \mathrm{div} \mathcal{S}(\pm(2\pi)^n \pi I + N)^{-1} \Psi \quad \text{in } \Omega_\pm \tag{3.289}$$

$$v_\pm = -\, \frac{1}{ik} r_{\Omega_\pm} \nabla \mathrm{div} \mathcal{S}(\pm(2\pi)^n \pi I + N)^{-1} \Psi \quad \text{in } \Omega_\pm \tag{3.290}$$

and to

$$E_\pm = r_{\Omega_\pm} \mathcal{S}\delta_\partial(\pm(2\pi)^n \pi I + N)^{-1} \Psi \quad \text{in } \Omega_\pm \tag{3.291}$$

$$H_\pm = *\mathscr{H}_\pm = \frac{1}{ik} r_{\Omega_\pm} \mathrm{curl} \mathcal{S}\delta_\partial(\pm(2\pi)^n \pi I + N)^{-1} \Psi \quad \text{in } \Omega_\pm \tag{3.292}$$

respectively. Equivalently, we derive from (3.285)–(3.286) that

$$p_\pm = ik r_{\Omega_\pm} \mathcal{S}(\pm(2\pi)^n \pi I + M)^{-1} \Psi \quad \text{in } \Omega_\pm \tag{3.293}$$

$$v_\pm = r_{\Omega_\pm} \nabla \mathcal{S}(\pm(2\pi)^n \pi I + M)^{-1} \Psi \quad \text{in } \Omega_\pm \tag{3.294}$$

for the case $(l, n) = (0, 3)$, while for the parameter pair $(l, n) = (1, 3)$ we obtain

$$E_\pm = ik r_{\Omega_\pm} \mathcal{S}(\pm(2\pi)^n I + M)^{-1} \Psi - \frac{1}{ik} \nabla \mathrm{div} \mathcal{S}(\pm(2\pi)^n \pi I + M)^{-1} \Psi \quad \text{in } \Omega_\pm \tag{3.295}$$

$$H_\pm = *\mathscr{H}_\pm = r_{\Omega_\pm} \mathrm{curl} \mathcal{S}(\pm(2\pi)^n \pi I + M)^{-1} \Psi \quad \text{in } \Omega_\pm \tag{3.296}$$

□

[19] Note that the convention in Reference [14] is such that the generalised electric field E is a differential form-valued function of order higher than that of the function $\mathscr{H}$.

Chapter 4

Variational Formulations

In Chapter 3, an investigation of the existence and uniqueness of solutions to acoustic and electromagnetic boundary value problems has been performed using the theory of layer potentials. This, however, is not the only possible approach, in that the *variational calculus* also provides a suitable framework for establishing existence and uniqueness in the present context. In general, methods involving maximisation of some sort are referred to as *variational*, and systems of equations resulting from applications of variational methods to boundary value problems are designated *variational formulations*. Having already developed the rather powerful theory of layer potentials, we need to justify our treatment of variational formulations in this chapter. Indeed, devoting a portion of the text to a second proof of existence and uniqueness of solutions may seem an exercise in redundancy. However, it should be noted that while the existence and uniqueness issues associated with acoustic and electromagnetic boundary value problems have already been addressed in Chapter 3, there remains the important question of convergence of numerical methods used to solve practical scattering problems. In light of the fact that these methods are typically based on a variational scheme called the *Method of Moments* (or MoM for short), it is of relevance to introduce the theoretical fundament of the variational approach. Time and space constraints prohibit us from presenting examples of the use of the basic variational calculus in an analysis of convergence of a particular numerical method. However, we do present a novel generalisation of a known theoretical result concerning the Method of Auxiliary Sources, which is a special case of the MoM and therefore, at least in principle, constitutes a variational method. The generalisation is performed in the framework of differential form-valued functions and associated operators.

We begin by proving the fundamental result of the variational calculus, namely the Lax-Milgram lemma. Thereafter, the Gårding inequality is shown, which, as illustrated in the following, facilitates the application of the Lax-Milgram lemma to existence, uniqueness and regularity analysis of boundary value problems. We next present the Method of Moments and, as an example of a practical numerical method utilised in solution of acoustic and electromagnetic scattering problems, we briefly discuss the Method of Auxiliary Sources (also referred to as the MAS) and show a generalisation of the associated convergence result presented in Reference [12]. To the best of the knowledge of the author, this generalisation has not been exhibited earlier.

4.1 The Lax-Milgram Lemma and the Gårding Inequality

In order to prove the main result of this section, namely the Lax-Milgram lemma, some preliminary work must be done.

DEFINITION 4.1.1
Let T be an operator with domain in a Hilbert space U. The *numerical range* $\nu(T)$ of T is defined [9] by

$$\nu(T) \equiv \{(Tu,u)|u \in D\{T\},\ ||u||_U = 1\} \tag{4.1}$$

The *lower bound* $m(T)$ of T is defined by

$$m(T) \equiv \inf\{\Re\{(Tu,u)\}|u \in D\{T\},\ ||u||_U = 1\} \tag{4.2}$$

□

We note that $\sigma(T) \subset \nu(T)$ and $\Re\{\sigma(T)\} \geq m(T)$, with T as defined above.

PROPOSITION 4.1.2
Let T be an operator with domain in a Hilbert space U. If the lower bound of T is well-defined and if $m(T) \geq \alpha > 0$, then T is a homeomorphism of $D\{T\}$ on $R\{T\}$. More precisely, T is injective and T^{-1} is bounded on $D\{T^{-1}\} = R\{T\}$ with $||T^{-1}|| \leq \alpha^{-1}$. If T furthermore is a closed operator, then its range $R\{T\}$ is a closed set.
Proof: If $m(T) \geq \alpha > 0$, it holds true that

$$\alpha \leq \Re(T\frac{u}{||u||_U}, \frac{u}{||u||_U}) \quad \forall u \in U\backslash\{0\} \tag{4.3}$$

such that, using the Cauchy-Schwartz inequality, we have

$$\alpha||u||_U^2 \leq \Re(Tu,u) \leq |(Tu,u)| \leq ||Tu||_U\, ||u||_U \quad \forall u \in D\{T\}\backslash\{0\} \tag{4.4}$$

Consequently, it is true that

$$||Tu||_U \geq \alpha||u||_U \quad \forall u \in D\{T\}\backslash\{0\} \tag{4.5}$$

and thus $\ker\{T\} = \{0\}$. Therefore, we may rewrite (4.5) as

$$||T^{-1}Tu||_U \leq \alpha^{-1}||Tu||_U \quad \forall u \in D\{T\}\backslash\{0\} \tag{4.6}$$

which implies $||T^{-1}|| \leq \alpha^{-1}$.
If, in addition, T is a closed operator, then T^{-1} is closed, so $D\{T^{-1}\} = R\{T\}$ is a closed set.

■

DEFINITION 4.1.3
Let V, U be two Hilbert spaces, V densely included in U with[1]

$$||u||_V \geq c||u||_U \quad \forall u \in V \quad \text{for some } c_0 > 0 \tag{4.7}$$

[1]The property (4.7) may be expressed by saying that the space V has a *stronger norm* than the space U. [11]

A sesquilinear form $s : V^2 \to \mathbb{C}$ is characterised as *V-elliptic* iff there exist constants $c > 0, k \in \mathbb{R}$ such that

$$\Re s(u,u) \geq c||u||_V^2 \quad \forall u \in V \tag{4.8}$$

It is characterised as *V-coercive* iff there exist constants $c > 0, k \in \mathbb{R}$ such that

$$\Re s(u,u) \geq c||u||_V^2 - k||u||_U^2 \quad \forall u \in V \tag{4.9}$$

□

The following theorem establishes existence and uniqueness of solutions to variational problems involving coercive sesquilinear forms. It is a variant of the Lax-Milgram lemma. References [10, 11, 28] each present a distinct version of the result.

THEOREM 4.1.4
The Lax-Milgram Lemma (The Variational Construction)
Consider two Hilbert spaces U, V with dense injection $V \hookrightarrow U$ and satisfying

$$\exists c_0 > 0 : \quad ||u||_V \geq c_0||u||_U \quad \forall u \in V \tag{4.10}$$

Let $s : V^2 \to \mathbb{C}$ be a continuous V-coercive sesquilinear form,

$$|s(u,v)| \leq C||u||_V||v||_V \quad \forall u, v \in V, \tag{4.11}$$

$$\exists c > 0, k \in \mathbb{R} : \ \Re s(u,u) \geq c||u||_V^2 - k||u||_U^2 \quad \forall u \in V \tag{4.12}$$

Furthermore, let

$$s^*(u,v) = \overline{s(v,u)} \quad \forall u, v \in V \tag{4.13}$$

Define the operator $S : D\{S\} \to U$ as follows:

$$D\{S\} \equiv \{u \in V \big| \exists f \in U : \ s(u,v) = (f,v)_U \quad \forall v \in V\}, \tag{4.14}$$

$$Su = f \quad \forall u \in D(S) \tag{4.15}$$

Finally, S^* is defined in a similar manner.

The operators $S + kI : D\{S\} \to U$, $S^* + kI : D\{S^*\} \to U$ are isomorphisms. In fact, the spectrum and the numerical range of the operators S, S^* lie in the sector W of the complex plane given by

$$W = \{z \in \mathbb{C} \mid \Re\{z\} \geq cc_0^2 - k, \ |\Im\{z\}| \leq Cc^{-1}(\Re\{z\} + k)\} \tag{4.16}$$

Proof: The dense injection $V \hookrightarrow U$ induces the dense injection $U \hookrightarrow V^*$. For any $k \in \mathbb{R}$, the sesquilinear forms defined by

$$s_k(u,v) \equiv s(u,v) + k(u,v)_U, \ s_k^*(u,v) \equiv \overline{s_k(v,u)} \quad \forall u, v \in V \tag{4.17}$$

induce the following continuous operators:

$$\begin{aligned} \mathscr{S}_k : V \to V^*, \quad \langle \mathscr{S}_k u, v\rangle_{V^*,V} \equiv s_k(u,v) \quad \forall u, v \in V \\ \mathscr{S}_k^* : V \to V^*, \quad \langle \mathscr{S}_k^* u, v\rangle_{V^*,V} \equiv s_k^*(u,v) \quad \forall u, v \in V \end{aligned} \tag{4.18}$$

By construction, the V-coerciveness of the form s implies V-ellipticity of s_k and s_k^* for all $k \in \mathbb{R}$. Hence, it is true that

$$|\langle \mathscr{S}_k u, u, \rangle_{V^*,V}| = |s_k(u,u)| \geq \Re\{s_k(u,u)\} \geq c||u||_V^2 \quad \forall u \in V, \ k \in \mathbb{R} \tag{4.19}$$

or simply

$$\left|\left\langle \mathscr{S}_k \frac{u}{||u||_V}, \frac{u}{||u||_V} \right\rangle_{V^*,V}\right| \geq c \quad \forall u \in V,\, k \in \mathbb{R} \tag{4.20}$$

An entirely similar relation is valid for the operator $\mathscr{S}_k^*$, $k \in \mathbb{R}$. Consequently, we have

$$m(\mathscr{S}_k) \geq c > 0,\; m(\mathscr{S}_k^*) \geq c > 0 \quad \forall k \in \mathbb{R} \tag{4.21}$$

and, by Proposition 4.1.2, the operators $\mathscr{S}_k, \mathscr{S}_k^* : V \to V^*$ are homeomorphisms of V on V^* for all $k \in \mathbb{R}$. Clearly, it holds true that

$$\langle \mathscr{S}_k u, v\rangle_{V^*,V} = s(u,v) + k(u,v)_U = ((S + kI)u, v)_U \quad \forall u \in D\{S\},\, v \in V,\, k \in \mathbb{R} \tag{4.22}$$

such that, for all $k \in \mathbb{R}$, the restriction of $\mathscr{S}_k$ to the space $D\{S\}$ is given by $\mathscr{S}_k|_{D\{S\}} = S + kI$. In a similar manner, it holds that $\mathscr{S}_k^*|_{D\{S^*\}} = S^* + kI$ for all $k \in \mathbb{R}$. But note that the space $D\{S\}$ introduced above has the equivalent definition

$$D\{S\} = \{u \in V | \mathscr{S}_k u \in U,\, k \in \mathbb{R}\} \tag{4.23}$$

Furthermore, a similar remark applies to the space $D\{S^*\}$. Functionals in U on V are continuous, since by assumption

$$|(v,u)_U| \leq ||u||_U ||v||_U \leq ||u||_U ||v||_V \quad \forall v \in V, u \in U \tag{4.24}$$

There exists thus for every $u \in U$ the dual vector $w \in V$, and we have, from the above discussion, that $\mathscr{S}_k w = (S + kI)w = u$. Again, a similar development can be made for the operator $S^* + kI$. Hence, the operator $S + kI$ ($S^* + kI$) is a bijection mapping $D\{S\}$ ($D\{S^*\}$) onto U. Being restrictions of duality operators, $S + kI$ and $S^* + kI$ are isometries.

We now prove the second assertion of the present theorem. Firstly, it is clear from the definition (4.14)–(4.15) and the assumptions (4.12), (4.10) that

$$\Re\{(Su,u)_U\} \geq c||u||_V^2 - k||u||_U^2 \geq (cc_0^2 - k)||u||_V^2 \quad \forall u \in V \tag{4.25}$$

Secondly, using the triangle inequality, the fact that the sesquilinear form s_k is continuous on $V \times V$ for all $k \in \mathbb{R}$, as well as the V-coercivity relation (4.12), we write

$$|\Im\{s_k(u,u)\}| \leq |s_k(u,u)| \leq C||u||_V^2 \leq Cc^{-1}\Re\{s_k(u,u)\} \quad \forall u \in V \tag{4.26}$$

Consequently, it holds true that

$$|\Im\{(Su,u)_U\}| = |\Im\{(S_k u,u)_U\}| \leq Cc^{-1}\Re\{s_k(u,u)\} \quad \forall u \in V \tag{4.27}$$

But then, for any eigenvector u_0 of the operator S with pertaining eigenvalue λ, it holds true that

$$|\Im\{\lambda\}|\, ||u_0||_U^2 = |\Im\{(Su_0,u_0)_U\}| \leq Cc^{-1}(\Re\{\lambda\} + k)||u_0||_U^2 \tag{4.28}$$

or simply

$$|\Im\{\lambda\}| \leq Cc^{-1}(\Re\{\lambda\} + k) \tag{4.29}$$

This completes the proof.

∎

The above discussed operator $\mathscr{S}_k$ is referred to as the *Lax-Milgram operator* associated with the triple (U, V, s_k), or as the *variational operator*. [10]

A comment is in order regarding the connection between the variational formalism presented here and the theory of Chapter 3. The Lax-Milgram lemma is an alternative representation of the Fredholm alternative presented in Corollary 2.6.18. As we have seen in Chapter 3, the Fredholm alternative is a consequence of the more general index stability theorem for Fredholm operators.

We now apply the theory of pseudodifferential operators to derive the Gårding's inequality. This is a result of fundamental importance in the analysis of existence and uniqueness of solutions to variational formulations of boundary value problems. According to Reference [36], L. Gårding originally obtained this result for the special case of differential operators.

THEOREM 4.1.5
The Gårding Inequality
Given $m \in \mathbb{R}$ and $\rho, \delta \in \mathbb{R}$ such that $0 \leq \delta < \rho \leq 1$, consider a pseudodifferential operator $p(x, D) \in OPS^m_{\rho,\delta}(\mathbb{R}^n)$. The following holds true:

$$\begin{gathered} \exists C > 0 :\ \Re\{p(x,\xi)\} \geq C|\xi|^m \quad \forall x, \xi \in \mathbb{R}^n, |\xi| \gg 1 \\ \Downarrow \\ \forall s \in \mathbb{R}\ \exists C_1, C_2 \in \mathbb{R}, C_1 > 0 : \\ \Re\{(p(x,D)u, u)\} \geq C_1 ||u||^2_{H^{m/2}(\mathbb{R}^n)} - C_2 ||u||^2_{H^s(\mathbb{R}^n)} \quad \forall u \in H^{m/2}(\mathbb{R}^n) \end{gathered} \tag{4.30}$$

Proof: Define $q(x, D) = \Lambda^{-m/2} p(x, D) \Lambda^{-m/2}$ on $\mathbb{R}^n$. Since $\Lambda^s \in OPS^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, we have $q(x, D) \in OPS^0 \subset OPS^0_{1,0}$ and hence, by Proposition 2.5.16, it is true that

$$q(x, D) \in OPS^0_{\rho,\delta}(\mathbb{R}^n) \tag{4.31}$$

According to Proposition 2.5.26, $q(x, D)$ maps $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. We now introduce the function $A : \mathbb{R}^{2n} \to \mathbb{R}$ as follows:

$$A(x,\xi) \equiv \left\{ \Re\{q(x,\xi)\} - \frac{C}{2} \right\}^{1/2} \quad \forall x, \xi \in \mathbb{R}^n \tag{4.32}$$

It is now seen by Remark 2.5.15 that

$$A(x, D) \in S^0_{\rho,\delta}(\mathbb{R}^n) \tag{4.33}$$

and we may hence write

$$A(x,D)^* A(x,D) = \Re\{q(x,D)\} - \frac{C}{2} I + r(x,D) \quad \text{on } \mathbb{R}^n \tag{4.34}$$

where

$$r(x, D) \in OPS^{-(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^n) \tag{4.35}$$

Thus, we have

$$\begin{aligned} \Re\{(q(x,D)u,u)_{L^2(\mathbb{R}^n)}\} = &||A(x,D)u||^2_{L^2(\mathbb{R}^n)} + \frac{C}{2}||u||^2_{L^2(\mathbb{R}^n)} - (r(x,D)u,u)_{L^2(\mathbb{R}^n)} \geq \\ &\frac{C}{2}||u||^2_{L^2(\mathbb{R}^n)} - C_2 ||u||^2_{H^{-(\rho-\delta)/2}(\mathbb{R}^n)} \quad \forall u \in L^2(\mathbb{R}^n) \end{aligned} \tag{4.36}$$

for some $C_2 > 0$, or equivalently

$$\Re\{(p(x,D)u,u)_{L^2(\mathbb{R}^n)}\} \geq C_1||u||^2_{H^{m/2}(\mathbb{R}^n)} - C_2||u||^2_{H^{-(\rho-\delta)/2}(\mathbb{R}^n)} \tag{4.37}$$

with $C_1, C_2 > 0$ and for all $u \in H^{m/2}(\mathbb{R}^n)$. In the case $s < -(\rho-\delta)/2$, it holds true that

$$||u||^2_{H^s(\mathbb{R}^n)} \leq \varepsilon||u||^2_{L^2(\mathbb{R}^n)} + C(\varepsilon)||u||^2_{H^{-(\rho-\delta)/2}(\mathbb{R}^n)} \tag{4.38}$$

and the result is seen to hold.

∎

4.2 The Integral Equation/Method of Moments Scheme

In this section, we present briefly the most common approach to practical numerical solution of acoustic and electromagnetic scattering problems, namely the so-called *Integral Equation/Method of Moments* scheme (or IE/MoM for short.)

Given the Hilbert spaces U, V, consider the inhomogeneous equation

$$Lu = g \tag{4.39}$$

comprising the operator $L \in \mathcal{L}(U,V)$. A *discretisation* of the problem (4.39) is effected by choosing a basis $\{\varphi_j\}_{j\in\mathbb{N}}$ in the space U and substituting for the unknown $u \in U$ the corresponding generalised Fourier expansion

$$u = \sum_{j\in\mathbb{N}} \mathscr{U}_j\varphi_j, \quad \mathscr{U}_j \in \mathbb{C}\ \forall j \in \mathbb{N} \tag{4.40}$$

Due to the continuity of the operator L, the resultant discretised equation reads

$$\sum_{j\in\mathbb{N}} \mathscr{U}_j L\varphi_j = g \tag{4.41}$$

We refer to the functions $\varphi_j, j \in \mathbb{N}$, as *basis vectors* or *expansion functions*. If the set $\{\psi_j\}_{j\in\mathbb{N}}$ constitutes a basis in V, equation (4.41) is clearly equivalent to

$$\left(\sum_{j\in\mathbb{N}} \mathscr{U}_j L\varphi_j - g, \psi_j\right) = 0 \quad \forall j \in \mathbb{N} \tag{4.42}$$

or, by the continuity of the inner product, to

$$\sum_{j\in\mathbb{N}} \mathscr{U}_j (L\varphi_j, \psi_j) = (g, \psi_j) \quad \forall j \in \mathbb{N} \tag{4.43}$$

Expression (4.43) is the weak, or variational, formulation of the problem (4.41). The process taking the problem (4.41) into the form (4.42) is commonly referred to as *testing*, and accordingly, the basis vectors $\psi_j, j \in \mathbb{N}$ are called the *test functions*, or *test vectors*. If expansion and test vectors are of the same type, the numerical scheme is called Galerkin.

In practical applications, only a finite number of basis and test vectors may be employed, and hence the exact representation of equation (4.40) is reduced to

$$u = \sum_{j=1}^{N} \mathscr{U}_j\varphi_j + R_u \tag{4.44}$$

where $N \in \mathbb{N}$ is finite, and the quantity $R_u \in U$ is the *residual*. Consequently, expression (4.41) is written

$$\sum_{j=1}^{N} \mathscr{U}_j L\varphi_j = g + R \tag{4.45}$$

with the residual $R \in V$ given by $R = LR_u$. Moreover, the testing procedure amounts to requiring the residual R to be orthogonal to a finite-dimensional (typically N-dimensional) subset $\overline{\text{span}(\psi_j,\ j \in \mathbb{N})}$ of U:

$$\sum_{j=1}^{N} \mathscr{U}_j (L\varphi_j, \psi_i) = (g, \psi_i) \quad \forall i \in \{1, \ldots, N\} \tag{4.46}$$

Note that the finite-dimensional testing transforms the equation (4.44) with N unknowns into a system of N equations with N unknowns. The obtained matrix-vector equation can be solved numerically by a variety of existing methods.

As shown in Chapter 3, the unique exact solution to the generalised exterior Maxwell boundary value problem (3.20) is of the form

$$E = \frac{(-1)^{l+1}}{ik} d\delta\mathcal{S}j + (-1)^{l(n-l+1)} * d\mathcal{S}m + (-1)^l ik\mathcal{S}j \quad \text{in } \mathbb{R}^n \backslash \Gamma \tag{4.47}$$

$$H = \frac{(-1)^{l(n-l)+1}}{ik} d\delta\mathcal{S}m + (-1)^{l(n-l+1)} * d\mathcal{S}j + (-1)^{l(n-l)+1} ik\mathcal{S}m \quad \text{in } \mathbb{R}^n \backslash \Gamma \tag{4.48}$$

where j, m are surface densities. The IE/MoM approach to a scattering problem consists in utilising the Stratton-Chu fomulae (4.47) and (4.48) to express the given boundary condition, whereafter the Method of Moments is applied to the resultant integral equation.[2] Hence, instead of looking directly for the solution fields, the original boundary value problem is reformulated and the unknown surface densities j, m are solved for.

4.3 The Method of Auxiliary Sources

In references [15, 16], we provide an applications–oriented description of the Method of Auxiliary Sources and discuss convergence issues associated with the method. Furthermore, we present specific computational examples along with numerical results, and list a selection of relevant background articles for the interested reader. Here, it suffices to say that the MAS is characterised by the choice of spatially impulsive basis and test vectors (Dirac delta functions.) The basis vectors are located inside the scatterer surface, and as the number of these vectors is increased for improved accuracy of numerical solution, the set of isolated currents approaches a continuous surface current enclosed by the scatterer. Reference [12] demonstrated the completeness, under certain circumstances, of this limit set of currents in terms of representing the scattered field outside the scatterer structure. Now follows our generalisation of this result to l-form-valued fields in $\mathbb{R}^n$:

[2] Clearly, the 'integral equation' part of the IE/MoM designation stems from the fact that the layer potentials involved in the Stratton-Chu formulae are integral operators.

THEOREM 4.3.1
A Generalised Convergence Theorem for the Method of Auxiliary Sources
Let there be given $n \in \mathbb{N}, l \in \mathbb{N}_0$ such that

$$l = \frac{n-1}{2} \tag{4.49}$$

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and $G \subset \Omega$ a compact set with the boundary ∂G being an $n-1$-dimensional C^1 hypersurface. At each point x of ∂G there can be defined two linearly independent tangential l-forms, say i_x and j_x. Define the set

$$Q = \left\{ i_x \middle| x \in \partial G \right\} \bigcup \left\{ j_x \middle| x \in \partial G \right\} \tag{4.50}$$

Denote by Φ the fundamental solution to the scalar Helmholtz equation in $\mathbb{R}^n$. For notational convenience, we set

$$\Phi_x(y) \equiv \Phi(x,y) \quad \forall y \in \mathbb{R}^n \backslash \{x\},\, x \in \partial G \tag{4.51}$$

Define

$$\begin{aligned} V_q(y) &\equiv *d(\Phi_{x_q} q)(y) \quad \forall y \in \mathbb{R}^n \backslash \{x_q\},\, \forall q \in Q \\ W_q(y) &\equiv \delta d(\Phi_{x_q} q)(y) \quad \forall y \in \mathbb{R}^n \backslash \{x_q\},\, \forall q \in Q \end{aligned} \tag{4.52}$$

where $x_q \in \partial G$ corresponds to the vector $q \in Q$. The set $\{V_q, W_q\}_{q \in Q}$ is dense in the space of tangential traces on $\partial\Omega$ of solutions to the Maxwell system.
Proof: Given $n \in \mathbb{N}, l \in \{0, \ldots, n\}, p \in]1, \infty[$, we define the following spaces:

$$\begin{aligned} D_l^p(d,\Omega) &= \{u \in L^p(\partial\Omega, \Lambda^l) | du \in L^p(\partial\Omega, \Lambda^{l+1})\} \\ D_l^p(\delta,\Omega) &= \{u \in L^p(\partial\Omega, \Lambda^l) | \delta u \in L^p(\partial\Omega, \Lambda^{l-1})\} \\ \mathfrak{X}_l^p(\Gamma) &= \{u \in B^{-\frac{1}{p},p}(\partial\Omega, \Lambda^l) | \exists w \in D_{l-1}^p(d,\Omega) : u = \nu \wedge w\} \\ \mathfrak{Y}_l^p(\partial\Omega) &= \{u \in B^{-\frac{1}{p},p}(\partial\Omega, \Lambda^l) | \exists w \in D_{l+1}^p(\delta,\Omega) : u = \nu \vee w\} \end{aligned} \tag{4.53}$$

In the expressions above, the symbol B denotes the usual scale of Besov spaces [22].

Since S is a C^1 hypersurface, tangential differential forms are well-defined on S. Clearly, it holds true that $\Phi \in C^\infty(\Omega_-) \cap L^2(\Omega_-)$, while $q \in C^\infty(\Omega_-, \Lambda^l)\, \forall q \in Q$. Therefore, for all $q \in Q$, we have

$$V_q = *d(\Phi_{x_q} q) \in C^\infty(\Omega_-, \Lambda^{n-l-1}) \cap L^2(\Omega_-, \Lambda^{n-l-1}) \tag{4.54}$$

and

$$dV_q = d{*}d(\Phi_{x_q} q) \in C^\infty(\Omega_-, \Lambda^{n-l}) \cap L^2(\Omega_-, \Lambda^{n-l}) \tag{4.55}$$

Furthermore, for all $q \in Q$, it holds true that

$$W_q = (-1)^{l(n-l)+n-1} \delta * V_q \in C^\infty(\Omega_-, \Lambda^l) \cap L^2(\Omega_-, \Lambda^l) \tag{4.56}$$

and

$$dW_q = (-1)^{l(n-l)+n-1} d\delta * V_q \in C^\infty(\Omega_-, \Lambda^{l+1}) \cap L^2(\Omega_-, \Lambda^{l+1}) \tag{4.57}$$

Summarising, it is seen that, for all $q \in Q$, we have $V_q \in D_{n-l-1}^2(d, \Omega_-)$ and $W_q \in D_l^2(d,\Omega)$. Therefore, it holds true that (recall that $2l = n-1$ was assumed)

$$\begin{gathered} \nu \wedge V_q \in \mathfrak{X}_{n-l}^2(\partial\Omega),\ \pi_\tau V_q = \nu \vee (\nu \wedge V_q) \in \mathfrak{Y}_{n-l-1}^2(\partial\Omega) = \mathfrak{Y}_l^2(\partial\Omega) \quad \forall q \in Q, \\ \nu \wedge W_q \in \mathfrak{X}_{l+1}^2(\partial\Omega),\ \pi_\tau W_q = \nu \vee (\nu \wedge W_q) \in \mathfrak{Y}_l^2(\partial\Omega) \quad \forall q \in Q \end{gathered} \tag{4.58}$$

Thus, the quantities

$$\langle \pi_\tau V_q, u\rangle,\ \langle \pi_\tau W_q, v\rangle \tag{4.59}$$

are well-defined for all q and for any $u, v \in \mathfrak{X}_l^2(\partial\Omega)$. Consider now a (either electric or magnetic) density $\psi \in \mathfrak{X}_l^2(\partial\Omega)$, together with the pertaining electromagnetic field (E_ψ, H_ψ).[3] It holds true that

$$I_q \equiv (\pi_\tau V_q(y), \overline{\psi})_{TL^2(\partial\Omega)} = \int\limits_{\partial\Omega} \langle \pi_\tau V_q(y), \psi(y)\rangle d\omega(y) = \int\limits_{\partial\Omega} \langle \nu \vee (\nu \wedge V_q)(y), \psi(y)\rangle d\omega(y) \tag{4.60}$$

Now, using (2.66), (4.52) and (2.71), we obtain

$$\begin{gathered} I_q = \int\limits_{\partial\Omega} \langle V_q, \nu \vee (\nu \wedge \psi)(y)\rangle d\omega(y) = \int\limits_{\partial\Omega} \langle V_q(y), \psi(y)\rangle d\omega(y) = \\ \int\limits_{\partial\Omega} \langle *d_y(\Phi_{x_q} q)(y), \psi(y)\rangle d\omega(y) = \int\limits_{\partial\Omega} \langle *((d_y\Phi_{x_q}) \wedge q)(y), \psi(y)\rangle d\omega(y) \end{gathered} \tag{4.61}$$

The first part of (2.62), as well as (2.66) and the first part of (2.70) are now used to get

$$\begin{gathered} I_q = (-1)^l \int\limits_{\partial\Omega} \langle (d_y\Phi_{x_q})(y) \vee *q, \psi(y)\rangle d\omega(y) = (-1)^l \int\limits_{\partial\Omega} \langle *q, (d_y\Phi_{x_q}) \wedge \psi(y)\rangle d\omega(y) = \\ (-1)^{l(n-l+1)} \int\limits_{\partial\Omega} \langle q, *((d_y\Phi_{x_q}) \wedge \psi)(y)\rangle d\omega(y) \end{gathered} \tag{4.62}$$

Due to the antisymmetry present in the form of the fundamental solution Φ_{x_q}, it holds true that

$$d_y\Phi_{x_q} = -d_x\Phi_{x_q} \tag{4.63}$$

This fact, together with (2.71), leads to

$$\begin{gathered} I_q = (-1)^{l(n-l+1)+1} \int\limits_{\partial\Omega} \langle q, *((d_x\Phi_{x_q}) \wedge \psi)(y)\rangle d\omega(y) = \\ (-1)^{l(n-l+1)+1} \int\limits_{\partial\Omega} \langle q, *d_x(\Phi_{x_q}\psi)(y)\rangle d\omega(y) = (-1)^{l(n-l+1)+1} \langle q, *d\mathcal{S}\psi(x_q)\rangle \end{gathered} \tag{4.64}$$

Recalling the Stratton-Chu representation formula (3.188), we now see that if ψ represents an electric surface current density, then the quantity $*d\mathcal{S}\psi(x_q)$ is proportional to the dual of the magnetic field emitted by the density ψ, evaluated at $x_q \in \partial G$, namely

$$*d\mathcal{S}\psi(x_q) \propto (H_\psi(x_q))^* \quad \forall q \in Q \tag{4.65}$$

If, alternatively, ψ represents a magnetic surface current density, then by the Stratton-Chu representation formula (3.187), the quantity $*d\mathcal{S}\psi(x_q)$ is proportional to the dual of the electric field emitted by the density ψ, evaluated at $x_q \in \partial G$, namely

$$*d\mathcal{S}\psi(x_q) \propto (E_\psi(x_q))^* \quad \forall q \in Q \tag{4.66}$$

[3] In physical terms, (E_ψ, H_ψ) is the field radiated, or emitted, by the (either electric or magnetic) surface current density ψ.

The following derivation is done in a similar manner as already shown above.

$$
\Pi_q \equiv (\pi_\tau W_q, \overline{\psi}) = \int_{\partial\Omega} \langle \pi_\tau W_q(y), \psi(y) \rangle d\omega(y) = \int_{\partial\Omega} \langle \nu \vee (\nu \wedge W_q)(y), \psi(y) \rangle d\omega(y) = \\
\int_{\partial\Omega} \langle W_q(y), \psi(y) \rangle d\omega(y) = \int_{\partial\Omega} \langle \delta_y d_y (\Phi_{x_q} q)(y), \psi(y) \rangle d\omega(y) \tag{4.67}
$$

By the definition of the Hodge Laplacian, $-\Delta = d\delta + \delta d$, we have furthermore

$$
\Pi_q = \int_{\partial\Omega} \langle (-\Delta_y + d_y \delta_y)(\Phi_{x_q} q)(y), \psi(y) \rangle d\omega(y) \tag{4.68}
$$

Since by definition $x_q \notin \partial\Omega$, it holds true that

$$
\Pi_q = \int_{\partial\Omega} \langle (k^2 + d_y \delta_y)(\Phi_{x_q} q)(y), \psi(y) \rangle d\omega(y) = \int_{\partial\Omega} \langle (k^2 + d_x \delta_x)(\Phi_{x_q} q)(y), \psi(y) \rangle d\omega(y) = \\
\langle q, k^2 \mathcal{S}\psi(x_q) + d\delta\mathcal{S}\psi(x_q) \rangle \tag{4.69}
$$

By virtue of the Stratton-Chu representation formula (3.187), it is evident that if ψ represents an electric surface current density, the quantity $k^2\mathcal{S}\psi(x_q) + d\delta\mathcal{S}\psi(x_q)$ is proportional to the dual of the electric field emitted by the density ψ and evaluated at x_q,

$$
k^2 \mathcal{S}\psi(x_q) + d\delta\mathcal{S}\psi(x_q) \propto (E_\psi(x_q))^* \quad \forall q \in Q \tag{4.70}
$$

Finally, if ψ represents a magnetic surface current density, then by the Stratton-Chu representation formula (3.188), the quantity $k^2\mathcal{S}\psi(x_q) + d\delta\mathcal{S}\psi(x_q)$ is proportional to the dual of the magnetic field emitted by ψ, namely

$$
k^2 \mathcal{S}\psi(x_q) + d\delta\mathcal{S}\psi(x_q) \propto (H_\psi(x_q))^* \quad \forall q \in Q \tag{4.71}
$$

■

References [2, 3] represent a very recent[4] practical confirmation of a special case of our result in Theorem 4.3.1. In these articles, it is demonstrated that the MAS is convergent on two-dimensional perfectly electrically conducting circular cylinders, regardless of the shape of the auxiliary surface and the polarisation of the incident field.

To the best of author's knowledge, the applied electromagnetic scattering community seems unaware of this result.[5] It is indicative that papers mentioning convergence properties of the Method of Auxiliary Sources typically state that, even for smooth scatterers, the auxiliary surface must enclose all the singularities of the exact scattered field pertaining to the problem at hand, since otherwise the MAS would be divergent. The standard argument is the supposed inability of any current distribution on the auxiliary surface to emit an electromagnetic field singular at a point not enclosed by the surface. This, however, is in clear disagreement with the result of Reference [12], as well as with our generalised version of that result. This situation could perhaps be explained

[4] In fact, the author was provided copies of these (as of yet unpublished) references in the final phase of writing of this text!

[5] In fact, at a conference recently attended by the author, the forum was not able to produce answer to the question of convergence of the Method of Auxiliary Sources even for canonical scatterers.

by noting that the latter results are merely of existence type, not illuminating the dependence of convergence rate of MAS upon the characteristics of a specific auxiliary surface employed. Hence, while convergence is expected theoretically for *all* choices of C^1 auxiliary surfaces inside the scatterer domain Ω, it may very well be the case that convergence rates corresponding to the different choices vary to the extent where a successful numerical implementation simply is not feasible in some instances. In fact, it is the experience of this author (see [15, 16]) that the convergence rate of a practical MAS implementation is highly sensitive to the choice of the auxiliary surface. Furthermore, it is plausible that numerical problems (e.g. slow convergence) are encountered specifically in situations where some of the exact scattered field singularities are not enclosed by the auxiliary surface, since only a finite number of auxiliary sources is employed at any one time in a practical numerical solution; trivially, a finite number of auxiliary sources indeed cannot reproduce the above mentioned singularities. The variation of the convergence rate of MAS with respect to the choice of auxiliary surface may also be due to computational limitations such as finite computational error and limited range of representable numbers on computers, as well as finite storage and processing resources available. Hence, it is possible that experience with practical numerical solution of electromagnetic scattering problems can lead to the belief that the observed numerical divergence of MAS for some specific choices of the auxiliary surface is in fact a genuine divergence, intrinsic in the mathematical setup of the problem. Although of no immediate practical value, a clarification of the above type with regard to the performance of MAS for different choices of auxiliary surfaces is, in the author's opinion, needed for a better understanding of the numerical scheme under consideration.

It is important to realise that the above mentioned convergence results do not a priori violate any postulate on the uniqueness of analytical continuation of the electromagnetic field inside the scatterer domain. An article is in writing which directs attention to the work in Reference [12], and also presents our generalised convergence result for the Method of Auxiliary Sources.

Chapter 5

Conclusion

In this chapter, a brief recapitulation is presented of the tasks accomplished in the course of our exposition. Moreover, a closing discussion is given, raising some questions as well as suggesting future work.

5.1 Accomplished Tasks

In this text, we have treated in a unified manner the existence, uniqueness and regularity of solutions to acoustic and electromagnetic boundary value problems comprising Lipschitz boundaries in an arbitrary number of dimensions. In essence, the task was to show invertibility properties of certain operators. The unified approach was possible due to the fact that the equation systems of the acoustic and the electromagnetic model could be recast, using the formalism of Clifford algebras, into special cases of a general system involving an n-dimensional l-form-valued field and an n-dimensional $l+1$-form-valued field, with $l \in \mathbb{N}_0, n \in \mathbb{N}, l \leq n$. The general system was, in turn, reduced to a Helmholtz system involving the Hodge Laplacian operator, and hence we have conducted an analysis of the well-posedness of boundary value problems more general than the acoustic and the electromagnetic cases. Two different approaches were presented which are traditionally utilised in the analysis of the (general) Helmholtz system on smooth boundaries, namely the theory of layer potentials and the variational scheme. The analysis of the smooth case was carried out using the layer potential theory in conjunction with the formalism of Fredholm operators, whereafter the more general Lipschitz case was treated using the Rellich-type relations and the Fredholm theory. In all cases, existence and uniqueness of solutions were established for all parameter values $l \in \mathbb{N}_0, n \in \mathbb{N}, l \leq n$. The general result is given in Corollary 3.4.17.

Also, the traditional Method of Auxiliary Sources (MAS) was generalised to treat scattering problems involving arbitrary l-form-valued fields and Lipschitz scatterers in n-dimensional spaces, with $l \in \mathbb{N}_0, n \in \mathbb{N}, l \leq n$. As a novel extension of a well-known proof by Hähner [12], it was shown that the generalised MAS is convergent for smooth scatterers and parameter values satisfying $l \in \mathbb{N}_0, n \in \mathbb{N}, l \leq n,\ l = (n-1)/2$, regardless of the choice of the auxiliary surface.

As we have seen, in order to arrive at a general boundary value problem comprising the acoustic and the electromagnetic problems as special cases, as well as to analyse the well-posedness and regularity of this general problem, we have made a number of generalisations of fundamental mathematical concepts. The formalism of Clifford algebras was introduced as a generalisation of the complex numbers; pseudodifferential operators generalised the notion of a differential operator; the Fredholm theory introduced a generalisation of the concept of bijectivity.

In the context of difficulty of proof, it is our impression that the change from the smooth to the general Lipschitz setting was not as severe as could be anticipated having in mind the generality of the Lipschitz class of surfaces.

5.2 The Closing Discussion

We have seen that the physically distinct acoustic and electromagnetic wave phenomena could in fact be described by a single model. One is prompted to ponder whether this general model is capable of describing additional types of wave phenomena, possibly of a fundamentally different nature than those considered here. Of course, the question remains whether observations will be made in nature which can be modeled by such an extension. Extrapolating, we predict that should such a phenomenon indeed exist, it will be a process allowing for the transport of energy, i.e. radiation should be allowed. Similarly, we should be able to observe other wave-like types of behavior, such as interference and diffraction, associated with this phenomenon. What about gravitational waves? If the case specified by $l = 2$ does indeed correspond to a model for gravitational waves, the question remains as to what the dual quantities $E, \mathscr{H}$ represent. With regards to possible modeling of a third natural wave phenomenon by the general wave system (3.7) using the pair $E \in \Lambda^2, \mathscr{H} \in \Lambda^3$, it is (at least) encouraging that the quantities $*E, *\mathscr{H}$ are, for $n = 3$, a 1-form and a 0-form, respectively. This namely leaves open the possibility of 'direct' representation of a force field-like and a potential-like physical quantity by the differential forms $E, \mathscr{H}$, respectively.

It was emphasised in the text that, despite of the possibility of unified modeling of acoustic and electromagnetic phenomena, even the modern literature (to the best of the knowledge of the author) maintained a clear distinction between the existence and uniqueness analysis of boundary value problems of the two varieties. Reference [28] served to illustrate this point.

Note that our analysis comprised both interior and exterior boundary value problems. In physical terms, we have treated both cavity problems and scattering problems.

Many questions arise after the common origin of the acoustic and electromagnetic equations has been established. Although these questions are of a philosophical nature and hence not directly relevant to the present exposition, we feel that at least some of them must be included here, for completeness.

May we interpret the presented general wave model in the sense that at most three physically disctinct wave phenomena can exist in a physical space of dimension three? Is it possible to construct an even more general wave model, which would incorporate acoustics and electromagnetism, but which would also allow for an accurate description of more than just three physically distinct wave processes? Does any of the three remaining known interactions in nature exhibit behaviour which could be described by the wave model, with $l = 2$? Is the demonstrated mathematical similarity of the acoustic and the electromagnetic model due to the already mentioned electromagnetic origin of acoustic waves? What about a combined 'electromagnetoacoustic' description by mixed terms in a Clifford algebra? We have, at least, an explanation of the fact that many direct analogies are observed in the acoustic and the electromagnetic domains, such as propagation and scattering, interference, existence of waveguides with modes, etc.

We have, in the course of our exposition, introduced the designation ’wave model’ for the general system of equations (3.7), due to the adequacy of this system in describing physically disctinct wave phenomena. In a consistent manner, we suggest the names ’0-waves’ and ’1-waves’ for the acoustic and the electromagnetic fields, respectively.

To the author’s knowledge, the most recent development in the treatment of existence, uniqueness and regularity of solution to elliptic PDE systems is the establishing of Besov regularity of layer potentials on Lipschitz domains in Riemannian manifolds. References [8, 25, 22, 27] provide detailed accounts on this subject. A natural course of action in the future would be to pursue similar results, e.g. with the aim of improving the numerical techniques of solution of this class of problems.

We perceive the result regarding the convergence of the Method of Auxiliary Sources for Lipschitz scatterers, regardless of the choice of the auxiliary surface, as being of great interest. For the smooth case alone, the author believes, based on his current experience in the field of applied electromagnetics, and in the light of the increasing interest in and application of the MAS in practise, that the (perhaps surprising) result of Hähner needs to be exposed to the electromagnetic scattering community much more efficiently than is the case at present. Similarly, numerical investigation is most certainly due of the implications of our generalised result concerning MAS convergence on practical application of the Method of Auxiliary Sources in electromagnetic scattering, particularly in connection with the localisation of the singularities of the exact scattered field.

We have not addressed the issue of eigenvalue solutions, i.e. the cases where k is an eigenvalue of the appropriate Laplacian operator. The important theory of these *scattering poles* [36] remains to be studied in the future.

Of course, a goal for future study is the generalisation and improvement of the analysis presented here by considering nonscalar layer potentials, as in references [22, 23].

Appendix A

Auxiliary Calculations

A.1 Calculation of Integrals for the Jump Relations and Regularity Analysis

It is our purpose in this section to evaluate the integrals

$$I_j = \int_{\mathbb{R}} \frac{\xi_n^j e^{ix_n\xi_n}}{\xi_n^2 + |\xi'|^2 + k^2} d\xi_n \tag{A.1}$$

for $x_n \in \mathbb{R}\backslash\{0\}$, $j \in \{0, 1, 2\}$. To this end, we follow the standard procedure and extend the integrands to the whole complex plane. Hence, we consider the integrals

$$\begin{aligned} I_0 = &\int_{\mathbb{R}} \frac{e^{ix_n z}}{z^2 + |\xi'|^2 + k^2} dz = \\ &\int_{\mathbb{R}} \frac{e^{ix_n z}}{2i\sqrt{|\xi'|^2 + k^2}} \left\{ \frac{1}{z - i\sqrt{|\xi'|^2 + k^2}} - \frac{1}{z + i\sqrt{|\xi'|^2 + k^2}} \right\} dz \end{aligned} \tag{A.2}$$

$$\begin{aligned} I_{j\in\{1,2\}} = &\int_{\mathbb{R}^n} \frac{z^j e^{ix_n z}}{z^2 + |\xi'|^2 + k^2} dz = \\ &\frac{1}{2}\int_{\mathbb{R}} z^{j-1} e^{ix_n z} \left\{ \frac{1}{z - i\sqrt{|\xi'|^2 + k^2}} + \frac{1}{z + i\sqrt{|\xi'|^2 + k^2}} \right\} dz \end{aligned} \tag{A.3}$$

where the straightforward decompositions

$$\begin{aligned} \frac{1}{z^2 + x} &= \frac{1}{2i\sqrt{x}} \left\{ \frac{1}{z - i\sqrt{x}} - \frac{1}{z + i\sqrt{x}} \right\} \quad \forall z \in \mathbb{C}\backslash\{\pm i\sqrt{x}\}, x \geq 0 \\ \frac{z^j}{z^2 + x} &= \frac{z^{j-1}}{2} \left\{ \frac{1}{z - i\sqrt{x}} + \frac{1}{z + i\sqrt{x}} \right\} \quad \forall z \in \mathbb{C}\backslash\{\pm i\sqrt{x}\}, x \geq 0, j \in \mathbb{R} \end{aligned} \tag{A.4}$$

have been used. Clearly, the integrand in I_0 has simple poles at $z = \pm i\sqrt{|\xi'|^2 + k^2}$. Furthermore, for $x_n > 0$ ($x_n < 0$), it is bounded and decaying in the upper (lower) complex half-plane. A simple application of the residue theory [34] now yields:

$$I_0^{\pm} = \pm 2\pi i \lim_{z \to \pm i\sqrt{|\xi'|^2+k^2}} \left\{ \frac{e^{ix_n z}}{z \pm i\sqrt{|\xi'|^2 + k^2}} \right\} = \frac{\pi e^{-|x_n|\sqrt{|\xi'|^2+k^2}}}{\sqrt{|\xi'|^2 + k^2}}, \quad x_n \gtrless 0 \tag{A.5}$$

Hence, we have

$$\lim_{x_n \searrow 0} \{I_0^+\} = \lim_{x_n \nearrow 0} \{I_0^-\} = \frac{\pi}{\sqrt{|\xi'|^2 + k^2}} \tag{A.6}$$

and I_0 can be extended to a continuous function of $x_n \in \mathbb{R}$. Each of the integrals $I_{j\in\{1,2\}}$ comprises integrands with simple poles and decaying in the upper (lower) complex half-plane for $x_n > 0$ ($x_n < 0$). Using the residue theory, we readily obtain

$$\begin{aligned} I_j^{\pm} = &\pm 2\pi i \lim_{z \to \pm i\sqrt{|\xi'|^2+k^2}} \left\{ \frac{z^{j-1} e^{ix_n z}}{2} \right\} = \\ &(\pm i)^j \pi (|\xi'|^2 + k^2)^{(j-1)/2} e^{-|x_n|\sqrt{|\xi'|^2+k^2}}, \quad x_n \gtrless 0, j \in \{1, 2\} \end{aligned} \tag{A.7}$$

Hence, the integral I_1 is discontinuous at $x = 0$, exhibiting a jump of magnitude $2\pi i$, while I_2 satisfies

$$\lim_{x_n \searrow 0} \{I_2^+\} = \lim_{x_n \nearrow 0} \{I_2^-\} = -\pi\sqrt{|\xi'|^2 + k^2} \tag{A.8}$$

and may thus be extended to a continuous function of $x_n \in \mathbb{R}$.

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