How to Twist a Knot

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Abstract

Take a strip of paper and "twist" it, tie a knot on it, and glue its ends together. Then you obtain a closed twisted and knotted strip. We shall use this as a model for a class of geometric objects which we shall call the class of closed strips. We present the twisting number of a closed strip as an invariant of ambient isotopy measuring the topological twist of the closed strip. We classify closed strips in euclidean 3-space by their knots, given by the center curves of the closed strips, and by their twisting numbers. Finally, we point out how the theory of polynomial invariants for links, in natural ways, leads to polynomial invariants for strip links.

The necessary elements from the theory of knots, links, and braids will be described and, moreover, we give a survey of polynomial link invariants.

When a closed strip is unorientable we shall call it a Möbius strip. We give a construction of flat analytic Möbius strips due to C. Chicone and N.J. Kalton. This construction provides an existence proof of flat analytic Möbius strips. They have also found an apparently new formula for the total torsion of closed space curves. We shall give a complete proof of this formula which provides an almost purely topological expression for the total torsion.

Preface

This Master of Science Thesis is a report on our work done at the Mathematical Institute, Technical University of Denmark, Lyngby, in the period 1st of August 1994 to 31st of January 1995. Our advisors were Professors Jens Gravesen and Steen Markvorsen.

We would like to thank Jens Gravesen and Steen Markvorsen for always having taken the time to help us when we have needed it and for giving us freedom to study the tropics we have taken interest in during our work.

Special thanks are send to:

The staff and students at the Mathematical Institute for the inspiring atmosphere and for being helpful when we have needed mathematical, computer, and mental advice.

Professor Vagn Lundsgaard Hansen for opening our eyes for geometry and topology in the early years of our studies.

Professor Bodil Branner for making us aware of “Immortality” by John Robinson, which we have used on the front page. “Immortality” is described in [Brown, 1994] and it is a 1-m-high bronze sculpture of a Möbius strip in the form of a trefoil knot. This sculpture stands in the foyer of the School of Mathematics, University of Wales, Bangor, U.K.

Carmen Chicone for sending the paper [Chicone & Kalton, 1984].

Birgitte Clausen for making our english more English.

Last but not least, our families and friends for support and understanding.

Thomas Randrup and Peter Røgen
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Contents

Chapter 1

Introduction

This introduction describes our motivation for the work reported in this thesis and how this thesis is organized. It also includes comments on prerequisites necessary for reading this thesis. We hope that the introduction will motivate you to read our thesis.

1.1 Motivation of the work

In a preliminary project, [Randrup & Røgen, 1994], we made an analysis of a method used by G. Schwarz for constructing flat analytic Möbius strips embedded in euclidean 3-space. In [Schwarz, 1990] there is given an explicit example of a flat, analytic, and algebraic Möbius strip embedded in euclidean 3-space.

This method for constructing flat analytic Möbius strips stems from the fact that a flat Möbius strip is uniquely determined by its center curve[1]. Hereby, the center curves are essential for the study of flat (analytic) Möbius strips.

In [Randrup & Røgen, 1994] we presented a family of sufficient conditions for a space curve to be the center curve of a flat (analytic) Möbius strip - and a family of sufficient conditions for a space curve to be the center curve of an orientable flat (analytic) closed strip[2]. With this family of sufficient conditions for a space curve to be the center curve of a Möbius strip as tool, we were able to extend G. Schwarz's example to a continuous family of flat (analytic) embeddings of the Möbius strip.

During our work [Randrup & Røgen, 1994] we noticed that continuous families of center curves for Möbius strips always are divided into parts in such a way that, for example, a center curve never is in the same connected component as

[1]For reference see e.g. [Randrup & Røgen, 1994], page 2, Theorem 2.

[2]These conditions are stated in Theorem 4 and Corollary 6 page 5 resp. page 6 in [Randrup & Røgen, 1994].

its mirror image. We observed the same behaviour with paper models, i.e., we were unable to deform a Möbius strip into its mirror image. To explain this we conjectured that every Möbius strip has a twisting number equal to one half of the linking number between the center curve and the boundary curve of the Möbius strip.

The reason for this conjecture is that a reflection of a Möbius strip in a plane changes the sign of its twisting number[3], whereby, the twisting number explains the above observations.

Consider a circle in euclidean 3-space with rulings rotating half of an odd integer number of times around the circle. This gives a non flat Möbius strip with an intuitive twisting number that agrees with the above definition of a twisting number. So as long as the center curve of the Möbius strip describes an unknot, it makes sense to talk about the twisting number of the Möbius strip. When the Möbius strip is knotted, it is not clear what its twisting number is. One could ask the question:

> What is the twisting number of the Möbius strip shown on the front page of this thesis?

We can (and will) answer this question!

1.2 Organization of our thesis

This thesis is (apart from this introduction) divided into three parts. The first part consists of Chapter 2. Afterwards, Chapter 3, Chapter 4, and Chapter 5 make up Part II, and Chapter 6 is the third part.

Part I: Construction of flat analytic Möbius strips has also been treated by C. Chicone and N.J. Kalton. In Chapter 2 we have included some of their work presented in [Chicone & Kalton, 1984] which consists of two independent but closely related parts.

The first part of [Chicone & Kalton, 1984] contains C. Chicone's and N.J. Kalton's construction of flat analytic Möbius strips in which an axis orthogonal to the rulings is essential. It is noteworthy that this axis has to have a total torsion that equals an odd integral multiple of π because in their construction the torsion of this axis is related to the rotation of the rulings.

The second part of [Chicone & Kalton, 1984] is used to develop an apparently new formula for the total torsion of closed space curves. As we are to prove that the twisting number is a topological invariant for closed strips, it is (in connection

[3] For reference see [Randrup & Røgen, 1994], page 14, Theorem 9.

with the above axis) interesting that this new formula for the total torsion of space curves is an almost purely topological expression. The precise connection is not clear to us, but represents an interesting topic for further work.

Part II: This part consist of three chapters. Chapter 3 describes the necessary elements from the theory of knots and links including some basic piecewise linear topology. In Chapter 4 we introduce geometric braids, the braid group, and closed braids. Finally, Chapter 5 is a survey of polynomial link invariants. For reading Chapter 4 and Chapter 5 it is necessary to have read Chapter 3.

Part III: In Chapter 6 we will prove that it is possible to define the twisting number of all closed strips, even if they are knotted. Note, that "all closed strips" include both unorientable closed strips, i.e., Möbius strips and orientable closed strips. We present, in Section 6.1, the twisting number of a closed strip as an invariant of ambient isotopy measuring the topological twist of the closed strip. In Section 6.2 we classify closed strips in euclidean 3-space by their knots, given by the center curves of the closed strips, and by their twisting numbers. Finally, in Section 6.3 we point out how known polynomial invariants for links, in natural ways, lead to polynomial invariants for strip links.

1.3 Prerequisites

Our aim has been that a postgraduate student with some training in differential geometry and homotopy theory should be able to read and understand (most of) this report. The amount of homotopy theory needed is as in [Greenberg, 1966], pp. 3-31. In Part I it is necessary to know classical differential geometry as in [Fabricius-Bjerre, 1987] or any other textbook on this subject. You can consider Part II as the background material which is necessary and sufficient for reading Part III. In fact, Chapter $i+2$ is the background material for Section 6.i for $i = 1, 2, 3$.

Chapter 2

Möbius strips and total torsion of curves

This chapter includes some of the work presented in [Chicone & Kalton, 1984] which consists of two independent but closely related parts as mentioned in Section 1.2.

In [Chicone & Kalton, 1984] C. Chicone and N.J. Kalton first give an existence proof of flat analytic Möbius strips in which axes orthogonal to the rulings are essential. On a Möbius strip those axes have to have a total torsion that equals an odd integral multiple π because in their construction the torsion of those axes are related to the rotation of the rulings. In Section 2.3 we make some comments on their construction in connection with another Möbius strip construction given by G. Schwarz in [Schwarz, 1990].

In [Chicone & Kalton, 1984] they also have developed an apparently new formula for the total torsion of closed space curves. Loosely speaking the formula states that the total torsion of a curve in euclidean 3-space with closed Gauss map is equal to an integral of a function over the unit 2-sphere, which takes integer values on each component of the complement to the closed unit tangent curve on the unit 2-sphere. We shall give the complete proof of this formula which is an almost purely topological expression.

2.1 Notation

Let $\mathbf{r}_a : \mathbb{R} \to \mathbb{R}^3$ denote a closed curve parametrized by arc length. We assume $\mathbf{r}_a$ is periodic with period T, $\mathbf{r}_a|_{[0,T[}$ has no self intersections, and that the curvature, κ, of $\mathbf{r}_a$ never vanishes, i.e., $\kappa > 0$. Then, as usual, we define

$$\mathbf{t} = \mathbf{r}'_a, \quad \mathbf{n} = \frac{\mathbf{r}''_a}{|\mathbf{r}''_a|}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{2.1}$$

The Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ along $\mathbf{r}_a$ satisfies the cross product relations

$$\mathbf{t} = \mathbf{n} \times \mathbf{b}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n} \tag{2.2}$$

and the Frenet formulas

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n} \\ \mathbf{n}' &= -\kappa \mathbf{t} + \tau \mathbf{b} \\ \mathbf{b}' &= -\tau \mathbf{n} \end{aligned} \tag{2.3}$$

where κ is the curvature and τ is the torsion of $\mathbf{r}_a$.

Remark 2.1 (Torsion and Frenet formulas) *We use the sign convention on the torsion of [Fabricius-Bjerre, 1987]. Therefore, the Frenet formulas are given as above and not as in [Chicone & Kalton, 1984] on page 5.*

2.2 A Möbius strip construction

The paper [Chicone & Kalton, 1984] contains a construction of a flat analytic Möbius strip. C. Chicone and N.J. Kalton construct a flat analytic embedding of the Möbius strip as a portion of a ruled surface parametrized by a mapping $F : \mathbb{R} \times]-\varepsilon, \varepsilon[\to \mathbb{R}^3$ of the form

$$F(s, t) = \mathbf{r}_a(s) + t\mathbf{r}_r(s)$$

where $\mathbf{r}_a$ is a curve satisfying the conditions stated in Section 2.1, $\mathbf{r}_r$ is a unit vector field along $\mathbf{r}_a$, $s \in \mathbb{R}$, and $|t| < \varepsilon$ for some $\varepsilon > 0$.

Here the rulings of the surface is the family of straight lines generated by the vector $\mathbf{r}_r$[1]. Since F is to be an embedding F must parametrize a regular surface, i.e, there must (locally) be a non-vanishing normal vector at each point of the surface (see [do Carmo, 1976] pp. 52-54). The normal vector of a surface is defined as the vector

$$\frac{\partial F}{\partial s} \times \frac{\partial F}{\partial t}.$$

Therefore, the surface described above has normal vector given by

$$(\mathbf{t}(s) + t\mathbf{r}_a'(s)) \times \mathbf{r}_r(s).$$

Thus, since the image of $\mathbf{r}_a$ is compact the condition

$$\forall s \in \mathbb{R} : \mathbf{t}(s) \times \mathbf{r}_r(s) \neq \mathbf{0}$$

[1]This is the reason why we use the index r on the vector $\mathbf{r}_r$.

ensures that there is an $\varepsilon > 0$ such that the strip $\{(s,t)|s \in \mathbb{R} \wedge |t| < \varepsilon\}$ is mapped to a regular surface by F. A closed curve which satisfies this condition is transversal to all the lines in the ruling and is called an axis of the ruled surface[2].

The ruled surface will be a flat surface (have Gauss curvature identically zero) if and only if it is developable , i.e., $\langle \mathbf{t}, \mathbf{r}_r \times \mathbf{r}_r' \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denote the usual scalar product in $\mathbb{R}^3$. For reference see [do Carmo, 1976] page 194.

If $\mathbf{r}_r = p\mathbf{t} + q\mathbf{n} + r\mathbf{b}$ then by use of the Frenet formulas (cf. (2.3) page 6)

$$\begin{aligned} \mathbf{r}_r' &= p'\mathbf{t} + p\kappa\mathbf{n} + q'\mathbf{n} - q\kappa\mathbf{t} + q\tau\mathbf{b} + r'\mathbf{b} - r\tau\mathbf{n} \\ &= (p' - q\kappa)\mathbf{t} + (p\kappa + q' - r\tau)\mathbf{n} + (q\tau + r')\mathbf{b} \end{aligned}$$

and

$$\begin{aligned} \langle \mathbf{t}, \mathbf{r}_r \times \mathbf{r}_r' \rangle &= \begin{vmatrix} 1 & p & p' - q\kappa \\ 0 & q & p\kappa + q' - r\tau \\ 0 & r & q\tau + r' \end{vmatrix} \\ &= q(q\tau + r') - r(p\kappa + q' - r\tau). \end{aligned}$$

Hence,

$$\begin{array}{rcl} \langle \mathbf{t}, \mathbf{r}_r \times \mathbf{r}_r' \rangle & = & 0 \\ & \Updownarrow & \\ qr' - rq' & = & -\tau(q^2 + r^2) + rp\kappa. \end{array} \tag{2.4}$$

Let cylinder coordinates be given by

$$q = R\cos\theta, \quad r = R\sin\theta, \quad \text{and} \quad p = p.$$

Differentiation gives

$$\begin{aligned} r' &= R\theta'\cos\theta + R'\sin\theta \\ q' &= -R\theta'\sin\theta + R'\cos\theta \end{aligned}$$

and then

$$\begin{aligned} qr' - rq' &= (R\cos\theta)(R\theta'\cos\theta + R'\sin\theta) - (R\sin\theta)(-R\theta'\sin\theta + R'\cos\theta) \\ &= R^2\theta'. \end{aligned}$$

Equation (2.4) may now be interpreted as

$$R^2\theta' = pR\kappa\sin\theta - \tau R^2, \tag{2.5}$$

[2]This is the reason why we use the index a on the curve $\mathbf{r}_a$.

where $\theta(s)$ is the angle $(q(s), r(s))$ makes with the positive r-axis. In this interpretation (q, r) are the coordinates of the projection of $\mathbf{r}_r$ into the normal plane spanned by the vectors $\mathbf{n}$ and $\mathbf{b}$ at each point along $\mathbf{r}_a$.

The closed curve $\mathbf{r}_a$ is periodic with period T implying $\mathbf{r}_a^{(n)}(s+T) = \mathbf{r}_a^{(n)}(s)$ and then it follows by (2.1) page 5 that the Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is periodic of period T. Since the Frenet frame is periodic of period T the rulings $\mathbf{r}_r$ will generate a Möbius strip provided

$$\mathbf{r}_r(s+T) = -\mathbf{r}_r(s)$$

which is equivalent to

$$\theta(s+T) = \theta(s) + (2n+1)\pi, n \in \mathbb{Z}, T > 0,$$

since this automatically ensures $p(s+T) = -p(s)$ by (2.5). This is the basic geometric idea; it leads to the following lemma.

Lemma 2.2 *Let $\mathbf{r}_a : \mathbb{R} \to \mathbb{R}^3$ be a smooth (resp. analytic) closed curve with non-vanishing curvature ($\kappa > 0$), torsion τ, periodic with period $T > 0$, and with no self intersections on $[0, T[$. Then the following two statements are equivalent:*

(1) There exists a smooth (resp. analytic) function $\theta : \mathbb{R} \to \mathbb{R}$ satisfying

$$\theta(s+T) = \theta(s) + (2n+1)\pi$$

for some integer n such that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(s) = \frac{\theta'(s) + \tau(s)}{\sin(\theta(s))}$$

is smooth (resp. analytic).

(2) The curve $\mathbf{r}_a$ is the axis of a regular developable flat embedding of the Möbius strip in $\mathbb{R}^3$ which is smooth (resp. analytic).

Remark 2.3 *In [Chicone & Kalton, 1984] on page 7 the above Lemma 2.2 is only formulated and proved with* (1) $\Rightarrow$ (2)*, but later on they use* (2) $\Rightarrow$ (1) *(cf. [Chicone & Kalton, 1984] p. 19).*

Proof:
(1) $\Rightarrow$ (2). Assume that the curve $\mathbf{r}_a : \mathbb{R} \to \mathbb{R}^3$ is smooth (resp. analytic), is periodic with period T, has no self intersections on $[0, T[$, is parametrized by arc length, and has non-vanishing curvature. Then, $\mathbf{r}_a(s+T) = \mathbf{r}_a(s)$, $|\mathbf{r}_a'(s)| = 1$, and $\mathbf{r}_a'' \neq \mathbf{0}$. We will construct $\mathbf{r}_r = p\mathbf{t} + q\mathbf{n} + r\mathbf{b}$ such that:

(A) $\mathbf{r}_r$ is smooth (resp. analytic).
(B) $|\mathbf{r}_r| = 1$.
(C) $\mathbf{t} \times \mathbf{r}_r \neq \mathbf{0}$ (regular surface).
(D) $\langle \mathbf{t}, \mathbf{r}_r \times \mathbf{r}_r' \rangle = 0$ (flatness).
(E) $\mathbf{r}_r(s+T) = -\mathbf{r}_r(s)$ (unorientable).

If these conditions are satisfied it is clear that for some $\varepsilon > 0$ the mapping

$$F(s,t) = \mathbf{r}_a(s) + t\mathbf{r}_r(s), s \in \mathbb{R}, |t| < \varepsilon,$$

will parametrize a regular developable flat Möbius strip which is smooth (resp. analytic).

If the function $\theta : \mathbb{R} \to \mathbb{R}$ exists and satisfies the hypothesis, we define the map p by

$$p(s) = \frac{f(s)}{\sqrt{\kappa(s)^2 + f(s)^2}}, s \in \mathbb{R},$$

where $f = (\theta' + \tau)(\sin\theta)^{-1}$. Then,

$$\begin{aligned} f(s+T) &= \frac{\theta'(s+T) + \tau(s+T)}{\sin(\theta(s+T))} \\ &= \frac{\theta'(s) + \tau(s)}{\sin(\theta(s) + (2n+1)\pi)} \\ &= -f(s) \end{aligned}$$

and, therefore, also $p(s+T) = -p(s)$. We observe that p is smooth (resp. analytic) and that

$$|p| = \frac{|f|}{\sqrt{\kappa^2 + f^2}} < 1.$$

Since $|p| < 1$ the map R given by

$$R(s) = \sqrt{1 - p(s)^2}, s \in \mathbb{R},$$

is smooth (resp. analytic) and so are the maps q, r defined by

$$q(s) = R(s)\cos(\theta(s)), s \in \mathbb{R},$$

and

$$r(s) = R(s)\sin(\theta(s)), s \in \mathbb{R}.$$

It remains to check that the conditions (A)-(E) stated above is satisfied with those maps p, q, and r.

(A) $\mathbf{r}_r = p\mathbf{t} + q\mathbf{n} + r\mathbf{b}$ is smooth (resp. analytic).
(B)

$$
\begin{aligned}
|\mathbf{r}_r|^2 &= p^2 + q^2 + r^2 \\
&= p^2 + (1-p^2)\cos^2\theta + (1-p^2)\sin^2\theta \\
&= 1.
\end{aligned}
$$

(C)

$$
\mathbf{t} \times \mathbf{r}_r = \begin{vmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \\ 1 & 0 & 0 \\ p & q & r \end{vmatrix} = -r\mathbf{n} + q\mathbf{b}
$$

and $|\mathbf{t} \times \mathbf{r}_r|^2 = |-r\mathbf{n} + q\mathbf{b}|^2 = 1 - p^2 \neq 0$, because $|p| < 1$. This implies that $\mathbf{t} \times \mathbf{r}_r \neq \mathbf{0}$.
(D) By use of (2.5) page 7 we see that

$$
f(s) = \frac{\theta'(s) + \tau(s)}{\sin(\theta(s))} = \frac{p(s)\kappa(s)}{R(s)}. \tag{2.6}
$$

Recall (2.4) page 7, i.e.,

$$
\begin{array}{rcl}
\langle \mathbf{t}, \mathbf{r}_r \times \mathbf{r}_r' \rangle & = & 0 \\
& \Updownarrow & \\
-\tau(q^2 + r^2) + rp\kappa. & = & -\tau(R^2\cos^2\theta + R^2\sin^2\theta) + R\sin\theta p\kappa \\
& = & R^2(-\tau + \dfrac{p\kappa\sin\theta}{R}), \text{ and by (2.6)} \\
& = & R^2(-\tau + f\sin\theta) \\
& = & R^2\theta' \\
& = & qr' - rq'.
\end{array}
$$

Finally,

(E) $\mathbf{r}_r(s+T) = -\mathbf{r}_r(s)$ as required, because the Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is periodic of period T and

$$\begin{aligned}
p(s+T) &= -p(s) \\
\\
q(s+T) &= R(s+T)\cos(\theta(s+T)) \\
&= R(s)\cos(\theta(s)+(2n+1)\pi) \\
&= -q(s) \\
\\
r(s+T) &= R(s+T)\sin(\theta(s+T)) \\
&= R(s)\sin(\theta(s)+(2n+1)\pi) \\
&= -r(s).
\end{aligned}$$

(2) $\Rightarrow$ (1). Assume the curve $\mathbf{r}_a$ is the axis of a regular developable flat embedding of the Möbius strip in $\mathbb{R}^3$ which is smooth (resp. analytic). Then, $\mathbf{t} \times \mathbf{r}_r \neq \mathbf{0}$, i.e., the vectors $\mathbf{t}$ and $\mathbf{r}_r$ are never parallel. Hence, $|p| \neq 1$, because, $\mathbf{r}_r$ is a unit vector field along $\mathbf{r}_a$. $|p| \neq 1$ implies $R \neq 0$, whereby,

$$f(s) = \frac{\theta'(s)+\tau(s)}{\sin(\theta(s))} = \frac{p(s)\kappa(s)}{R(s)}$$

is smooth (resp. analytic). This proves the lemma. □

To give the proof found by C. Chicone and N.J. Kalton for existence of a flat analytic embedding of the Möbius strip in $\mathbb{R}^3$ the following technical lemma is essential.

Lemma 2.4 *Let $a \neq 0$ be a real number. Then there is an analytic 2π-periodic function $v : \mathbb{R} \to \mathbb{R}$ such that*

$$I = \int_0^{2\pi} \frac{v'(t)}{\sqrt{a^2 + v'(t)^2}} dt$$

if and only if $I \in]-2\pi, 2\pi[$.

Proof: "$\Rightarrow$". For all analytic 2π-periodic functions v we for all $t \in \mathbb{R}$ have that

$$\left| \frac{v'(t)}{\sqrt{a^2 + v'(t)^2}} \right| < 1.$$

Hence, $I \in]-2\pi, 2\pi[$.
"$\Leftarrow$". Let $w : \mathbb{R} \to \mathbb{R}$ be an analytic 2π-periodic function not identically equal to

zero and define $I_w : \mathbb{R} \to]-2\pi, 2\pi[$ by

$$I_w(\lambda) = \int_0^{2\pi} \frac{\lambda w(t)}{\sqrt{a^2 + (\lambda w(t))^2}} dt.$$

Note, that $I_w : \mathbb{R} \to]-2\pi, 2\pi[$ is odd and continuous and that the integrand is dominated by the constant function 1. Hence, Lebesgue's Dominated Convergence Theorem (see e.g. [Madsen, 1975] p. 47) states

$$\lim_{\lambda\to\infty} I_w(\lambda) = \int_0^{2\pi} \operatorname{sgn}(w(t))dt = A - B,$$

where A denotes the length of the intervals $(w^{-1}(\mathbb{R}_+)) \cap [0, 2\pi]$ and B denotes the length of the intervals $(w^{-1}(\mathbb{R}_-)) \cap [0, 2\pi]$.

Let $\delta \in]0, 2\pi[$ be given. Choose a continuous and piecewise C^1 function $g : \mathbb{R} \to \mathbb{R}$ fulfilling

g is 2π-periodic,

$\int_0^{2\pi} g(t)dt = 0$, and

the length of $g^{-1}(\mathbb{R}_+) \cap [0, 2\pi]$ exceeds $2\pi - \delta$.

For any $\varepsilon > 0$ there is a trigonometric polynomial $P_\varepsilon : \mathbb{R} \to \mathbb{R}$, such that for all t in $\mathbb{R}$ the polynomial P_ε fulfills $|P_\varepsilon(t) - g(t)| < \varepsilon$. For reference see [KU-noter, 1981/1982], p. V.73., Theorem 6.7. The function $w_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$w_\varepsilon(t) = P_\varepsilon(t) - \frac{1}{2\pi}\int_0^{2\pi} P_\varepsilon(s)ds$$

satisfies

$$\begin{aligned}
|w_\varepsilon(t) - g(t)| &= \left|P_\varepsilon(t) - g(t) - \tfrac{1}{2\pi}\left[\int_0^{2\pi} g(s)ds + \int_0^{2\pi}(P_\varepsilon(s) - g(s))ds\right]\right| \\
&\le |P_\varepsilon(t) - g(t)| + 0 + \tfrac{1}{2\pi}\left|\int_0^{2\pi}(P_\varepsilon(s) - g(s))ds\right| \\
&\le |P_\varepsilon(t) - g(t)| + \tfrac{1}{2\pi}\int_0^{2\pi}|P_\varepsilon(s) - g(s)|\,ds \\
&< \varepsilon + \tfrac{1}{2\pi}\int_0^{2\pi}\varepsilon ds \\
&= 2\varepsilon.
\end{aligned}$$

Especially, we for all t in $\mathbb{R}$ have that $w_\varepsilon(t) > g(t) - 2\varepsilon$. Thus for $\varepsilon > 0$ sufficiently small we have that the length of the intervals $(w_\varepsilon^{-1}(\mathbb{R}_+)) \cap [0, 2\pi]$ is bigger than the length of $((g - 2\varepsilon)^{-1}(\mathbb{R}_+)) \cap [0, 2\pi]$, which again is bigger than $2\pi - 2\delta$. This ensures that $I_{w_\varepsilon}(\lambda)$ exceeds $(2\pi - 2\delta) - (2\delta)$ for λ big enough, i.e., there exists a real λ such that $I_{w_\varepsilon}(\lambda) = 2\pi - 4\delta$.

Finally, define the function $v : \mathbb{R} \to \mathbb{R}$ by

$$v(t) = \int_0^t \lambda w_\varepsilon(s) ds.$$

The function v is analytic and 2π-periodic. We can for all $I \in [0, 2\pi[$ choose v such that $I_{v'}(1) = I$. On the other hand, if we choose $-v$ then $I_{-v'}(1) = -I$ and the proof is completed. □

Theorem 2.5 *There exists an analytic flat Möbius strip embedded in* $\mathbb{E}^3$.

Proof: Assume $\mathbf{r}_a : \mathbb{R} \to \mathbb{R}^3$ is an analytic closed curve of length T with non-vanishing curvature, total torsion π, and parametrized by arc length, such that $\mathbf{r}_a|_{[0,T[}$ has no self intersections. Note, that by the total torsion we mean the integrated torsion of one traversion of the curve $\mathbf{r}_a$, i.e., the total torsion of $\mathbf{r}_a$ is given by

$$\int_0^T \tau(s) ds.$$

Now, defining the function $\theta : \mathbb{R} \to \mathbb{R}$ by

$$\theta(s) = -\int_0^s \tau(t) dt$$

we observe that

$$\theta(s + T) = \theta(s) - \pi$$

and that

$$\theta'(s) = -\tau(s)$$

for all $s \in \mathbb{R}$. Hence, the function θ is analytic, because $\mathbf{r}_a$ is analytic. Then the function $f : \mathbb{R} \to \mathbb{R}$ from Lemma 2.2 is here given by

$$f(s) = \frac{\theta'(s) + \tau(s)}{\sin(\theta(s))} \equiv 0,$$

which is an analytic function. Thus the proof will be completed by Lemma 2.2 if there exists an analytic closed curve $\mathbf{r}_a : \mathbb{R} \to \mathbb{R}^3$ with properties as described above.

Let $\tilde{\mathbf{r}}_a : \mathbb{R} \to \mathbb{R}^3$ be an analytic closed curve of the form

$$\tilde{\mathbf{r}}_a(t) = (a\cos t, a\sin t, v(t)),$$

where $v : \mathbb{R} \to \mathbb{R}^3$ is an analytic 2π-periodic function and $a > 0$. We draw the attention to the fact that the torsion of $\tilde{\mathbf{r}}_a$ is given by (see e.g. [Fabricius-Bjerre, 1987] p. 62)

$$\tau(t) = \frac{[\tilde{\mathbf{r}}_a'\tilde{\mathbf{r}}_a''\tilde{\mathbf{r}}_a'''](t)}{|\tilde{\mathbf{r}}_a' \times \tilde{\mathbf{r}}_a''|^2(t)}$$

and, therefore, in order to calculate the total torsion of $\mathbf{r}_a$ we do the following calculations.

$$\tilde{\mathbf{r}}_a'(t) = (-a\sin t, a\cos t, v'(t))$$

$$\tilde{\mathbf{r}}_a''(t) = (-a\cos t, -a\sin t, v''(t))$$

$$\tilde{\mathbf{r}}_a'''(t) = (a\sin t, -a\cos t, v'''(t))$$

$$(\tilde{\mathbf{r}}_a' \times \tilde{\mathbf{r}}_a'')(t) = (a(v''(t)\cos t + v'(t)\sin t), a(v''(t)\sin t - v'(t)\cos t), a^2)$$

$$\begin{aligned}|\tilde{\mathbf{r}}_a' \times \tilde{\mathbf{r}}_a''|^2(t) &= a^2(v''(t)^2\cos^2 t + v'(t)^2\sin^2 t + 2v'(t)v''(t)\cos t\sin t \\ &\quad + v''(t)^2\sin^2 t + v'(t)^2\cos^2 t - 2v'(t)v''(t)\cos t\sin t) + a^4 \\ &= a^2(v''(t)^2 + v'(t)^2 + a^2)\end{aligned}$$

Finally,

$$\begin{aligned}[\tilde{\mathbf{r}}_a'\tilde{\mathbf{r}}_a''\tilde{\mathbf{r}}_a'''](t) &= ((\tilde{\mathbf{r}}_a' \times \tilde{\mathbf{r}}_a'') \cdot \tilde{\mathbf{r}}_a''')(t) \\ &= a^2(v''(t)\cos t\sin t + v'(t)\sin^2 t - v''(t)\cos t\sin t + v'(t)\cos^2 t) \\ &\quad + a^2 v'''(t) \\ &= a^2(v'(t) + v'''(t)).\end{aligned}$$

The curve $\tilde{\mathbf{r}}_a$ has non-vanishing curvature, because $|\tilde{\mathbf{r}}_a' \times \tilde{\mathbf{r}}_a''| \neq 0$.

Set $t = \rho(s)$, where $\rho'(s) > 0$ for all s in $\mathbb{R}$, so the reparametrization of $\tilde{\mathbf{r}}_a$, denoted $\mathbf{r}_a(s) = \tilde{\mathbf{r}}_a(\rho(s))$, is parametrized by arc length, s. To be able to change integral variable we compute

$$|\tilde{\mathbf{r}}_a'(t)|^2 = a^2 + v'(t)^2$$

and then

$$\rho'(\rho^{-1}(t)) = \frac{1}{|\tilde{\mathbf{r}}_a'(t)|} = \frac{1}{\sqrt{a^2 + v'(t)^2}}.$$

We are now ready to calculate the total torsion of $\mathbf{r}_a$.

$$\int_0^T \tau(\rho(s))ds = \int_0^{2\pi} \frac{[\tilde{\mathbf{r}}_a' \tilde{\mathbf{r}}_a'' \tilde{\mathbf{r}}_a'''](t)}{|\tilde{\mathbf{r}}_a' \times \tilde{\mathbf{r}}_a''|^2(t)\rho'(\rho^{-1}(t))} dt$$

$$= \int_0^{2\pi} \frac{a^2(v'(t)+v'''(t))\sqrt{a^2+v'(t)^2}}{a^2(a^2+v'(t)^2+v''(t)^2)} dt, \quad \text{since } v'' \text{ is } 2\pi\text{−periodic}$$

$$= \int_0^{2\pi} \left[\frac{(v'(t)+v'''(t))\sqrt{a^2+v'(t)^2}}{a^2+v'(t)^2+v''(t)^2} \right.$$

$$\left. - \frac{d}{dt}\left(\mathrm{Arctan}\left(\frac{v''(t)}{\sqrt{a^2+v'(t)^2}}\right)\right)\right] dt$$

$$= \int_0^{2\pi} \frac{v'(t)}{\sqrt{a^2+v'(t)^2}} dt.$$

The last equality in the above calculation of the total torsion is valid since

$$\begin{aligned}
& \frac{(v'(t)+v'''(t))\sqrt{a^2+v'(t)^2}}{a^2+v'(t)^2+v''(t)^2} - \frac{d}{dt}\left(\mathrm{Arctan}\left(\frac{v''(t)}{\sqrt{a^2+v'(t)^2}}\right)\right) \\
= \; & \frac{(v'(t)+v'''(t))\sqrt{a^2+v'(t)^2}}{a^2+v'(t)^2+v''(t)^2} \\
& - \; \frac{1}{1+\frac{v''(t)^2}{a^2+v'(t)^2}}\left(\frac{v'''(t)}{\sqrt{a^2+v'(t)^2}} - \frac{2v''(t)^2 v'(t)}{2(a^2+v'(t)^2)^{3/2}}\right) \\
= \; & \frac{(v'(t)+v'''(t))\sqrt{a^2+v'(t)^2}}{a^2+v'(t)^2+v''(t)^2} \\
& - \left(\frac{v'''(t)\sqrt{a^2+v'(t)^2}}{a^2+v'(t)^2+v''(t)^2} - \frac{a^2+v'(t)^2}{a^2+v'(t)^2+v''(t)^2}\frac{v''(t)^2 v'(t)}{(a^2+v'(t)^2)^{3/2}}\right) \\
= \; & \frac{(v'(t)+v'''(t))(a^2+v'(t)^2) - v'''(t)(a^2+v'(t)^2) + v''(t)^2 v'(t)}{(a^2+v'(t)^2+v''(t)^2)\sqrt{a^2+v'(t)^2}}
\end{aligned}$$

$$= \frac{v'(t)}{\sqrt{a^2 + v'(t)^2}}.$$

According to Lemma 2.4 there exist an analytic 2π-periodic function v such that the total torsion of $\mathbf{r}_a$ equals π. This completes the proof. □

2.2.1 Properties of axes of Möbius strips

In order to give descriptions of axes of smooth (resp. analytic) Möbius strips C. Chicone and N.J. Kalton give two theorems in the paper [Chicone & Kalton, 1984] which together states

Theorem 2.6 *Let $\mathbf{r}_a$ be a closed smooth (resp. analytic) curve in $\mathbb{R}^3$ with non-vanishing curvature and no self intersections. Then $\mathbf{r}_a$ is the axis of a smooth (resp. analytic) ruled developable flat Möbius strip if and only if $\mathbf{r}_a$ does not lie in a plane.*

Proof: See Theorem 2.4. and Theorem 2.5. in [Chicone & Kalton, 1984] pp. 13-18.

If one to the assumptions given in Theorem 2.6 on the axis of a Möbius strip add the assumption that the axis is orthogonal to the rulings everywhere one can obtain a much more interesting theorem. Namely,

Theorem 2.7 *Let $\mathbf{r}_a$ be a closed smooth (resp. analytic) curve in $\mathbb{R}^3$ with non-vanishing curvature and no self intersections. Then $\mathbf{r}_a$ is an axis of a ruled developable flat Möbius strip with smooth (resp. analytic) rulings everywhere orthogonal to $\mathbf{r}_a$ if and only if the total torsion of $\mathbf{r}_a$ is an odd multiple of π.*

Proof: Let $\mathbf{r}_a : \mathbb{R} \to \mathbb{R}^3$ be a T-periodic axis of a ruled developable flat Möbius strip. Consider a parametrization of the surface of the form

$$F(s,t) = \mathbf{r}_a(s) + t\mathbf{r}_r(s),\, s \in \mathbb{R},\, |t| < \varepsilon,\, \varepsilon > 0,$$

where $\mathbf{r}_r = p\mathbf{t} + q\mathbf{n} + r\mathbf{b}$. If κ is the curvature and τ the torsion of the curve $\mathbf{r}_a$, then by Lemma 2.2 (recall Remark 2.3) and by (2.6) page 10 there will be a solution of the differential equation

$$\theta'(s) = \frac{p(s)\kappa(s)}{R(s)} \sin(\theta(s)) - \tau(s),$$

which satisfies $\theta(s+T) = \theta(s) + (2n+1)\pi$ for an integer n, where T is the period of $\mathbf{r}_a$. The tangent vector, $\mathbf{t}$, of $\mathbf{r}_a$ is orthogonal to the ruling $\mathbf{r}_r$, i.e., $p \equiv 0$. Hence, the function θ is actually a solution of the differential equation

$$\theta'(s) = -\tau(s).$$

The total torsion of $\mathbf{r}_a$ is an odd multiple of π, because

$$\int_0^T \tau(s)ds = -\int_0^T \theta'(s)ds = -\theta(0+T) + \theta(0) = -(2n+1)\pi.$$

On the other hand, if the total torsion of $\mathbf{r}_a$ is an odd multiple of π the proof of Theorem 2.5 shows that there is a ruled developable flat Möbius strip of the required form. □

2.3 Another Möbius strip construction

There are two problems with the construction of flat analytic Möbius strip due to C. Chicone and N.J. Kalton given in Section 2.2.

The first problem is that they only have given a proof of existence of a flat analytic embedding of the Möbius strip (cf. Theorem 2.5), and it seems that their method can not be used to construct a flat analytic embedding of the Möbius strip explicitly.

The second problem is that their method gives no control of the shape of the Möbius strip when it has been cut and rolled out in the plane.

In order to find flat analytic Möbius strips made from a rectangular piece of paper one has to use other methods. One other method is used by G. Schwarz (see [Schwarz, 1990]). The basic idea in his method is that the center curve of a piecewise ruled developable embedding of the Möbius strip is not only an axis of the ruled developable surface, it is also a geodesic. Actually, a flat Möbius strip is uniquely determined by its center curve[3].

In [Schwarz, 1990] G. Schwarz presents an analytic and algebraic curve in euclidean 3-space that is the center curve of a flat, analytic, and algebraic embedding of the Möbius strip into euclidean 3-space. His method also provides an isometric embedding such that one has total control of the shape of the Möbius strip when it has been cut and rolled out in the plane.

In a preliminary project, [Randrup & Røgen, 1994], we made an analysis of this construction. We proved that there is a strong connection between the methods for construction of Möbius strips introduced by C. Chicone and N.J. Kalton

[3] For reference see e.g. [Randrup & Røgen, 1994], page 2, Theorem 2.

resp. G. Schwarz. This follows from the fact that there is, at most, one closed curve on a Möbius strip that has the property to be orthogonal to all rulings and the property to close after one traversal of the Möbius strip. For reference see [Randrup & Røgen, 1994], page 14, Theorem 10. In other words, for each Möbius strip there is one unique closed space curve that is orthogonal to all the rulings as well as closes after one traversal of the (total) ruled surface. However, it is not certain that the Möbius strip is (or can be chosen) wide enough to contain this closed space curve. Whence, we are not certain that the orthogonal curve and the center curve describe the same knot (if the center curve describes a knot). Furthermore, it is not certain that the space curve orthogonal to all the rulings has non-vanishing curvature as demanded in Theorem 2.7[4].

2.4 The total torsion of curves in $\mathbb{R}^3$

In the paper [Chicone & Kalton, 1984] there is an apparently new formula for the total torsion of a closed curve or more generally a curve with closed Gauss map. We will in Section 2.4.4 prove this formula (see Theorem 2.15). As mentioned earlier this formula states that the total torsion of a curve in $\mathbb{R}^3$ with closed Gauss map is equal to an integral of a function over S^2, which takes integer values on each component of the complement to the closed unit tangent curve on S^2. We need some preparations which is contained in the next three sections.

2.4.1 A calculation of curvature and torsion

Let $\mathbf{r} = \mathbf{r}(s), s \in [0, L]$, denote a space curve parametrized by arc length. We assume throughout that the curvature of $\mathbf{r}$ never vanishes. Then the tangent curve $\mathbf{t}(s) = \mathbf{r}'(s), s \in [0, L]$, describes a curve on the unit 2-sphere and $\mathbf{t}'$ is never zero[5]. We select a point P on the unit 2-sphere so that the tangent curve $\mathbf{t}$ never visits either P or its antipode, which we denote by $-P$, i.e.,

$$\forall s \in [0, L] : \mathbf{t}(s) \notin \{P, -P\}.$$

Select an orthonormal base, $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, in $\mathbb{R}^3$ so that the point P is the north pole on the unit 2-sphere, i.e., $P = (0, 0, 1)$ with respect to $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. With respect to this choice we write

$$\mathbf{t}(s) = (u(s), v(s), w(s)), s \in [0, L].$$

[4]This demand might very well be unnecessary.

[5]This could also be formulated by using the Gauss map.

The tangent vector $\mathbf{t}$ is a unit vector, i.e.,

$$|\mathbf{t}|^2 = u^2 + v^2 + w^2 = 1 \tag{2.7}$$

and differentiating this equation we obtain

$$u'u + v'v + w'w = 0. \tag{2.8}$$

Define the vector $\mathbf{P} = (0, 0, 1)$ and the vector $\mathbf{N}(s) = \mathbf{t}(s) \times \mathbf{P} = (v(s), -u(s), 0)$, $s \in [0, L]$. Then, $(\mathbf{t}, \mathbf{P}, \mathbf{N})$ is a moving frame along $\mathbf{r}_a$ which is not orthonormal. We have

$$\begin{aligned} \mathbf{t} \times \mathbf{P} &= \mathbf{N}, \\ \mathbf{t} \times \mathbf{N} &= (uw, vw, -u^2 - v^2) \\ &= (uw, vw, w^2 - 1) \\ &= w\mathbf{t} - \mathbf{P}, \quad \text{and} \\ \mathbf{P} \times \mathbf{N} &= (u, v, 0) \\ &= \mathbf{t} - w\mathbf{P}. \end{aligned} \tag{2.9}$$

Let $\mathbf{A}$ denote the matrix which shifts from $(\mathbf{t}, \mathbf{P}, \mathbf{N})$ to $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. Because,

$$\left.\begin{array}{rcl} \mathbf{t} &=& (u, v, w) \\ \mathbf{P} &=& (0, 0, 1) \\ \mathbf{N} &=& (v, -u, 0) \end{array}\right\} \quad \text{with respect to } (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$$

the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} u & 0 & v \\ v & 0 & -u \\ w & 1 & 0 \end{bmatrix}.$$

Consequently,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} u & v & 0 \\ -uw & -vw & u^2 + v^2 \\ v & -u & 0 \end{bmatrix}$$

is the matrix which shifts from $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ to $(\mathbf{t}, \mathbf{P}, \mathbf{N})$, where the determinant of the matrix $\mathbf{A}$ is given by

$$\det(\mathbf{A}) = u^2 + v^2 = 1 - w^2.$$

In the following we will use

$$\alpha = \frac{u'v - v'u}{u^2 + v^2}$$

and

$$\beta = \frac{w'}{1 - w^2}$$

to obtain an easy notation. It is notable that both fractions have non-vanishing denominators by the hypothesis on the point P.

Differentiating the tangent curve $\mathbf{t}(s)$ gives (u', v', w') which are the coordinates for $\mathbf{t}'$ with respect to $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. Hence,

$$\mathbf{A}^{-1} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}$$

are the coordinates for $\mathbf{t}'$ with respect to $(\mathbf{t}, \mathbf{P}, \mathbf{N})$. The following calculation give these coordinates explicit, where (2.7) and (2.8) are used.

$$\begin{aligned}
\mathbf{t}' &= \mathbf{A}^{-1} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \\
&= \frac{1}{\det(\mathbf{A})}(u'u + v'v, -wu'u - wv'v + (u^2 + v^2)w', u'v - v'u) \\
&= \frac{1}{\det(\mathbf{A})}(-w'w, -w(-w'w) + (u^2 + v^2)w', u'v - v'u) \\
&= (\frac{-w'w}{1 - w^2}, \frac{w'(w^2 + u^2 + v^2)}{1 - w^2}, \frac{u'v - v'u}{u^2 + v^2}) \\
&= (-\beta w, \beta, \alpha)
\end{aligned}$$

The vector $\mathbf{N}' = (v', -u', 0)$ are the coordinates for $\mathbf{N}'$ with respect to $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. Hence,

$$\mathbf{A}^{-1} \begin{bmatrix} v' \\ -u' \\ 0 \end{bmatrix}$$

are the coordinates for $\mathbf{N}'$ with respect to $(\mathbf{t}, \mathbf{P}, \mathbf{N})$. Below we calculate these coordinates explicitly.

$$\begin{aligned}
\mathbf{N}' &= \mathbf{A}^{-1} \begin{bmatrix} v' \\ -u' \\ 0 \end{bmatrix} \\
&= \frac{1}{\det(\mathbf{A})}(v'u - u'v, -wv'u + wu'v, v'v + u'u) \\
&= (-\alpha, \frac{w(u'v - v'u)}{u^2 + v^2}, \frac{-w'w}{1 - w^2}) \\
&= (-\alpha, w\alpha, -w\beta)
\end{aligned}$$

Summarizing we have

$$\begin{aligned} \mathbf{t}' &= -\beta w\mathbf{t} + \beta\mathbf{P} + \alpha\mathbf{N} \\ \mathbf{P}' &= \mathbf{0} \\ \mathbf{N}' &= -\alpha\mathbf{t} + \alpha w\mathbf{P} - \beta w\mathbf{N} \end{aligned} \tag{2.10}$$

where

$$\alpha = \frac{u'v - v'u}{u^2 + v^2}$$

and

$$\beta = \frac{w'}{1 - w^2}.$$

The following lemma implicitly states that the curvature and the torsion of a space curve parametrized by arc length are given by the tangent map of the curve.

Lemma 2.8 *For a curve $\mathbf{r} : I \to \mathbb{R}^3$ parametrized by arc length the curvature, κ, and the torsion, τ, are given by*

$$\begin{aligned} \kappa &= \sqrt{(\alpha^2 + \beta^2)(1 - w^2)} \text{ and} \\ \tau &= \frac{\beta\alpha' - \alpha\beta'}{\alpha^2 + \beta^2} - \alpha w, \end{aligned}$$

where α and β are given as above and the unit tangent vector field of $\mathbf{r}$ is $\mathbf{t} = (u, v, w)$.

Remark 2.9 *In the paper [Chicone & Kalton, 1984] there is an error. They have the wrong sign on the term $\frac{\beta\alpha' - \alpha\beta'}{\alpha^2+\beta^2}$. For reference see [Chicone & Kalton, 1984] page 22 remembering Remark 2.1.*

Proof: For the calculation of the curvature, κ, and the torsion, τ, we need to know the scalar products of the vectors in the frame $(\mathbf{t}, \mathbf{P}, \mathbf{N})$. Noticing that

$$\begin{aligned} \mathbf{t} &= (u, v, w), \\ \mathbf{P} &= (0, 0, 1), \quad \text{and} \\ \mathbf{N} &= (v, -u, 0), \end{aligned}$$

one easily finds

$$\left.\begin{aligned} \langle \mathbf{t}, \mathbf{t} \rangle &= \langle \mathbf{P}, \mathbf{P} \rangle = 1 \\ \langle \mathbf{t}, \mathbf{N} \rangle &= \langle \mathbf{P}, \mathbf{N} \rangle = 0 \\ \langle \mathbf{t}, \mathbf{P} \rangle &= w \\ \langle \mathbf{N}, \mathbf{N} \rangle &= u^2 + v^2 = 1 - w^2 \ \ \text{(see (2.7) page 19)} \end{aligned}\right\} \tag{2.11}$$

Using the fact $\mathbf{t}' = \kappa\mathbf{n}$ from the Frenet formulas page 6 we now find κ^2 by the calculation

$$\begin{aligned}\kappa^2 &= \langle \kappa\mathbf{n}, \kappa\mathbf{n}\rangle \\ &= \langle \mathbf{t}', \mathbf{t}'\rangle, \text{ and by (2.10) page 21} \\ &= \langle -\beta w\mathbf{t} + \beta\mathbf{P} + \alpha\mathbf{N}, -\beta w\mathbf{t} + \beta\mathbf{P} + \alpha\mathbf{N}\rangle \\ &= \beta^2 w^2 - 2\beta^2 w\langle \mathbf{t}, \mathbf{P}\rangle + \beta^2 + \alpha^2\langle \mathbf{N}, \mathbf{N}\rangle \\ &= \beta^2 w^2 - 2\beta^2 ww + \beta^2 + \alpha^2(1 - w^2) \\ &= (\alpha^2 + \beta^2)(1 - w^2).\end{aligned}$$

Hence,

$$\kappa = \sqrt{(\alpha^2 + \beta^2)(1 - w^2)},$$

since κ is positive.

To be able to calculate the torsion we start with

$$\begin{aligned}\kappa\mathbf{b} &= \kappa(\mathbf{t} \times \mathbf{n}) \\ &= \mathbf{t} \times (\kappa\mathbf{n}), \text{ and by (2.3) page 6} \\ &= \mathbf{t} \times \mathbf{t}', \text{ and by (2.10) page 21} \\ &= \mathbf{t} \times (-\beta w\mathbf{t} + \beta\mathbf{P} + \alpha\mathbf{N}) \\ &= \beta\mathbf{t} \times \mathbf{P} + \alpha\mathbf{t} \times \mathbf{N}, \text{ and by (2.9) page 19} \\ &= \beta\mathbf{N} + \alpha(w\mathbf{t} - \mathbf{P}).\end{aligned}$$

Hence,

$$\kappa\mathbf{b} = \alpha w\mathbf{t} - \alpha\mathbf{P} + \beta\mathbf{N}. \tag{2.12}$$

The Frenet formulas page 6 include the equation $\mathbf{b}' = -\tau\mathbf{n}$. Whence, an expression for the torsion can be obtained by using the fact

$$\tau = -\frac{1}{\kappa^2}\langle \kappa\mathbf{b}', \kappa\mathbf{n}\rangle$$

and

$$(\kappa\mathbf{b})' = \kappa'\mathbf{b} + \kappa\mathbf{b}' \iff \kappa\mathbf{b}' = (\kappa\mathbf{b})' - \kappa'\mathbf{b}.$$

Finally, we are ready for the calculation of the torsion, where we use the equations

(2.3), (2.10), (2.11), and (2.12).

$$
\begin{aligned}
\tau &= -\frac{1}{\kappa^2}\langle \kappa \mathbf{b}', \kappa \mathbf{n}\rangle \\
&= -\frac{1}{\kappa^2}\langle (\kappa \mathbf{b})' - \kappa' \mathbf{b}, \kappa \mathbf{n}\rangle, \ \text{and by orthonormality of } \mathbf{n} \text{ and } \mathbf{b} \\
&= -\frac{1}{\kappa^2}\langle (\kappa \mathbf{b})', \kappa \mathbf{n}\rangle \\
&= -\frac{1}{\kappa^2}\langle \alpha' w \mathbf{t} + \alpha w' \mathbf{t} + \alpha w \kappa \mathbf{n} - \alpha' \mathbf{P} - \alpha \mathbf{0} + \beta' \mathbf{N} \\
&\quad + \beta(-\alpha \mathbf{t} + \alpha w \mathbf{P} - \beta w \mathbf{N}), \kappa \mathbf{n}\rangle, \ \text{and by orthonormality of } \mathbf{t} \text{ and } \mathbf{n} \\
&= -\alpha w - \frac{1}{\kappa^2}\langle -\alpha' \mathbf{P} + \beta' \mathbf{N} + \beta \alpha w \mathbf{P} - \beta^2 w \mathbf{N}, \kappa \mathbf{n}\rangle \\
&= -\alpha w - \frac{1}{\kappa^2}\langle -\alpha' \mathbf{P} + \beta' \mathbf{N} + \beta \alpha w \mathbf{P} - \beta^2 w \mathbf{N}, \mathbf{t}'\rangle \\
&= -\alpha w - \frac{1}{\kappa^2}\langle -\alpha' \mathbf{P} + \beta' \mathbf{N} + \beta \alpha w \mathbf{P} - \beta^2 w \mathbf{N}, -\beta w \mathbf{t} + \beta \mathbf{P} + \alpha \mathbf{N}\rangle \\
&= -\alpha w - \frac{1}{\kappa^2}\langle (-\alpha' + \beta \alpha w)\mathbf{P} + (\beta' - \beta^2 w)\mathbf{N}, -\beta w \mathbf{t} + \beta \mathbf{P} + \alpha \mathbf{N}\rangle \\
&= -\alpha w - \frac{1}{\kappa^2}[(-\alpha' + \beta \alpha w)(-\beta w)w + (-\alpha' + \beta \alpha w)\beta + (\beta' - \beta^2 w)\alpha(1 - w^2)] \\
&= -\alpha w - \frac{1}{\kappa^2}[\beta \alpha' w^2 - \beta^2 \alpha w^3 - \beta \alpha' + \beta^2 \alpha w + \beta' \alpha - \beta' \alpha w^2 - \beta^2 \alpha w + \beta^2 \alpha w^3] \\
&= -\alpha w - \frac{(\alpha \beta' - \beta \alpha')(1 - w^2)}{(\alpha^2 + \beta^2)(1 - w^2)} \\
&= -\alpha w - \frac{\alpha \beta' - \beta \alpha'}{\alpha^2 + \beta^2}.
\end{aligned}
$$

□

2.4.2 Index and rotation index of plane curves

We are going to define the index and the rotation index of plane curves. To do this we need to introduce the notion of degree. More precise the degree theory for continuous maps of the unit circle.

Let the map $\Pi : \mathbb{R} \to S^1$ be the covering of the unit circle, S^1, by the real line $\mathbb{R}$ given by

$$\Pi(x) = (\cos x, \sin x), x \in \mathbb{R}.$$

Let $f_\psi : [0, T] \to S^1$ be a continuous map such that $f_\psi(0) = f_\psi(T) = y \in S^1$. Thus, $f_\psi([0, T])$ is a closed curve at y in S^1 which can be lifted into a unique

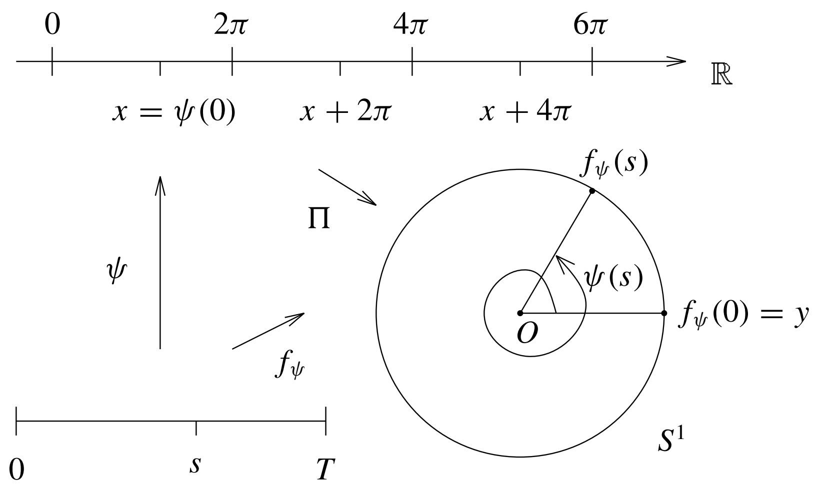

Figure 2.1: The relation between the maps f_ψ, Π, and ψ

function $\psi : [0, T] \to \mathbb{R}$, starting at a point $x \in \mathbb{R}$ with $\Pi(x) = y$. For reference see [do Carmo, 1976], page 376, Proposition 2. Since $\Pi(\psi(0)) = \Pi(\psi(T))$, the difference $\psi(T) - \psi(0)$ is an integral multiple of 2π. The integer $\deg(f_\psi)$ given by

$$\psi(T) - \psi(0) = 2\pi \deg(f_\psi)$$

is called the degree of f_ψ.

Remark 2.10 (Degree) *This definition of the degree is independent of choices of the points x and y (see [do Carmo, 1976] page 391). In other words, the degree is well-defined. For a general introduction of the degree of a map see e.g. [Flanders, 1963] pp. 77-78 or [Hansen, 1968] p. 276.*

Intuitively, the integer $\deg(f_\psi)$ is the number of times that f_ψ wraps $[0, T]$ around S^1. Notice that the function $\psi(s)$ is a continuous determination of the positive angle that the fixed vector $f_\psi(0) - O$ makes with the vector $f_\psi(s) - O$, $s \in [0, T]$, where $O = (0, 0)$. See Figure 2.1.

Furthermore, we state (for reference see [do Carmo, 1976] pp. 391-392):

- The degree of a map is invariant under homotopy.

- If f_ψ is differentiable, the degree of f_ψ can be expressed by an integral as below.

$$\deg(f_\psi) = \frac{1}{2\pi} \int_0^T \psi'(s) ds. \tag{2.13}$$

Given a plane curve and a point disjoint from the curve one can define the number of times the curve makes a complete turn around this point by

Definition 2.11 (Indices of curves in pointed planes) *Let* $\mathbf{r} : [0, T] \to \mathbb{R}^2$ *be a plane, continuous closed curve and let* P *be a point in the complement to* $\mathbf{r}([0, T])$*, i.e.,* $P \in \mathbb{R}^2 \backslash \mathbf{r}([0, T])$. *Let* $f_\psi : [0, T] \to S^1$ *be given by*

$$f_\psi(t) = \frac{\mathbf{r}(t) - P}{|\mathbf{r}(t) - P|}, t \in [0, T].$$

The degree of f_ψ *is called the index of the curve* $\mathbf{r}$ *relative to* P *and it is denoted by* $\mathrm{Index}(\mathbf{r}, P)$.

Remark 2.12 (Index) *This definition of the index of a curve is also known as the winding number. For reference see [do Carmo, 1976] page 392.*

The degree is continuous as a function of P. Hence, the index relative to P is constant when P runs in a connected component of $\mathbb{R}^2 \backslash \mathbf{r}([0, T])$.

Definition 2.13 (Rotation indices of plane curves) *Let* $\mathbf{r} : [0, T] \to \mathbb{R}^2$ *be a plane, regular closed curve and let* $f_\psi : [0, T] \to S^1$ *be given by*

$$f_\psi(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}, t \in [0, T].$$

The degree of f_ψ *is called the rotation index of the curve* $\mathbf{r}$ *and it is denoted by* $\mathrm{Index}_R(\mathbf{r})$.

Intuitively, the rotation index of a closed curve is the number of times the normalized tangent of the curve makes a complete turn. To define the rotation index of a plane curve it is only necessary to assume that $\frac{\mathbf{r}'}{|\mathbf{r}'|}$ is a closed curve.

2.4.3 Stereographic projections

Let (θ, φ) be spherical coordinates on the unit 2-sphere , i.e., for

$$(u, v, w) \in S^2 \subset \mathbb{R}^3$$

we have

$$\left.\begin{aligned} u &= \sin\theta\cos\varphi \\ v &= \sin\theta\sin\varphi \\ w &= \cos\theta \end{aligned}\right\} \tag{2.14}$$

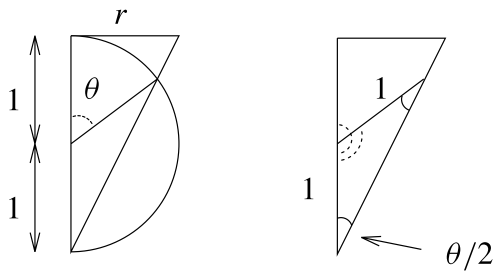

Figure 2.2: The stereographic projection

where $0 < \theta < \pi$ and $0 < \varphi < 2\pi$. There is nothing special about the choice of the half circle that has been cut out of the sphere. To make spherical coordinates well-defined, any other half circle will do. This means that you do not have to worry about shift of spherical coordinate systems because they all look the same.

We denote the stereographic projection of the unit 2-sphere onto the tangent plane of P, $T_P S^2$, by $\mathrm{Proj}_P : S^2 \backslash \{-P\} \to T_P S^2$ and the stereographic projection is given by

$$\mathrm{Proj}_P(\theta, \varphi) = (2\tan(\theta/2), \varphi),$$

where the right hand side is polar coordinates (r, φ) in the tangent plane. See Figure 2.2.

2.4.4 A new formula for the total torsion

This section contains the apparently new formula for the total torsion of a closed curve (or a curve with closed Gauss map) found by C. Chicone and N.J. Kalton, [Chicone & Kalton, 1984]. First we prove a theorem involving index and rotation index, which is necessary for the proof of Theorem 2.15. In [Chicone & Kalton, 1984] page 23 the theorem is used, but without proof or reference.

Theorem 2.14 *Let $\Gamma : [0, T] \to S^2$ be a regular closed curve which divide the unit 2-sphere in finitely many components. Denote the complement to $\Gamma([0, T])$ by Ω. There exists a continuous map $f : \Omega \to \mathbb{Z}$ satisfying*

$$f(Q) = 2\,\mathrm{Index}(\mathrm{Proj}_P(\Gamma), \mathrm{Proj}_P(Q)) - \mathrm{Index}_R(\mathrm{Proj}_P(\Gamma)),\ Q \in \Omega,$$

for $Q \neq -P$, where $-P \in \Omega$.

Proof: The problem is whether this map f is independent of the P used to define it. If f is independent of P then f obviously is defined on all of $\Omega = S^2 \backslash \Gamma([0, T])$.

Let $\Gamma : [0, T] \to S^2$ be a closed regular curve which divide the unit 2-sphere in finitely many components and let Q be an arbitrary point on Ω. Choose P_0 and P_1 on S^2 such that $-P_0, -P_1 \in \Omega \backslash \{Q\}$ and consider the two integers n_i, $i = 0, 1$, given by

$$n_i \quad = \quad 2\,\mathrm{Index}(\mathrm{Proj}_{P_i}(\Gamma), \mathrm{Proj}_{P_i}(Q)) - \mathrm{Index}_R(\mathrm{Proj}_{P_i}(\Gamma)). \qquad (2.15)$$

Note, that at this point it is necessary that Γ divide the unit 2-sphere in only finitely many components. Otherwise, the integers are not well-defined.

If $-P_0$ and $-P_1$ lie in the same connected component of Ω, then Proj_{P_1} simply is a deformation of Proj_{P_0}. Hence,

$$\mathrm{Index}_R(\mathrm{Proj}_{P_0}(\Gamma)) = \mathrm{Index}_R(\mathrm{Proj}_{P_1}(\Gamma))$$

and since the image of Q undergo the same deformation as the image of Γ we also have that

$$\mathrm{Index}(\mathrm{Proj}_{P_0}(\Gamma), \mathrm{Proj}_{P_0}(Q)) = \mathrm{Index}(\mathrm{Proj}_{P_1}(\Gamma), \mathrm{Proj}_{P_1}(Q)).$$

Thus, we conclude that the two integers n_0 and n_1 are equal if $-P_0$ and $-P_1$ lie in the same connected component of Ω.

Assume next that $-P_0$ and $-P_1$ lie in neighbouring components of Ω. The above ensures that $-P_0$ and $-P_1$ can be chosen arbitrarily close to each other. Choose $-P_0$ and $-P_1$, so that they have a common neighbourhood U fulfilling:

- $Q \notin \overline{U}$.

- $\Gamma \cap \overline{U}$ have only one segment, S.

To obtain an easier notation we at first assume that this segment, S, is traversed only once, i.e., $\Gamma^{-1}(U) = \Gamma^{-1}(S)$ has only one component.

To investigate if $n_0 = n_1$, we now split the terms in there right hand sides of (2.15) into the contributions from U and from $S^2 \backslash U$. With some abuse of notation we let

$$k_i = \mathrm{Index}(\mathrm{Proj}_{P_i}(\Gamma), \mathrm{Proj}_{P_i}(Q))|_{(S^2 \backslash U)}, i = 0, 1,$$

and

$$m_i = \mathrm{Index}_R(\mathrm{Proj}_{P_i}(\Gamma))|_{(S^2 \backslash U)}, i = 0, 1,$$

be the contributions from $S^2 \backslash U$. See Figure 2.3 and Figure 2.4. As P_0 and P_1 can be chosen arbitrarily close to each other k_0 and k_1 resp. m_0 and m_1 can be assumed to be almost identical. We can, without loss of generality, assume that the

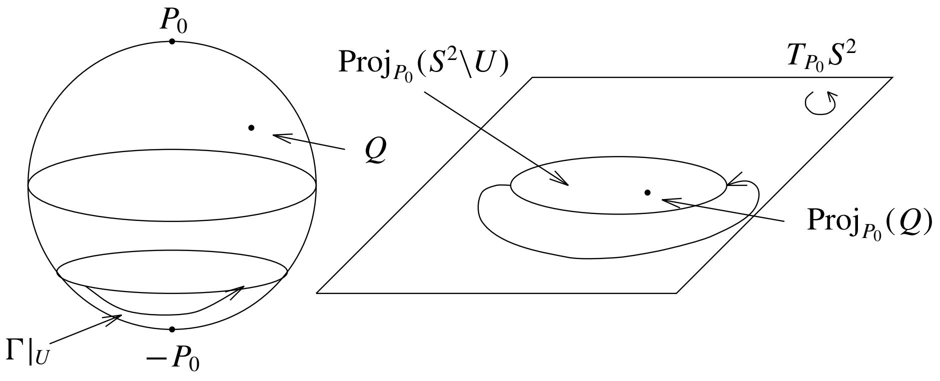

Figure 2.3: Stereographic projection

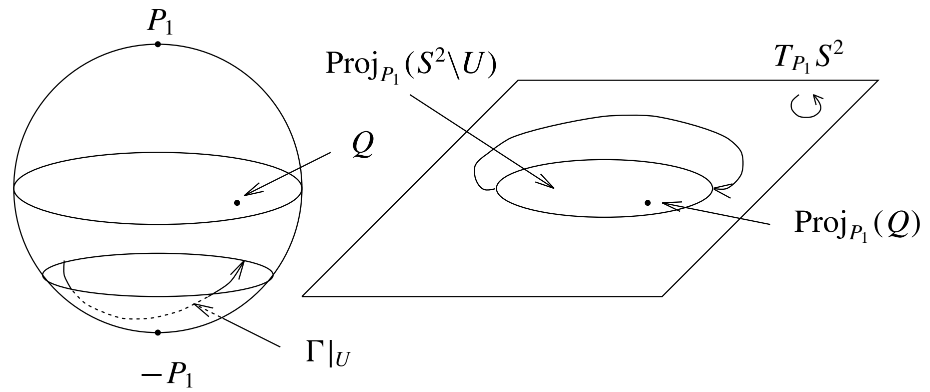

Figure 2.4: Stereographic projection

projections of Γ have parallel tangent vectors in the same direction when entering and leaving U. In the situations sketched in Figure 2.3 and Figure 2.4 we obtain

$$\begin{aligned}\Delta\,\text{Index} &= \text{Index}(\text{Proj}_{P_1}(\Gamma),\text{Proj}_{P_1}(Q)) - \text{Index}(\text{Proj}_{P_0}(\Gamma),\text{Proj}_{P_0}(Q))\\ &\approx -\frac{1}{2}+k_1-(\frac{1}{2}+k_0)\\ &\approx -1\end{aligned}$$

and

$$\begin{aligned}\Delta\,\text{Index}_R &= \text{Index}_R(\text{Proj}_{P_1}(\Gamma)) - \text{Index}_R(\text{Proj}_{P_0}(\Gamma))\\ &\approx -1+m_1-(1+m_0)\\ &\approx -2.\end{aligned}$$

Whence, the relation between the integers n_0 and n_1 is

$$n_1 \approx n_0 + 2\Delta\,\text{Index} - \Delta\,\text{Index}_R \approx n_0 + 2 - 2 = n_0$$

and the conclusion is that $n_0 = n_1$.

If the segment of Γ in U is traversed more than once the conclusion obvious still holds, because one just has to multiply with the number of times the curve is traversed. Since the unit 2-sphere is arc-wise connected the proof is complete. $\square$

The sphere expression for the torsion given in Lemma 2.8 has to be used in spherical coordinates on the unit 2-sphere. For this we need to calculate α and β in spherical coordinates. Differentiation of the coordinate functions of the embedding (cf. (2.14) page 25) gives

$$\begin{aligned}u' &= -\varphi'\sin\theta\sin\varphi + \theta'\cos\theta\cos\varphi,\\ v' &= \varphi'\sin\theta\cos\varphi + \theta'\cos\theta\sin\varphi, \quad\text{and}\\ w' &= -\theta'\sin\theta.\end{aligned}$$

Insertion in to (2.10) page 21 now gives α as

$$\begin{aligned}\alpha &= \frac{vu'-uv'}{u^2+v^2}\\ &= \frac{\sin\theta\sin\varphi(-\varphi'\sin\theta\sin\varphi+\theta'\cos\theta\cos\varphi)}{\sin^2\theta}\\ &\quad -\frac{\sin\theta\cos\varphi(\varphi'\sin\theta\cos\varphi+\theta'\cos\theta\sin\varphi)}{\sin^2\theta}\\ &= \frac{\varphi'[-\sin^2\theta\sin^2\varphi - sin^2\theta\cos^2\varphi]}{\sin^2\theta}\\ &= -\varphi' \qquad (2.16)\end{aligned}$$

and β as

$$\beta = \frac{w'}{1-w^2} = \frac{-\theta' \sin\theta}{1-\cos^2\theta} = \frac{-\theta'}{\sin\theta}. \tag{2.17}$$

Now we are ready to prove the formula (mentioned earlier) for the total torsion of space curves with closed Gauss map found by C. Chicone and N.J. Kalton.

Theorem 2.15 *Let* $\mathbf{r}$ *be a curve in* $\mathbb{R}^3$ *with non-vanishing curvature and parametrized by arc length such that the tangent curve of* $\mathbf{r}$, $\Gamma : [0, T] \to S^2\backslash\{-P\}$, *is a closed curve which divide the unit* 2*-sphere in finitely many components. Let the map*

$$\mathrm{Proj}_P : S^2\backslash\{-P\} \to T_P S^2$$

be the stereographic projection described in Section 2.4.3 and let $\gamma = \mathrm{Proj}_P(\Gamma)$. *Then the total torsion of* $\mathbf{r}$ *is given by the formula*

$$\int_0^T \tau(s)ds = -\frac{1}{2}\int_{Q\in S^2}[2\,\mathrm{Index}(\gamma, \mathrm{Proj}_P(Q)) - \mathrm{Index}_R(\gamma)]dA.$$

Proof: Assume $\mathbf{r}$ is a curve in $\mathbb{R}^3$ with non-vanishing curvature and parametrized by arc length such that the tangent curve of $\mathbf{r}$, $\Gamma : [0, T] \to S^2\backslash\{-P\}$, is a closed curve which divide the unit 2-sphere in finitely many components. Set $\gamma = \mathrm{Proj}_P(\Gamma)$, where the map Proj_P is the stereographic projection described in Section 2.4.3. Then, γ has the polar coordinates

$$(2\tan(\theta/2), \varphi), 0 < \theta < \pi.$$

Remembering the expression for the torsion stated in Lemma 2.8 we first compute

$$I_1 = \int_0^T \frac{\beta\alpha' - \alpha\beta'}{\alpha^2 + \beta^2} ds.$$

Let γ_1 be the plane curve in the tangent plane of P with polar coordinates $r = \theta(s)$ and $\varphi = \varphi(s)$, $s \in [0, T]$. Since the curve γ_1 just is a radial rescaling of γ we have

$$\mathrm{Index}_R(\gamma) = \mathrm{Index}_R(\gamma_1)$$

and

$$\mathrm{Index}(\gamma, 0) = \mathrm{Index}(\gamma_1, 0),$$

where the point 0 is the projection of the point P on the tangent plane of P.

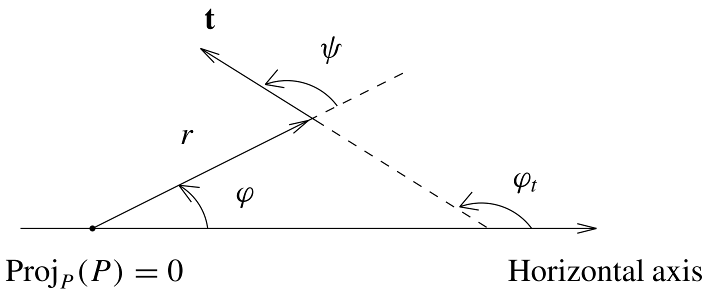

Figure 2.5: The angles in the tangent plane

Let φ_t be the angle the tangent vector makes with the horizontal axis[6] in the tangent plane of P and let ψ be the angle between the radius vector and the tangent vector. See Figure 2.5. Then, by the angle relation in a triangle,

$$\begin{aligned} \varphi + \psi + (\pi - \varphi_t) &= \pi \\ &\Updownarrow \\ \varphi_t &= \psi + \varphi \end{aligned}$$

and, therefore,

$$\varphi_t' = \psi' + \varphi'. \tag{2.18}$$

Furthermore, by use of (2.13) page 24, (2.18), Definition 2.11, and Definition 2.13 we have

$$\begin{aligned} \mathrm{Index}_R(\gamma_1) - \mathrm{Index}(\gamma_1, 0) &= \frac{1}{2\pi}\int_0^T {\varphi_t}'(s)ds - \frac{1}{2\pi}\int_0^T \varphi'(s)ds \\ &= \frac{1}{2\pi}\int_0^T [\psi'(s) + \varphi'(s)]ds - \frac{1}{2\pi}\int_0^T \varphi'(s)ds \\ &= \frac{1}{2\pi}\int_0^T \psi'(s)ds \\ &= \mathrm{Index}(\gamma_2, 0), \end{aligned}$$

where γ_2 is a plane curve. We claim[7] that the plane curve γ_2 can be chosen to be the plane curve with Cartesian coordinates $(\theta', \theta\varphi')$. To show this we compute

$$\cos\psi = \frac{\langle \mathbf{v}, \mathbf{t}\rangle}{|\mathbf{v}||\mathbf{t}|}$$

[6]The axis with $\varphi = 0$.
[7]In [Chicone & Kalton, 1984] p. 24 this is not proved.

and

$$\sin\psi = -\cos(\psi + \frac{\pi}{2}) = -\frac{\langle \mathbf{v}, \hat{\mathbf{t}} \rangle}{|\mathbf{v}||\mathbf{t}|},$$

where $\mathbf{v}$ is the radius vector and $\mathbf{t}$ the tangent vector of the curve γ_1. The Cartesian coordinates of γ_1 are given by

$$(\theta \cos\varphi, \theta \sin\varphi), 0 < \theta < \pi.$$

Hence,

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{t} \rangle &= (\theta \cos\varphi, \theta \sin\varphi) \cdot (\theta' \cos\varphi - \theta\varphi' \sin\varphi, \theta' \sin\varphi + \theta\varphi' \cos\varphi) \\
&= \theta\theta' \cos^2\varphi - \theta^2\varphi' \cos\varphi \sin\varphi + \theta\theta' \sin^2\varphi + \theta^2\varphi' \sin\varphi \cos\varphi \\
&= \theta\theta',
\end{aligned}$$

$$\begin{aligned}
\langle \mathbf{v}, \hat{\mathbf{t}} \rangle &= (\theta \cos\varphi, \theta \sin\varphi) \cdot (-\theta' \sin\varphi - \theta\varphi' \cos\varphi, \theta' \cos\varphi - \theta\varphi' \sin\varphi) \\
&= -\theta\theta' \cos\varphi \sin\varphi - \theta^2\varphi' \cos^2\varphi + \theta\theta' \sin\varphi \cos\varphi - \theta^2\varphi' \sin^2\varphi \\
&= -\theta^2\varphi',
\end{aligned}$$

$$|\mathbf{v}|^2 = \theta^2, \quad \text{and}$$

$$\begin{aligned}
|\mathbf{t}|^2 &= (\theta')^2 \cos^2\varphi + (\theta\varphi')^2 \sin^2\varphi - 2\theta\theta'\varphi' \cos\varphi \sin\varphi \\
&\quad + (\theta')^2 \sin^2\varphi + (\theta\varphi')^2 \cos^2\varphi + 2\theta\theta'\varphi' \cos\varphi \sin\varphi \\
&= (\theta')^2 + (\theta\varphi')^2.
\end{aligned}$$

Consequently,

$$\cos\psi = \frac{\theta'}{\sqrt{(\theta')^2 + (\theta\varphi')^2}}$$

and

$$\sin\psi = \frac{\theta\varphi'}{\sqrt{(\theta')^2 + (\theta\varphi')^2}}.$$

Then, it follows that the plane curve γ_2 can be chosen to be the plane curve with Cartesian coordinates $(\theta', \theta\varphi')$, because $f_\psi = \Pi(\psi) = (\cos\psi, \sin\psi)$ cf. page 23 and by Definition 2.11 is $f_\psi = \frac{\gamma_2}{|\gamma_2|}$.

Since $0 < \theta < \pi$ the plane curve γ_3 with Cartesian coordinates $(\frac{\theta'}{\sin\theta}, \varphi')$ is just a rescaling of the plane curve γ_2, therefore, we have

$$\text{Index}(\gamma_3, 0) = \text{Index}(\gamma_2, 0).$$

By (2.16) and (2.17) page 30 we observe that the curve γ_3 in fact have Cartesian coordinates given by $(-\beta, -\alpha)$. The plane curve γ_3 determines the angle ϑ from its polar coordinate system and this angle is given by

$$\vartheta = \operatorname{Arctan}\left(\frac{-\alpha}{-\beta}\right)$$

and then

$$\begin{aligned}\vartheta' &= \frac{\frac{\alpha'}{\beta} - \frac{\alpha\beta'}{\beta^2}}{1 + (\frac{\alpha}{\beta})^2} \\ &= \frac{\beta\alpha' - \alpha\beta'}{\alpha^2 + \beta^2}.\end{aligned}$$

By (2.13) page 24 we have

$$\begin{aligned}\operatorname{Index}(\gamma_3, 0) &= \frac{1}{2\pi}\int_0^T \vartheta' ds \\ &= \frac{1}{2\pi}\int_0^T \frac{\beta\alpha' - \alpha\beta'}{\alpha^2 + \beta^2} ds\end{aligned}$$

and then we see that the integral I_1 is given by[8]

$$\begin{aligned}I_1 &= \int_0^T \frac{\beta\alpha' - \alpha\beta'}{\alpha^2 + \beta^2} ds \\ &= 2\pi \operatorname{Index}(\gamma_3, 0) \\ &= 2\pi \operatorname{Index}(\gamma_2, 0) \\ &= 2\pi(\operatorname{Index}_R(\gamma_1) - \operatorname{Index}(\gamma_1, 0)) \\ &= 2\pi(\operatorname{Index}_R(\gamma) - \operatorname{Index}(\gamma, 0)).\end{aligned}$$

This completes the integration of the term $\frac{\beta\alpha' - \alpha\beta'}{\alpha^2+\beta^2}$ of the torsion. We now proceed with

$$\begin{aligned}I_2 &= \int_0^T w\alpha ds, \quad \text{and by (2.16) page 29} \\ &= -\int_0^T w\varphi' ds.\end{aligned}$$

We recognize the integral I_2 as a line integral so that

$$\begin{aligned}I_2 &= -\int_{\gamma_1} w d\varphi, \quad \text{and by (2.14) page 25} \\ &= -\int_{\gamma_1} \cos\theta d\varphi.\end{aligned}$$

[8] In [Chicone & Kalton, 1984] the last line on page 24 there is an error. The integral I_1 have the wrong sign. This actually annul their error on the torsion (cf. Remark 2.9 page 21).

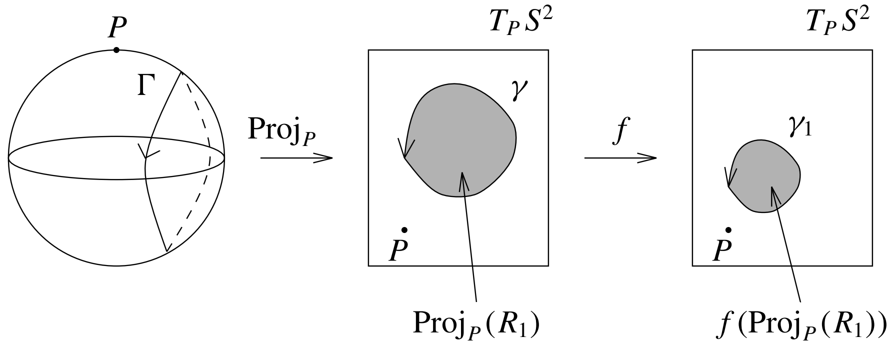

Figure 2.6: Projection into the tangent plane and then the radial rescaling

We claim

$$I_2 = \int_{S^2} \operatorname{Index}(\gamma, \operatorname{Proj}_P(Q))dA - 2\pi \operatorname{Index}(\gamma, 0),$$

where $\gamma = \operatorname{Proj}_P(\Gamma)$.

To show that this formula is valid for any closed curve Γ, which divide the unit 2-sphere in finitely many components[9], it is only necessary to establish it for a simple closed curve; the general case follows by induction on the number of components of $\Omega = S^2 \backslash \Gamma([0, T])$. So, suppose Γ is a simple closed curve. Then, Ω has two components which we denote by R_1 and R_2. There are two cases; either $\operatorname{Index}(\gamma, 0)$ is equal to zero or not equal to zero.

Assume $\operatorname{Index}(\gamma, 0)$ is equal to zero. Let R_1 be the component of Ω which corresponds to the bounded component of the complement to the plane curve γ. As mentioned earlier (page 30) the plane curves γ is a radial rescaling of γ_1. Then, there exists an orientation preserving homeomorphism $f : T_P S^2 \to T_P S^2$ such that $f(\gamma) = \gamma_1$. In other words, the map f realizes the radial rescaling. See Figure 2.6. Whence,

$$\begin{aligned} n &= \operatorname{Index}(\gamma_1, f(\operatorname{Proj}_P(Q))) \\ &= \operatorname{Index}(f(\gamma), f(\operatorname{Proj}_P(Q))) \\ &= \operatorname{Index}(\gamma, \operatorname{Proj}_P(Q)) \\ &= \begin{cases} \pm 1 & , \text{ for } Q \in R_1 \\ 0 & , \text{ for } Q \in R_2 \end{cases} \end{aligned}$$

We apply Green's Theorem (see e.g. [Hansen, 1968] page 240) to obtain

$$I_2 = -\int_{\gamma_1} \cos\theta d\varphi = n \int\int_{f(\operatorname{Proj}_P(R_1))} \sin\theta d\theta d\varphi.$$

9 According to C. Chicone and N.J. Kalton this is fulfilled if Γ is a piecewise analytic closed curve.

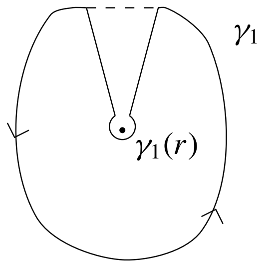

Figure 2.7: The modified curve

We recognize that $\sin\theta d\theta d\varphi$ is the element of 2-dimensional volume on S^2. Denote the element of 2-dimensional volume on S^2 by dA. Then, it follows that

$$\begin{aligned} I_2 &= n \int\int_{f(\mathrm{Proj}_P(R_1))} \sin\theta d\theta d\varphi \\ &= \int_{R_1} n dA + \int_{R_2} 0 dA \\ &= \int_{S^2} \mathrm{Index}(\gamma, \mathrm{Proj}_P(Q)) dA. \end{aligned}$$

If $\mathrm{Index}(\gamma, 0)$ not is equal to zero we apply Green's Theorem by modifying γ_1 to be the boundary curve of a slit annular region with inner boundary a small circle $\gamma_1(r)$ of radius r around P, where $r = \theta$ (see page 30). The modified curve, $\gamma_1 + \gamma_1(r)$, is showed on Figure 2.7 and fulfills

$$\mathrm{Index}(\gamma_1 + \gamma_1(r), 0) = 0.$$

Then, in the limit we obtain

$$\lim_{r\to 0^+} (n \int_{\gamma_1(r)} \cos\theta d\varphi) - \int_{\gamma_1} \cos\theta d\varphi = \int_{S^2} \mathrm{Index}(\gamma, \mathrm{Proj}_P(Q)) dA,$$

where

$$\begin{aligned} \lim_{r\to 0^+} (n \int_{\gamma_1(r)} \cos\theta d\varphi) &= \lim_{r\to 0^+} (n \int_0^{2\pi - r} \cos\theta d\varphi) \\ &= \lim_{r=\theta\to 0^+} (n[\varphi\cos\theta]_0^{2\pi - r}) \\ &= 2\pi\ \mathrm{Index}(\gamma, 0). \end{aligned}$$

Thus, in either case,

$$I_2 = \int_{S^2} \mathrm{Index}(\gamma, \mathrm{Proj}_P(Q)) dA - 2\pi\ \mathrm{Index}(\gamma, 0).$$

By Lemma 2.8 page 21 the torsion of $\mathbf{r}$ is given by

$$\tau = \frac{\beta\alpha' - \alpha\beta'}{\alpha^2 + \beta^2} - w\alpha$$

and then the following calculation gives the desired formula.

$$\begin{aligned}
\int_0^T \tau(s)ds &= \int_0^T (\frac{\beta\alpha' - \alpha\beta'}{\alpha^2 + \beta^2} - w\alpha)ds \\
&= I_1 - I_2 \\
&= 2\pi[\mathrm{Index}_R(\gamma) - \mathrm{Index}(\gamma, 0)] \\
&\quad - [\int_{S^2} \mathrm{Index}(\gamma, \mathrm{Proj}_P(Q))dA - 2\pi \,\mathrm{Index}(\gamma, 0)] \\
&= -[\int_{S^2} \mathrm{Index}(\gamma, \mathrm{Proj}_P(Q))dA - 2\pi \,\mathrm{Index}_R(\gamma)] \\
&= -[\int_{S^2} \mathrm{Index}(\gamma, \mathrm{Proj}_P(Q))dA - \mathrm{Index}_R(\gamma) \int_{S^2} \frac{1}{2} dA] \\
&= -\frac{1}{2} \int_{S^2} [2\,\mathrm{Index}(\gamma, \mathrm{Proj}_P(Q)) - \mathrm{Index}_R(\gamma)]dA.
\end{aligned}$$

Finally, by Theorem 2.14 the integer

$$2\,\mathrm{Index}(\gamma, \mathrm{Proj}_P(Q)) - \mathrm{Index}_R(\gamma),\ Q \in \Omega,$$

is well-defined on all of $\Omega = S^2 \backslash \Gamma([0, T])$. On $\Gamma([0, T])$ we set

$$2\,\mathrm{Index}(\gamma, \mathrm{Proj}_P(Q)) - \mathrm{Index}_R(\gamma) = 0,\ Q \in \Gamma([0, T]).$$

This completes the proof of the theorem. □

Chapter 3

Links and equivalences of links

The aim of this chapter is to introduce links and the basic tools for studying this class of geometric objects. A link is an embedding of a disjoint union of unit circles into euclidean 3-space or into the unit 3-sphere. It is not the individual link that takes interest but the class of links that it can be deformed into by non-damaging deformations. There are several types of non-damaging deformations and therefore, there are several equivalence relations among links. It turn out that all the equivalence relations among links we introduce are equivalent. This is stated in the two main theorems in this chapter: Theorem 3.35 and Theorem 3.42, which we shall prove. The reason why we describe several equivalent relations among links is that each of them has their own advantages.

It is necessary to restrict the class of links to the class of piecewise linear links. Therefore, we first introduce some basic piecewise linear topology.

3.1 Piecewise linear topology

We shall describe the standard building blocks called simplices, which pieces together to a spaces called polyhedra on which it is natural to define piecewise linear maps.

3.1.1 Simplices

Points in $\mathbb{E}^m$ will be denoted V_i, $0 \leq i \leq n \leq m$. We need to consider points $V_0, V_1, \ldots, V_n$ in $\mathbb{E}^m$, such that the n vectors $V_1 - V_0, V_2 - V_0, \ldots, V_n - V_0$ are linearly independent, i.e.,

$$\dim(\operatorname{span}\{V_1 - V_0, V_2 - V_0, \ldots, V_n - V_0\}) = n.$$

Then we say that the $(n + 1)$ points $V_0, V_1, \ldots, V_n$ are in general position.

Remark 3.1 (General position) *For a more detailed treatment of general position see [Brøndsted, 1983] or [Rockafellar, 1970]. These books contain the basic concepts of affine sets and convex sets.*

Definition 3.2 (Convexity and convex hulls) *A set $V \subset \mathbb{E}^m$ is convex if for each $V_0, V_1 \in V$, the set V contains the segment*

$$\begin{aligned}[V_0V_1] &= \{\alpha_0 V_0 + \alpha_1 V_1 \mid \alpha_0, \alpha_1 \geq 0, \alpha_0 + \alpha_1 = 1\} \\ &= \{\alpha V_0 + (1-\alpha)V_1 \mid \alpha \in [0,1]\}.\end{aligned}$$

The convex hull of a set $V \subset \mathbb{E}^m$ is the smallest convex subset of $\mathbb{E}^m$ that contains V, i.e., the intersection of all convex subsets of $\mathbb{E}^m$ that contain V.

Definition 3.3 (Convex combinations) *A convex combination of points $V_0, V_1, \ldots, V_n$ in $\mathbb{E}^m$ is a linear combination*

$$\alpha_0 V_0 + \cdots + \alpha_n V_n,$$

where $\alpha_0 + \cdots + \alpha_n = 1$ and $\alpha_0, \ldots, \alpha_n \geq 0$.

Theorem 3.4 *Let $V = \{V_0, V_1, \ldots, V_n\}$ be a set of $(n+1)$ points in $\mathbb{E}^m$. The convex hull of the set V consists of all points obtained as convex combinations of the points $V_0, V_1, \ldots, V_n$.*

Proof: See [Rockafellar, 1970] page 12, Corollary 2.3.1.

Definition 3.5 (Simplices and subsimplices) *Let $V = \{V_0, V_1, \ldots, V_n\}$ be a set of $(n+1)$ points in general position in $\mathbb{E}^m$, $n \leq m$. The n-dimensional simplex (or just n-simplex), $s^n = [V_0V_1 \ldots V_n]$, is the convex hull of V.*

Let $W \subset V$ be a subset of V consisting of $(i+1)$ different points. The i-dimensional subsimplex (or i-subsimplex), $s^i, 0 \leq i \leq n$, of the n-simplex s^n is the convex hull of W.

The $(i+1)$ different points in the subset W of V (cf. Definition 3.5) are in general position, because when n vectors $V_1 - V_0, V_2 - V_0, \ldots, V_n - V_0$ are linearly independent a subset consisting of i vectors also will be linearly independent. Therefore, an i-subsimplex, $s^i, 0 \leq i \leq n$, of the n-simplex, s^n, actually is a simplex of lower dimension (or the same simplex in the case $i = n$). In other words, an i-subsimplex is itself an i-simplex. This is the reason why we call it i-subsimplex. Most authors call the here defined i-subsimplex for a face (or i-face) of the simplex, e.g., [Armstrong, 1983] page 120, [Moise, 1977] page 3.

Figure 3.1 shows the simplices of dimensions 0, 1, 2, and 3. The geometry of a 3-simplex, $s^3 = [V_0V_1V_2V_3]$, is as follows: The 0-subsimplices are just points and

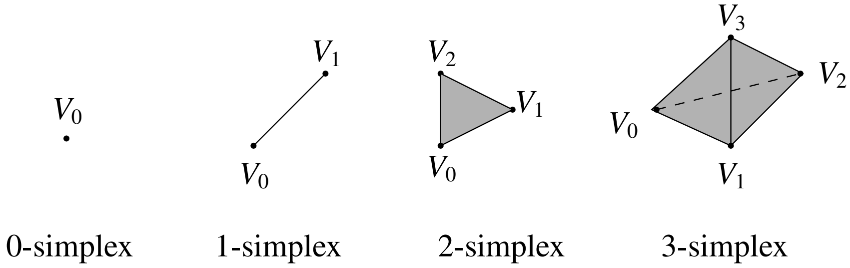

Figure 3.1: Some low dimensional simplices

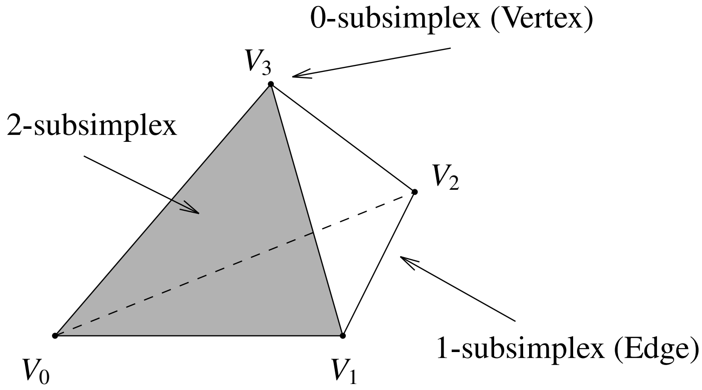

Figure 3.2: A 3-simplex, $s^3 = [V_0 V_1 V_2 V_3]$

they are often called the vertices of the simplex. The 1-subsimplices are closed line segments, and they are called edges of the simplex. The 2-subsimplices are (full) triangles and the 3-simplex, s^3, is the (full) tetrahedron. See Figure 3.2.

Let $v \in \mathbb{E}^m$ be a point in the n-simplex $s^n = [V_0V_1 \dots V_n]$, $n \leq m$. Hence, it follows from Theorem 3.4 that v can be written on the form

$$v = \alpha_0 V_0 + \cdots + \alpha_n V_n \in s^n, \tag{3.1}$$

where $\alpha_0 + \cdots + \alpha_n = 1$ and $\alpha_0, \dots, \alpha_n \geq 0$.

Definition 3.6 (Barycentric coordinates) *Let the point v be given as above. The coordinates $(\alpha_0, \alpha_1, \dots, \alpha_n)$ for v are called the barycentric coordinates for v in the n-simplex, $s^n = [V_0V_1 \dots V_n]$.*

The coordinates $(\alpha_0, \alpha_1, \dots, \alpha_n)$ are well-defined, i.e., unique cf. [Rockafellar, 1970] page 7. The coordinates $(\alpha_0, \alpha_1, \dots, \alpha_n)$ are also called the barycentric coordinates for v if we omit the restriction $\alpha_0, \dots, \alpha_n \geq 0$, but then the point v could be outside the n-simplex, s^n.

3.1.2 Complexes and polyhedra

The following definitions make a piecewise linear structure on subsets of euclidean spaces.

Definition 3.7 (Complexes) *A complex is a collection $\mathcal{C}$ of simplices in $\mathbb{E}^m$, such that:*

- *$\mathcal{C}$ contains all subsimplices of all elements of $\mathcal{C}$.*
- *If $s^i, s^j \in \mathcal{C}$ and $s^i \cap s^j \neq \emptyset$, $0 \leq i, j \leq m$, then $s^i \cap s^j$ is a subsimplex of both of s^i and of s^j.*
- *(In case $\mathcal{C}$ is infinite) Every s^i, $0 \leq i \leq m$, in $\mathcal{C}$ lies in an open set U which intersects only a finite number of elements of $\mathcal{C}$.*

Remark 3.8 (Complexes) *A complex is by some authors call simplices complex. We prefer the short notation as in [Moise, 1977].*

Definition 3.9 (Polyhedra) *Let $\mathcal{C}$ be a complex. Then $|\mathcal{C}|$ denotes the union of the elements of $\mathcal{C}$, with the subspace topology induced by the topology of $\mathbb{E}^m$. The set $|\mathcal{C}|$ is called a polyhedron.*

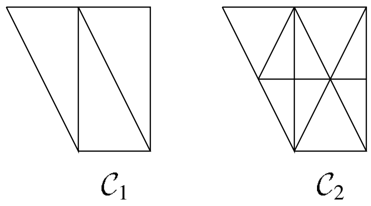

Figure 3.3: $|\mathcal{C}_1| = |\mathcal{C}_2|$, but $\mathcal{C}_1 \neq \mathcal{C}_2$. The complex $\mathcal{C}_2$ is a subdivision of the complex $\mathcal{C}_1$

The polyhedron, $|\mathcal{C}|$, is the topological space underlying the complex $\mathcal{C}$. We shall think of $|\mathcal{C}|$ ambiguously, as either a subset of $\mathbb{E}^m$ or a space. The difference between $\mathcal{C}$ and $|\mathcal{C}|$ are shown in Figure 3.3, where the polyhedra $|\mathcal{C}_1|$ and $|\mathcal{C}_2|$ are equal but the complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ are different.

Definition 3.10 (Subdivision of complexes) *Let $\mathcal{C}$ and $\mathcal{C}'$ be complexes, in the same space $\mathbb{E}^m$. $\mathcal{C}'$ is a subdivision (or subcomplex) of $\mathcal{C}$, if*

- *the polyhedra $|\mathcal{C}'|$ and $|\mathcal{C}|$ are equal, i.e., $|\mathcal{C}'| = |\mathcal{C}|$.*
- *every element of $\mathcal{C}'$ is a subset of some element of $\mathcal{C}$.*

Figure 3.3 shows an example of a subdivision of a complex.

Definition 3.11 (Simplex stars) *Let $S = \{s_1, \dots, s_k\}$ be a subset of a complex, $\mathcal{C}$, consisting of k simplices. Let $\mathcal{S}$ be the subset of the complex, $\mathcal{C}$, consisting of all simplices which have a common subsimplex with one of the simplices from S, i.e.,*

$$\mathcal{S} = \{s \in \mathcal{C} | S \cap s \neq \emptyset\}.$$

The simplex star of S, $\mathrm{star}(S)$*, is the polyhedron $|\mathcal{S}|$.*

Our reason for introducing the simplex star of S is that it is a neighbourhood of S which after a subdivision of the original complex can be made to an arbitrarily small neighbourhood of S.

3.1.3 Piecewise linear maps

We shall use the word “linear” instead of “affine” because this is widespread used. Therefore, “linear” always means linear in the affine sense.

Definition 3.12 (Linear maps and simplicial maps of simplices) *Let s^i and s^j be simplices in $\mathbb{E}^m$, $0 \leq i, j \leq m$. Let v be a point in s^i. A map $f : s^i \to s^j$ is linear if the coordinates of the point $f(v)$ are linear of those of v. If also vertices of s^i are mapped onto vertices of s^j then f is simplicial.*

Remark 3.13 (Maps) *It is throughout an implied condition for maps to be continuous.*

Definition 3.14 (Simplicial maps) *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be complexes in $\mathbb{E}^m$ and let f be a map from $|\mathcal{C}_1|$ to $|\mathcal{C}_2|$. If each restriction of f to a simplex, $f|_{s^i}$ for $s^i \in \mathcal{C}_1, 0 \leq i \leq m$, is simplicial, then f is simplicial.*

The coordinates mentioned in Definition 3.12 could be barycentric coordinates or Cartesian coordinates. If the point $v \in \mathbb{E}^m$ is given as in equation (3.1) on page 40, then the linear map $f : s^n \to s^j, 0 \leq j, n \leq m$, will be given by

$$f(v) = f(\sum_{i=0}^{n} \alpha_i V_i) = \sum_{i=0}^{n} \alpha_i f(V_i).$$

Definition 3.15 (Piecewise linear maps) *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be complexes in $\mathbb{E}^m$ and let f be a map from $|\mathcal{C}_1|$ to $|\mathcal{C}_2|$. If there is a subdivision $\mathcal{C}_1'$ of $\mathcal{C}_1$ such that each restriction of f to a simplex from the subcomplex, $f|_{s^i}$ for $s^i \in \mathcal{C}_1', 0 \leq i \leq m$, maps s^i linearly into a simplex of $\mathcal{C}_2$, then f is piecewise linear.*

3.2 Knots and links

The remainder of this chapter is an introduction to the theory of knots and links based on the book [Burde & Zieschang, 1985].

3.2.1 Isotopies of knots and links

Knots and links are topological embeddings, therefore, we shall recall the definition of a topological embedding.

Definition 3.16 (Topological embeddings) *Let X and Y be Hausdorff spaces. A map $f : X \to Y$ is called a topological embedding if $f : X \to f(X)$ is a homeomorphism.*

Unless otherwise indicated all embeddings are topological embedding, i.e., C^0-embeddings.

Definition 3.17 (Links and knots) *A link with n components is an embedding of a disjoint union of n unit circles into euclidean 3-space or into the unit 3-sphere. A link with one component is called a knot.*

Since a knot simply is a link with one component we will formulate theorems etc. for links without explicit to mention that they also holds for knots. In accordens with our physical intuition it is not the single embedding that takes our interest but suitable classes containing non-damaging deformations of its members. Speaking of intuition - a link in S^3 can be imagined as a link in $\mathbb{E}^3 \cup \{\infty\}$. For technical reasons it is often convenient to consider links in the one-point compactification of $\mathbb{E}^3$, S^3. From a technical point of view a link in $\mathbb{E}^3$, therefore, is a link in S^3 avoiding one point where this point "seen" from $\mathbb{E}^3$ is infinity. We are going to make the same abuse of language as normally done to avoid complicating the notation. A link will be an embedding or the image of the embedding, and additional a link will be an embedding or a class of embeddings.

Definition 3.18 (Isotopies) *Let X and Y be Hausdorff spaces and let I be the unit interval,* $[0, 1]$. *Two embeddings f_0, $f_1 : X \to Y$ are isotopic if there is an embedding $F : X \times I \to Y \times I$, such that*

$$F(x,t) = (f(x,t), t), x \in X, t \in I,$$

with $f(x, 0) = f_0$ and $f(x, 1) = f_1$.

Note, that every $f_t(x) = f(x,t)$ in the above definition are embeddings $f_t : X \to Y$. The above F is called a level-preserving isotopy connecting f_0 and f_1. A more or less surprising fact is that all embeddings of the unit circle into euclidean 3-space are isotopic. The reason for this is that any area where knotting occurs can be contracted continuously to a point[1]. The proper isotopy for links is ambient isotopy.

Definition 3.19 (Ambient isotopies) *Let X and Y be Hausdorff spaces and let I be the unit interval,* $[0, 1]$. *Two embeddings, f_0, $f_1 : X \to Y$ are ambient isotopic if there is a level-preserving isotopy $H : Y \times I \to Y \times I$ satisfying*

$$H(y,t) = (h_t(y), t), y \in Y, t \in I,$$

with $f_1 = h_1 \circ f_0$ and $h_0 = id_Y$. H is called an ambient isotopy.

The above definition ensures that the complements of the two embeddings in Y are ambient isotopic. The family $h_t : Y \to Y, 0 \leq t \leq 1$, is a continuous family of homeomorphisms deforming f_0 and the space around f_0, continuous into f_1 and the space around f_1. Ambient isotopy allows a thin three-dimensional string to follow the embeddings during deformation. In this sense ambient isotopy forces one-dimensional links to behave as if they were three-dimensional, except for the

[1]For reference see [Burde & Zieschang, 1985] p. 2.

fact that they behave like arbitrary thin strings. You can carry on "knotting" links. In fact, a topological embedding of the unit circles into euclidean 3-space can be quit bizarre. There can be an infinite sequence of similar meshes converging to a limit point at which the knot is called wild[2]. To avoid such behaviour we give

Definition 3.20 (Piecewise linear links) *A piecewise linear link with n components is a piecewise linear embedding of a disjoint union of n unit circles into euclidean 3-space or into the unit* 3*-sphere.*

Definition 3.21 (Tame and wild links) *A link in* $\mathbb{E}^3$ *or* S^3 *is called a tame link if it is ambient isotopic to a piecewise linear link. A link is a wild link if it is not tame.*

The category of piecewise linear links and piecewise linear ambient isotopies is essential for the most fundamental results about links and it contains the basic tools for links. For piecewise linear links in S^3 (or $\mathbb{E}^3$) there is a natural restriction of Definition 3.19 given by

Definition 3.22 (Piecewise linear ambient isotopies) *Let* $|\mathcal{C}_1|$ *and* $|\mathcal{C}_2|$ *be two compact polyhedra with finite complexes* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*. Two piecewise linear embeddings* $f_0 : |\mathcal{C}_1| \to |\mathcal{C}_2|$ *and* $f_1 : |\mathcal{C}_1| \to |\mathcal{C}_2|$ *are piecewise linear ambient isotopic if there exists a level-preserving piecewise linear isotopy* $H : |\mathcal{C}_2| \times I \to |\mathcal{C}_2| \times I$ *satisfying*

$$H(y,t) = (h_t(y), t),\ y \in |\mathcal{C}_2|, t \in I,$$

with $f_1 = h_1 \circ f_0$ *and* $h_0 = id_{|\mathcal{C}_2|}$.

The following theorem states that the classes of piecewise linear ambient isotopic knots are the same as the classes of tame ambient isotopic knots. We have not been able to find a reference which prove this result for links. Anyway, later one[3] we shall only use this result for knots.

Theorem 3.23 *Let* L_0 *and* L_1 *be two tame knots, and let* K_0 *and* K_1 *be two piecewise linear knots, where* K_0 *respective* K_1 *is ambient isotopic to* L_0 *respective* L_1*. Then the following statements are equivalent:*

(1) L_0 *and* L_1 *are ambient isotopic.*

(2) K_0 *and* K_1 *are piecewise linear ambient isotopic.*

Proof: (2) $\Rightarrow$ (1) is trivial. (1) $\Rightarrow$ (2) see [Burde & Zieschang, 1985], page 39, Corollary 3.16.

[2]For reference see [Burde & Zieschang, 1985] pp. 2-3.
[3]In Chapter 6 in the proof of Theorem 6.15.

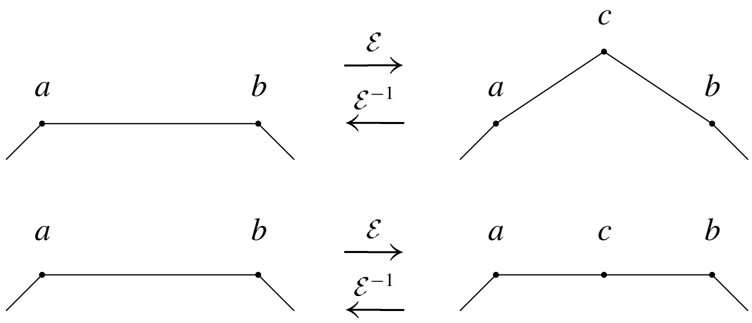

Figure 3.4: A normal elementary deformation and a degenerated elementary deformation

3.2.2 Elementary deformations

A very useful kind of deformation of piecewise linear links is called elementary deformations. To be able to define elementary deformations we use the notation on piecewise linear links they have as polyhedra. We recall that the closed line segments on a piecewise linear link are called edges and their end points vertices.

Let L be a piecewise linear link in $\mathbb{E}^3$ or S^3 and let $[ab]$ be an edge on L with vertices a and b. Let c be a point in $\mathbb{E}^3$ or S^3 different from a and b and consider the (possibly degenerate) 2-simplex $[abc]$. Suppose that $[abc]$ intersects the piecewise linear link L in exactly the edge $[ab]$, i.e.,

$$L \cap [abc] = [ab].$$

Then we say that the elementary deformation $\mathcal{E} = \mathcal{E}^c_{ab}$ is applicable to L and we define a new piecewise linear link, $\mathcal{E}^c_{ab}L$, by

$$\mathcal{E}^c_{ab}L = (L \backslash [ab]) \cup [ac] \cup [cb].$$

In other words, to obtain $\mathcal{E}^c_{ab}L$ we deform L along the triangle $[abc]$ by removing the edge $[ab]$ and substituting with the two new edges $[ac]$ and $[cb]$. We shall refer to both $\mathcal{E}^c_{ab}$ and its inverse $(\mathcal{E}^c_{ab})^{-1}$ as an elementary deformation or as an $\mathcal{E}$-move. See Figure 3.4. We can now define a relation among piecewise linear links.

Definition 3.24 (Combinatorial equivalence) *Two piecewise linear links L and L' in $\mathbb{E}^3$ or S^3 are said to be combinatorially equivalent if one can be transformed into the other by a finite sequence of elementary deformations.*

It is easy to check that the relation combinatorial equivalence is reflexive, symmetric, and transitive. Hence, combinatorial equivalence is an equivalence relation among piecewise linear links.

Until now the following has happened: We began with a class of tame links which by definition contains a piecewise linear link. Piecewise linear ambient isotopy has been restricted to elementary deformations. One of the main result in this chapter, Theorem 3.35, shows that during the above restriction nothing has been lost. We need a few definitions, lemmas, and theorems before we prove Theorem 3.35. We start with

Definition 3.25 (Deformations) *Let X be a Hausdorff space. An orientation preserving homeomorphism $f : X \to X$ isotopic to the identity is called a deformation.*

Lemma 3.26 *Let $\mathcal{E}_{ab}^{c}$ be an elementary deformation applicable to a piecewise linear link, L. There exists an orientation preserving piecewise linear homeomorphism $f : S^3 \to S^3$ realizing $\mathcal{E}_{ab}^{c}$, i.e.,*

$$f(L) = \mathcal{E}_{ab}^{c} L.$$

Proof: Let L be a piecewise linear link in S^3. Suppose that the elementary deformation $\mathcal{E}_{ab}^{c}$ is applicable to L, i.e.,

$$[abc] \cap L = [ab].$$

There exists three points, V_1, V_2, and V_3, such that

$$([aV_1V_2V_3] \cup [bV_1V_2V_3]) \cap L = [ab]$$

and

$$[abc] \subset [aV_1V_2V_3] \cup [bV_1V_2V_3].$$

These points exist, because otherwise the elementary deformation, $\mathcal{E}_{ab}^{c}$, was not applicable to L. See Figure 3.5.

Denote the point in the intersection between $[ab]$ and $[V_1V_2V_3]$ by V_0. The simplex star of V_0, $\mathrm{star}(V_0)$, contains six 3-simplices, which have to undergo the following piecewise linear deformation:

$$\begin{aligned}
[V_0V_2V_3a] &\to [cV_2V_3a] \\
[V_0V_1V_2a] &\to [cV_1V_2a] \\
[V_0V_1V_3a] &\to [cV_1V_3a] \\
[V_0V_2V_3b] &\to [cV_2V_3b] \\
[V_0V_1V_2b] &\to [cV_1V_2b] \\
[V_0V_1V_3b] &\to [cV_1V_3b]
\end{aligned}$$

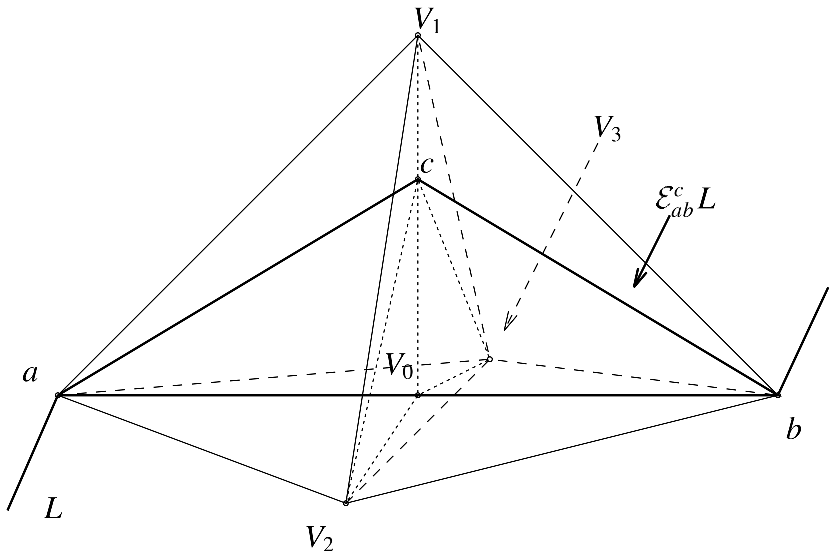

Figure 3.5: An elementary deformation of a piecewise linear link

This deformation is an orientation preserving piecewise linear homeomorphism $f : S^3 \to S^3$ which leaves fixed S^3 minus $[aV_1V_2V_3] \cup [bV_1V_2V_3]$. In other words, f realizes the elementary deformation, $\mathcal{E}^c_{ab}$. □

The following corollary shows that the name elementary deformation is well-chosen.

Corollary 3.27 (of Theorem 3.35) *An elementary deformation can be realized by a deformation.*

Proof: Lemma 3.26 gives an orientation preserving piecewise linear homeomorphism, f. In the part (1) $\Rightarrow$ (2) on page 53 in the proof of Theorem 3.35 there is constructed an isotopy from f to the identity. □

Remark 3.28 *Corollary 3.27 is also a corollary of Fisher's theorem stating that an orientation preserving homeomorphism $f : S^m \to S^m$ is isotopic to the identity, i.e., a deformation. For reference see [Fisher, 1960].*

3.2.3 The Alexander trick

The proof of Theorem 3.31, which is given below, is our completion of a sketch proof in [Burde & Zieschang, 1985] pp. 5-6. The construction of a level-preserving piecewise linear ambient isotopy proving the theorem is known as the

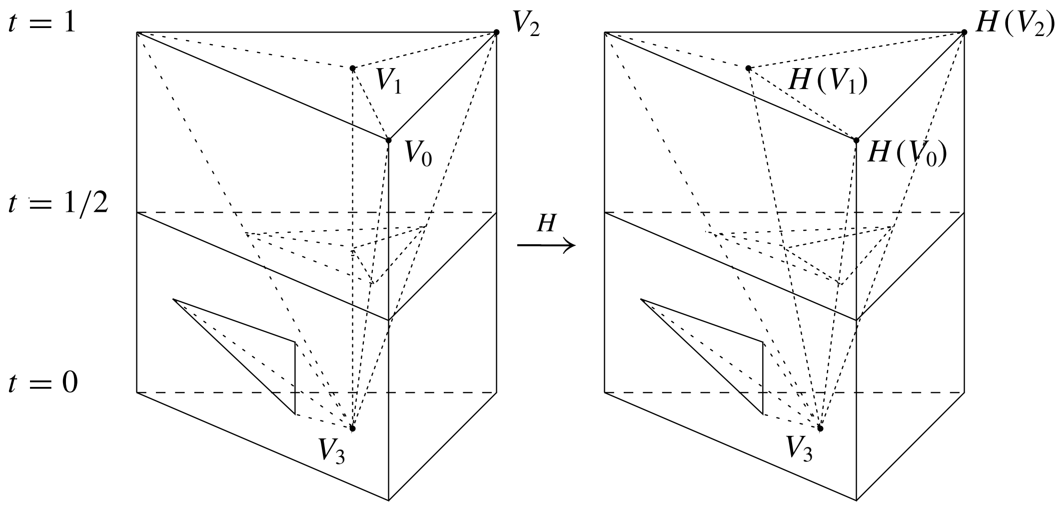

Figure 3.6: A piecewise linear ambient isotopy H in the case $n = 2$

Alexander trick (see [Burde & Zieschang, 1985] p. 6 and [Rourke & Sanderson, 1972] p. 37). First we define a combinatorial n-ball and state the piecewise linear Schönflies theorem.

Definition 3.29 (Combinatorial n-balls) *A combinatorial n-ball is a set which is piecewise linear homeomorphic to a n-simplex.*

Theorem 3.30 (The piecewise linear Schönflies theorem) *Let $\iota : S^2 \to S^3$ be a piecewise linear embedding. Then*

$$S^3 = B_1 \cup B_2, \quad \iota(S^2) = B_1 \cap B_2 = \partial B_1 = \partial B_2,$$

where $B_i, i = 1, 2$, is a combinatorial 3-ball.

Proof: A proof can be found in [Moise, 1977] pp. 122-125.

Theorem 3.31 (Alexander-Tietze) *Let B be a combinatorial n-ball and let $f : B \to B$ be a piecewise linear homeomorphism on B keeping the boundary of B, ∂B, fixed. Then f is isotopic to the identity by a piecewise linear ambient isotopy keeping the boundary fixed.*

Proof: Let B be a combinatorial n-ball and I the unit interval, $[0, 1]$. Let $f : B \to B$ be a piecewise linear homeomorphism on B keeping the boundary of B, ∂B, fixed. The desired level-preserving piecewise linear ambient isotopy $H : B \times I \to B \times I$ has to have the following restriction to the boundary of $B \times I$:

$$H(x,t)\mid_{\partial(B\times I)}= \begin{cases} (x,0) & , \text{ for } x \in B, t = 0 \\ (x,t) & , \text{ for } x \in \partial B, t \in]0,1[\\ (f(x),1) & , \text{ for } x \in B, t = 1 \end{cases}$$

Let V_{n+1} be an interior point of $B \times \{0\}$. Every point $v = (x,t) \in B \times I$ in $B \times I$ lies on a straight segment in $B \times I$ joining V_{n+1} to a variable point V on $(\partial B) \times I \cup B \times \{1\}$. See Figure 3.6. In other words, v lies on the edge $[V_{n+1}V]$. We claim that $H(x,t)\mid_{\partial(B\times I)}$ extend linearly on these edges to a piecewise linear ambient isotopy which then by construction has the desired properties.

If $V \in (\partial B) \times I$ then H is the identity on both vertices in the edge $[V_{n+1}V]$. Hence,

$$H\mid_{[V_{n+1}V]}(v) = v \text{ for } v \in [V_{n+1}V] \text{ and } V \in (\partial B) \times I.$$

The case remaining to be considered is when V lies on $B \times \{1\}$. For the n-simplex $[V_0V_1 \ldots V_n]$ in $B \times \{1\}$ consider the $(n+1)$-simplex $[V_0V_1 \ldots V_nV_{n+1}]$ in $B \times I$. See Figure 3.6. Let $V \in [V_0V_1 \ldots V_n]$ have the barycentric coordinates $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ in the n-simplex, $[V_0V_1 \ldots V_n]$, i.e.,

$$V = \sum_{i=0}^{n} \alpha_i V_i,$$

where $\alpha_0 + \cdots + \alpha_n = 1$ and $\alpha_0, \ldots, \alpha_n \geq 0$. Denote the point on the edge $[V_{n+1}V]$ in the t-level by v, i.e., $\{v\} = (B \times \{t\}) \cap [V_{n+1}V], t \in I$. The point v has the barycentric coordinates $(1-t, t)$, $t \in I$, in the 1-simplex, $[V_{n+1}V]$, i.e.,

$$v = (1-t)V_{n+1} + tV.$$

Hence,

$$v = tV + (1-t)V_{n+1} = t\sum_{i=0}^{n} \alpha_i V_i + (1-t)V_{n+1} = \sum_{i=0}^{n+1} \alpha_i' V_i.$$

That $(\alpha_0', \alpha_1', \ldots, \alpha_{n+1}')$ are barycentric coordinates in the $(n+1)$-simplex, $[V_0V_1 \ldots V_nV_{n+1}]$, follows from the calculation

$$\sum_{i=0}^{n+1} \alpha_i' = t\sum_{i=0}^{n} \alpha_i + (1-t) = t1 + (1-t) = 1$$

and the fact that $\alpha_i' \geq 0$ for all i. Therefore, the point v has the barycentric coordinates $(\alpha_0', \alpha_1', \ldots, \alpha_{n+1}')$ in the $(n+1)$-simplex, $[V_0V_1 \ldots V_nV_{n+1}]$.

Because f is a piecewise linear homeomorphism, there exists a complex $\mathcal{C}$ on B so that f on each n-simplex in $\mathcal{C}$, $[V_0V_1 \dots V_n]$, is of the form

$$f(V) = f(\sum_{i=0}^{n} \alpha_i V_i) = \sum_{i=0}^{n} \alpha_i f(V_i),$$

where $(\alpha_0, \dots, \alpha_n)$ are the barycentric coordinates for $V \in [V_0 \dots V_n]$. The edge through V_{n+1} and $f(V)$ has one intersection $H(v)$ with the t-level, $B \times \{t\}$, given by

$$H(v) = H(\sum_{i=0}^{n+1} \alpha'_i V_i) = t\sum_{i=0}^{n} \alpha_i f(V_i) + (1-t)V_{n+1} = \sum_{i=0}^{n} \alpha'_i f(V_i) + \alpha'_{n+1} V_{n+1}.$$

The restriction of H to the $(n+1)$-simplex, $[V_0V_1 \dots V_n V_{n+1}]$, has the form

$$H(\sum_{i=0}^{n+1} \alpha'_i V_i) = \sum_{i=0}^{n+1} \alpha'_i H(V_i),$$

where $H(V_i) = f(V_i), i = 1, \dots, n$, and $H(V_{n+1}) = V_{n+1}$. Whence, H is piecewise linear.

The fact that $\alpha'_{n+1} = 1 - t$ is preserved by H and that each $H(V_i)$ lies in the same level as V_i ensure that H is level-preserving. Knowing that H is level-preserving it is easy seen that H is a homeomorphism, because on the piece of $B \times \{t\}$ where H is not the identity it is simply a scaling of the homeomorphism f (see again Figure 3.6). This proves the theorem. □

3.2.4 Equivalence of equivalences

The fundamental theorem given below states that two piecewise linear links are piecewise linear ambient isotopic if and only if they are combinatorially equivalent. The next two lemmas are concerned with moving a point in a simplex and between simplices.

Lemma 3.32 *Let s^3 be a 3-simplex in the unit 3-sphere, S^3, and let P_0, P_1 be interior points in s^3. There exists a piecewise linear ambient isotopy $H : S^3 \times I \to S^3 \times I$ moving P_0 into P_1, which leaves $\overline{S^3 \backslash s^3}$ fixed.*

Proof: According to Theorem 3.31 it is sufficient to find a piecewise linear orientation preserving homeomorphism $f : s^3 \to s^3$ fulfilling

$$f(P_0) = P_1$$

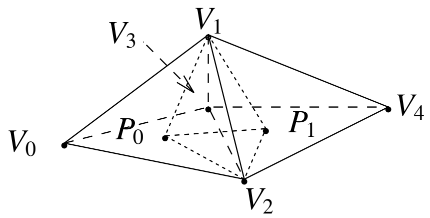

Figure 3.7: The 3-simplices used to move between neighbouring simplices

and

$$f(v) = v \text{ for } v \in \partial s^3.$$

Divide the 3-simplex, $s^3 = [V_0V_1V_2V_3]$, into the simplices $[P_0V_1V_2V_3]$, $[V_0P_0V_2V_3]$, $[V_0V_1P_0V_3]$, and $[V_0V_1V_2P_0]$. Define f on the first of the new simplices by

$$f\,|_{[P_0V_1V_2V_3]}\,(\alpha_0 P_0 + \sum_{i=1}^{3} \alpha_i V_i) = \alpha_0 P_1 + \sum_{i=1}^{3} \alpha_i V_i$$

and do the analogous on the other new simplices. This completes the proof. □

Lemma 3.33 *Let there be given two points P_0 and P_1 in the unit 3-sphere, S^3, such that P_0 is an interior point of a 3-simplex, s^3, and P_1 is an interior point of a 3-simplex, s'^3. There exist a piecewise linear ambient isotopy $H : S^3 \times I \to S^3 \times I$ moving P_0 into P_1, which leaves the closure of S^3 minus a chain of 3-simplices connecting P_0 and P_1 fixed.*

Proof: If $s^3 = s'^3$ this is simply Lemma 3.32.

Assume that $s^3 = [V_0V_1V_2V_3]$ and $s'^3 = [V_1V_2V_3V_4]$ have a common 2-subsimplex, $[V_1V_2V_3]$. See Figure 3.7. Inside $s^3 = [V_0V_1V_2V_3]$ one can, by use of Lemma 3.32, move P_0 to an interior point of the 3-simplex, $[P_0V_1V_2P_1]$, denoted by $P_{1/3}$. Analogous move $P_{1/3}$ to an interior point of $[P_0V_1V_2P_1] \cap [V_1V_2V_3V_4]$ called $P_{2/3}$. Finally, move $P_{2/3}$ into P_1 inside $[V_1V_2V_3V_4]$. Note, that all these moves leave the complement to $[V_0V_1V_2V_3] \cup [V_1V_2V_3V_4]$ fixed (cf. Lemma 3.32).

Since any two 3-simplices in the unit 3-sphere can be joined by a chain of 3-simplices the proof is complete. □

Remark 3.34 *Consider Figure 3.7. Note, that under the deformation constructed in the proof of Lemma 3.33 the 1-simplex $[V_2P_0]$ is deformed into the 1-simplex $[V_2P_1]$ and the 2-simplex $[V_1V_2P_0]$ is deformed into the 2-simplex $[V_1V_2P_1]$.*

We are now ready to state and prove the main theorem of this section.

Theorem 3.35 (Equivalence of equivalences) *Given two piecewise linear links L_0 and L_1 in S^3 the following statements are equivalent:*

(1) There is an orientation preserving piecewise linear homeomorphism $f : S^3 \to S^3$ which deforms L_0 onto L_1, i.e., $f(L_0) = L_1$.

(2) L_0 and L_1 are piecewise linear ambient isotopic.

(3) L_0 and L_1 are combinatorially equivalent.

Remark 3.36 *This proof is our completion of a short proof for knots in [Burde & Zieschang, 1985] pp. 6-7. Note, that this proof generalizes to links without changes. We have tried to structure the proof by splitting it up in three lemmas (Lemma 3.26, 3.32, 3.33) and one sublemma (Sublemma 3.37). Our main work on this proof has been to construct maps which have properties claimed by the authors in [Burde & Zieschang, 1985].*

Sublemma 3.37 *Assume property (1) in Theorem 3.35 holds. Then there is a piecewise linear ambient isotopy $H : S^3 \times I \to S^3 \times I$ given by*

$$H(x, t) = (h_t(x), t), x \in S^3, t \in I,$$

such that:

- *$h_1 f$ leaves fixed a 3-simplex, $[V_0 V_1 V_2 V_3]$, which, at most, has common 2-simplices with the simplex star of L_0 and L_1.*
- *H leaves fixed S^3 minus a chain of 3-simplices connecting $f(V_0)$ and V_0. Especially, H leaves fixed the links L_0 and L_1.*

Proof: [of sublemma] Assume property (1) in Theorem 3.35 holds, i.e., there is an orientation preserving piecewise linear homeomorphism $f : S^3 \to S^3$ which deforms L_0 onto L_1.

Let V_0 be any interior point of a 3-simplex, s_1^3. The point $f(V_0)$ is an interior point in some 3-simplex, s_2^3, in S^3, because f is a piecewise linear homeomorphism. We can assume that the simplices s_1^3 and s_2^3 both, at most, have common 2-simplices with the simplex star of L_0 and L_1. If this is not the case we can simply subdivide the original complex on S^3.

By Lemma 3.33 the points $f(V_0)$ and V_0 can be joint by a piecewise linear ambient isotopy H^0, which leaves fixed S^3 minus a chain of 3-simplices connecting $f(V_0)$ and V_0. This means that

$$h_1^0 f(V_0) = V_0$$

and that $h_1^0 f$ is an orientation preserving piecewise linear homeomorphism. The chain can be assumed to be disjoint from the two links L_0 and L_1.

Next choose a point V_1 in the interior of the simplex star of V_0, $\mathrm{star}(V_0)$, such that $h_1^0 f(V_1)$ also is in the interior of the simplex star of V_0. This can be done, because the map $h_1^0 f$ has V_0 as a fixpoint and it is continuous. By use of Lemma 3.33 the point $h_1^0 f(V_1)$ can be moved onto V_1, so that the 1-simplex, $[V_0 \; h_1^0 f(V_1)]$, according to Remark 3.34 is moved onto $[V_0 V_1]$. Hence, there is a piecewise linear ambient isotopy H^1 such that

$$h_1^1 h_1^0 f([V_0 V_1]) = [V_0 V_1]$$

and such that $h_1^1 h_1^0 f$ is an orientation preserving piecewise linear homeomorphism.

Similar arguments give a piecewise linear ambient isotopy H^2 such that

$$h_1^2 h_1^1 h_1^0 f([V_0 V_1 V_2]) = [V_0 V_1 V_2]$$

and such that $h_1^2 h_1^1 h_1^0 f$ is an orientation preserving piecewise linear homeomorphism.

Let V_3 be an interior point in one of the two 3-simplices, which have $[V_0 V_1 V_2]$ as 2-subsimplex, and so that the point $h_1^2 h_1^1 h_1^0 f(V_3)$ is in the same simplex as V_3. There exists such points because the map $h_1^2 h_1^1 h_1^0 f$ is continuous and orientation preserving. By Lemma 3.32 there exist an orientation preserving piecewise linear homeomorphism, h_1^3, moving $h_1^2 h_1^1 h_1^0 f(V_3)$ onto V_3, and by the construction in the proof of Lemma 3.32 the map $h_1^3 h_1^2 h_1^1 h_1^0 f$ leaves fixed the 3-simplex, $[V_0 V_1 V_2 V_3]$.

Since all the maps h_1^3, h_1^2, h_1^1, and h_1^0 are the identity outside the chain of 3-simplices, used to connect the points V_0 and $f(V_0)$, the piecewise linear ambient isotopy $H = H^3 H^2 H^1 H^0$ has the required properties. This ends the proof of the sublemma. $\square$

Proof: [of theorem]
(1) $\Rightarrow$ (2): Assume property (1) in Theorem 3.35 holds, i.e., there is an orientation preserving piecewise linear homeomorphism $f : S^3 \to S^3$ which deforms L_0 onto L_1. Sublemma 3.37 gives a piecewise linear ambient isotopy $H : S^3 \times I \to S^3 \times I$ such that $h_1 f$ leaves fixed a 3-simplex, $[V_0 V_1 V_2 V_3]$.

By Theorem 3.30 the complement of the 3-simplex, $[V_0 V_1 V_2 V_3]$, is a combinatorial 3-ball, B, given by

$$B = \overline{S^3 \backslash [V_0 V_1 V_2 V_3]}.$$

Let $g : S^3 \to S^3$ be given by $g = h_1^3 h_1^2 h_1^1 h_1^0$. We can apply Theorem 3.31 to $gf|_B$, because,

$$gf|_{\partial B} = id$$

and $gf|_B$ is an orientation preserving homeomorphism. Whence, $gf|_B$ is piecewise linear ambient isotopic to the identity on B by a piecewise linear ambient isotopy $\tilde{H} : B \times I \to B \times I$. The piecewise linear ambient isotopy $\tilde{H}$ extents by the identity on $[V_0V_1V_2V_3] \times I$ to a piecewise linear ambient isotopy on $S^3 \times I$ satisfying

$$\begin{aligned} \tilde{h}_1(L_0) &= gf(L_0) = L_1 \text{ and} \\ \tilde{h}_0 &= id \text{ on } S^3. \end{aligned}$$

In other words, L_0 and L_1 are piecewise linear ambient isotopic.

(2) $\Rightarrow$ (1): Assume property (2) in Theorem 3.35 holds, i.e., L_0 and L_1 are piecewise linear ambient isotopic by the isotopy $H : S^3 \times I \to S^3 \times I$ given by

$$H(x, t) = (h_t(x), t), x \in S^3, t \in I.$$

Then, $f := h_1$ is an orientation preserving piecewise linear homeomorphism $f : S^3 \to S^3$ deforming L_0 onto L_1.

(1) $\Rightarrow$ (3): Assume property (1) in Theorem 3.35 holds, i.e., there is an orientation preserving piecewise linear homeomorphism $f : S^3 \to S^3$ which deforms L_0 onto L_1.

Let the map $g : S^3 \to S^3$ and the 3-simplex, $s^3 = [V_0V_1V_2V_3]$, be given as in the proof of (1) $\Rightarrow$ (2), where s^3 is chosen such that it, at most, has a common 2-simplex with the simplex star of L_0 and L_1. Hence, $(gf)^{-1}$ is an orientation preserving homeomorphism leaving fixed the 3-simplex, s^3.

Let P be a point in the interior of the 3-simplex, s^3. Consider $S^3\backslash\{P\}$ as euclidean 3-space, $\mathbb{E}^3$. Let $\mathcal{T}$ be a translation of $\mathbb{E}^3$ which moves L_1 into $s^3\backslash\{P\}$. The translation of L_1 can be obtained as a sequence of elementary deformations, $\mathcal{E}_1, \dots, \mathcal{E}_k$, used on L_1 (see Figure 3.8), i.e.,

$$L_1 \to \mathcal{E}_1 L_1 \to \cdots \to (\prod_{i=1}^{j} \mathcal{E}_i) L_1 \to \cdots \to (\prod_{i=1}^{k} \mathcal{E}_i) L_1,$$

where

$$\mathcal{T}(L_1) = (\prod_{i=1}^{k} \mathcal{E}_i) L_1.$$

This ensures by Definition 3.24 that L_1 and $\mathcal{T}(L_1)$ are combinatorially equivalent. The piecewise linear map $(gf)^{-1}$ is a homeomorphism. Therefore,

$$(gf)^{-1}(\mathcal{E}_1), \dots, (gf)^{-1}(\mathcal{E}_k)$$

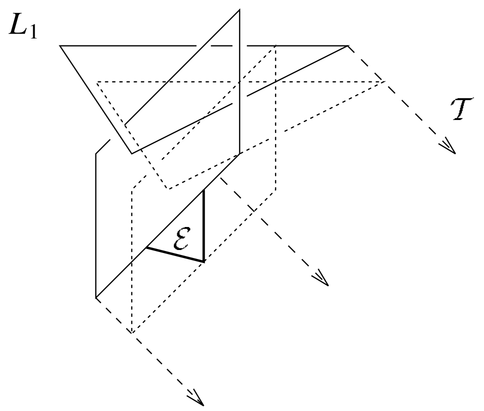

Figure 3.8: The translation of L_1

are elementary deformations. These elementary deformations used on the link L_0 give the following sequence

$$L_0 = (gf)^{-1}(L_1) \to (gf)^{-1}(\mathcal{E}_1 L_1) \to \cdots \to (gf)^{-1}((\prod_{i=1}^{j} \mathcal{E}_i) L_1) \to \cdots$$

$$\cdots \to (gf)^{-1}((\prod_{i=1}^{k} \mathcal{E}_i) L_1) = (gf)^{-1}(\mathcal{T}(L_1)).$$

Hence, L_0 and $(gf)^{-1}(\mathcal{T}(L_1))$ are combinatorially equivalent.

The map $(gf)^{-1}$ leaves fixed the 3-simplex, s^3, containing the link $\mathcal{T}(L_1)$ such that

$$\mathcal{T}(L_1) = (gf)^{-1}(\mathcal{T}(L_1)).$$

We have obtained that:

- L_0 and $\mathcal{T}(L_1)$ are combinatorially equivalent.
- $\mathcal{T}(L_1)$ and L_1 are combinatorially equivalent.

An equivalence relation is transitive - whence, L_0 and L_1 are combinatorially equivalent.

(3) $\Rightarrow$ (1): Assume property (3) in Theorem 3.35 holds, i.e., L_0 and L_1 are combinatorially equivalent under a sequence of elementary deformations, $\mathcal{E}_1, \ldots, \mathcal{E}_k$. By Lemma 3.26 this sequence gives k orientation preserving piecewise linear

homeomorphisms $f_i : S^3 \to S^3$, $1 \le i \le k$, where each $f_i : S^3 \to S^3$ realizes the elementary deformation $\mathcal{E}_i$. Finally, the map $f : S^3 \to S^3$ given by

$$f = f_k \cdots f_1,$$

is an orientation preserving piecewise linear homeomorphism which deforms L_0 onto L_1, i.e., $f(L_0) = L_1$.

We have now proved (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3). Hence, (2) $\Leftrightarrow$ (3). This completes the proof of Theorem 3.35. $\square$

3.2.5 Link projections and Reidemeister moves

The data that determine a link are often given by a projection of the link onto a plane called the projection plane. We shall start by defining which projections there are allowed.

Definition 3.38 (Multiple points) *Let* Proj *be a projection of euclidean* 3*-space,* $\mathbb{E}^3$*, into a plane* E^2 *in* $\mathbb{E}^3$ *and let* L *be a link in* $\mathbb{E}^3$*. A point* $P \in \mathrm{Proj}(L) \subset E^2$ *whose pre-image* $\mathrm{Proj}^{-1}(P)$ *contains more than one point is called a multiple point.*

Definition 3.39 (Regular projections) *Let* Proj *be a projection of euclidean* 3*-space,* $\mathbb{E}^3$*, into a plane* E^2 *in* $\mathbb{E}^3$*. The projection of a piecewise linear link,* L*, is called regular if:*

- *There are only finitely many multiple points,* $\{P_i | 1 \le i \le n\}$.
- *All multiple points are double points, i.e,* $\mathrm{Proj}^{-1}(P_i)$ *contains two points for all* i.
- *No vertex of* L *is mapped onto a double point.*

The projection of a link does not determine the link, but if at every double point in a regular projection the overcrossing line is marked, the link can be reconstructed from the projection. We shall refer to a regular projection of a link, where the overcrossing lines are marked, as a link diagram of the link. See Figure 3.9.

There are sufficiently many regular projections. In fact, if we identify the space of all projections with the unit 2-sphere we have

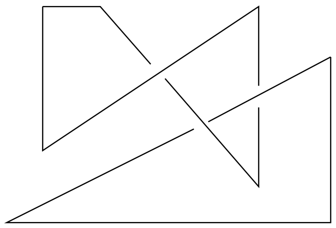

Figure 3.9: A link diagram of a link with two components (The Hopf link)

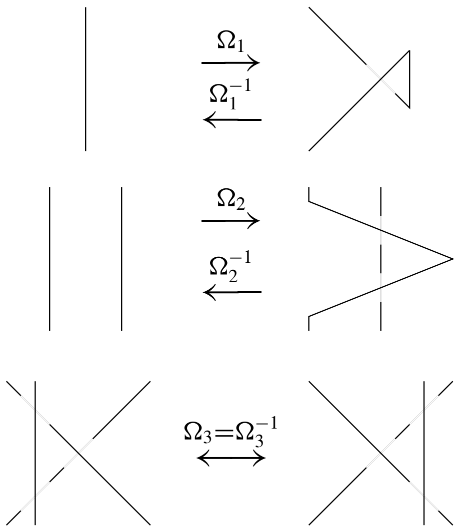

Figure 3.10: The Reidemeister moves

Theorem 3.40 *The set of all regular projections is open and dense in the space of all projections.*

Proof: See [Burde & Zieschang, 1985], page 8, 1.12 Proposition.

In the next definition it is necessary to know a new kind of moves[4], namely, the Reidemeister moves. These moves are described in Figure 3.10 and are called Reidemeister moves of type I, type II, and type III. The operations $\Omega_i^{\pm 1}$, for $i = 1, 2, 3$, effect only local changes in the link diagram.

Definition 3.41 (Equivalent link diagrams) *Two link diagrams are called equivalent, if one can be transformed into the other by a finite sequence of Reidemeister moves of type I, type II, and type III.*

[4]In Section 3.2.2 we introduced the $\mathcal{E}$-moves.

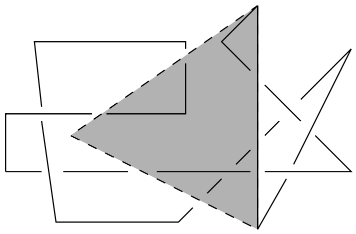

Figure 3.11:

Theorem 3.42 *Two piecewise linear links are piecewise linear ambient isotopic if and only if all their diagrams are equivalent.*

Proof: The first step in the proof will be to verify that any two regular projections Proj_0 and Proj_1 of the same piecewise linear link, L, are connected by the operations $\Omega_i^{\pm 1}$, $i = 1, 2, 3$. The space of regular projections is now identified with the unit 2-sphere, $S^2 \subset \mathbb{E}^3$. Therefore, let Proj_0 and Proj_1 be represented by two points on S^2 and choose on S^2 a polygonal path s from Proj_0 to Proj_1 transversal to the lines of singular projections on S^2. When such a line is crossed the diagram will be changed by an operation $\Omega_i^{\pm 1}$, the actual type depending on the type of singularity corresponding to the line that is crossed.

It remains to show that for a fixed projection equivalent links possess equivalent diagrams. According to Theorem 3.35 it suffices to show that an $\mathcal{E}$-move (cf. page 45) induces the operations $\Omega_i^{\pm 1}$ on the projection. By use of Figure 3.11 this is easily verified. $\square$

It will turn out in Chapter 5 that link diagrams and equivalence of these, i.e., link diagrams and Reidemeister moves, are good tools for defining and proving existence of link invariants. There is one other equivalence relation attached to link diagrams (not to links) that often is used in Chapter 5. Two link diagrams are said to be regular isotopic if one of the link diagrams can be transformed into the other link diagram by a finite sequence of Reidemeister moves of the types II and III (not type I). A motivation for the terminology regular isotopy is in the words of L.H. Kauffman in [Kauffman, 1990] pp. 448-449:

> Two immersions of the circle into the plane are said to be regularly homotopic if there is a time-parameter (t) family of immersions of the circle that restricts to one map at $t = 0$ and to the other at $t = 1$ (for t varying in the interval from 0 to 1). It is required that the family be differentiable in the variable t. This means that the shadows[5] of

[5] A regular projection where over- and under-crossings not are indicated.

the Reidemeister type II and III moves can be seen as regular homotopies, but that the shadow of the type I move cannot be a regular homotopy, since the contraction of the loop would violate differentiability. Therefore any regular isotopy of link diagrams projects to a regular homotopy of the underlying plane curve(s).

Chapter 4

Braids and links

We need to introduce braids and closed braids. The reason for this is that closed braids can be seen as link diagrams, whereby, the notion of closed braids turns the topological problem of classifying link types in euclidean 3-space into an algebraic problem involving the family of braid groups. Our introduction (in Section 4.1 and Section 4.2) to geometric braids, the braid group, and closed braids is based on the book [Hansen, 1989].

4.1 Geometric braids and the braid group

We identify the euclidean 3-space, $\mathbb{E}^3$, with the 3-dimensional real number space, $\mathbb{R}^3$, by choosing a coordinate system with coordinates (x, y, z), in which the Z-axis points vertically downwards. See Figure 4.1.

Consider two horizontal parallel planes in $\mathbb{E}^3$ with constant z-coordinates z_0 resp. z_1, where $z_0 < z_1$. We call the plane $z = z_0$ the upper plane and the plane $z = z_1$ the lower plane. Mark n different points $P_1, \ldots, P_n$ on a line in the upper plane and project them orthogonally onto the lower plane to the points $P'_1, \ldots, P'_n$.

Definition 4.1 (Geometric braids) *A geometric braid on n strings (or a n-braid), β, is a system of n embedded arcs, $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$, in $\mathbb{E}^3$, where the ith arc, $\mathcal{A}_i$, connects the point P_i on the upper plane to the point $P_{\tau(i)}$ on the lower plane for some permutation τ of the integers in the set $\{1, \ldots, n\}$, such that:*

- *Each arc, $\mathcal{A}_i$, intersects each intermediate parallel plane between the upper and the lower plane exactly once.*
- *The arcs $\mathcal{A}_1, \ldots, \mathcal{A}_n$ intersect each intermediate parallel plane between the upper and the lower plane in exactly n different points.*

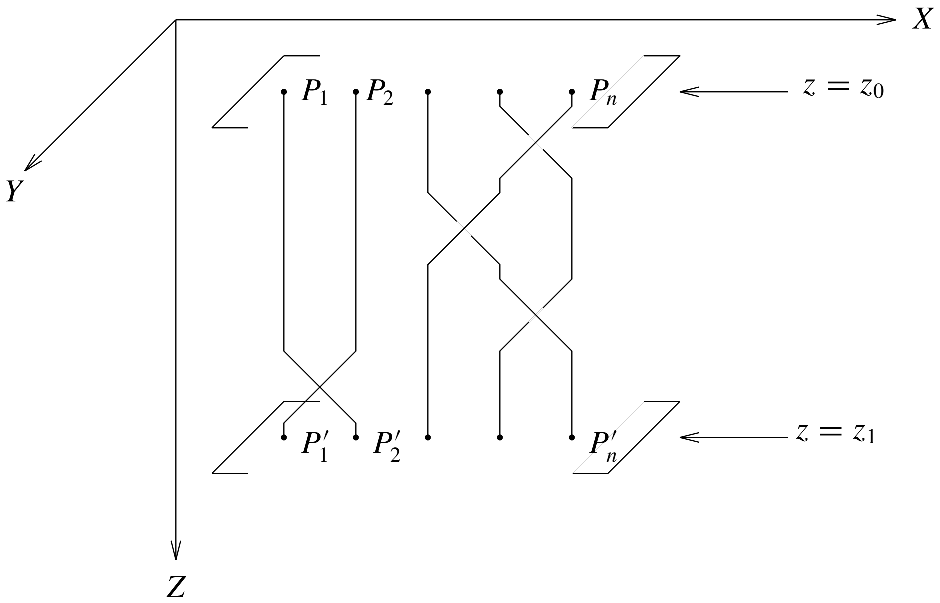

Figure 4.1: A geometric braid

The permutation τ *is called the permutation of the braid. The arc* $\mathcal{A}_i$ *is called the* i*th string in the braid.*

We think of an arc in $\mathbb{E}^3$ as the image of an embedding $\mathcal{A}_i$ of the unit interval $[0, 1]$ into $\mathbb{E}^3$. We use the same notation for the arc and the corresponding embedding. As indicated, we think of a braid as hanging downwards.

To make braids into a useful concept we need to define equivalence of braids.

Definition 4.2 (Equivalence of braids) *Two* n*-braids* $\mathcal{A}^0 = \{\mathcal{A}_1^0, \ldots, \mathcal{A}_n^0\}$ *and* $\mathcal{A}^1 = \{\mathcal{A}_1^1, \ldots, \mathcal{A}_n^1\}$ *are called equivalent (or homotopic), if there is a homotopy through geometric braids from* $\mathcal{A}^0$ *to* $\mathcal{A}^1$*, i.e., if there exist* n *continuous maps*

$$F_i : [0, 1] \times [0, 1] \to \mathbb{E}^3, \quad 1 \le i \le n,$$

such that

$$\left.\begin{aligned} F_i(t, 0) &= \mathcal{A}_i^0(t) \\ F_i(t, 1) &= \mathcal{A}_i^1(t) \end{aligned}\right\} \quad 0 \le t \le 1, \quad 0 \le i \le n,$$

$$\left.\begin{aligned} F_i(0, s) &= P_i \\ F_i(1, s) &= P'_{\tau(i)} \end{aligned}\right\} \quad 0 \le s \le 1, \quad 0 \le i \le n,$$

and such that if we define $\mathcal{A}_i^s : [0, 1] \to \mathbb{E}^3$ *by* $\mathcal{A}_i^s = F_i(t, s)$*, then* $\mathcal{A}^s = \{\mathcal{A}_1^s, \ldots, \mathcal{A}_n^s\}$ *is a geometric* n*-braid for each* $0 \le s \le 1$.

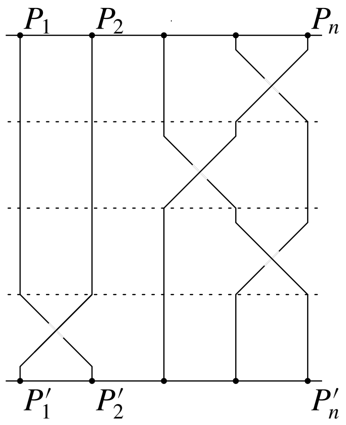

Figure 4.2: Projection of braid

Remark 4.3 *A consequence of Definition 4.2 is that equivalent braids have the same permutation τ.*

We shall not distinguish in notation between the equivalence class of a braid and the braid itself.

After a homotopy, we can (and will) assume that a braid, β, is piecewise linear and that we get transversal crossings of the arcs if we project the braid orthogonally onto the plane in $\mathbb{E}^3$ containing the points $P_1, \dots, P_n, P_1', \dots, P_n'$, i.e., the projection must be a regular projection (cf. Definition 3.39). By this regular projection we get a standard picture of a braid. Figure 4.2 is the standard picture of the braid, β, shown in Figure 4.1. Also note that (up to equivalence) we may assume that the crossings of strings occur on different levels, and that over- and under-crossings of strings must be indicated[1].

In Figure 4.2, we have indicated that a braid can be resolved into elementary braids, in which all strings except a neighbouring pair of strings go straight from the top to the bottom and the neighbouring pair just interchange.

For $1 \le i \le n-1$, we denote by σ_i that elementary geometric n-braid, in which the ith string just overcrosses the $(i+1)$st string once and all other strings go straight from the top to the bottom. See Figure 4.3.

Let $B(n)$ denote the set of all equivalence classes of geometric n-braids. It turns out that this set can be equipped with a natural group structure, which we now define.

Let β_1 and β_2 be geometric n-braids. Then we define the product (composition) of β_1 and β_2, denoted $\beta_1 \cdot \beta_2$, as follows: First hang the braid β_2 under the braid

[1]As over- and under-crossings are indicated in link diagrams cf. Section 3.2.5.

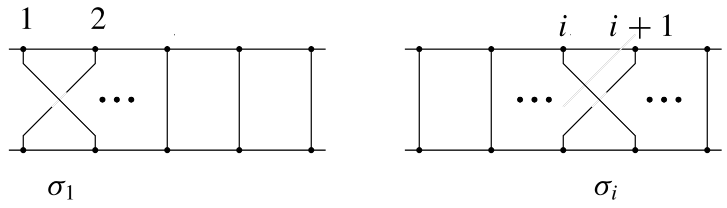

Figure 4.3: Elementary braids

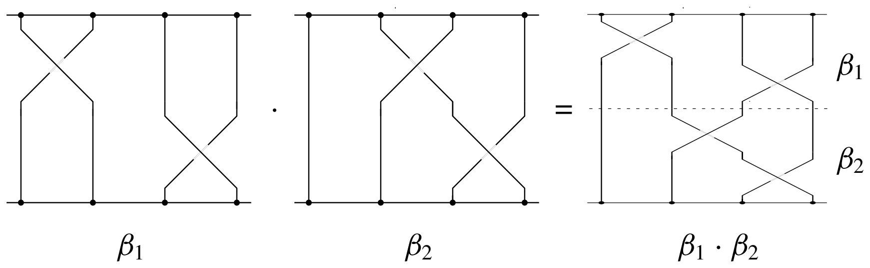

Figure 4.4: Product of braids

β_1 by attaching the lower plane of β_1 to the upper plane of β_2. Then remove the plane along which the braids β_1 and β_2 are attached to each other. Now squeeze the resulting system of arcs (strings) to lie between the plane $z = z_0$ and $z = z_1$, and we have the braid $\beta_1 \cdot \beta_2$. See Figure 4.4.

If we substitute homotopic braids β_1' and β_2' for β_1 and β_2 respectively, then it easy to prove that the product braids $\beta_1' \cdot \beta_2'$ and $\beta_1 \cdot \beta_2$ are homotopic. Thus the product is well-defined on equivalence classes of n-braids and induces a product in $B(n)$.

The trivial n-braid, ϵ, is the n-braid in which all strings just go straight from the upper plane to the lower plane. The projection of ϵ is shown in Figure 4.5. It is easily seen that the equivalence class of ϵ is a neutral element for the product in $B(n)$.

The inverse braid β^{-1} of the braid β is defined as the mirror image of β with

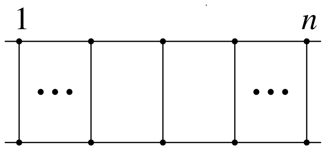

Figure 4.5: The trivial n-braid ϵ

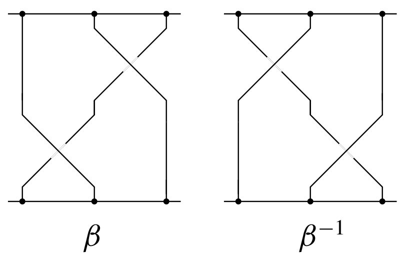

Figure 4.6: A braid and its inverse

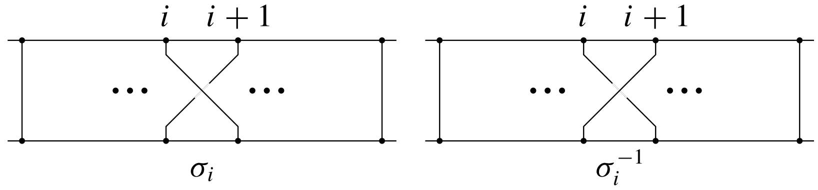

Figure 4.7: An elementary braid and its inverse

respect to a horizontal plane between the upper plane and the lower plane. The projections of β and β^{-1} are shown in Figure 4.6.

It is not difficult to see that the equivalence class of β^{-1} is well-defined from that of β and that the product braids $\beta \cdot \beta^{-1}$ and $\beta^{-1} \cdot \beta$ are homotopic to the trivial braid. Therefore, the equivalence class of β^{-1} is the inverse element in $B(n)$ to the equivalence class of β.

For the elementary n-braid, σ_i, $1 \leq i \leq n-1$, we get the inverse braid, σ_i^{-1}, by substituting in the standard projection the overcrossing of the $(i+1)$st string by the ith string with an undercrossing. See Figure 4.7.

It is now easy to prove that with the above product, neutral element, and inverse elements, the set of equivalence classes of geometric n-braids, $B(n)$, is actually a group. This group is called the (Artin) braid group of braids on n strings.

As already indicated in Figure 4.2, it is intuitively clear that the equivalence class of any n-braid can be written as a product of elementary n-braids, σ_i, $1 \leq i \leq n-1$, and their inverses. In other words, the elementary n-braids, $\sigma_1, \dots, \sigma_{n-1}$, generate the group $B(n)$.

We shall now look for relations among the elements in $B(n)$. First we notice that if the condition $|i-j| \geq 2$ for $1 \leq i, j \leq n-1$ is fulfilled, then - since the pair consisting of the ith and the $(i+1)$st string does not interfere with the pair consisting of the jth and the $(j+1)$st string - we get the following relation

$$\sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i \quad \text{for} \quad |i-j| \geq 2, 1 \leq i, j \leq n-1.$$

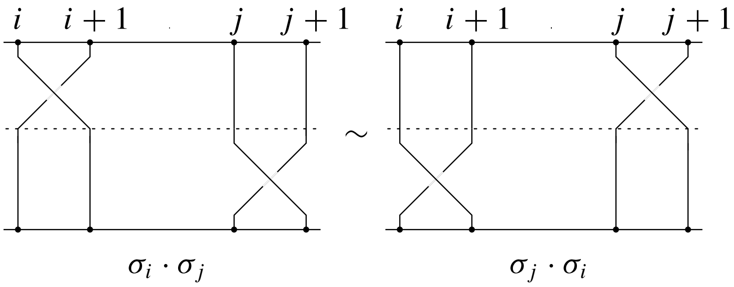

Figure 4.8: A relation among elementary braids

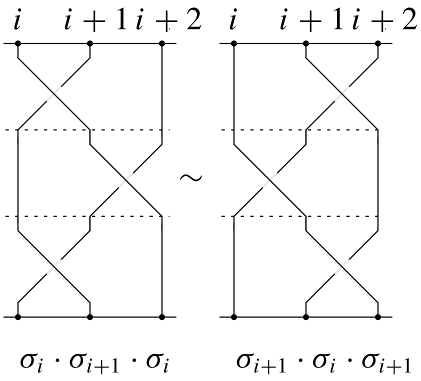

Figure 4.9: Another relation among elementary braids

Figure 4.8 illustrates the above relation.

As illustrated in Figure 4.9, we also have the following relation in $B(n)$

$$\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} \quad \text{for} \quad 1 \le i \le n-2.$$

The two relations above generate all relations among the elements in $B(n)$. We state the result in

Theorem 4.4 (Artin's presentation of the braid groups) *The group $B(n)$ of geometric braids on n strings admits a presentation with generators: $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ and defining relations:*

$$\begin{aligned} \sigma_i \cdot \sigma_j &= \sigma_j \cdot \sigma_i \quad \text{for} \quad |i-j| \ge 2, 1 \le i, j \le n-1. && (4.1) \\ \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i &= \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} \quad \text{for} \quad 1 \le i \le n-2. && (4.2) \end{aligned}$$

Proof: It is highly nontrivial to prove Theorem 4.4. For a proof see [Hansen, 1989] pp. 18-24 or see [Artin, 1925], where the original proof can be found.

Remark 4.5 *From now on we will omit the dot which denote the composition in $B(n)$, i.e., we write $\sigma_i \sigma_j$ for $\sigma_i \cdot \sigma_j$.*

4.2 The connection between links and braids

In this section we shall present the theorem of A.A. Markov on combinatorial equivalence of closed braids. We need some preparations.

Consider a piecewise linear link, L, in euclidean 3-space, $\mathbb{E}^3$. Let l be an arbitrary, but henceforth fixed, line in $\mathbb{E}^3$, which does not meet L. We shall refer to l as the axis for L.

Definition 4.6 (General position of links) *A piecewise linear link, L, is said to be in general position with respect to the axis l if none of its edges are coplaner with l.*

As the following lemma shows, we only have to worry about links in general position.

Lemma 4.7 *Every piecewise linear link is combinatorially equivalent to some piecewise linear link in general position.*

Proof: If $[ab]$ is an edge on a piecewise linear link, L, which is coplaner with the axis l, then we can choose a point c in $\mathbb{E}^3$ outside this plane and change the edge $[ab]$ of L to $[ac] \cup [cb]$ by an elementary deformation, $\mathcal{E}^c_{ab}$. If the point c is chosen sufficiently close to the edge $[ab]$, then the elementary deformation, $\mathcal{E}^c_{ab}$, will be an applicable elementary deformation. This procedure can be carried out for all the finitely many edges on L, which are coplaner with l, without interfering with the rest of the link. Then one obtain a new piecewise linear link, L', which is in general position. By Definition 3.24 L and L' are combinatorially equivalent. This proves the lemma. $\square$

We suppose from now on that the piecewise linear link, L, is in general position with respect to the axis l. Suppose furthermore, that L is oriented, i.e., each component of the link has a preferred direction attached. An orientation of the link assigns an orientation to every edge of the link. Fix also an orientation of the axis, l.

The orientation of the piecewise linear link, L, and the axis, l, enable us to divide the edges on L in positive and negative edges. We call an edge $[ab]$ on L positive, when the half plane, M, determined by l and a point on the edge $[ab]$ turns on a right hand screw around l, when the point on $[ab]$ moves along the edge in the positive direction determined by the orientation of L. Call the edge, $[ab]$,

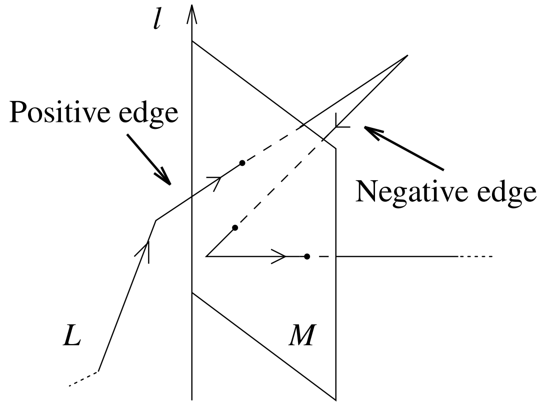

Figure 4.10: Determine signs on edges of an oriented link

negative, when the half plane M turns on a left hand screw around l, when $[ab]$ is traversed in the positive direction. See Figure 4.10.

Definition 4.8 (Closed braids) *A closed braid in* $\mathbb{E}^3$ *is an oriented piecewise linear link, L, which admits an oriented axis, l, with respect to which all edges of L are positive.*

Let β be a braid on n-strings. Now let l be a suitable axis placed behind the braid, i.e., placed behind the projection plane in $\mathbb{E}^3$ and close the braid around the axis by identifying the initial points and the end points of the braid. Then we obtain a closed braid from the braid β. Conversely, if we take a closed braid and cut it open along a half plane determined by an (oriented) axis l, and then fold it out in the projection plane, we get back the braid β.

The closed braids are exactly the piecewise linear links in $\mathbb{E}^3$, which arise by closing (open) braids as above[2]. The closed braid obtained from the open braid $\beta \in B(n)$ will be denoted by $\hat{\beta}$.

Theorem 4.9 (J.W. Alexander) *Every piecewise linear link in* $\mathbb{E}^3$ *is combinatorially equivalent to a closed braid.*

Proof: See [Hansen, 1989] pp. 55-58. The main idea of the proof is:

- Orientate the piecewise linear link and stick an axis through it.

[2]For reference see [Hansen, 1989], page 57.

- Deform the link so that it everywhere runs the same way around the axis. This can be done like this: If a small piece of the oriented link runs the wrong way, it can be changed to a curve going almost ones the right way around the axis. And then big pieces of the oriented link running the wrong way around the axis are cut up in smaller ones.

- Now every half-plane with the axis as edge will intersect the link the same number of times. Rotating a half-plane the right way around the axis gives you exactly a braid that closes up to a deformation of the original piecewise linear link.

□

We can now state the theorem of A.A. Markov that (as mentioned in the introduction of this chapter) turns the topological problem of classifying link types in $\mathbb{E}^3$ into an algebraic problem involving the family of braid groups. In the theorem we write (β, n) to specify the number of strings in a braid $\beta \in B(n)$. Furthermore, note that a braid word γ is a n-braid of the form

$$\gamma = \sigma_{i_1}^{\nu_1}\sigma_{i_2}^{\nu_2}\cdots\sigma_{i_k}^{\nu_k}, \text{ where } \nu_s \in \{\pm 1\} \text{ and } 1 \le i_s \le n-1 \text{ for } 1 \le s \le k.$$

Theorem 4.10 (A.A. Markov) *Let $\hat{\beta}$ and $\hat{\beta}'$ be two closed braids in E^3, with braid representatives (β, n) and (β', n'). Then $\hat{\beta}$ is combinatorially equivalent to $\hat{\beta}'$ if and only if there is a finite sequence of moves*

$$(\beta, n) = (\beta_0, n_0) \to (\beta_1, n_1) \to \cdots \to (\beta_{s-1}, n_{s-1}) \to (\beta_s, n_s) = (\beta', n')$$

joining (β, n) to (β', n'), such that for each $0 \le i < s$, the braid (β_{i+1}, n_{i+1}) can be obtained from its predecessor (β_i, n_i) by applying one of the following moves:

$\mathcal{M}_1$: *Replace β_i by any other braid in $B(n_i)$, which is conjugate to β_i. Set $n_{i+1} = n_i$.*

$\mathcal{M}_2$: *Replace (β_i, n_i) by $(\beta_i\sigma_{n_i}^{\pm 1}, n_i + 1)$; or if $\beta_i = \gamma\sigma_{n_i-1}^{\pm 1}$, where the braid word γ only involves the generators $\sigma_1, \ldots, \sigma_{n_i-2}$, replace (β_i, n_i) by $(\gamma, n_i - 1)$.*

Proof: For a sketch of the proof see [Hansen, 1989] pp. 64-65. A detailed proof can be found in [Birman, 1974] pp. 48-67.

Definition 4.11 (Markov moves) *The moves $\mathcal{M}_1$ and $\mathcal{M}_2$ in Theorem 4.10 are called Markov moves.*

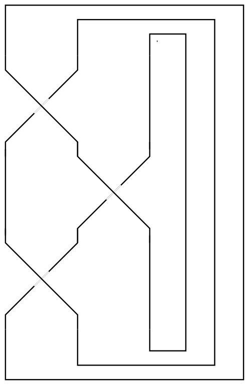

Figure 4.11: A closed braid

4.3 Closed braids and link diagrams

Consider a closed braid obtained by closing a braid as shown on Figure 4.11. Then we obtain a link diagram[3] describing the closed braid. We will now explain how the defining relations of the braid group and the Markov moves can be seen as Reidemeister moves on this link diagram of the closed braid.

The closed braids $\hat{\beta}$ and $\widehat{\beta\sigma_n^{\pm 1}}$ obtained by closing the n-braid β resp. the $(n+1)$-braids $\beta\sigma_n^{\pm 1}$ are as link diagrams connected by Reidemeister moves of type I. Using the notation of Reidemeister moves from Figure 3.10 on page 57 we have

$$\widehat{\beta\sigma_n^{\pm 1}} = \Omega_{\mp 1}(\hat{\beta}),$$

where $\Omega_{\mp 1}$ are as described[4] in Figure 5.4 on page 75.

As $(\beta, n) \to (\beta\sigma_n^{\pm 1}, n+1)$ are Markov moves of type $\mathcal{M}_2$ we have that all Markov moves of type $\mathcal{M}_2$ correspond to Reidemeister moves of type I on the link diagram.

The braid identity $\sigma_i\sigma_i^{-1} = \epsilon$ for $1 \leq i \leq n-1$ is a Reidemeister move of type II on the link diagram.

A Markov move of type $\mathcal{M}_1$, i.e., a conjugation $(\beta, n) \to (\gamma\beta\gamma^{-1}, n)$ is the identity for closed braids that corresponds to the identity $\sigma_i\sigma_i^{-1} = \epsilon$ (for $1 \leq i \leq n-1$) for braids, because we have the calculation

$$\widehat{\gamma\beta\gamma^{-1}} = \widehat{\beta\gamma^{-1}\gamma} = \widehat{\beta\epsilon} = \hat{\beta}.$$

Hence, a Markov move of type $\mathcal{M}_1$ is a Reidemeister move of type II on the link diagram.

---

[3]Cf. Section 3.2.5.

[4]We shall introduce this notation in Section 5.1.

The braid relation

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i \leq n-2$$

is a Reidemeister move of type III on the link diagram. See Figure 4.9 on page 66.

Finally, the braid relation

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2, 1 \leq i, j \leq n-1$$

is a deformation of the link diagram that does not change the link diagram's crossings. This deformation can be seen as four Reidemeister moves of type II. See Figure 4.8 on page 66.

Chapter 5

Link invariants

Considering two links as equivalent if they are ambient isotopic forces the link invariants to be invariant under ambient isotopy. According to Theorem 3.35 and Theorem 3.42 a link invariant is also an invariant defined on link diagrams invariant under the three types of Reidemeister moves. However, the other equivalence relation called regular isotopy attached to link diagrams (not to links) will also be considered. Our interest in regular isotopy is caused by the fact that regular isotopy of link diagrams can be lifted to ambient isotopy for oriented links using the writhe or self-writhe of a link diagram[1].

Consider the situation that you are given two link diagrams and you are to tell whether they are regular projections of the same link or not, i.e., whether they are ambient isotopic or not. Until now, we know from Theorem 3.42 that this question is equivalent to the question: Is there a finite sequence of Reidemeister moves connecting the two link diagrams? Unfortunately, there are no known ways to determine such a sequence of Reidemeister moves. There are not even a notation on link diagrams on which such a hypothetical algorithm could work[2].

In the above situation one would start to count the number of components in the two link diagrams to see if they are alike. Because, if the two link diagrams have distinct numbers of components they are obviously not ambient isotopic. One could also compute the linking numbers between the components of the two link diagrams to see if this could distinct between them. In other words, we need link invariants because they make it possible to distinguish between links.

In the latest ten years there has been found many new link invariants that are good at distinguishing links. All these link invariants are polynomials attached to link diagrams. We shall use this chapter to introduce these polynomial in-

[1]See Lemma 5.12 and Remark 5.13.

[2]At this point there still might be hope using closed braids, but we will not investigate this. Here we would like to mention that it is J.S. Birman's stated dream to classify links via braids (cf. [Birman, 1991], page 52).

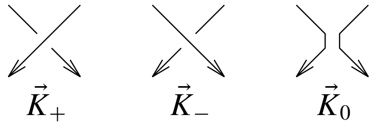

Figure 5.1: A notation for oriented link diagrams

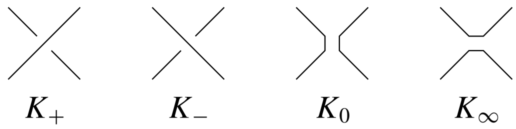

Figure 5.2: A notation for unoriented link diagrams

variants and show how they are gathered in two families, in the sense that there is one polynomial in each family that generates all the others. These two generating polynomials are known as the Kauffman polynomial and the HOMFLY polynomial.

This chapter is based on works of L.H. Kauffman in the article [Kauffman, 1990] and the book [Kauffman, 1991]. L.H. Kauffman uses hand drawings but we have chosen to use a notation similar to the notation in [Brandt, Lickorish & Millett, 1986] and [Hansen, 1990]. The next section introduces our notation.

5.1 Notation

To distinguish between oriented link diagrams and unoriented link diagrams we will use a vector arrow to indicate that a link diagram is oriented. Hence, the link diagram K is unoriented and the link diagram $\vec{K}$ is oriented. This leads to the notation $\mathcal{D}$ for the space of all link diagrams and the notation $\vec{\mathcal{D}}$ for the space of all oriented link diagrams.

Working with a link invariant defined by all the links's link diagrams, we often have to consider what happens to the invariant when, for example, one of its crossings is changed. For this we need an easy notation for link diagrams that only differ inside a neighbourhood. This notation is given in the figures 5.1, 5.2, 5.3, and 5.4.

Definition 5.1 *Let K be a link in $\mathbb{E}^3$ that consists of two links, K_1 and K_2, one on each side of a plane in $\mathbb{E}^3$. Then the link K resp. K's link diagrams are called split link resp. split link diagrams and we write $K = K_1 \sqcup K_2$.*

Remark 5.2 *All links that are ambient isotopic to split links are also called split links.*

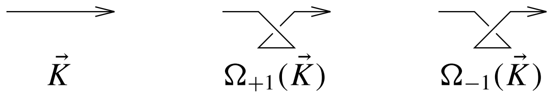

Figure 5.3: A notation for oriented link diagrams

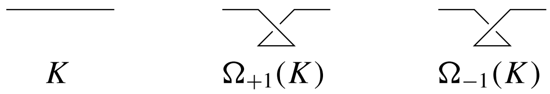

Figure 5.4: A notation for unoriented link diagrams

Let $\mathcal{R}$ be a ring and $S : \mathcal{D} \to \mathcal{R}$ or $S : \vec{\mathcal{D}} \to \mathcal{R}$ be an attachment of an element of the ring $\mathcal{R}$ to each unoriented or oriented link diagram. That S is an invariant of regular resp. ambient isotopy means that S attaches the same element in the ring $\mathcal{R}$ to all regular resp. ambient isotopic link diagrams. As many link invariants are polynomial invariants the above ring, $\mathcal{R}$, often is the set of polynomials in the commuting variables $z_1, \dots, z_n$ with integer coefficients. The set of these polynomial is denoted by $\mathbb{Z}[z_1, \dots, z_n]$. Using both a variable and its inverse one obtain Laurent polynomials. For example $\mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ is the set of Laurent polynomials over the integers with commuting independent variables α and z.

5.2 Linking number

An orientation of a link in $\mathbb{E}^3$ induces an orientation on its crossings in its link diagrams. This orientation labels each of these oriented crossings with a sign plus (+) or minus (-) as shown on Figure 5.5. Note, that in the positive crossing both curves make a right hand screw around the other curve and in the negative crossing the curves make left hand screws with each other.

The notation on links that only differ inside a neighbourhood given by the figures 5.1, 5.3, and 5.4 (not Figure 5.2) simply indicate the sign of the crossing that has been added to the link diagram. Note, that the crossings in Figure 5.4 will have the indicated signs for any orientation of the link.

Definition 5.3 (Linking number) *Let $\vec{c}_1$ and $\vec{c}_2$, $\vec{c}_1 \neq \vec{c}_2$, be two components of an oriented link diagram. The linking number,* $\mathrm{link}(\vec{c}_1, \vec{c}_2)$*, is defined by the formula*

$$\mathrm{link}(\vec{c}_1, \vec{c}_2) = \frac{1}{2} \sum_{p \in \vec{c}_1 \cap \vec{c}_2} \delta(p),$$

where $\vec{c}_1 \cap \vec{c}_2$ denotes the set of crossings of $\vec{c}_1$ with $\vec{c}_2$ and $\delta(p)$ denotes the sign of the crossing, i.e., ± 1.

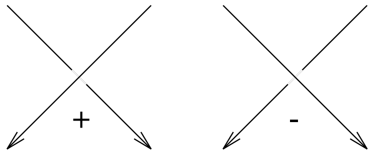

Figure 5.5: Orientation of crossings

Theorem 5.4 *The linking number of two different components in an oriented link diagram given by Definition 5.3 is an invariant of ambient isotopy.*

Proof: If a Reidemeister move does not involve both of the two components it has no effect on the linking number. Hence, all Reidemeister move of type I has no effect on the linking number. A Reidemeister move of type II adds one positive and one negative crossing to the linking number, leaving the linking number invariant. A type III move simply permutes the addends in the linking number which has no effect on the linking number $\square$

The linking number(s) between the components of a link is apparently the only nontrivial link invariant[3] that is known how to be defined and calculated directly in $\mathbb{E}^3$, especially without use of link diagrams. That the linking number can be defined on links in $\mathbb{E}^3$ follows from [Flanders, 1963] p. 80.

Given the number of components and the linking number as tools one can not tell the difference between the link diagram on Figure 5.6 and the link diagram $\vec{\bigcirc} \sqcup \vec{\bigcirc}$, where $\vec{\bigcirc}$ is an oriented canonical unknot diagram[4]. Therefore, for two components in a link to be separable it is necessary, but not sufficient, that they have linking number zero.

For closed braids (cf. Definition 4.8 page 68) the linking number is given by

Theorem 5.5 (Linking number of two components in a closed braid) *Let $\vec{c}_1$ and $\vec{c}_2$, $\vec{c}_1 \neq \vec{c}_2$, be two components of the closed n-braid*

$$\hat{\beta} = (\sigma_{i_1}^{\nu_1}\sigma_{i_2}^{\nu_2}\cdots\sigma_{i_k}^{\nu_k})\widehat{}\,,\text{ where } \nu_s \in \{\pm 1\} \text{ and } 1 \leq i_s \leq n-1 \text{ for } 1 \leq s \leq k.$$

The linking number between $\vec{c}_1$ and $\vec{c}_2$, $\operatorname{link}(\vec{c}_1, \vec{c}_2)$, is given by the formula

$$\operatorname{link}(\vec{c}_1, \vec{c}_2) = -\frac{1}{2}\sum_{s \in J} \nu_s,$$

[3]The number of components in a link is here thought of as a trivial link invariant.

[4]The notation $\bigcirc$ is standard and we have chosen to use it even though it violate our attention to make it clear that link diagrams are piecewise linear. In other words, we will not use a notation like $\lozenge$ or $\triangle$.

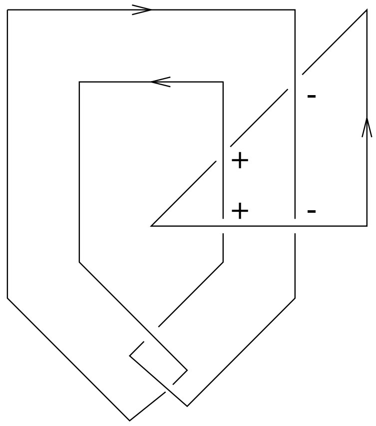

Figure 5.6: Two inseparable components with linking number zero

where the set J contains the index s if the elementary braid $\sigma_{i_s}^{\nu_s}$ in $\hat{\beta}$ acts on both the components, $\vec{c}_1$ and $\vec{c}_2$.

Remark 5.6 *It is not necessary to prove that the linking number is well-defined once more but we want to give a proof using closed braids. We do this to demonstrate the analogies between Reidemeister moves on one side and on the other side Markov moves and braid group equivalences mentioned in Section 4.3.*

Proof: To ensure that the linking number is well-defined it is enough to show that it is invariant under Markov moves and braid group equivalences.

The braid group equivalence $\sigma_i \sigma_i^{-1} = \epsilon$ has no effect on the linking number. The braid group relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$, only move two crossings in the closed braid , hence, it has no influence on the linking number. The braid group relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ permutes the addends in the sum leaving the linking number invariant. It is now proven that the linking number is well-defined on braids, as the linking number between two strings.

For the first kind of Markov move $\mathcal{M}_1$ (conjugation) it is sufficient to conjugate the original braid with an elementary braid $\sigma_i^{\pm 1}$.

There are two cases. In the first case neither $\sigma_i^{\pm 1}$ over the original braid nor $\sigma_i^{\mp 1}$ under the original braid act on both components of the link, so they have no effect on the linking number. In the second case both the generator and its inverse act on the two components of the link, but together they have the effect $\mp 1/2 \pm 1/2 = 0$ on the linking number.

The second kind of Markov move, $\mathcal{M}_2$, adds or removes an elementary braid $\sigma_n^{\pm 1}$ which only act on one component of the link. Hence, a $\mathcal{M}_2$ move has no effect on the linking number.

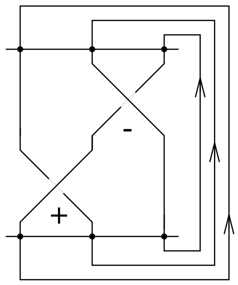

Figure 5.7: The closed 3-braid $\widehat{\sigma_2\sigma_1^{-1}}$

The linking number is now proven to be invariant under Markov moves and braid group equivalences. It remains to check that it is consistent with Definition 5.3. An easy way to see this is to look at a picture of a closed n-braid and note that the braid are running downwards. See Figure 5.7. Hence, σ_i is labeled (-) and σ_i^{-1} is labeled (+) for all $1 \leq i \leq n-1$. $\square$

The reason why there is a minus in the formula in Theorem 5.5 is that we follow the notation on the elementary braids of [Birman, 1974] and [Hansen, 1989], therefore, the elementary braid σ_i will be considered as a crossing labeled (-). If we had used the notation in [Kauffman, 1991], where σ_i correspond to our σ_i^{-1}, the minus in Theorem 5.5 could have been omitted.

5.2.1 The linking number of a half twist

Consider the 3-braid $\xi = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$ shown on Figure 5.8. The closed braid $\hat{\xi}$ has two components. The second string is a component, $\vec{c_1}$, by itself and the first and third string together make up one component, $\vec{c}_2$. By use of Theorem 5.5 we find that

$$\text{link}(\vec{c}_1, \vec{c}_2) = -\frac{1}{2}(-1-1) = 1.$$

Take a paper strip and let it hang downwards. Twist the strip half a twist through a right hand screw moving downwards. Now the center curve of the strip (drawn on both sides) together with the boundary curves describe the braid ξ. The closed strip is a Möbius strip (with a half twist). In this sense we shall think of ξ as a half twist.

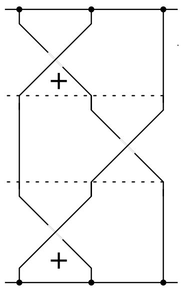

Figure 5.8: $\xi = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$

5.3 Writhe and Self-writhe

In order to construct polynomial invariants for links we need to introduce the writhe and the self-writhe of oriented resp. unoriented link diagrams. The writhe and the self-writhe are only invariant under regular isotopy[5], hence, they can not be defined on links.

Definition 5.7 (Writhe of oriented link diagrams) *The writhe* $\mathrm{w} : \vec{\mathcal{D}} \to \mathbb{Z}$ *of oriented link diagrams is defined by the formula*

$$\mathrm{w}(\vec{K}) = \sum_p \delta(p), \quad \text{for } \vec{K} \in \vec{\mathcal{D}},$$

where the sum is over the set of crossings p *in the oriented link diagram* $\vec{K}$ *and* $\delta(p)$ *denotes the sign of the crossing* p*, i.e.,* ± 1.

The writhe of an oriented link diagram is also known in the literature as the twisting number of an oriented link diagram (for references see [Hansen, 1990] p. 22, [Kauffman, 1991] p. 19, and [Kauffman, 1990] p. 420). We are going to define the twisting number[6] as the twist of a closed strip as in Section 5.2.1, hence, we only will refer to the writhe as the writhe.

Theorem 5.8 *The writhe* $\mathrm{w} : \vec{\mathcal{D}} \to \mathbb{Z}$ *of oriented link diagrams is invariant under regular isotopy.*

Proof: A Reidemeister move of type II adds or removes two crossings from the sum. One of these two crossings is labeled plus and the other is labeled

[5]See page 58.
[6]Definition 6.6, page 99.

minus. Hence, a Reidemeister move of type II has no effect on the writhe. A Reidemeister move of type III can only permute the addends in the sum leaving the sum invariant. □

The self-writhe is the amount of the writhe that comes from the self-crossings in a link diagram. This cause

Definition 5.9 (Self-crossings) *A crossing of one component with itself in a link diagram is called a self-crossing.*

Definition 5.10 (Self-writhe of unoriented link diagrams) *The self-writhe* $\mathrm{s} : \mathcal{D} \to \mathbb{Z}$ *of unoriented link diagrams is defined by the formula*

$$\mathrm{s}(K) = \sum_p \delta(p), \quad \text{for } K \in \mathcal{D},$$

where the sum is over the set of self-crossings p in the unoriented link diagram K and $\delta(p)$ denotes the sign (± 1) of the self-crossing p for some orientation of of the link diagram.

Theorem 5.11 *The self-writhe* $\mathrm{s} : \mathcal{D} \to \mathbb{Z}$ *of unoriented link diagrams is well-defined and invariant under regular isotopy.*

Proof: The signs of self-crossings are not changed by a change of the orientation of the self crossing component (or any other component). Hence, the self-writhe $\mathrm{s}(K)$ of an unoriented link diagram K, $\mathrm{s}(K)$, is independent of the orientation used to define it.

Invariants under Reidemeister moves of type II and III follows from the proof of Theorem 5.8 noticing that now it is not even sure that the Reidemeister moves have effects adding up to zero on the sum. □

A Reidemeister move of type I adds or removes a self-crossing in a link diagram. This makes it impossible for both the writhe and the self-writhe to be invariant under ambient isotopy.

5.4 The Kauffman polynomial

We shall in this section define the Kauffman polynomial which is an invariant of ambient isotopy. The Kauffman polynomial is given as a lift of another polynomial which only is an invariant of regular isotopy. This other polynomial is called the L-polynomial. The lift, or normalization, of maps invariant under regular isotopy to maps invariant under ambient isotopy, is contained in

Lemma 5.12 (Lift of isotopy) *Let $\mathcal{R}$ be a ring, and α an invertible element of $\mathcal{R}$. Let $R : \vec{\mathcal{D}} \to \mathcal{R}$ be a map invariant under regular isotopy for oriented link diagrams satisfying*

$$R[\Omega_+(\vec{K})] = \alpha R[\vec{K}] \text{ and}$$
$$R[\Omega_-(\vec{K})] = \alpha^{-1} R[\vec{K}], \text{ for } \vec{K} \in \vec{\mathcal{D}}.$$

Then the map $S : \vec{\mathcal{D}} \to \mathcal{R}$ given by

$$S[\vec{K}] = \alpha^{-\mathrm{w}(\vec{K})} R[\vec{K}], \text{ for } \vec{K} \in \mathcal{D}$$

is an invariant of regular isotopy for oriented link diagrams, where $\mathrm{w} : \vec{\mathcal{D}} \to \mathbb{Z}$ *is the writhe.*

Remark 5.13 *In Lemma 5.12 the writhe of an oriented link diagram can be exchanged with the self-writhe of an unoriented link diagram. This will also be proved in the following proof.*

Proof: Let $\vec{K}$ be any oriented link diagram. Since the writhe of $\vec{K}$, $\mathrm{w}(\vec{K})$, is an invariant of regular isotopy it follows that also $\alpha^{-\mathrm{w}(\vec{K})}$ is an invariant of regular isotopy. Hence, the map $S[\vec{K}]$, being the product of two regular invariants, itself is invariant under regular isotopy.

It remains to check for invariance under Reidemeister moves of type I. Since the writhe fulfills the equations

$$\mathrm{w}(\Omega_\pm(\vec{K})) = \pm 1 + \mathrm{w}(\vec{K}),$$

invariance under Reidemeister moves of type I follows from the calculations

$$\begin{aligned} S[\Omega_\pm(\vec{K})] &= \alpha^{-\mathrm{w}(\Omega_\pm(\vec{K}))} R[\Omega_\pm(\vec{K})] \\ &= \alpha^{-(\pm 1 + \mathrm{w}(\vec{K}))} \alpha^{\pm 1} R[\vec{K}] \\ &= \alpha^{-\mathrm{w}(\vec{K})} R[\vec{K}] \\ &= S[\vec{K}]. \end{aligned}$$

This ends the proof of the lemma.

To prove Remark 5.13 it is only necessary to observe that the self-writhe , s, is, just like the writhe, invariant under regular isotopy and that the self-writhe (analogous to the writhe) fulfills the equations

$$\mathrm{s}(\Omega_\pm(K)) = \pm 1 + \mathrm{s}(K), \text{ for } K \in \mathcal{D},$$

since the crossings made by the Reidemeister moves $\Omega_{\pm 1}$ are self-crossings. This ends the proof of the remark. □

Remark 5.14 *Lemma 5.12 is also valid if the map $R : \vec{\mathcal{D}} \to \mathcal{R}$ is independent of the link diagrams orientation. Hence, Lemma 5.12 is also applicable to maps defined on unoriented link diagrams. Using the self-writhe for the normalization it is possible to obtain an invariant (of ambient isotopy) for unoriented links if one can find a map invariant under regular isotopy for unoriented link diagrams.*

Lemma 5.12 was introduced by Kauffman to lift the polynomial given by the next theorem to a polynomial invariant under ambient isotopy.

Theorem 5.15 (L-polynomial) *Let the map $L : \mathcal{D} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ be given by the axioms:*

$(L1)$*: If K and K' are regular isotopic, then $L[K] = L[K']$.*

$(L2)$*: $L[\bigcirc] = 1$, where $\bigcirc$ is the canonical unoriented unknot diagram.*

$(L3)$*: $L[K_+] + L[K_-] = z(L[K_0] + L[K_\infty])$.*

$(L4)$*: $L[\Omega_{\pm 1}(K)] = \alpha^{\pm 1} L[K]$.*

The map L is uniquely well-defined and gives a Laurent polynomial, $L[K]$, for every unoriented link diagram, K. The polynomial $L[K]$ is called the L-polynomial.

Proof: See [Kauffman, 1990] pp. 452-465.

The Dubrovnik polynomial $D : \mathcal{D} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ has the same axioms as the L-polynomial expect that instead of (L3) it has the axiom

$(D3)$: $D[K_+] - D[K_-] = z(D[K_0] - D[K_\infty])$.

Since the Dubrovnik polynomial and the L-polynomial are isomorphic[7] and the parametrization of the L-polynomial is the most natural for our purposes we will only use the L-polynomial from now on.

Definition 5.16 (Kauffman polynomial) *The Kauffman polynomial $F : \vec{\mathcal{D}} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ is given by the equation*

$$F[\vec{K}](\alpha, z) = \alpha^{-\mathrm{w}(\vec{K})} L[K](\alpha, z), \quad \text{for } \vec{K} \in \vec{\mathcal{D}},$$

where $L : \mathcal{D} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ is defined by forgetting the orientation of $\vec{K}$ and $\mathrm{w} : \vec{\mathcal{D}} \to \mathbb{Z}$ is the writhe.

[7] For reference see [Kauffman, 1990] p. 466.

Theorem 5.17 *The Kauffman polynomial* $F : \vec{\mathcal{D}} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ *is an invariant of ambient isotopy for oriented links.*

Proof: According to Remark 5.14 the lemma Lemma 5.12 is applicable to maps invariant under regular isotopy for unoriented link diagrams with the property $(L4)$, i.e., Lemma 5.12 is applicable to L. This proves the theorem. □

5.5 Related polynomials I

This section describes some polynomials which are generated by the Kauffman polynomial and some other polynomials with close relations to the Kauffman polynomial.

5.5.1 The Q-polynomial

If the α in the definition of the L-polynomial is chosen to be the neutral element we are to obtain a polynomial invariant not only under regular isotopy but also under ambient isotopy called the Q-polynomial. R.D. Brandt, W.B.R. Lickorish, and K.C. Millett published in 1986 the article [Brandt, Lickorish & Millett, 1986] in which they state and prove the following theorem

Theorem 5.18 (Q-polynomial) *Let the map* $Q : \mathcal{D} \to \mathbb{Z}[z, z^{-1}]$ *be given by the axioms:*

$(Q1)$*: If* K *and* K' *are ambient isotopic, then* $Q[K] = Q[K']$.

$(Q2)$*:* $Q[\bigcirc] = 1$, *where* $\bigcirc$ *is the canonical unoriented unknot diagram.*

$(Q3)$*:* $Q[K_+] + Q[K_-] = z(Q[K_0] + Q[K_\infty])$.

The map Q *is uniquely well-defined and gives a Laurent polynomial,* $Q[K]$, *for every unoriented link diagram,* K. *The polynomial* $Q[K]$ *is called the* Q*-polynomial.*

The existence of the Q-polynomial can now be proved by

Theorem 5.19 *The* Q*-polynomial* $Q : \mathcal{D} \to \mathbb{Z}[z, z^{-1}]$ *is given by*

$$Q[K](z) = L[K](1, z), \quad \text{for } K \in \mathcal{D},$$

where $L[K]$ *is the* L*-polynomial.*

Proof: From Theorem 5.15 we have that the polynomial $Q'[K]$ given by

$$Q'[K](z) = L[K](1, z)$$

fulfills the two axioms $(Q2)$ and $(Q3)$. Using the lift $\alpha^{-\mathrm{w}(\vec{K})}$, that lifted the L-polynomial to the Kauffman polynomial, on the Q'-polynomial we by Lemma 5.12 obtain that

$$Q''[\vec{K}] = 1^{-\mathrm{w}(\vec{K})} Q'[K] = Q'(K)$$

is invariant under ambient isotopy. As the writhe not is used to define $Q'' = Q'$ there is no need for an orientation of the link diagram. In other words, $Q'[K]$ also fulfills axiom $(Q1)$. $\square$

The above proof implies that the Q-polynomial is contained in both the Kauffman polynomial and in the L-polynomial.

5.5.2 The bracket polynomial and its normalization

We start by defining the bracket polynomial and use this to define the normalized bracket polynomial. This normalization uses the writhe. The bracket polynomial is an invariant of regular isotopy and the normalized bracket polynomial is an invariant of ambient isotopy. The bracket polynomial is given by

Theorem 5.20 (Bracket polynomial) *Let the map $\langle\ \rangle : \mathcal{D} \to \mathbb{Z}[A, A^{-1}]$ be given by the axioms:*

$\langle 1 \rangle$: If K and K' are regular isotopic, then $\langle K \rangle = \langle K' \rangle$.

$\langle 2 \rangle$: $\langle \bigcirc \rangle = 1$, where $\bigcirc$ is the canonical unoriented unknot diagram.

$\langle 3A \rangle$: $\langle K_+ \rangle = A\langle K_0 \rangle + A^{-1}\langle K_\infty \rangle$.

$\langle 3B \rangle$: $\langle K_- \rangle = A\langle K_\infty \rangle + A^{-1}\langle K_0 \rangle$.

$\langle 4 \rangle$: $\langle K \sqcup \bigcirc \rangle = -(A^2 + A^{-2})\langle K \rangle$.

The map $\langle\ \rangle$ is uniquely well-defined and gives a Laurent polynomial, $\langle K \rangle$, for every unoriented link diagram, K. The polynomial $\langle K \rangle$ is called the bracket polynomial.

Remark 5.21 *As the axiom $\langle 3B \rangle$ simply corresponds to a 90 degree turn of the link diagram in the axiom $\langle 3A \rangle$ (see Figure 5.2 page 74) these two axioms are equivalent.*

A direct proof of Theorem 5.20 is contained in Definition 2.1. together with Proposition 2.5. in [Kauffman, 1987] page 396 resp. page 398. The existence of the bracket polynomial can now be proved by

Theorem 5.22 *The bracket polynomial* $\langle\ \rangle : \mathcal{D} \to \mathbb{Z}[A, A^{-1}]$ *is given by*

$$\langle K\rangle(A) = L[K](-A^3, A + A^{-1}), \text{ for } K \in \mathcal{D},$$

where $L[K]$ *is the* L*-polynomial.*

Proof: From Theorem 5.15 we have that the Laurent polynomial $\langle\ \rangle' : \mathcal{D} \to \mathbb{Z}[A, A^{-1}]$ given by

$$\langle K\rangle'(A) = L[K](-A^3, A + A^{-1})$$

fulfills the axioms:

$\langle 1\rangle'$: If K and K' are regular isotopic, then $\langle K\rangle' = \langle K'\rangle'$.
$\langle 2\rangle'$: $\langle \bigcirc \rangle' = 1$, where $\bigcirc$ is the canonical unoriented unknot diagram.
$\langle 3\rangle'$: $\langle K_+\rangle' + \langle K_-\rangle' = (A + A^{-1})(\langle K_0\rangle' + \langle K_\infty\rangle')$.
$\langle 4\rangle'$: $\langle \Omega_{\pm 1}(K)\rangle' = (-A^3)^{\pm 1}\langle K\rangle'$.

The first two axioms for $\langle\ \rangle'$ are identical to the axioms for the bracket polynomial $\langle\ \rangle$, i.e., $\langle 1\rangle$ and $\langle 2\rangle$ are fulfilled by $\langle\ \rangle'$. The axiom $\langle 3\rangle'$ for the map $\langle\ \rangle'$ is apparently weaker than the two equivalent axioms $\langle 3A\rangle$ and $\langle 3B\rangle$ for the bracket polynomial, but they are actually all equivalent as we shall prove now.

Let $K \in \mathcal{D}$ be an unoriented link diagram. Then there exists a Laurent polynomial in the variable A denoted Pol_K such that

$$\langle K_+\rangle' = A\langle K_0\rangle' + A^{-1}\langle K_\infty\rangle' + \mathrm{Pol}_K\,.$$

A 90 degree turn of the link diagram K now gives the identity

$$\langle K_-\rangle' = A\langle K_\infty\rangle' + A^{-1}\langle K_0\rangle' + \mathrm{Pol}_K\,.$$

Addition of these two identities gives

$$\langle K_+\rangle' + \langle K_-\rangle' = (A + A^{-1})(\langle K_0\rangle' + \langle K_\infty\rangle') + 2\,\mathrm{Pol}_K,$$

which together with axiom $\langle 3\rangle'$ imply that Pol_K is equal to zero for every unoriented link diagram K. Hence, the here defined map $\langle\ \rangle'$ fulfills the two equivalent axioms $\langle 3A\rangle$ and $\langle 3B\rangle$ for the bracket polynomial.

It is well known that the bracket polynomial fulfills the two equations in $\langle 4\rangle'$. This is e.g. stated in [Kauffman, 1987], p. 398, in Proposition 2.5.[8]. We will now prove that $\langle 4\rangle'$ and $\langle 4\rangle$ in fact are equivalent when you assume $\langle 1\rangle$, $\langle 2\rangle$, and $\langle 3\rangle$.

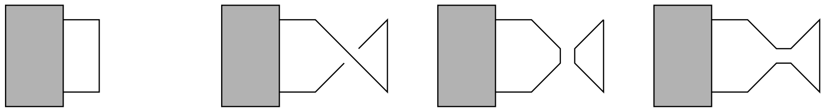

Figure 5.9:

Let K, K_+, K_0, and K_∞ be given as in Figure 5.9. Note, that $K_0 = K \sqcup \bigcirc$ and $K_\infty = K$. From $\langle 4 \rangle'$ we know that

$$\langle K_+ \rangle' = (-A^3)^{+1} \langle K \rangle'$$

and we obtain

$$\begin{aligned} -A^3 \langle K \rangle' &= \langle K_+ \rangle' \\ &= A \langle K_0 \rangle' + A^{-1} \langle K_\infty \rangle' \\ &= A \langle K \sqcup \bigcirc \rangle' + A^{-1} \langle K \rangle' \\ &\Updownarrow \\ \langle K \sqcup \bigcirc \rangle' &= -A^2 \langle K \rangle' - A^{-2} \langle K \rangle'. \end{aligned}$$

This completes the proof. □

The bracket polynomial is an invariant of regular isotopy and fulfills $\langle 4 \rangle'$ according to the above proof. Hence, the regular invariance of the bracket polynomial can by Lemma 5.12 be lifted to ambient invariance of the normalized bracket polynomial. This both explains the following definition and proves the below Theorem 5.24.

Definition 5.23 (Normalized bracket polynomial) *The normalized bracket polynomial $\mathcal{L} : \vec{\mathcal{D}} \to \mathbb{Z}[A, A^{-1}]$ is given by the equation*

$$\mathcal{L}[\vec{K}](A) = (-A^3)^{-\mathrm{w}(\vec{K})} \langle K \rangle (A), \quad \text{for } \vec{K} \in \vec{\mathcal{D}},$$

where the bracket polynomial, $\langle\ \rangle : \mathcal{D} \to \mathbb{Z}[A, A^{-1}]$, is defined by forgetting the orientation of the link diagram $\vec{K}$ and $\mathrm{w} : \vec{\mathcal{D}} \to \mathbb{Z}$ is the writhe.

Theorem 5.24 *The normalized bracket polynomial $\mathcal{L} : \vec{\mathcal{D}} \to \mathbb{Z}[A, A^{-1}]$ is an invariant of ambient isotopy for oriented links.*

[8]Unfortunately, there are two diagrams of $\Omega_{-1}(K)$ instead of one of $\Omega_{-1}(K)$ and one of $\Omega_{+1}(K)$ in this reference.

The relation between the normalized bracket polynomial, $\mathcal{L}$, and the Kauffman polynomial, F, is

$$\begin{aligned}\mathcal{L}[\vec{K}](A) &= (-A^3)^{-\mathrm{w}(\vec{K})} L[K](-A^3, A + A^{-1}) \\ &= F[\vec{K}](-A^3, A + A^{-1}).\end{aligned}$$

5.5.3 The original Jones polynomial

V.F.R. Jones found in Spring 1984 the later denoted Jones polynomial, which is an invariant of ambient isotopy for oriented link diagrams. To be able to prove that the Jones polynomial is given by a change of variable in the normalized bracket polynomial we first have to describe the normalized bracket polynomial in an axiomatic way analogous to e.g. Theorem 5.15 page 82.

Theorem 5.25 (Normalized bracket polynomial) *Let the map $\mathcal{L} : \vec{\mathcal{D}} \to \mathbb{Z}[A, A^{-1}]$ be given by the axioms:*

($\mathcal{L}1$): If $\vec{K}$ and $\vec{K}'$ are ambient isotopic, then $\mathcal{L}[\vec{K}] = \mathcal{L}[\vec{K}']$.

($\mathcal{L}2$): $\mathcal{L}[\vec{\bigcirc}] = 1$, where $\vec{\bigcirc}$ is an oriented canonical unknot diagram.

($\mathcal{L}3$): $A^4\mathcal{L}[\vec{K}_+] - A^{-4}\mathcal{L}[\vec{K}_-] = (A^{-2} - A^2)\mathcal{L}[\vec{K}_0]$.

The map $\mathcal{L}$ is uniquely well-defined and gives a Laurent polynomial, $\mathcal{L}[\vec{K}]$, for every oriented link diagram, $\vec{K}$. The polynomial $\mathcal{L}[\vec{K}]$ is called the normalized bracket polynomial.

Proof: Let $\mathcal{L}' : \vec{\mathcal{D}} \to \mathbb{Z}[A, A^{-1}]$ be given by the equation

$$\mathcal{L}'[\vec{K}](A) = (-A^3)^{-\mathrm{w}(\vec{K})}\langle K\rangle(A), \quad \text{for } \vec{K} \in \vec{\mathcal{D}},$$

where the bracket polynomial, $\langle\ \rangle : \mathcal{D} \to \mathbb{Z}[A, A^{-1}]$, is defined by forgetting the orientation of the link diagrams. By Theorem 5.24 the map $\mathcal{L}'$ fulfills the axiom ($\mathcal{L}1$). The second axiom for $\mathcal{L}$ is also valid for $\mathcal{L}'$ since

$$\mathcal{L}'[\vec{\bigcirc}] = (-A^3)^{\pm 0}\langle\bigcirc\rangle = 1,$$

where $\pm$ indicate the two possible orientations of the canonical unknot diagram.

Since $\mathcal{L}$ and $\mathcal{L}'$ needs oriented link diagrams we have to omit the link diagram K_∞, that is used in the third axiom for the bracket polynomial. From the axioms $\langle 3A\rangle$ and $\langle 3B\rangle$ for the bracket polynomial we have

$$\begin{aligned} A\langle K_+\rangle &= A^2\langle K_0\rangle + \langle K_\infty\rangle \quad \text{and} \\ A^{-1}\langle K_-\rangle &= A^{-2}\langle K_0\rangle + \langle K_\infty\rangle. \end{aligned}$$

Subtraction now gives

$$A\langle K_+\rangle - A^{-1}\langle K_-\rangle = (A^2 - A^{-2})\langle K_0\rangle.$$

Thinking of normalization we multiply with $\alpha^{-w} = (-A^3)^{-w}$, where $w = \mathrm{w}(\vec{K}_0)$. This gives

$$A(-A^3)^{-w}\langle K_+\rangle - A^{-1}(-A^3)^{-w}\langle K_-\rangle = (A^2 - A^{-2})(-A^3)^{-w}\langle K_0\rangle$$

or equivalent

$$\begin{aligned}&- A^4(-A^3)^{-(w+1)}\langle K_+\rangle + A^{-4}(-A^3)^{-(w-1)}\langle K_-\rangle \\ &= (A^2 - A^{-2})(-A^3)^{-w}\langle K_0\rangle.\end{aligned}$$

As $w = \mathrm{w}(\vec{K}_0)$, $w + 1 = \mathrm{w}(\vec{K}_+)$, and $w - 1 = \mathrm{w}(\vec{K}_-)$ we finally have the third axiom for $\mathcal{L}$ as

$$-A^4\mathcal{L}[\vec{K}_+] + A^{-4}\mathcal{L}[\vec{K}_-] = (A^2 - A^{-2})\mathcal{L}[\vec{K}_0].$$

By the construction of lift of isotopy in Lemma 5.12 the last axiom for the bracket polynomial does not give any demands on the normalization of the bracket polynomial. □

The Jones polynomial is now simply given by a formal change of variable, $A = t^{-1/4}$, in the normalized bracket polynomial. Setting $V[\vec{K}](t) = \mathcal{L}[\vec{K}](t^{-1/4})$ we obtain

Theorem 5.26 (Jones polynomial) *Let the map $V : \vec{\mathcal{D}} \to \mathbb{Z}[\sqrt{t}, \sqrt{t}^{-1}]$ be given by the axioms:*

($V1$): If $\vec{K}$ and $\vec{K}'$ are ambient isotopic, then $V[\vec{K}] = V[\vec{K}']$.

($V2$): $V[\vec{\bigcirc}] = 1$, where $\vec{\bigcirc}$ is an oriented canonical unknot diagram.

($V3$): $t^{-1}V[\vec{K}_+] - tV[\vec{K}_-] = (\sqrt{t} - \frac{1}{\sqrt{t}})V[\vec{K}_0]$.

The map V is uniquely well-defined and gives a Laurent polynomial in the variable $\sqrt{t}$, denoted by $V[\vec{K}]$, for every oriented link diagram, $\vec{K}$. The polynomial $V[\vec{K}]$ is called the Jones polynomial.

5.5.4 A family of polynomials I

We have in the above sections described six polynomials which are invariant under regular or ambient isotopy for oriented or unoriented link diagrams. These polynomials are all closely related to the most general of them: The Kauffman polynomial. The relations in the family are repeated in the following diagram, where all the polynomials in the first row are invariant under ambient isotopy and the polynomials in the second row are invariant under regular isotopy (The polynomial given by $L[K](1, z)$ is also invariant under ambient isotopy).

$$\begin{array}{ccccccc} V[\vec{K}](t) & \overset{i_0}{\leftrightarrow} & \mathcal{L}[\vec{K}](A) & \overset{i_1}{\hookrightarrow} & F[\vec{K}](\alpha, z) & \overset{i_2}{\hookleftarrow} & Q[K](z) \\ & & \uparrow \phi_1 & & \uparrow \phi_2 & & \updownarrow \phi_3 \\ & & \langle K \rangle (A) & \overset{i_1}{\hookrightarrow} & L[K](\alpha, z) & \overset{i_2}{\hookleftarrow} & L[K](1, z), \end{array}$$

where the isomorphism i_0 and the inclusions i_j, $j = 1, 2$, are determined by

$$\begin{aligned} i_0 : \quad & A = t^{-1/4} \\ i_1 : \; & \begin{cases} \alpha = -A^3 \\ z = A + A^{-1} \end{cases} \\ i_2 : \; & \begin{cases} \alpha = 1 \\ z = z \end{cases} \end{aligned}$$

and the lifts ϕ_j, $j = 1, 2, 3$, are multiplication with resp.

$$(-A^3)^{-\mathrm{w}(\vec{K})}, \quad (\alpha)^{-\mathrm{w}(\vec{K})} \;, \quad \text{and} \;\; 1.$$

The history of the family is briefly:

- V.F.R. Jones discovered the Jones polynomial in Spring 1984. For this achievement V.F.R. Jones received the Fields medal at the World Congress for Mathematicians in Kyoto, August 1990.

- In the year 1986 R.D. Brandt, W.B.R. Lickorish, and K.C. Millett published the article [Brandt, Lickorish & Millett, 1986] in which the Q-polynomial was described for the first time.

- L.H. Kauffman's article [Kauffman, 1987] describes a state model for the Jones polynomial, whereby, the construction of bracket polynomial and their normalization gives a simple proof of the existence of the Jones polynomial. Furthermore, it gives a simple proof of the reversing property of the Jones polynomial[9].

- In the year 1990 the Kauffman polynomial was introduced in the article [Kauffman, 1990] and, hereby, there was found a link invariant that generalizes both the Jones polynomial and the Q-polynomial.

5.5.5 The U-polynomial

There is one more polynomial link invariant called the U-polynomial[10] that has a close relation to the Kauffman polynomial. This polynomial has the property to be an invariant of ambient isotopy for unoriented link diagrams common with the Q-polynomial. The U-polynomial is given by

Theorem 5.27 (The U-polynomial) *The U-polynomial $U : \mathcal{D} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ is given by the equation*

$$U[K](\alpha, z) = (-A^3)^{-\mathrm{s}(K)} L[K](\alpha, z), \quad \text{for } K \in \mathcal{D},$$

is an invariant of ambient isotopy for unoriented link diagrams, where $L : \mathcal{D} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ is given in Theorem 5.15 and $\mathrm{s} : \mathcal{D} \to \mathbb{Z}$ is the self-writhe.

Proof: This unoriented normalization is described in Remark 5.14. □

The unoriented normalization can also be applied to the bracket polynomial but that would just give a restriction of the U-polynomial.

5.6 The HOMFLY polynomial

The HOMFLY polynomial has its name from the first letter in the surnames of its discovers: Hoste, Ocneanu, Millet, Freyd, Lickorish, and Yetter. The HOMFLY polynomial was according to [Hansen, 1990] p. 14 and [Kauffman, 1991] p. 52 also, independent of the above, discovered by J.H. Przytycki and P. Traczyk.

[9] We shall return to the reversing property of the Jones polynomial and give this short proof in Section 6.3.

[10] For reference see [Kauffman, 1990] p. 428.

Theorem 5.28 (The HOMFLY polynomial) *Let the map $P : \vec{\mathcal{D}} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ be given by the axioms:*

$(P1)$*: If $\vec{K}$ and $\vec{K}'$ are ambient isotopic, then $P[\vec{K}] = P[\vec{K}']$.*

$(P2)$*: $P[\vec{\bigcirc}] = 1$, where $\vec{\bigcirc}$ is an oriented canonical unknot diagram.*

$(P3)$*: $\alpha P[\vec{K}_+] - \alpha^{-1} P[\vec{K}_-] = z P[\vec{K}_0]$.*

The map P is uniquely well-defined and gives a Laurent polynomial, $P[\vec{K}]$, for every oriented link diagram, $\vec{K}$. The polynomial $P[\vec{K}]$ is called the HOMFLY polynomial.

Proof: See [Freyd et al., 1985].

In order to prove that the HOMFLY polynomial, like the Kauffman polynomial, is the normalization of a polynomial invariant under regular isotopy, we start by defining the candidate in

Definition 5.29 (H-polynomial) *The H-polynomial $H : \vec{\mathcal{D}} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ is given by the equation*

$$P[\vec{K}](\alpha, z) = \alpha^{-\mathrm{w}(\vec{K})} H[\vec{K}](\alpha, z), \quad \text{for } \vec{K} \in \vec{\mathcal{D}},$$

where $P : \vec{\mathcal{D}} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ is the HOMFLY polynomial and $\mathrm{w} : \vec{\mathcal{D}} \to \mathbb{Z}$ is the writhe.

Theorem 5.30 (H-polynomial) *Let the map $H : \vec{\mathcal{D}} \to \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ be given by the axioms:*

$(H1)$*: If $\vec{K}$ and $\vec{K}'$ are regular isotopic, then $H[\vec{K}] = H[\vec{K}']$.*

$(H2)$*: $H[\vec{\bigcirc}] = 1$, where $\vec{\bigcirc}$ an oriented canonical unknot diagram*

$(H3)$*: $H[\vec{K}_+] - H[\vec{K}_-] = z H[\vec{K}_0]$.*

$(H4)$*: $H[\Omega_{\pm 1}(\vec{K})] = \alpha^{\pm 1} H[\vec{K}]$.*

The map H is uniquely well-defined and gives a Laurent polynomial, $H[\vec{K}]$, for every oriented link diagram, $\vec{K}$. The polynomial $H[\vec{K}]$ is called the H-polynomial and it is identical to the map defined in Definition 5.29.

Proof: Let $H' : \vec{\mathcal{D}} \rightarrow \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$ be given as in Definition 5.29. By construction H' fulfills the axioms $(H1)$ and $(H4)$.

The calculations

$$\begin{aligned} H'[\vec{\bigcirc}] &= \alpha^{+\,\mathrm{w}(\vec{\bigcirc})} P[\vec{\bigcirc}] \\ &= \alpha^{\pm 0} P[\vec{\bigcirc}] \\ &= 1, \end{aligned}$$

where $\pm$ indicates the two possible orientations of the canonical unknot diagram, proves that H' fulfills axiom $(H2)$.

The third axiom is also valid since

$$\begin{gathered} \alpha P[\vec{K}_+] - \alpha^{-1} P[\vec{K}_-] = z P[K_0] \\ \Updownarrow \\ \alpha \alpha^{\mathrm{w}(\vec{K}_0)} P[\vec{K}_+] - \alpha^{-1} \alpha^{\mathrm{w}(\vec{K}_0)} P[\vec{K}_-] = z \alpha^{\mathrm{w}(\vec{K}_0)} P[K_0] \\ \Updownarrow \\ \alpha^{\mathrm{w}(\vec{K}_+)} P[\vec{K}_+] - \alpha^{\mathrm{w}(\vec{K}_-)} P[\vec{K}_-] = z \alpha^{\mathrm{w}(\vec{K}_0)} P[K_0] \\ \Updownarrow \\ H'[\vec{K}_+] - H'[\vec{K}_-] = z H'[K_0]. \end{gathered}$$

This completes the proof. □

5.7 Related polynomials II

By analogy with Section 5.5 this section describes briefly some polynomials which are related to the HOMFLY polynomial.

5.7.1 The Alexander polynomial

The first polynomial link invariant was found by J.W. Alexander about 1928. The axiomatic description of the Alexander polynomial due to J.H. Conway was first published in 1969[11]. The Alexander polynomial has in the parametrization[12] used in e.g. [Kauffman, 1991] page 174 exactly the same relation to the HOMFLY polynomial as the Q-polynomial has to the Kauffman polynomial, i.e., the Alexander polynomial is obtained from the HOMFLY polynomial by replacing α with the neutral element. This proves the following

[11] For reference see [Hansen, 1990] p. 12.

[12] Instead of the variable z used in Theorem 5.31 the variable $(t^{-1/2} - t^{1/2})$ often is used.

Theorem 5.31 (The Alexander polynomial) *Let the map $\nabla : \vec{\mathcal{D}} \to \mathbb{Z}[z, z^{-1}]$ be given by the axioms:*

(∇1)*: If $\vec{K}$ and $\vec{K}'$ are ambient isotopic, then $\nabla[\vec{K}] = \nabla[\vec{K}']$.*

(∇2)*: $\nabla[\vec{\bigcirc}] = 1$, where $\vec{\bigcirc}$ is an oriented canonical unknot diagram.*

(∇3)*: $\nabla[\vec{K}_+] - \nabla[\vec{K}_-] = z\nabla[\vec{K}_0]$.*

The map ∇ is uniquely well-defined and gives a Laurent polynomial, $\nabla[\vec{K}]$, for every oriented link diagram, $\vec{K}$. The polynomial $\nabla[\vec{K}]$ is called the Alexander polynomial.

5.7.2 The bracket and normalized bracket polynomials

From the axioms for the HOMFLY polynomial page 5.28 and the axioms for the normalized bracket polynomial page 5.25 it follows immediately that these polynomials are connected by

Theorem 5.32 *The normalized bracket polynomial $\mathcal{L} : \vec{\mathcal{D}} \to \mathbb{Z}[A, A^{-1}]$ is given by*

$$\mathcal{L}[\vec{K}](A) = P[\vec{K}](A^4, A^{-2} - A^2), \text{ for } K \in \vec{\mathcal{D}},$$

where $P[\vec{K}]$ is the HOMFLY polynomial.

This leads to

Theorem 5.33 *The bracket polynomial $\langle\ \rangle : \mathcal{D} \to \mathbb{Z}[A, A^{-1}]$ is given by*

$$\langle K\rangle(A) = (-A)^{-\mathrm{w}(\vec{K})} H[\vec{K}](A^4, A^{-2} - A^2), \text{ for } K \in \mathcal{D},$$

where $H[\vec{K}]$ is the H-polynomial and $\mathrm{w} : \vec{\mathcal{D}} \to \mathbb{Z}$ is the writhe for any orientation of the unoriented link diagram K.

Proof: Let $\vec{K} \in \vec{\mathcal{D}}$ be an oriented link diagram and K be the unoriented link diagram obtained from the oriented link diagram $\vec{K}$ by forgetting its orientation. The direct calculation

$$\begin{aligned}\langle K\rangle(A) &= (-A^3)^{\mathrm{w}(\vec{K})}\mathcal{L}[\vec{K}](A)\\ &= (-A^3)^{\mathrm{w}(\vec{K})}P[\vec{K}](A^4, A^{-2} - A^2)\\ &= (-A^3)^{\mathrm{w}(\vec{K})}(A^4)^{-\mathrm{w}(\vec{K})}H[\vec{K}](A^4, A^{-2} - A^2)\\ &= (-A)^{-\mathrm{w}(\vec{K})}H[\vec{K}](A^4, A^{-2} - A^2)\end{aligned}$$

completes the proof. □

5.7.3 A family of polynomials II

We have in the above sections described six polynomials which are invariant under regular or ambient isotopy for oriented or unoriented link diagrams. These polynomials are all closely related to the most general of them: The HOMFLY polynomial. Note, that the structure of the family is analogous to the structure of the Kauffman family. The relations in the HOMFLY family are repeated in the following diagram, where all the polynomials in the first row are invariant under ambient isotopy and all the polynomials in the second row are invariant under regular isotopy (The polynomial given by $H[\vec{K}](1,z)$ is also invariant under ambient isotopy).

$$\begin{array}{ccccccc}
V[\vec{K}](t) & \overset{i_0}{\leftrightarrow} & \mathcal{L}[\vec{K}](A) & \overset{i_1}{\hookrightarrow} & P[\vec{K}](\alpha, z) & \overset{i_2}{\hookleftarrow} & \nabla[K](z) \\
 & & \uparrow \phi_1 & & \uparrow \phi_2 & & \updownarrow \phi_3 \\
 & & \langle K \rangle(A) & \overset{i_3}{\hookrightarrow} & H[\vec{K}](\alpha, z) & \overset{i_2}{\hookleftarrow} & H[\vec{K}](1, z),
\end{array}$$

where the isomorphism i_0 and the inclusions i_j, $j = 1, 2, 3$, are determined by

$$\begin{aligned}
&i_0 : \quad A = t^{-1/4} \\
&i_1 : \begin{cases} \alpha = A^4 \\ z = A^2 + A^{-2} \end{cases} \\
&i_2 : \begin{cases} \alpha = 1 \\ z = z \end{cases} \\
&i_3 : \langle K \rangle(A) = (-A)^{-\mathrm{w}(\vec{K})} H[\vec{K}](A^4, A^{-2} - A^2)
\end{aligned}$$

and the lifts ϕ_j, $j = 1, 2, 3$, are multiplication with resp.

$$(-A^3)^{-\mathrm{w}(\vec{K})}, \quad (\alpha)^{-\mathrm{w}(\vec{K})}\ , \text{ and } 1.$$

The history of the family is briefly:

- The Alexander polynomial was the first polynomial link invariant and it was found by J.W. Alexander about 1928.
- In 1969 J.H. Conway found the elegant axiomatic description of the Alexander polynomial which is a powerful tool for calculating the Alexander polynomial.

- V.F.R. Jones discovered the Jones polynomial in 1984.
- In 1985 a generalization of both the Alexander polynomial and the Jones polynomial was published and was later denoted the HOMFLY polynomial.
- L.H. Kauffman's article [Kauffman, 1987] introduces the bracket polynomial and its normalization which is isomorphic to the Jones polynomial.
- The H-polynomial was introduced by L.H. Kauffman in [Kauffman, 1991] p. 54 to show that the HOMFLY polynomial also can be thought of as a lift of a polynomial invariant under regular isotopy.

Chapter 6

Strips and strip link invariants

Take a strip of paper and "twist" it, tie a knot on it, and glue its ends together. Then you obtain a closed twisted and knotted strip. We shall use this as a model for a class of geometric objects which we shall call the class of closed strips.

On a closed strip we are particularly interested in its boundary curve(s) and in its center curve. These curves make it possible to define the twisting number of the closed strip and the knot of the closed strip. The twisting number is an invariant of ambient isotopy measuring the topological twist of the closed strip. The knot of the closed strip is the knot of its center curve.

Our aim is to find a closed braid representation of the link consisting of the center curve and the boundary curve(s). Hereby, we actually can classify closed strips. The main theorem in this chapter states:

> A closed strip is classified by its twisting number and by the knot given by the center curve of the closed strip.

In the last section of this chapter we will explain how most of the polynomials introduced in Chapter 5 in natural ways give or are polynomial invariants for strip links.

6.1 Twisting number

We need an exact definition of strips and closed strips in euclidean 3-space. This is contained in

Definition 6.1 (Strips and closed strips) *A strip in* $\mathbb{E}^3$ *is the image of an embedding* f *of* $[0, 1] \times [0, 1]$ *into* $\mathbb{E}^3$.

A closed strip in $\mathbb{E}^3$ is a strip where the edges are pointwise identified in one of the following ways:

$$f(0,t) = f(1,1-t), \quad t \in [0,1]. \tag{6.1}$$
$$f(0,t) = f(1,t), \quad t \in [0,1]. \tag{6.2}$$

Remark 6.2 *Instead of using the identifications $t \mapsto (1-t)$ resp. $t \mapsto t$ we could have used any other orientation reversing homeomorphism resp. orientation preserving homeomorphism because these are homotopic to $t \mapsto (1-t)$ resp. $t \mapsto t$.*

An ambient isotopy of a strip or a closed strip is still a strip or a closed strip. Hence, both classes of geometric objects are closed under ambient isotopy. To specify the center curve on a closed strip we give

Definition 6.3 (Center curves) *Let M be a closed strip in $\mathbb{E}^3$. The center curve on the strip M is the curve given by*

$$c(t) = f(t, \frac{1}{2}), \quad t \in [0,1],$$

where $f : [0,1] \times [0,1] \to \mathbb{E}^3$ is the embedding from Definition 6.1.

Remark 6.4 (Center curves) *The center curve will in both cases (6.1) and (6.2) close up after one turn around the closed strip.*

We will assume that the embedding in Definition 6.1 is piecewise linear or continuous differentiable such that the center curve and the boundary curve(s) each are tame knots. When the embedding is piecewise linear the tameness follows by Definition 3.21. In the case when the embedding is continuous differentiable the tameness of the knots are ensured by

Theorem 6.5 *If a knot parametrized by arc length is of class C^1, then it is tame.*

Proof: A proof is given in appendix I in [Crowell & Fox, 1963] pp. 147-152.

In Definition 6.1 we see that there are two topological classes of closed strips in $\mathbb{E}^3$. The closed strips in the first class (6.1) of closed strips are not orientable, because they are all embeddings of the Möbius strip. Hence, each closed strip has one center curve, c_1, and one boundary curve, c_2. These two closed curves on a closed strip, M, give a link with two components, c_1 and c_2. The closed strips in the other class (6.2) of closed strips are orientable, hence, each closed strip has one center curve, c_1, and two boundary curves, c_2 and c_3. Then each closed strip, M, from the class (6.2) gives a link with three components, c_1, c_2, and c_3.

Choose a direction in which a closed strip is traversed. Consider the link obtained from this closed strip as just described. The direction in which the closed strip is traversed induces a common orientation of the components, c_i, of the link. We shall denote each of these oriented components by $\vec{c}_i$ for $i = 1, 2, (3)$. For such a link we can compute its linking number(s), because it is tame and by use of this define the twisting number of the closed strip. We shall think of a twist as described in Section 5.2.1.

It is now possible to define the twisting number of a closed strip.

Definition 6.6 (Twisting number) *Let M be a closed strip in $\mathbb{E}^3$. Then the twisting number of M,* twist(M)*, is given by the formula*

$$\text{twist}(M) = \begin{cases} \frac{1}{2}\,\text{link}(\vec{c}_1, \vec{c}_2), & \textit{if } M \textit{ is unorientable} \\ \frac{1}{2}\,\text{link}(\vec{c}_1, \vec{c}_2) + \frac{1}{2}\,\text{link}(\vec{c}_1, \vec{c}_3), & \textit{if } M \textit{ is orientable} \end{cases}$$

where $\vec{c}_i$ for $i = 1, 2, (3)$ are given as above.

Theorem 6.7 *Let M be a closed strip in $\mathbb{E}^3$. The twisting number of M,* twist(M)*, is well-defined and an invariant of ambient isotopy.*

Proof: Let M be a closed strip in $\mathbb{E}^3$. Then M has two directions in which it can be traversed. Hence, to prove that the twisting number is well-defined, we must prove that it is invariant under shift of the direction in which M is traversed.

If we shift the direction in which M is traversed, both the orientation of the center curve and the orientation(s) of the boundary curve(s) are reversed. This has no effect on the signs of the crossings (in a link diagram) between the center curve and the boundary curve(s). Whence, the twisting number is well-defined.

By Theorem 5.4 the linking number is an invariant of ambient isotopy, therefore, the twisting number of M also is an invariant of ambient isotopy. □

In connection with Section 5.2.1 we now observe that

$$\text{twist}(\widehat{\xi^n}) = \frac{n}{2}, \quad \text{for } n \in \mathbb{Z},$$

where $\widehat{\xi^n}$ is considered as a closed strip by analogy with the explanation in Section 5.2.1.

This means, that all the orientable unknotted closed strips have integer twisting numbers and all the unorientable unknotted closed strips have twisting numbers that equals one half of an odd integer. Hence, we have divided each of the topological classes of closed strips into countable many subclasses.

In the next section we shall subdivide each of these subclasses according to the knottedness of the center curve.

6.2 Knotted closed strips in $\mathbb{E}^3$

Let a closed strip, $M \subset \mathbb{E}^3$, be contained in a thin tube around the center curve. We consider this tube as a knot. Recall, that a knot is a link with one component. This knot is combinatorially equivalent to a closed braid, $\hat{\beta}$, cf. Theorem 4.9 on page 68. To the closed braid, $\hat{\beta}$, there is a corresponding braid, β, on n strings. These strings are now tubes each containing a piece of the closed strip, M, and we can in analogy with the construction in Section 4.1 define σ_j, for $1 \le j \le n-1$, as that elementary n-braid in which the jth tube string overcrosses the $(j+1)$st tube string.

We need a family of homomorphisms all called $\Psi : B(n) \to B(3n)$ for each n defined by

$$\begin{aligned}\Psi(\sigma_j) \quad = \quad & \sigma_{1+3(j-1)}^{-1}\sigma_{2+3(j-1)}^{-1}\sigma_{1+3(j-1)}^{-1}\sigma_{4+3(j-1)}^{-1}\sigma_{5+3(j-1)}^{-1}\sigma_{4+3(j-1)}^{-1} \\ & \sigma_{3+3(j-1)}\sigma_{4+3(j-1)}\sigma_{5+3(j-1)}\sigma_{2+3(j-1)}\sigma_{3+3(j-1)}\sigma_{4+3(j-1)} \\ & \sigma_{1+3(j-1)}\sigma_{2+3(j-1)}\sigma_{3+3(j-1)}, \quad \text{for } j = 1, \ldots, n-1.\end{aligned}$$

We will use the notation $\boldsymbol{\sigma}_j$ for the $3n$-braid word $\Psi(\sigma_j)$.

A good way to think of the braid word $\boldsymbol{\sigma}_j$ is the following: Think of the total braid on $3n$ strings as n strips hanging down (analogous to the definition of geometric n-braids). The $3n$ strings are the center and boundary curves of the n strips hanging down. Then $\boldsymbol{\sigma}_j$ first twist the jth and the $(j+1)$st strip one half time in the positive direction. Afterwards, it interchanges the jth and the $(j+1)$st strip by lifting the jth strip over the $(j+1)$st strip. See Figure 6.1 and compare with Figure 4.3 on page 64.

In order to obtain homomorphisms we have to define

$$\begin{aligned}\Psi(\sigma_i^{\nu_i}\sigma_j^{\nu_j}) &= \Psi(\sigma_i^{\nu_i})\Psi(\sigma_j^{\nu_j}) \\ &= \boldsymbol{\sigma}_i^{\nu_i}\boldsymbol{\sigma}_j^{\nu_j},\end{aligned}$$

for $1 \le i, j \le n-1$ and $\nu_i, \nu_j \in \{\pm 1\}$, where the braid word $\boldsymbol{\sigma}_j^{-1}$ is the inverse of the braid word $\boldsymbol{\sigma}_j$. See Figure 6.2.

To verify that we have obtained a family of homomorphisms, we have to check that the defining relations of $B(n)$ factor through the defining relations of $B(3n)$. The braid group $B(n)$ has the defining relation (cf. Theorem 4.4)

$$\sigma_i\sigma_j = \sigma_i\sigma_j \ \text{ for } \ |i-j| > 1.$$

When the indices i and j fulfil the condition $|i-j| > 1$, then all the indices k_i and k_j of the (normal) elementary $3n$-braids in the braid words $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_j$ will fulfil

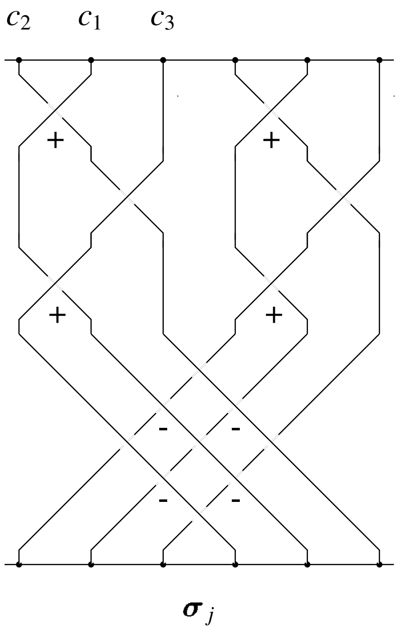

Figure 6.1: A picture of $\Psi(\sigma_j) = \boldsymbol{\sigma}_j$ consisting of two half twist, ξ, and under these a cross

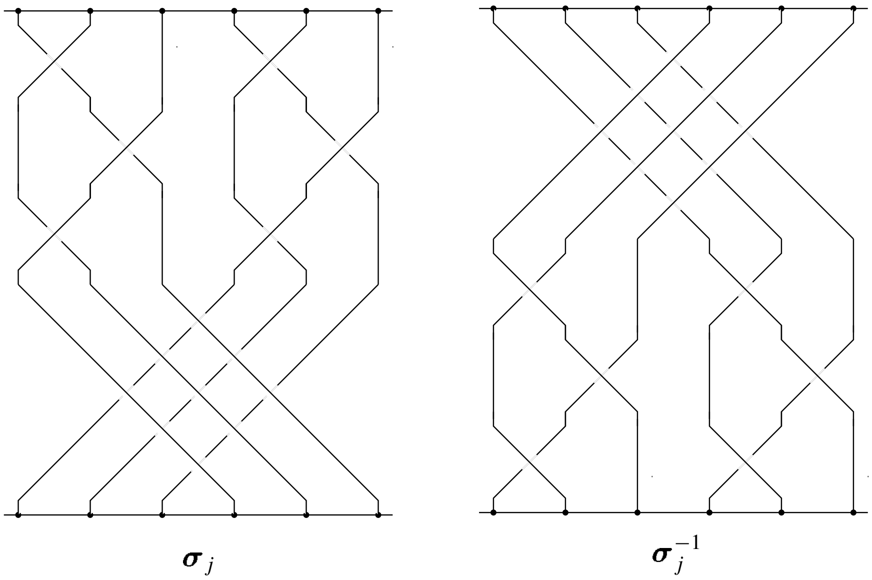

Figure 6.2: A picture of $\boldsymbol{\sigma}_j$ and its inverse $\boldsymbol{\sigma}_j^{-1}$

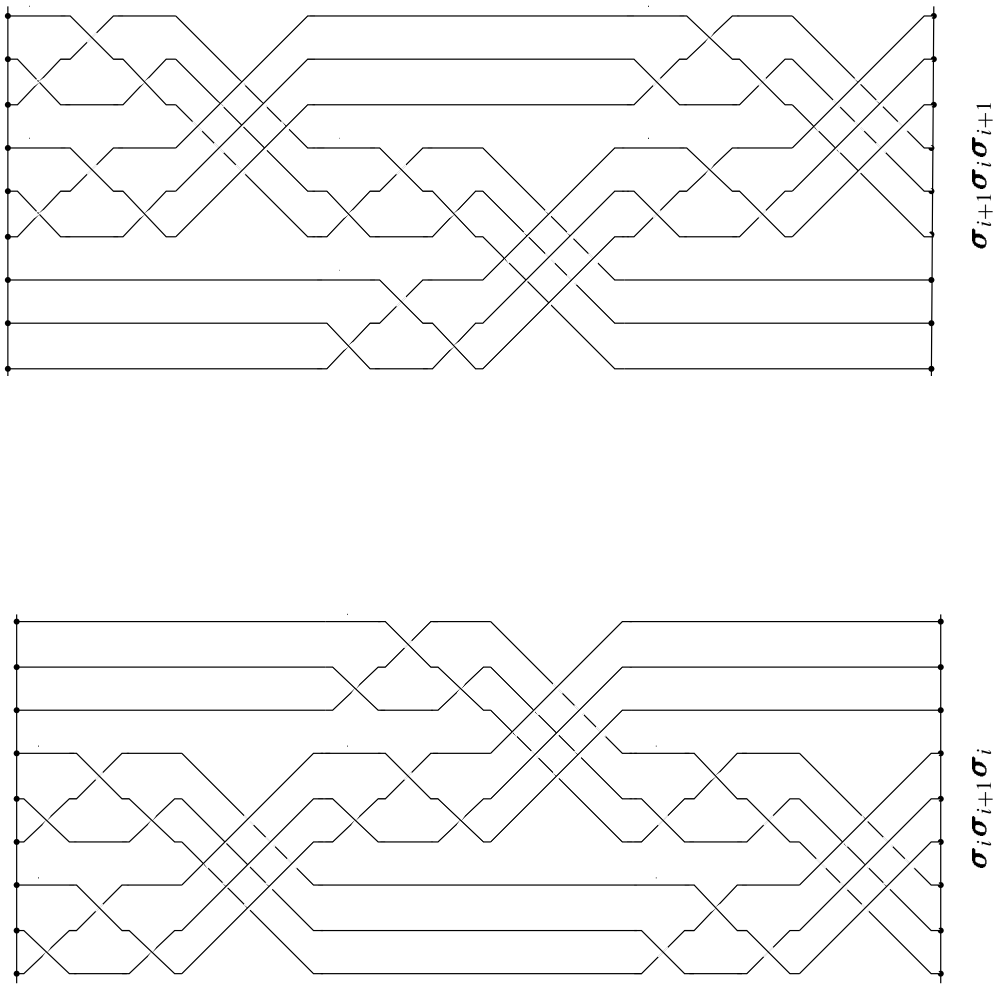

Figure 6.3: A braid relation for tube strings

the condition $|k_i - k_j| > 3$. Hence,

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_i\boldsymbol{\sigma}_j \text{ for } |i-j|>1$$
$$\Updownarrow$$
$$\Psi(\sigma_i\sigma_j) = \Psi(\sigma_i\sigma_j) \text{ for } |i-j|>1.$$

The last defining relation for the braid group $B(n)$ is

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \le i \le n-2.$$

This relation is fulfilled by the tube strings, hence, it is also fulfilled by the closed $3n$-braid contained in the tube strings. See the figures on Figure 6.3. To prove this algebraicly on the $3n$ stringed braid it is necessary to move three strings across fifteen crossings using the analogous defining relation for $B(3n)$ each time. This is a straight forward calculation, but it is very tedious and this is why we shall omit it.

We summarize the above in

Theorem 6.8 *For each $n \geq 2$ the map Ψ from $B(n)$ to $B(3n)$ defined by*

$$\begin{aligned}\Psi(\sigma_j) &= \sigma_{1+3(j-1)}^{-1}\sigma_{2+3(j-1)}^{-1}\sigma_{1+3(j-1)}^{-1}\sigma_{4+3(j-1)}^{-1}\sigma_{5+3(j-1)}^{-1}\sigma_{4+3(j-1)}^{-1}\\ &\quad \sigma_{3+3(j-1)}\sigma_{4+3(j-1)}\sigma_{5+3(j-1)}\sigma_{2+3(j-1)}\sigma_{3+3(j-1)}\sigma_{4+3(j-1)}\\ &\quad \sigma_{1+3(j-1)}\sigma_{2+3(j-1)}\sigma_{3+3(j-1)}, \text{ for } j = 1, \ldots, n-1,\end{aligned}$$

is a homomorphism.

Since Ψ is a homomorphism we will use the notation $\boldsymbol{\beta}$ for the $3n$-braid word $\Psi(\beta)$, where β is a n-braid, i.e., $\Psi(\beta) = \boldsymbol{\beta}$.

Given a twisting number, T, and a representation of a knot we will now make a closed braid describing the center and boundary curve(s) of a closed strip with this knot and twisting number. Let the knot be represented by a braid β on n tube strings. We define a braid $\gamma(\beta, T) \in B(3n)$ on $3n$ strings by

$$\gamma(\beta, T) = \xi^{2T}\boldsymbol{\sigma}_{i_1}^{\pm 1} \cdots \boldsymbol{\sigma}_{i_k}^{\pm 1} = \xi^{2T}\boldsymbol{\beta},$$

where

$$\boldsymbol{\beta} = \boldsymbol{\sigma}_{i_1}^{\pm 1} \cdots \boldsymbol{\sigma}_{i_k}^{\pm 1} \text{ for } 1 \leq i_1, \ldots, i_k \leq n-1$$

is the braid word corresponding to the generators in the braid word of the knot and $\xi = (\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})$ is the braid from Section 5.2.1.

The braid words ξ and $\boldsymbol{\sigma}_j^{\pm 1}$ define in a natural way a piece of a strip resp. two pieces of strip (cf. Section 5.2.1 and Figure 6.2). Hence, the closed braid $\hat{\gamma}(\beta, T)$, obtained by closing the braid $\gamma(\beta, T)$, defines a closed strip. With some abuse of notation we will use the notation

$$\text{twist}(\hat{\gamma})$$

for the twisting number of the closed strip defined by the closed braid $\hat{\gamma}$.

First we prove that the closed braid $\hat{\gamma}(\beta, T)$ has twisting number T as indicated in the notation.

Theorem 6.9 *Let β be a braid representation of a knot. The twisting number of the closed strip defined by the closed braid $\hat{\gamma} = \hat{\gamma}(\beta, T) = (\xi^{2T}\boldsymbol{\beta})\hat{}$ is T, i.e.,*

$$\text{twist}(\hat{\gamma}) = T.$$

Proof: First assume that $\hat{\gamma}$ has three components. Figure 6.1 shows the link diagram of the braid $\boldsymbol{\sigma}_j$ for a fixed $j = 1, \dots, n-1$. Denote the components of $\hat{\gamma}$ by $\vec{c}_1$, $\vec{c}_2$, and $\vec{c}_3$ as on the figure. On the figure the crossings between $\vec{c}_1$ and $\vec{c}_2$ resp. $\vec{c}_1$ and $\vec{c}_3$ are labeled[1] with plus (+) resp. minus (-) according to the convention on Figure 5.5, page 76. By Definition 5.3 the contributions to the linking numbers $\mathrm{link}(\vec{c}_1, \vec{c}_2)$ and $\mathrm{link}(\vec{c}_1, \vec{c}_3)$ from $\boldsymbol{\sigma}_j$ are

$$\mathrm{link}(\vec{c}_1, \vec{c}_2)|\boldsymbol{\sigma}_j = \frac{1}{2}(+1 - 1 + 1 - 1) = 0$$

and

$$\mathrm{link}(\vec{c}_1, \vec{c}_3)|\boldsymbol{\sigma}_j = \frac{1}{2}(+1 - 1 + 1 - 1) = 0.$$

It follows that the contribution to the twisting number of $\hat{\gamma}$ that comes from $\boldsymbol{\sigma}_j$ is

$$\begin{aligned}\mathrm{twist}(\hat{\gamma})|\boldsymbol{\sigma}_j &= \frac{1}{2}\,\mathrm{link}(\vec{c}_1, \vec{c}_2)|\boldsymbol{\sigma}_j + \frac{1}{2}\,\mathrm{link}(\vec{c}_1, \vec{c}_3)|\boldsymbol{\sigma}_j \\ &= 0.\end{aligned}$$

If the closed braid $\hat{\gamma}$ only has two components the contribution to the twisting number of $\hat{\gamma}$ from $\boldsymbol{\sigma}_j$ is exactly one half of the sum of the two linking numbers standing above, because $\vec{c}_2$ and $\vec{c}_3$ now both are parts of the same component of the closed braid $\hat{\gamma}$. In either case we have

$$\mathrm{twist}(\hat{\gamma})|\boldsymbol{\sigma}_j = 0.$$

Similar we also have

$$\mathrm{twist}(\hat{\gamma})|\boldsymbol{\sigma}_j{}^{-1} = 0.$$

According to Section 5.2.1 the contribution to the twisting number of $\hat{\gamma}$ from the braid $\xi = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$ is exactly one half.

The conclusion is that

$$\begin{aligned}\mathrm{twist}(\hat{\gamma}(\beta, T)) &= \mathrm{twist}(\widehat{\xi^{2T}\boldsymbol{\beta}}) \\ &= \mathrm{twist}((\xi^{2T}\boldsymbol{\sigma}_{i_1}^{\pm 1} \cdots \boldsymbol{\sigma}_{i_k}^{\pm 1})\hat{}\,) \\ &= \mathrm{twist}(\widehat{\xi^{2T}}) \\ &= 2T(1/2) \\ &= T.\end{aligned}$$

This completes the proof. □

[1]The labels are the same either the orientation of the link diagram is vertically downwards or upwards. See also Theorem 6.7.

In connection with the proof of Theorem 6.9 it is notable that we have isolated one structure of the closed strip; its twist, from another structure; its knot, in the braid word

$$\gamma(\beta, T) = \xi^{2T}\boldsymbol{\beta}.$$

At the moment it is only proven that $\hat{\gamma}$ maps a half integer T and one representation β of a knot into a closed braid $\hat{\gamma}(\beta, T)$. We will prove that $\hat{\gamma}(\beta, T)$ is independent of the representation of the knot. We shall first prove some results about the braid words $\boldsymbol{\sigma}_j^{\pm 1}$.

We need to recall the relations in the braid group $B(n)$ from page 66. Namely,

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for} \quad |i-j| \geq 2, 1 \leq i, j \leq n-1. \tag{6.3}$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2. \tag{6.4}$$

The relations can, after conjugation, be written on the form:

$$\sigma_j\sigma_i^{-1} = \sigma_i^{-1}\sigma_j \quad \text{for} \quad |i-j| \geq 2, 1 \leq i, j \leq n-1. \tag{6.5}$$
$$\sigma_{i+1}^{-1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i^{-1} \quad \text{for} \quad 1 \leq i \leq n-2. \tag{6.6}$$

In the following proofs the relations in the braid group $B(n)$ will most often be used as in (6.5) and (6.6), but we will make the references back to Theorem 4.4. In the proofs we underline the braid words where the relations will be used or where there will be some other kind of changes.

The first theorem states loosely speaking - that the closed braid $\hat{\boldsymbol{\sigma}}_1{}^{\pm 1}$ on 6 strings is combinatorially equivalent to the trivial closed braid on 3 strings. The exact statement is contained in

Theorem 6.10 *Let β be a $(3j)$-braid. The links obtained by closing the $(3j)$-braid β and the $(3j+3)$-braids $\beta\boldsymbol{\sigma}_j^{\pm 1}$ are combinatorially equivalent.*

Proof: We only prove that the links obtained by closing the $(3j)$-braid β and the $(3j+3)$-braid $\beta\boldsymbol{\sigma}_j$ are combinatorially equivalent, because the case with the $(3j+3)$-braid word $\beta\boldsymbol{\sigma}_j^{-1}$ is similar.

In the following calculation we will write $\sigma_i^{\pm 1}$ for $\sigma_{i+3j-3}^{\pm 1}$. Furthermore, we will overline the braid words which occur after the changes.

$$(\beta\boldsymbol{\sigma}_1, 3j+3)$$

$$= (\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_4^{-1}\sigma_5^{-1}\underline{\sigma_4^{-1}\sigma_3\sigma_4}\sigma_5\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3, 3j+3) \quad \text{, and by (4.2) page 66}$$

$$= (\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_4^{-1}\underline{\sigma_5^{-1}\overline{\sigma_3\sigma_4\sigma_3^{-1}}\sigma_5}\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3, 3j+3) \quad \text{, and by (4.1) page 66}$$

$$= (\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_4^{-1}\overline{\sigma_3\underline{\sigma_5^{-1}\sigma_4\sigma_5}\sigma_3^{-1}}\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3, 3j+3) \quad \text{, and by (4.2)}$$

$$= (\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_4^{-1}\sigma_3\overline{\sigma_4\sigma_5\sigma_4^{-1}}\sigma_3^{-1}\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3, 3j+3)$$

Note, that the elementary braids in the $(3j)$-braid β have indices lower than the indices in the elementary braids $\sigma_3, \sigma_4,$ and σ_5, since this is necessary in the following, where we shall use Markov moves of type $\mathcal{M}_2$ (cf. Theorem 4.10).

Then, by $\mathcal{M}_1$ and $\mathcal{M}_2$ page 69 we have

$$(\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_4^{-1}\sigma_3\sigma_4\underline{\sigma_5}\sigma_4^{-1}\sigma_3^{-1}\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3, 3j+3)$$

$$\to (\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\underline{\sigma_4^{-1}\sigma_3\sigma_4\sigma_4^{-1}\sigma_3^{-1}\sigma_2}\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3, 3j+2) \quad \text{, and by } (\sigma_i\sigma_i^{-1} = \epsilon) \text{ and (4.1)}$$

$$= (\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\overline{\sigma_2\underline{\sigma_4^{-1}}}\underline{\sigma_3\sigma_4}\sigma_1\sigma_2\sigma_3, 3j+2) \quad \text{, and by (4.2)}$$

$$= (\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\overline{\sigma_3\underline{\sigma_4}\sigma_3^{-1}}\sigma_1\sigma_2\sigma_3, 3j+2) \quad \text{, and by } \mathcal{M}_1 \text{ and } \mathcal{M}_2$$

$$\to (\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\underline{\sigma_3\sigma_3^{-1}}\sigma_1\sigma_2\underline{\sigma_3}, 3j+1) \quad \text{, and by } (\sigma_i\sigma_i^{-1} = \epsilon) \text{ and } \mathcal{M}_2$$

$$\to (\beta\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\underline{\sigma_2\sigma_1\sigma_2}, 3j) \quad \text{, and by (4.2)}$$

$$= (\beta\underline{\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\overline{\sigma_1\sigma_2\sigma_1}}, 3j) \quad \text{, and by } (\sigma_i\sigma_i^{-1} = \epsilon)$$

$$= (\beta, 3j).$$

The proof of Theorem 6.10 is completed. □

The following lemma is a technical lemma that ensures that a half twist, $\xi = \sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\sigma_{i+1}^{-1}$, $i = 3(j-1)$, can pass one string in the cross of $\boldsymbol{\sigma}_j$. For a geometric picture of the cross see Figure 6.1 on page 101.

Lemma 6.11 *Let $B(n)$ be the group of braids on n strings and let i be an integer such that $0 \leq i \leq n-4$. The braid word $\sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\sigma_{i+1}^{-1}\sigma_{i+3}\sigma_{i+2}\sigma_{i+1}$ and the braid word $\sigma_{i+3}\sigma_{i+2}\sigma_{i+1}\sigma_{i+2}^{-1}\sigma_{i+3}^{-1}\sigma_{i+2}^{-1}$ give the same braid.*

Proof:

$$\begin{aligned}
&\sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\underline{\sigma_{i+1}^{-1}\sigma_{i+3}}\sigma_{i+2}\sigma_{i+1} && \text{, and by (4.1) page 66}\\
&= \sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\sigma_{i+3}\underline{\sigma_{i+1}^{-1}\sigma_{i+2}\sigma_{i+1}} && \text{, and by (4.2) page 66}\\
&= \sigma_{i+1}^{-1}\underline{\sigma_{i+2}^{-1}\sigma_{i+3}\sigma_{i+2}}\sigma_{i+1}\sigma_{i+2}^{-1} && \text{, and by (4.2)}\\
&= \sigma_{i+1}^{-1}\sigma_{i+3}\sigma_{i+2}\underline{\sigma_{i+3}^{-1}\sigma_{i+1}}\sigma_{i+2}^{-1} && \text{, and by (4.1)}\\
&= \underline{\sigma_{i+1}^{-1}\sigma_{i+3}}\sigma_{i+2}\sigma_{i+1}\sigma_{i+3}^{-1}\sigma_{i+2}^{-1} && \text{, and by (4.1)}\\
&= \sigma_{i+3}\underline{\sigma_{i+1}^{-1}\sigma_{i+2}\sigma_{i+1}}\sigma_{i+3}^{-1}\sigma_{i+2}^{-1} && \text{, and by (4.2)}\\
&= \sigma_{i+3}\sigma_{i+2}\sigma_{i+1}\sigma_{i+2}^{-1}\sigma_{i+3}^{-1}\sigma_{i+2}^{-1}.
\end{aligned}$$

□

The above lemma will in the proof of the next theorem be used three times to move the half twist ξ through the overlying part of the cross of the braid word $\boldsymbol{\sigma}_j$.

Theorem 6.12 *Let $B(3n)$ be the group of braids on $3n$ strings and let j be an integer such that $1 \leq j \leq n-1$. In the braid word $\boldsymbol{\sigma}_j$ the twist*

$$\xi = \sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\sigma_{i+1}^{-1}, \quad i = 3(j-1),$$

can pass the cross freely, i.e.,

$$\beta_1\underline{(\sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\sigma_{i+1}^{-1})}(\sigma_{i+4}^{-1}\sigma_{i+5}^{-1}\sigma_{i+4}^{-1})(\sigma_{i+3}\sigma_{i+4}\sigma_{i+5})(\sigma_{i+2}\sigma_{i+3}\sigma_{i+4})(\sigma_{i+1}\sigma_{i+2}\sigma_{i+3})\beta_2$$

$$= \beta_1(\sigma_{i+4}^{-1}\sigma_{i+5}^{-1}\sigma_{i+4}^{-1})(\sigma_{i+3}\sigma_{i+4}\sigma_{i+5})(\sigma_{i+2}\sigma_{i+3}\sigma_{i+4})(\sigma_{i+1}\sigma_{i+2}\sigma_{i+3})\overline{(\sigma_{i+4}^{-1}\sigma_{i+5}^{-1}\sigma_{i+4}^{-1})}\beta_2,$$

where β_1 and β_2 are $3n$-braids.

Proof: In the following we will write $\sigma_i^{\pm 1}$ for $\sigma_{i+k}^{\pm 1}$. By use of (4.1) page 66 we find that

$$(\sigma_3\sigma_4\sigma_5)(\sigma_2\sigma_3\sigma_4)(\sigma_1\sigma_2\sigma_3) \quad = \quad (\sigma_3\sigma_2\sigma_1)(\sigma_4\sigma_3\sigma_2)(\sigma_5\sigma_4\sigma_3). \tag{6.7}$$

The following calculation completes the proof.

$$
\begin{aligned}
&\beta_1(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})(\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1})\underline{(\sigma_3\sigma_4\sigma_5)(\sigma_2\sigma_3\sigma_4)(\sigma_1\sigma_2\sigma_3)}\beta_2 && \text{, and by (6.7)}\\
&= \beta_1\underline{(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})(\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1})}(\sigma_3\sigma_2\sigma_1)(\sigma_4\sigma_3\sigma_2)(\sigma_5\sigma_4\sigma_3)\beta_2 && \text{, and by (4.1) page 66}\\
&= \beta_1(\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1})\underline{(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})(\sigma_3\sigma_2\sigma_1)}(\sigma_4\sigma_3\sigma_2)(\sigma_5\sigma_4\sigma_3)\beta_2 && \text{, and by Lemma 6.11}\\
&= \beta_1(\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1})(\sigma_3\sigma_2\sigma_1)\underline{(\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1})(\sigma_4\sigma_3\sigma_2)}(\sigma_5\sigma_4\sigma_3)\beta_2 && \text{, and by Lemma 6.11}\\
&= \beta_1(\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1})(\sigma_3\sigma_2\sigma_1)(\sigma_4\sigma_3\sigma_2)\underline{(\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1})(\sigma_5\sigma_4\sigma_3)}\beta_2 && \text{, and by Lemma 6.11}\\
&= \beta_1(\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1})(\sigma_3\sigma_2\sigma_1)(\sigma_4\sigma_3\sigma_2)(\sigma_5\sigma_4\sigma_3)(\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1})\beta_2.
\end{aligned}
$$

□

Remark 6.13 (To Lemma 6.11 and Theorem 6.12) *By use of symmetries, we have results analogous to Lemma 6.11 and Theorem 6.12 in the case where ξ passes through the underlying part of the cross of $\boldsymbol{\sigma}_j$ and in the two cases where ξ passes through the cross in $\boldsymbol{\sigma}_j^{-1}$. In other words, the half twist, ξ, can be moved freely around in the braid word $\boldsymbol{\sigma}_j$ resp. $\boldsymbol{\sigma}_j^{-1}$.*

We are now ready to prove that the closed braids $\hat{\gamma}(\beta, T)$ are independent of representation of the knots.

Theorem 6.14 *Let β be a representation of a knot and T a half integer. The closed braid $\hat{\gamma}(\beta, T)$ is independent of the representation of the knot, β.*

Proof: Let β and β' be two braids representing the same knot. Theorem 6.8 ensures that $\gamma(\beta, T)$ is independent of braid group relations. By Theorem 4.10, there exists a finite sequence of Markov moves connecting β to β'. Hence, it is sufficient to prove that $\hat{\gamma}(\beta, T)$ is combinatorially equivalent to $\hat{\gamma}(\beta', T)$ if β and β' only differ by one Markov move of type $\mathcal{M}_1$ resp. of type $\mathcal{M}_2$.

If the Markov move is of type $\mathcal{M}_1$ it is a conjugation, i.e., $\beta' = \omega^{-1}\beta\omega$ (cf. Theorem 4.10). Hence,

$$
\begin{aligned}
\gamma(\beta', T) &= \xi^{2T}\boldsymbol{\beta}'\\
&= \xi^{2T}\boldsymbol{\omega}^{-1}\boldsymbol{\beta}\boldsymbol{\omega}.
\end{aligned}
$$

The knot, described by β', is connected and Remark 6.13 ensures that the twist can be moved freely around between the braid words $\boldsymbol{\sigma}_j^{\pm 1}$ in the closed braid $\hat{\gamma}(\beta', T)$. We therefore also have

$$\hat{\gamma}(\beta', T) = (\boldsymbol{\omega}^{-1}\xi^{2T}\boldsymbol{\beta}\boldsymbol{\omega})\hat{}\,.$$

This means that $\gamma(\beta', T)$ is combinatorially equivalent to $\boldsymbol{\omega}^{-1}\xi^{2T}\boldsymbol{\beta}\boldsymbol{\omega}$ which simply is a conjugation of $\xi^{2T}\boldsymbol{\beta}$. Since a conjugation is a Markov move of type $\mathcal{M}_1$, and

$$\xi^{2T}\boldsymbol{\beta} = \gamma(\beta, T)$$

it is proved that $\hat{\gamma}(\beta, T)$ is combinatorially equivalent to $\hat{\gamma}(\beta', T)$ when β and β' only differ by one Markov move of type $\mathcal{M}_1$.

If the Markov move is of type $\mathcal{M}_2$, we have, $\beta' = \beta\sigma_n^{\pm 1}$. That the underlying braid words are equivalent follows from Theorem 6.10. This completes the proof. □

The next theorem is the main theorem. In the proof of this theorem we show that a closed strip is ambient isotopic to a closed strip defined by a closed braid $\hat{\gamma}(\beta, T)$. This proves together with Theorem 6.9 and Theorem 6.14 the main theorem. This means that we have succeeded in classifying closed strips by their twisting numbers and by their knots given by the center curves of the closed strips. Note, that we can explicitly give a closed braid representation from each class, namely, the closed braid $\hat{\gamma}(\beta, T)$ of the form $\hat{\gamma}(\beta, T) = \widehat{\xi^{2T}\boldsymbol{\beta}}$.

Theorem 6.15 (Classification of closed strips) *A closed strip is classified by its twisting number and by the knot given by the center curve of the closed strip.*

Proof: Let M be a closed strip. Choose a direction in which M is traversed and let $\vec{c}$ be the center curve of M.

By the general assumption that M is either piecewise linear or continuous differentiable it follows by Definition 3.21 resp. Theorem 6.5 that $\vec{c}$ is a tame knot. Therefore, we can assume that $\vec{c}$ is piecewise linear. In this connection note that Theorem 3.23 ensures that the classes of piecewise linear knots are the same as the classes of tame knots. According to Theorem 4.9 the center curve $\vec{c}$ is combinatorially equivalent to a closed braid $\hat{\beta}$ and by Theorem 3.35 there is an ambient isotopy, H, deforming $\vec{c}$ into this closed braid. This ambient isotopy also deforms M into a new closed strip, which by Definition 3.19 or Definition 3.22 is denoted by $h_1(M)$.

By Definition 4.8 the center curve of the new closed strip $h_1(\vec{c})$ is in general position with respect to an oriented axis l so that all edges of $h_1(\vec{c})$ are positive with respect l. The strip $h_1(M)$ can by a deformation, f, inside a tubular neighbourhood of $h_1(M)$ be deformed such that the center curve and boundary curve(s) of the strip $f(h_1(M))$ are in general position with respect to the oriented axis l and so that all edges of the center curve and boundary curve(s) are positive with respect l. In other words, the center curve and boundary curve(s) of the strip $f(h_1(M))$ is a closed braid, $\hat{\omega}$.

In the case where the closed strip $h_1(M)$ is continuous differentiable this deformation can be done by decreasing the width of the strip and using compactness.

If the closed strip $h_1(M)$ is piecewise linear it is ambient isotopic to a continuous differentiable strip by the following arguments:

Step 1: First remove a small disk around each vertex in $h_1(M)$. The remaining of the edges in $h_1(M)$ can now be smoothed inside a neighbourhood of each edge. This gives a smooth approximation of $h_1(M)$ minus the disks. At each of the places where the disks were - we now put a plane with a normal vector that e.g. is the average of the normal vectors of the triangles (in a triangulation of $h_1(M)$) that had the removed vertex as vertex. From the smooth approximation of $h_1(M)$ minus the disks and the planes we obtain by a partition of unity argument a smooth approximation of the whole piecewise linear closed strip $h_1(M)$. Denote this approximation by N.

Step 2: To obtain an ambient isotopy deforming N into $h_1(M)$ we can use the following construction. Divide N into finitely many orientable pieces, N_i. Choose a tubular neighbourhood of each N_i containing the corresponding piece of $h_1(M)$, given by a neighbourhood of the zero level in the normal bundle of N_i. By the flow, along a suitable vector field defined by the normal vector field of N_i and a bump function such that the vector field tends to zero at the boundary of the tubular neighbourhood of N_i, we by a partition of unity argument obtain an ambient isotopy deforming all of N into $h_1(M)$. This ambient isotopy is the identity outside the tubular neighbourhood of N. A similar construction for a point is given in [Hansen, 1989] in Sublemma 2.2. on page 11.

We have now obtained a closed braid, $\hat{\omega}$, consisting of the center and boundary curve(s) of the strip $f(h_1(M))$, which lies inside tube strings. Consider the braid inside a piece of tube string ignoring the rest of the tube. This braid can only be of the form ξ^n, because otherwise there are self-intersection in the strip $f(h_1(M))$ which can not occur under the above deformation of M.

Consider the tube braid obtained by opening the closed tube braid. By analogy with elementary braids (see e.g. Figure 4.2) there are two cases in the projection of the braid. Namely, a piece of tube string that has no crossings with the other tube strings and a crossing between two tube strings.

By Remark 6.13 the twist of the strip can be moved freely around inside the tube strings, because the ambient isotopy that moved the braid word ξ around also is a deformation of the strip. Hence, the strip M can be deformed into the standard position that is given by

$$\hat{\gamma}(\beta, T) = \widehat{\xi^{2T}\boldsymbol{\beta}},$$

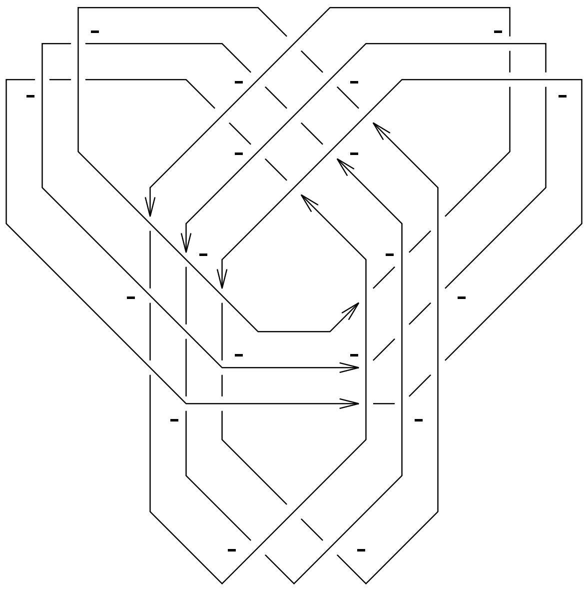

Figure 6.4: An oriented link diagram of the link consisting of the center and the boundary curve of the sculpture "Immortality" by John Robinson

where β is the tube string braid and T is the amount of twist that is left over after each crossing has received its measured portion of twist. By Theorem 6.9 and Theorem 6.14 the notation $\hat{\gamma}(\beta, T)$ is well-defined and the closed strip M is classified by the knot that β represents and by the twisting number T. $\square$

6.2.1 The picture on the front page

The picture on the front page is a sculpture titled "Immortality" by John Robinson mentioned in [Brown, 1994]. This sculpture can be seen as a closed strip. Figure 6.4 shows an oriented link diagram of the link consisting of the center and the boundary curve of that closed strip.

The classification of "Immortality" is determined by the fact that its center curve is the left handed trefoil knot and that it has the twisting number

$$\begin{aligned}\text{twist(Immortality)} &= \frac{1}{2}\,\text{link(center curve, boundary curve)}\\ &= \frac{1}{2}\frac{1}{2}(-4\cdot 3 - 3\cdot 2)\\ &= \frac{1}{4}(-18)\\ &= -\frac{9}{2}.\end{aligned}$$

Note, that each "crossing" contains minus one twist and each "loop" contains minus one half twist.

The left handed trefoil knot has the closed 2-braid representation $\sigma_1\sigma_1\sigma_1$. Hence, the link consisting of the center and the boundary curve of "Immortality" is combinatorially equivalent to the closed braid

$$\hat{\gamma}(\sigma_1\sigma_1\sigma_1, -\frac{9}{2}) = (\xi^{-9}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1)\hat{}\,.$$

6.3 Polynomials for strip links

The Q-polynomial and the U-polynomial are both invariants under ambient isotopy for unoriented links[2], hence, they both induce an invariant for closed strips given by the polynomial for the unoriented link consisting of the center curve and the boundary curve(s) of the closed strip. The property known as the reversing property of the Jones polynomial[3] lead us to formulate and prove that the Jones polynomial also gives an invariant of ambient isotopy for closed strips, even though the Jones polynomial are defined on oriented links. This is stated in Theorem 6.18.

To state and prove the reversing property of the Jones polynomial we need a notation for the sum of one component of a link's linking numbers with all the other components. This is called the total linking number of a component of an oriented link and is given by

Definition 6.16 (Total linking number) *Let $\vec{c}$ be a component of an oriented link diagram $\vec{K}$. The total linking number of $\vec{c}$,* $\text{link}(\vec{c}, \vec{K}\backslash\vec{c})$*, is defined by the formula*

$$\text{link}(\vec{c}, \vec{K}\backslash\vec{c}) = \frac{1}{2}\sum_{p\in\vec{c}\cap(\vec{K}\backslash\vec{c})} \delta(p),$$

[2]See Theorem 5.18 and Theorem 5.27.

[3]For reference see [Kauffman, 1991] page 51.

where $\vec{c} \cap (\vec{K} \backslash \vec{c})$ denotes the set of crossings between $\vec{c}$ and any other component of $\vec{K}$ and $\delta(p)$ denotes the sign of the crossing, i.e., ± 1.

By use of the total linking number the twisting number of a closed strip can be defined as follows:

Let M be a closed strip in $\mathbb{E}^3$ and let $\vec{K}$ be one of the two oriented links consisting of M's center curve and boundary curve(s). Let $\vec{c}_1$ denote the center curve of M. Then the twisting number of M is given by

$$\text{twist}\,(M) = \frac{1}{2}\,\text{link}(\vec{c}_1, \vec{K} \backslash \vec{c}_1).$$

Theorem 6.17 (The reversing property of the Jones polynomial) *Let $\vec{c}$ be a component of an oriented link diagram $\vec{K}$ and let $\vec{K}'$ be the oriented link diagram obtained from $\vec{K}$ by reversing the orientation of the component $\vec{c}$. Then,*

$$V[\vec{K}'](t) = t^{-3\,\text{link}(\vec{c}, \vec{K} \backslash \vec{c})} V[\vec{K}](t),$$

where $V[\vec{K}]$ and $V[\vec{K}']$ are the respective Jones polynomials.

Proof: Let λ denote the total linking number of the component $\vec{c}$ in $\vec{K}$, i.e., $\lambda = \text{link}(\vec{c}, \vec{K} \backslash \vec{c})$. When the orientation of $\vec{c}$ is reversed only the signs of the crossings between $\vec{c}$ and $\vec{K} \backslash \vec{c}$ will change. See Figure 6.5. Therefore, the writhe (cf. Definition 5.7) of the two link diagrams $\vec{K}$ and $\vec{K}'$ are related by the formula

$$\mathrm{w}(\vec{K}') + 2\left(\frac{1}{2}\sum_{p \in \vec{c} \cap (\vec{K} \backslash \vec{c})} \delta(p)\right) = \mathrm{w}(\vec{K}) - 2\left(\frac{1}{2}\sum_{p \in \vec{c} \cap (\vec{K} \backslash \vec{c})} \delta(p)\right)$$

$$\Updownarrow$$

$$\mathrm{w}(\vec{K}') = \mathrm{w}(\vec{K}) - 4\lambda.$$

Let $K = K'$ be the unoriented link diagram obtained from $\vec{K}$ or $\vec{K}'$ by forgetting the orientation. Recall, that $\langle K \rangle$ denote the bracket polynomial (see Theorem 5.20) and that $\mathcal{L}[\vec{K}]$ denote the normalized bracket polynomial (see Definition 5.23). Then,

$$\begin{aligned}
\mathcal{L}[\vec{K}'](A) &= (-A^3)^{-\mathrm{w}(\vec{K}')} \langle K' \rangle (A) \\
&= (-A^3)^{-\mathrm{w}(\vec{K}')} \langle K \rangle (A) \\
&= (-A^3)^{-\mathrm{w}(\vec{K}) + 4\lambda} \langle K \rangle (A) \\
&= (-A^3)^{4\lambda} \mathcal{L}[\vec{K}](A).
\end{aligned}$$

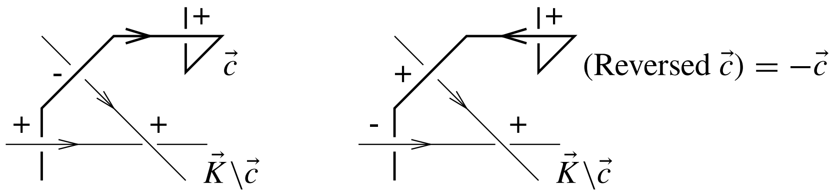

Figure 6.5: Reversing orientation of one component

Remembering that $V[\vec{K}](t) = \mathcal{L}[\vec{K}](t^{-1/4})$ (see page 88) we obtain

$$\begin{aligned} V[\vec{K}'](t) &= \mathcal{L}[\vec{K}'](t^{-1/4}) \\ &= t^{-3\lambda}\mathcal{L}[\vec{K}](t^{-1/4}) \\ &= t^{-3\lambda}V[\vec{K}](t) \\ &= t^{-3\,\mathrm{link}(\vec{c},\vec{K}\backslash\vec{c})}V[\vec{K}](t). \end{aligned}$$

This completes the proof □

The reversing property of the Jones polynomial can now be used to prove the next theorem that ensures that the Jones polynomial for the link, consisting of the center and boundary curves of a closed strips, is invariant under ambient isotopy and is independent of the way the strip is traversed.

Theorem 6.18 *Let M be a closed strip in $\mathbb{E}^3$ and let $\vec{K}$ be an oriented link, consisting of M's center and boundary curves, with orientation induced from the direction in which M is traversed. Then the Jones polynomial for $\vec{K}$, $V[\vec{K}](t)$, is invariant under shift of the direction in which M is traversed.*

Proof: Let M be a closed strip in $\mathbb{E}^3$ and let K be the link consisting of M's center curve, c_1, and boundary curve, c_2, or boundary curves, c_2 and c_3. Choose a direction in which M is traversed. This gives an orientation of the link K now denoted $\vec{K} = \vec{c}_1 \cup \vec{c}_2$ resp. $\vec{K} = \vec{c}_1 \cup \vec{c}_2 \cup \vec{c}_3$. We shall use a notation for reversing orientation on components of $\vec{K}$. This notation is that $-\vec{c}_i$ denote the component with the opposite orientation as $\vec{c}_i$ for $i = 1, 2, (3)$ and that $-\vec{K}$ denote the link where all components has the opposite orientation as in $\vec{K}$.

Suppose that K has two components such that $\vec{K} = \vec{c}_1 \cup \vec{c}_2$ and set

$$\begin{aligned} \lambda &= \mathrm{link}(\vec{c}_1, \vec{K}\backslash\vec{c}_1) \\ &= \mathrm{link}(\vec{c}_1, \vec{c}_2). \end{aligned}$$

Then,

$$-\lambda = \text{link}(-\vec{c}_1, \vec{c}_2).$$

According to the reversing property of the Jones polynomial we have

$$\begin{aligned} V[\vec{K}](t) &= V[\vec{c}_1 \cup \vec{c}_2](t) \\ &= t^{3\lambda} V[-\vec{c}_1 \cup \vec{c}_2](t) \\ &= t^{3\lambda} t^{3(-\lambda)} V[-\vec{c}_1 \cup -\vec{c}_2](t) \\ &= V[-\vec{K}](t). \end{aligned}$$

Suppose that K has three components[4]. Let

$$\begin{aligned} \lambda_{12} &= \text{link}(\vec{c}_1, \vec{c}_2), \\ \lambda_{13} &= \text{link}(\vec{c}_1, \vec{c}_3), \quad \text{and} \\ \lambda_{23} &= \text{link}(\vec{c}_2, \vec{c}_3). \end{aligned}$$

Then the following calculation, where the reversing property of the Jones polynomial is used three times, completes the proof.

$$\begin{aligned} V[\vec{K}](t) &= V[\vec{c}_1 \cup \vec{c}_2 \cup \vec{c}_3](t) \\ &= t^{3(\lambda_{12}+\lambda_{13})} V[-\vec{c}_1 \cup \vec{c}_2 \cup \vec{c}_3](t) \\ &= t^{3(\lambda_{12}+\lambda_{13})} t^{3(-\lambda_{12}+\lambda_{23})} V[-\vec{c}_1 \cup -\vec{c}_2 \cup \vec{c}_3](t) \\ &= t^{3(\lambda_{12}+\lambda_{13})} t^{3(-\lambda_{12}+\lambda_{23})} t^{3(-\lambda_{13}-\lambda_{23})} V[-\vec{c}_1 \cup -\vec{c}_2 \cup -\vec{c}_3](t) \\ &= V[-\vec{c}_1 \cup -\vec{c}_2 \cup -\vec{c}_3](t) \\ &= V[-\vec{K}](t). \end{aligned}$$

□

Consider n disjoint closed strips in $\mathbb{E}^3$. To be able to distinguish this geometric object from a link made of closed curves we shall denote this new class of geometric objects by the class of strip links. Our intention is to describe how the class of strip links differ from the class of (curve) links.

The Reidemeister moves of type II and type III for strips are similar to the Reidemeister moves of type II and type III for curves. See Figure 6.6 and compare with Figure 3.10. The Reidemeister move of type I for curves and the Reidemeister move of type I for strips are not alike! The reason is that when you stretch a curl on a strip you obtain a full twist on the strip[5]. This is also shown

[4]By induction on the number of components in a link we could prove that the Jones polynomial of an oriented link is invariant under a shift of the orientation of all components in the link. We shall not do this, because we focus on strips.

[5]Try this!

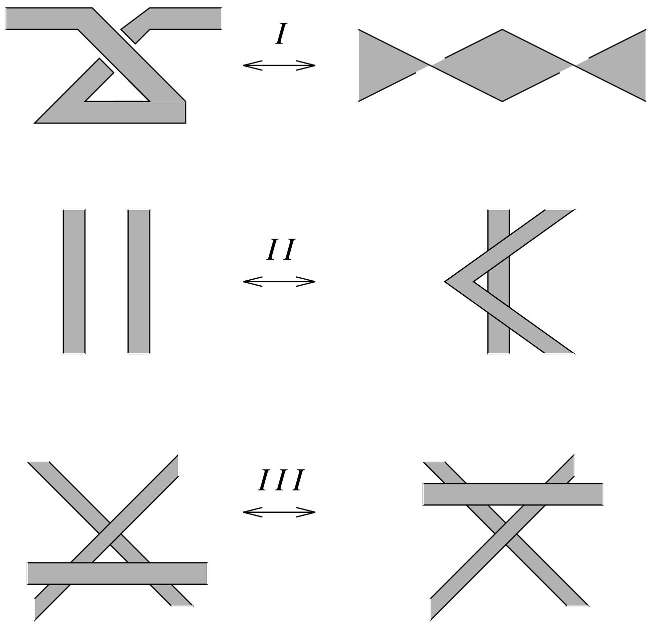

Figure 6.6: Reidemeister moves for strips

on Figure 6.6. Consider the polynomials from Chapter 5, which only are invariant under regular isotopy. All these polynomials have the following behaviour[6] under a Reidemeister move of type I:

$$R[\Omega_{\pm 1}(\vec{K})] = \alpha^{\pm 1} R[\vec{K}]$$

This behaviour is strongly connected to strip links. As mentioned by L.H. Kauffman in [Kauffman, 1990] page 452 and in [Kauffman, 1991] page 55 this can be seen as the axiom

$$R[curl] = R[\pm 1 \ \ twist] = \alpha^{\pm 1} R[0 \ \ twist],$$

where $curl$, $+1 \ \ twist$, and $0 \ \ twist$ are the strip link diagrams that only differ as shown on Figure 6.7. It is noteworthy that the α-variable measures the twisting in the corresponding part of the strip which is showed in Figure 6.7 and that the twisting number of this part equals the exponent of the variable α.

By the above discussion we see that all the polynomials invariant under regular isotopy (introduced in Chapter 5) can be seen as invariants for orientable closed strips. These polynomials are the bracket polynomial, the L-polynomial, and the H-polynomial.

[6]See Lemma 5.12.

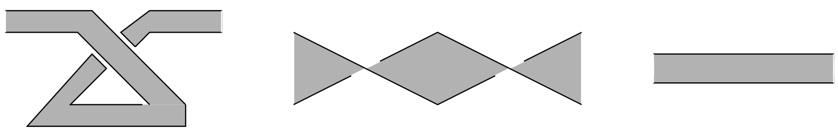

Figure 6.7: A notation on strip link diagrams

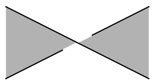

Figure 6.8: A notation on strip link diagrams

To obtain invariants for unorientable closed strips we suggest to add the axiom

$$R[\pm\frac{1}{2}\ twist] = \alpha^{\pm\frac{1}{2}} R[0\ twist]$$

to the defining relations for all the above mentioned polynomials. See Figure 6.8 for the notation of $+\frac{1}{2}\ twist$. This is the only possible way to extend these polynomial invariants for orientable strip links to polynomial invariants for all strip links (i.e. also to strip links with unorientable components). Indeed, one has to permit the following calculation

$$\begin{aligned} R[\pm 1\ twist] &= R[\pm\frac{1}{2}\ twist\ \ \pm\frac{1}{2}\ twist] \\ &= \alpha^{\pm\frac{1}{2}} R[\pm\frac{1}{2}\ twist] \\ &= \alpha^{\pm 1} R[0\ twist]. \end{aligned}$$

Furthermore, from the classification theorem for closed strips (Theorem 6.15) it is clear that this extension is possible, since the half twist commutes with crossings (Remark 6.13), i.e., that this last axiom for the polynomials is independent of the other axioms for the polynomials.

In this connection we have been (and still are) very interested in the constructions in the preprints [Hennings, 1989_1] and [Hennings, 1989_2] with the titles:

A polynomial invariant for oriented banded links.
A polynomial invariant for unoriented banded links.

Unfortunately, we have not been able to procure these preprints during the time we have worked on this thesis.

In Section 6.2 we found a closed braid representation of the link from a strip link with one component, i.e., a closed strip. To obtain a closed braid representation of the link from a strip link one has to be aware of the fact that it is only self-crossings (cf. Definition 5.9) in the strip link that has to be represented by the braid words $\boldsymbol{\sigma}_j$ and $\boldsymbol{\sigma}_j^{-1}$ (from Figure 6.2 on page 101). Crossings between two different components in the strip link are to be represented only as the cross from the same figure, i.e., without the braid words ξ. At last, the twisting number T of each component of the strip link isolated seen is represented as ξ^{2T} somewhere in their own components, because by Remark 6.13 ξ can be moved freely around.

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