Contributions to Geometric Knot Theory of Curves and Surfaces

Peter Røgen

Ph.D. Thesis

January 1999

Contributions to Geometric Knot Theory of Curves and Surfaces

PETER RØGEN

Department of Mathematics
Technical University of Denmark

Title of Thesis:

Contributions to Geometric Knot Theory of Curves and Surfaces

Graduate Student:

Peter Røgen

Supervisors:

Vagn Lundsgaard Hansen and Steen Markvorsen

Address:

Department of Mathematics
Technical University of Denmark
Building 303
DK-2800 Lyngby
DENMARK

Email:

Peter.Roegen@mat.dtu.dk
V.L.Hansen@mat.dtu.dk
S.Markvorsen@mat.dtu.dk

Thesis submitted in partial fulfilment of the requirements for the ph.d.-degree at the Technical University of Denmark.

Preface

This Ph.D. thesis is the result of three years work begun in February 1995, finished in January 1999, and interrupted by two half-year maternity leaves; one with my daughter Rebecca and one with my son Rasmus Emil. The work has been carried out at the Department of Mathematics, Technical University of Denmark under a DTU grant.

I sincerely thank my supervisors Professor Vagn Lundsgaard Hansen and Professor Steen Markvorsen for their support and for always taking time to discus mathematical as well as more personal matters, when needed. Moreover, I appreciate their confidence in letting me steer the direction of my research.

Special thanks go to Professor Herman Gluck, who I visited for one month during spring 1997 at the Department of Mathematics, University of Pennsylvania, Philadelphia, USA. I thank Herman Gluck for posing questions which started and guided my interests in flat surfaces and their boundaries, and for introducing me to the members of the department, including other visitors.

I would also like to thank:

> Jason Cantarella, who is a Ph.D. student of Herman Gluck, for sharing his very constructive ideas concerning isotopies of flat closed strip-like surfaces with me.
>
> Professor Robert Greene, for his always inspiring talks when visiting MAT, DTU, and for making me aware of the works of Herman Gluck and Liu-Hua Pan on the curvature sensitive version of Knot Theory.
>
> Carmen Chicone and his wife for making my visit at the Department of Mathematics, University of Missouri, Colombia, USA, very pleasant.

I thank my friends and colleagues in the staff of the Department of Mathematics, DTU, for the inspiring atmosphere and their help on many occasions.

My children, Rebecca and Rasmus, I thank you for bearing with an occasionally absent minded father and for bringing joy and life. Most of all I thank my wife, Birte, for having our children, for support, and for taking over for me when traveling or busy - but mostly, for being my wife.

Lyngby, January 28, 1999
Peter Røgen

Summary

The subject of this Ph.D. thesis is geometric knot theory of curves and (primarily flat) surfaces in three-dimensional euclidean space.

The integral of the geodesic curvature of a closed curve on the unit 2-sphere equals the sum of the areas of the connected components of its complement weighted by a half integer valued index. When applying this formula to closed indicatrices, defined by the Frenet frame of a space curve, uniform proofs of classic as well as new integral geometric results are obtained.

In 1950, W. Fenchel formulated the question: *Is a closed curve on the unit* 2*-sphere, with integrated geodesic curvature equal to an integral multiple of* 2π*, a principal normal indicatrix of a closed space curve?* A family of examples, showing that the general answer is in the negative, is presented. However, a necessary and sufficient condition for a closed curve on the unit 2-sphere to be a principal normal indicatrix of a closed space curve with, what we will call, positive regularity index is given. This includes all closed space curves with non-vanishing curvature and their principal normal indicatrices.

Two (purely) knot theoretical results are given. One is an upper bound on the unknotting number of any knot with a given shadow. The other result concerns the set of all numerical knot invariants equipped with the compact-open topology, and states: *The Vassiliev knot invariants constitute a dense subset if and only if they separate knots*.

The isotopy classes of compact flat surfaces with nonempty boundaries in 3-space are described by the following result: *In* 3*-space, any compact surface with nonempty boundary is isotopic to a flat surface. Two such flat surfaces are isotopic through ordinary surfaces if and only if they are isotopic through flat surfaces.* Combined with Seifert surfaces this gives that *any knot can be deformed until it bounds a flat embedded orientable surface*. This result calls for a geometric characterization of knots and links bounding flat (immersed) surfaces.

Finally, two results that point towards such a characterization are emphasized. The first result is that *the number of* 3*-singular points,* i.e. points of either vanishing curvature or of vanishing torsion, *on the boundary of a compact flat immersed surface is at least twice the absolute value of the Euler characteristic of the surface.* The second result is a set of necessary and, in a slightly weakened sense, sufficient conditions for a simple closed curve to be, what we will call, a generic boundary of a flat immersed surface without planar regions. These conditions are easily generalized to the problem of whether or not a link is a generic boundary of a flat surface without planar regions.

Dansk resumé

Emnet for denne ph.d. afhandling er geometrisk knudeteori for kurver og (primært lokalt udfoldelige) flader i det tre-dimensionelle euklidiske rum.

Integralet af den geodætiske krumning af en lukket kurve på enheds to-sfæren er lig summen af arealerne af sammenhængskomponenterne af kurvens komplement vægtet med et halv-talligt indeks. Ved anvendelse af dette resultat på lukkede indikatricer, defineret ved det ledsagende koordinatsystem af en rumkurve, opnås uniforme beviser for såvel klassiske som nye integralgeometriske resultater.

I 1950 formulerede W. Fenchel spørgsmålet: *Lad der være givet en lukket kurve på enheds to-sfæren med integreret geodætisk krumning et heltalligt multipla af* 2π. *Er denne kurve da lig billedet af hovednormalerne til en lukket rumkurve?* Det generelle svar er afkræftende, men her gives en nødvendig og tilstrækkelig betingelse for at en lukket sfærisk kurve er billedet af hovednormalerne til en lukket rumkurve med (hvad der her kaldes) positivt regularitets indeks. Dette inkluderer alle lukkede rumkurver med positiv krumning og deres hovednormal billeder.

Afhandlingen indeholder to (rent) knudetoretiske resultater. (1): En øvre grænse på opløsningstallet (unknotting number) af enhver knude med en given skygge. (2) *I mængden af alle numeriske knudeinvarianter, udstyret med den kompakt-åbne topologi, udgør Vassiliev knudeinvarianterne en tæt delmængde hvis og kun hvis de separerer knuder*.

Isotopiklasserne af kompakte lokalt udfoldelige flader med ikke forsvindende rand er fastlagt ved: *I det tre-dimensionelle euklidiske rum er enhver kompakt flade med ikke forsvindende rand isotop med en lokalt udfoldelig flade. To sådanne lokalt udfoldelige flader er isotope gennem ordinære flader, hvis og kun hvis de er isotope gennem lokalt udfoldelige flader.* Kombineret med Seifert flader fås, at *enhver knude kan deformeres til at afgrænse en lokalt udfoldelig indlagt orienterbar flade*. Dette resultat åbner for en geometrisk karakterisering af knuder og lænker, der afgrænser lokalt udfoldelige flader.

Sluttelig fremhæves to resultater, der peger frem mod en sådan karakterisering. Det første er, at *antallet af* 3*-singulære punkter*, dvs, punkter hvori enten krumning eller torsion er lig nul, *på randen af en lokalt udfoldelig immerseret flade er større end eller lig med to gange den numeriske værdi af Euler karakteristikken af fladen.* Det andet resultat består af et sæt nødvendige og, i en svagere forstand, tilstrækkelige betingelser for, at en simpel lukket rumkurve er (hvad her kaldes) en generisk rand af en lokalt udfoldelig immerseret flade, der ikke indeholder et plant område. Disse betingelser overføres umiddelbart til problemet om hvorvidt en lænke er en generisk rand af en lokalt udfoldelig immerseret flade uden plane områder.

Contents

Chapter 1

Introduction

This thesis contains two main parts. The first part concerns the differential geometry of closed space curves and is divided into Chapters 2-4. The second part consists of the Chapters 6 and 7 and concerns compact flat surfaces, i.e., compact surfaces with Gaussian curvature identically equal to zero, and their non-vanishing boundaries. The two parts are separated by Chapter 5, the only purely knot theoretical chapter of the thesis. The main result in Chapter 5 is that the set of Vassiliev knot invariants constitute a dense set of knot invariants (as a subspace of the set of all knot invariants equipped with the compact-open topology) if and only if they separate knots. A motivating example using the twist sequence of $(2, 2i + 1)$ torus knots is given.

1.1 The differential geometry of closed space curves

By the differential geometry of a space curve we mean its Frenet Apparatus in case the curve has non-vanishing curveture. In Chapter 2 we introduce the Frenet Apparatus as well as a generalization considered by W. Fenchel in [10]. A closed space curve with non-vanishing curvature defines, via its Frenet Frame, three closed curves on the unit 2-sphere: the spherical indicatrices of its tangent vectors, of its principal normal vectors, and of its binormal vectors. Hereby, the study of the differential geometry of a space curve can often be reduced to the study of curves on the unit 2-sphere.

The main result in Chapter 3 is that integrated geodesic curvature of a (non-simple) closed curve on the unit 2-sphere equals a half integer weighted sum of the areas of the connected components of the complement of the curve. These weights, that give a spherical analogy to the winding number of closed plane curves, are determined by Gauss-Bonnet's theorem after cutting the curve into simple closed sub-curves.

If the spherical curve is the tangent indicatrix of a space curve, a new short proof of a formula for integrated torsion, presented in the unpublished manuscript [5] by C. Chicone and N.J. Kalton, is obtained. Applying this result to the principal normal indicatrix, a theorem by Jacobi, stating that *a simple closed principal normal indicatrix of a closed space curve with non-vanishing curvature bisects the unit* 2*-sphere*, is generalized to non-simple principal normal indicatrices. Moreover, this generalization of Jacobi's theorem is shown to be equivalent to the classical result that *the integrated torsion of a closed curve lying on a sphere is zero*. Some errors in the literature, concerning generalizations of Jacobi's Theorem, are corrected, or, pointed out.

As a spinoff, it is shown that a factorization of a knot diagram into simple closed subcurves defines an immersed disc with the knot as boundary. This is then used to give an upper bound on the unknotting number of knots with a given shadow.

In the article [9] of 1930, W. Fenchel characterizes all closed curves on the 2-sphere that occur as a tangent indicatrix of a closed space curve. In [10] of 1950, he characterizes all closed curves on the 2-sphere that occur as a binormal indicatrix of a closed space curve. W. Fenchel concludes the article [10] by posing an open question about which closed curves on the 2-sphere are principal normal indicatrices of closed space curves. Chapter 4 contains a set of answers to this question.

A theorem due to J. Weiner [49], which has been proved by curve theoretical methods by L. Jizhi and W. Youning in [24] and by B. Solomon in [45], implies that a principal normal indicatrix of a closed space curve with non-vanishing curvature has integrated geodesic curvature zero and contains no subarc with integrated geodesic curvature π. This corollary is not pointed out in any other reference! The main result in Chapter 4 is that the inverse problem always has solutions if one allows zero and negative curvature of space curves, and we explain why this is not true if non-vanishing curvature is required. This answers affirmatively the question posed by W. Fenchel in [10], under the above assumptions, but in general the question must be answered in the negative as shown by examples.

1.2 Flat compact surfaces and their boundaries

The single main source of inspiration behind the studies of flat compact surfaces and their boundaries in 3-space, reported here, is the papers [16] and [17] by Herman Gluck and Liu-Hua Pan introducing the curvature sensitive version of knot theory. The main results in these two papers are

(1) Any two smooth simple closed curves in 3-space each having nowhere vanishing curvature, can be deformed into one another through a one-parameter family of such curves if and only if they have the same knot type and the same self-linking number[1].

(2) In 3-space, any compact orientable surface with nonempty boundary can be deformed into one with positive curvature. Any two such surfaces with positive curvature can be deformed into one another through surfaces of positive curvature if and only if they can be deformed into one another through ordinary surfaces, preserving their natural orientations.

[1]The self-linking number of a given curve is the linking number between the given curve and a curve obtained by slightly pushing the given curve along the principal normals.

(3) If a smooth knot is the boundary of a compact orientable surface of positive curvature, then its self-linking number is zero.

(4) If two smooth knots together bounds a compact orientable surface of positive curvature, then their self-linking numbers are equal.

(5) In 3-space there exist simple closed curves with nowhere vanishing curvature and self-linking number zero, which do not bound any compact orientable surface of positive curvature.

(6) In 3-space any simple closed curve with nowhere vanishing curvature and self-linking number zero, can be deformed through such curves until it bounds a compact orientable surface of positive curvature.

By (1) the curvature sensitive version of knot theory equals traditional knot theory plus self-linking numbers. Their main result, (2), describes the isotopy classes of compact curvature surfaces with positive curvature. A smooth curve on a positive curvature surface has nowhere vanishing curvature. Hence, considering knots with nowhere vanishing curvature, it is natural to consider surfaces with positive curvature as an akin to Seifert surfaces in traditional knot theory. However, the gap between (5) and (6) opens the question: *Which positive curvature knots bound positive curvature surfaces?* The following question, motivated by the Four Vertex Theorem, is raised by H. Rosenberg and might help narrowing this gap: *Does every curve bounding a surface of positive curvature in* 3*-space have four vertices, i.e., points where the torsion vanishes?* For further questions and conjectures concerning positive curvature (hyper)surfaces see [13].

Here we study flat, i.e., zero Gaussian curvature, surfaces which are compact and have nonempty boundaries. The first result, Theorem 6.1, shows that the isotopy classes of flat surfaces are in one-one correspondence with the isotopy classes of ordinary compact, orientable as well as non-orientable, surfaces with nonempty boundary. Theorem 6.1 asserts:

> In 3-space, any compact surface with nonempty boundary is isotopic to a flat surface. Two such flat surfaces are isotopic through ordinary surfaces if and only if they are isotopic through flat surfaces.

Chapter 6 is mainly devoted to the proof of this result - but ends with a discussion of the isotopy classes of negative curvature surfaces. This discussion leads to Conjecture 6.8 stating:

> In 3-space, any compact surface with nonempty boundary is isotopic to a negative curvature surface. Any two such negative curvature surfaces, S_1 and S_2, are isotopic through negative curvature surfaces if and only if there exists an isotopy through ordinary surfaces between S_1 and S_2, such that, for each simple closed curve on S_1, this curve and its image on S_2, under this isotopy, have equal rotational indices with respect to the principal directions on the respective surfaces.

An immediate corollary of Theorem 6.1, Corollary 7.1, states:

> Any simple closed space curve can be deformed until it bounds an orientable compact embedded flat surface.

However, Theorem 7.28 asserts

> There exist simple closed curves in 3-space that do not bound any flat compact immersed surface.

Thus a gap, as in the case of positive curvature surfaces, opens and raises the question about *which knots (or links) bound flat compact surfaces?*

Our first result in this direction, Theorem 7.2, applies to both flat as well as positive curvature surfaces and states:

> Let $\gamma = \partial S$ be a simple closed curve bounding an immersed orientable non-negative curvature surface S of genus $\mathrm{g}(S)$ (and Euler characteristic $\chi(S)$) immersed in 3-space. Then the total curvature of γ is bigger than or equal to $2\pi|\chi(S)| = 2\pi|2\,\mathrm{g}(S) - 1|$.

Applied to the set of torus knots, it follows that this lower bound on the total curvature of a knot bounding an embedded orientable non-negative curvature surface can be arbitrarily much larger than the infimum of curvature needed for the knot to have its knot type.

A simple smooth closed curve on a flat surface may have vanishing curvature and need therefore not to have a well-defined self-linking number. Moreover, if a curve with non-vanishing curvature bounds a flat surface then it may have any given self-linking number. See Footnote 5 in Section 7.6. Hence, concerning flat surfaces instead of poitive curvature surfaces there are no analogies to the above results (3) and (4).

A simple closed space curve bounding a compact flat surface may have points with zero curvature. Hence, instead of counting its vertices (zero torsion points) one is lead to count its 3-singular points. A 3-simgular point is a point on a curve in which its first three derivaties are linear dependent. On a regular curve, a 3-singular point thus is either a point of zero curvature, or, if not, then of zero torsion. As an important step in our analysis of boundaries of flat compact surfaces we prove the optimal Theorem 7.4, that states:

> The boundary (knot or link) of a flat and compact (immersed) surface in 3-space with Euler characteristic $\chi(S)$ and p planar regions has at least $2(|\chi(S)| + p)$ 3-singular points.

A connection between this generalized vertex theorem and the Four Vertex Theorem, [44], and a generalization thereof given in [40], is pointed out in Section 7.4. This is done by studying the number and multiplicities of 3-singular points on a simple closed curve bounding two distinct flat discs. See Section 7.4 for further detailes.

The thesis concludes, by giving a set of necessary and, in a slightly weakened sense, sufficient conditions for a simple closed curve to be, what we call, a generic boundary of a flat immersed surface without planar regions. In fact the solutions are given via explicit parametrizations and these conditions are easily generalized to the problem of a link being a generic boundary of a flat surface without planar regions.

Chapter 2

The Frenet Apparatus

2.1 The Frenet's equations

In this section we introduce the Frenet Apparatus. For this purpose let $\gamma : I \to \mathbb{R}^3$ be a space curve parametrized by arclength s. Assume furthermore, that γ is three times continuous differentiable (C^3 on shorthand). Denote differentiation with respect to arclength by a prime; then $\gamma' = \mathbf{t}$ is a unit tangent vector. The curvature of γ at $\gamma(s)$, denoted $\kappa(s)$, is the length of $\mathbf{t}'(s)$. If $\kappa(s) \neq 0$, the unit principal normal vector to γ at $\gamma(s)$, denoted $\mathbf{n}(s)$, is defined by $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$ and one may supply to an o.n. basis in $\mathbb{R}^3$ by adding the unit binormal vector $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$, where $\times$ is the vector product in 3-space with usual orientation. The triple $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ is referred to as the Frenet frame of γ at $\gamma(s)$. If γ has non-vanishing curvature, the evolution of the Frenet frame along γ is given by the Frenet equations:

$$\begin{array}{rcrcr} \mathbf{t}' & = & & \kappa\mathbf{n} & \\ \mathbf{n}' & = & -\kappa\mathbf{t} & & +\tau\mathbf{b}, \\ \mathbf{b}' & = & & -\tau\mathbf{n} & \end{array}$$

where τ is the torsion of γ. Quoting [16], page 3: "These equations appeared in the thesis of F. Frenet (Toulouse, 1847), an abstract of which was published in 1852, and were independently discovered by J. A. Serret in 1851."

2.2 The Fenchel assumption

In this section we explain the generalization of the Frenet Apparatus given by W. Fenchel in [10] pp. 44–45 in the case of closed space curves. Let $\gamma : \mathbb{R}/l\mathbb{Z} \to \mathbb{R}^3$ be a closed space curve of class C^4, parametrized by arc length s, with only finitely many zero points for the curvature. Whenever $\kappa > 0$, the first of the Frenet equations, $\mathbf{t}' = \kappa\mathbf{n}$, defines the principal normal, $\mathbf{n}$. From now on we assume that there exists a continuous vector field N along γ, such that, $N(s)$ equals plus or minus $\mathbf{n}(s)$ whenever $\kappa > 0$. A sufficient condition for the existence of such a vector field N is that if $\kappa(s) = 0$ then the third or fourth derivative, or a higher derivative of γ in case γ is more than four times continuous differentiable, is non-zero at this point. The equivalent condition, given a regular parametrization, not necessarily by arc length, is given in [33], p. 512. Exchanging $\mathbf{n}$ by N in the first Frenet equation we get $\mathbf{t}' = kN$, where k can be both positive and negative. Finally, we define $B = \mathbf{t} \times N$ and denote the unit tangent vector by the capital letter T. Using this new trihedral T, N, and B we get the usual Frenet Equations

$$\begin{array}{rcrcc} T' & = & & kN & \\ N' & = & -kT & & +\tau B, \\ B' & = & & -\tau N & \end{array}$$

where k, N, and B are defined up to a common global change of sign.

From now on, we furthermore assume that curvature and torsion only vanish a finite number of times and not simultaneously. We call this assumption *the Fenchel Assumption*. By the Fenchel Assumption the new principal normal N sweeps out a regular C^2 curve on the unit 2-sphere. In Chapter 4 we give answers to a question, raised by W. Fenchel in [10], about which closed spherical curves are the image of the principal normals of a closed space curve. This problem has a more "natural" setting with the Fenchel Assumption replacing the more usual assumption of space curves having non-vanishing curvature. The technique used in Chapter 4 requires that N sweeps out a regular C^2 curve on the unit 2-sphere. This is why we have chosen to work with with the assumption of γ being C^4, even though C^3 is enough to explain the Fenchel Assumption.

2.3 The indicatrices

A curve with Frenet frame defines three curves on the unit 2-sphere, namely, the spherical indicatrices of the tangent vectors, of the principal normal vectors, and of the binormal vectors. Denote these curves by $\Gamma_{\mathbf{t}}$, $\Gamma_{\mathbf{n}}$, and $\Gamma_{\mathbf{b}}$ respectively. The length of the tangent indicatrix (tantrix in [45] or Gauss image), the principal normal indicatrix, resp., the binormal indicatrix is

$$|\Gamma_{\mathbf{t}}| = \kappa(C) = \int_C \kappa(s)ds$$
$$|\Gamma_{\mathbf{n}}| = \omega(C) = \int_C \sqrt{\kappa^2(s) + \tau^2(s)}ds$$
$$|\Gamma_{\mathbf{b}}| = |\tau|(C) = \int_C |\tau(s)|ds.$$

Under the Fenchel Assumption, given in Section 2.2, the above equations hold with κ replaced by $|k|$ and they are independent of the global choice of sing.

It is not standard to use the notation, $|\tau|(C)$, for the total torsion of a space curve - but it is convenient as $\tau(C)$ hereby can denote the integral of the torsion of C with respect to arc-length, s, i.e.,

$$\tau(C) = \int_C \tau(s)ds.$$

2.4 On the Frenet's equations

The Fundamental Theorem of the differential geometry of space curves states: "*A space curve with non-vanishing curvature is, up to euclidean motions, uniquely given by its curvature and torsion as functions of arc length*". Up to the common global change of sign on k, N, and B this is also holds under the Fenchel assumption.

There are two "longstanding" problems concerning the differential geometry of space curves:

To Solve the Frenet equations.

To give necessary and sufficient conditions on a pair of periodic functions to be curvature and torsion of a closed space curve.

A solution of the first problem gives, naturally, a solution to the latter problem. In [41] and [21] this is done, but in both references the solution of the Frenet's equations is expressed in terms of a infinite sum of matrices involving infinitely repeated integrations. This is not a satisfactory solution to the problems, which also is remarked in [21], but it is, to my knowledge, not known if there exist general closed form solutions to the Frenet's equations involving only finitely many calculations.

In [19] it is proven that explicit solutions (in terms of the exponential of the integral of the curvature matrix) can be obtained, restricting to curves in $\mathbb{R}^3$, if and only if torsion is a constant multiple of the curvature, which, also in [19], is proven to give the set of generalized Helices, i.e., curves who's tangents make constant angle with a fixed vector in $\mathbb{R}^3$.

We use the remainder of this section to elaborate one these two problems. Let σ denote arclength of the tangent indicatrix. If s denotes arclength of the space curve then $\frac{d\sigma}{ds} = \kappa$. Assuming non-vanishing curvature of the space curve we in the two first of the Frenet's equations may divide by κ and if we now let prime indicate differentiation with respect to σ we get

$$\begin{aligned}\mathbf{t}' &= \mathbf{n}\\ \mathbf{t}'' &= \mathbf{n}' = -\mathbf{t} + \frac{\tau}{\kappa}\mathbf{b} = -\mathbf{t} + \frac{\tau}{\kappa}\mathbf{t}\times\mathbf{t}'.\end{aligned}$$

The unit vector $\mathbf{t}\times\mathbf{t}'$ is a tangent vector to the unit 2-sphere at the point $\mathbf{t}$ and it is orthogonal to $\mathbf{t}'$. Hence, the fraction $\frac{\tau}{\kappa}$ equals the geodesic curvature, κ_g, of the tangent indicatrix. A curve on the unit 2-sphere is, up to rotations, uniquely given by its geodesic curvature through the evolutional equations for curves on the unit 2-sphere, i.e., three non-linear coupled differential equations $\mathbf{t}'' = -\mathbf{t} + \kappa_g\mathbf{t}\times\mathbf{t}'$. Hence, as suggested by W. Fenchel in [10] one may instead of solving the Frenet's Equations solve the evolutional equations for curves on the unit 2-sphere to give necessary and sufficient conditions on a periodic function κ_g to be the geodesic curvature of a closed curve on the unit 2-sphere. However, in [29] it is stated that "*In the class of simple natural integral expressions the necessary conditions are either quite empty or are far from sufficient ones*" - for a more precise statement consult [29].

Let $\kappa > 0$ and τ be periodic functions with period L in the variable s, which is to be the arclength of a space curve, and define a diffeomorphism by $\sigma(s) = \int_0^s \kappa(s)ds$. Furthermore, set $\kappa_g(\sigma) = \frac{\tau(s(\sigma))}{\kappa(s(\sigma))}$ and assume that a solution $\mathbf{t}(\sigma)$ (and hence all solutions) to the evolutional equations for curves on the unit 2-sphere with this κ_g is closed. Then the desired space curve is given by

$$\mathbf{r}(s) - \mathbf{r}(0) = \int_0^s \mathbf{t}(s)ds = \int_0^{\sigma(s)} \frac{\mathbf{t}(\sigma)}{\kappa(s(\sigma))}d\sigma.$$

Hence, this space curve is closed if and only if $\int_0^{\int_0^L \kappa ds} \frac{\mathbf{t}(\sigma)}{\kappa(s(\sigma))}d\sigma = \mathbf{0}$ and it is simple if and only if $\int_a^b \frac{\mathbf{t}(\sigma)}{\kappa(s(\sigma))}d\sigma \neq \mathbf{0}$ for all $|a-b| < \int_0^L \kappa(s)ds$. This approach to the original problems has the advantage that there is only one function, the geodesic curvature, steering the $\mathbf{t}$-solutions not two and then the space curve is given by quadrature. If it is possible to give more explicit (and hand-able) conditions on curvature>0 and torsion to give a *simple* closed curve, then by the theorem from [16] stating: *Any two smooth simple closed curves in* 3*-space each having nowhere vanishing curvature, can be deformed into one another through a one-parameter family of such curves if and only if they have the same knot type and the same self-linking number*[1], one would get access to a classification of knots with self-linking numbers attached, by curvature and torsion.

We end this elaboration by deriving one third order linear differential equation, which for generic κ_g (assuming that κ_g and its derivative do not vanish simultaneously), is equivalent to the evolutional equations for curves on the unit 2-sphere[2].

$$\begin{aligned}
\mathbf{t}'' &= -\mathbf{t} + \kappa_g \mathbf{t} \times \mathbf{t}' \\
&\Downarrow \\
\mathbf{t}''' &= -\mathbf{t}' + \kappa_g' \mathbf{t} \times \mathbf{t}' + \kappa_g \mathbf{t} \times \mathbf{t}'' \\
&= -\mathbf{t}' + \kappa_g' \mathbf{t} \times \mathbf{t}' + \kappa_g \mathbf{t} \times (-\mathbf{t} + \kappa_g \mathbf{t} \times \mathbf{t}') \\
&= -\mathbf{t}' + \kappa_g' \mathbf{t} \times \mathbf{t}' - \kappa_g^2 \mathbf{t}'
\end{aligned}$$

Hence, we have the equations

$$\begin{aligned}
\kappa_g' \mathbf{t}'' &= -\kappa_g' \mathbf{t} + \kappa_g \kappa_g' \mathbf{t} \times \mathbf{t}' \\
\kappa_g \mathbf{t}''' &= -\kappa_g \mathbf{t}' + \kappa_g \kappa_g' \mathbf{t} \times \mathbf{t}' - \kappa_g^3 \mathbf{t}'.
\end{aligned}$$

And by subtracting the first equation from the second equation we get

$$\kappa_g \mathbf{t}''' - \kappa_g' \mathbf{t}'' + (\kappa_g + \kappa_g^3)\mathbf{t}' - \kappa_g' \mathbf{t} = \mathbf{0},$$

[1] The self-linking number of a given curve is the linking number between the given curve and a curve obtained by slightly pushing the given curve along the principal normals.

[2] This equation also appears in [20].

where the three coordinate equations are uncoupled. Thus we are left with only one third order linear differential equation and the initial conditions can eg. be chosen from

$$\begin{aligned} \mathbf{t}(0) &= (1, 0, 0) \\ \mathbf{t}'(0) &= (0, 1, 0) \\ \mathbf{t}''(0) &= (-1, 0, \kappa_g(0)). \end{aligned}$$

Hereby, periodicity of the first coordinate of $\mathbf{t}$ implies that $\mathbf{t}$ is closed, periodicity of the second coordinate implies smoothness of $\mathbf{t}$ at $\mathbf{t}(0)$, and it is unnecessary to examine the last coordinate of $\mathbf{t}$.

In this setting the solutions corresponding to explicit solutions to the Frenet equations in [19] are obtained as follows. If torsion is a constant multiple of curvature then the geodesic curvature of the tangent indicatrix, $\kappa_g = \tau/\kappa$, is constant. Hence, the solutions to $\kappa_g \mathbf{t}''' - \kappa_g' \mathbf{t}'' + (\kappa_g + \kappa_g^3)\mathbf{t}' - \kappa_g' \mathbf{t} = \mathbf{0}$ with the appropriate initial conditions are pieces of circles on the unit 2-sphere of (euclidean) radius $1/\sqrt{1+\kappa_g^2}$. Which obviously characterizes generalized Helices.

Chapter 3

Gauss-Bonnet's Theorem and closed Frenet frames[1]

3.1 Introduction

The main result in this chapter is that integrated geodesic curvature of a (non-simple) closed curve on the unit 2-sphere equals a half integer weighted sum of the areas of the connected components of the complement of the curve. These weights give a spherical analogy to the winding number of closed plane curves and are found using Gauss-Bonnet's theorem after cutting the curve into simple closed sub-curves.

If the spherical curve is the tangent indicatrix of a space curve we obtain a new short proof of a formula for integrated torsion presented in the unpublished manuscript [5] by C. Chicone and N.J. Kalton. It was the observation, that their formula for the integrated torsion does not depend on the space curve itself but only on its tangent indicatrix, that lead to the wish of finding a proof, as the one presented here, that reveals this.

Applying this result to the principal normal indicatrix a theorem by Jacobi stating that *a simple closed principal normal indicatrix of a closed space curve with non-vanishing curvature bisects the unit* 2*-sphere* is generalized to non-simple principal normal indicatrices. Some errors in the literature, concerning generalizations of Jacobi's Theorem, are corrected or pointed out.

As a spinoff, we show that a factorization of a knot diagram into simple closed sub-curves defines an immersed disc with the knot as boundary and use this to give an upper bound on the unknotting number of knots with a given shadow.

Our main topic is closed curves on the unit 2-sphere that arise as indicatrices of space curves. This explains

Definition 3.1 *We say that the tangent indicatrix,* $\Gamma_{\mathbf{t}}$*, of a regular curve* $\mathbf{r} : [0, L] \to \mathbb{R}^3$ *with non-vanishing curvature, is closed if* $\mathbf{t}(0) = \mathbf{t}(L)$ *and all derivatives of* $\Gamma_{\mathbf{t}}$ *agree in*

[1] All results in this chapter, with the exception of Remarks 3.25 and 3.31 and Section 3.9, are contained in the article [37].

this point. With similar definitions of closed principal normal and binormal indicatrices we say that a space curve has closed Frenet frame if its indicatrices are closed.

An example of a non-closed curve with closed Frenet frame is a suitable piece of a circular or generalized (cf. Section 2.4) Helix. This explains half of the title of this chapter. I order to use Gauss-Bonnet's theorem on closed Frenet frames we have to factorize non-simple curves into simple closed sub-curves. Due to an error in the literature the next section is devoted to this factorization.

3.2 SCS-factorization of closed curves

In this section we consider closed continuous curves $\Gamma : \mathbb{S}^1 \to \mathbb{T}$ in a topological space $\mathbb{T}$. We assume that the curves are not constant in any interval of $\mathbb{S}^1$.

Assume there exists a closed interval $I \in \mathbb{S}^1$ such that when we identify the endpoints of I, then $\Gamma|I$ is a simple closed curve in $\mathbb{T}$. By an elementary factorization of the curve Γ, we mean a splitting of Γ into the above simple closed curve S and into the rest of Γ, denoted by $\tilde{\Gamma}$. For an elementary factorization of Γ we write $\Gamma \to S + \tilde{\Gamma}$, where $S = \Gamma|I$ (identifying the endpoints of I) is a simple closed sub-curve of Γ and $\tilde{\Gamma} = \Gamma|(\mathbb{S}^1 \setminus \mathrm{int}(I))$ (identifying the endpoints of $\mathbb{S}^1$ minus the interior of I) is the rest of Γ.

Definition 3.2 *Let $\Gamma : \mathbb{S}^1 \to \mathbb{T}$ be a closed continuous curve in a topological space $\mathbb{T}$. If there exists a finite number of elementary factorizations*

$$\Gamma = \Gamma_0 \to S_1 + \Gamma_1 \to S_1 + S_2 + \Gamma_2 \to \cdots \to S_1 + S_2 + \cdots + S_{n-1} + \Gamma_{n-1}$$

*such that Γ_{n-1} is a simple closed curve S_n, then we say the curve Γ possesses a simple closed sub-curve factorization (*scs*-factorization). In this case we simply write $\Gamma \to S_1 + \cdots + S_n$ and we say that the* scs*-factorization is of order n.*

The scs-factorizations are obviously preserved under homeomorphism of the ambient space. This is basically why we give the definition in a general topological space.

By the simple closed sub-curve number (scs-number), $\mathrm{scsn}(\Gamma)$, of Γ we mean the minimum of the orders of the scs-factorizations that the curve Γ possesses. If the curve Γ does not possess an scs-factorization, then we set $\mathrm{scsn}(\Gamma) = +\infty$.

Remark 3.3 *In [42] an analogue to our* scs*-number, which is crucial for this paper, is defined using an algorithm. This algorithm starts in an arbitrary point P_0 on the curve Γ_0 and traverses the curve until the first pre-traversed point P_1 is reached. Then the simple closed sub-curve S_1, from the first time P_1 lies on the curve Γ_0 to the second time P_1 lies on the curve Γ_0, is excluded from Γ_0. Now, apply the elementary factorization $\Gamma_0 \to S_1 + \Gamma_1$ and mark Γ_1 with the starting point P_1. If this iterative process stops after l steps, then Γ_0 (or more correctly the pair (Γ_0, P_0)) is said to have "Umlaufszahl" l.*

In [42] it is claimed that the "Umlaufszahl" is independent of the starting point. This is false! On Figure 3.1 is a curve with "Umlaufszahl" 3 *if traversion starts at the*

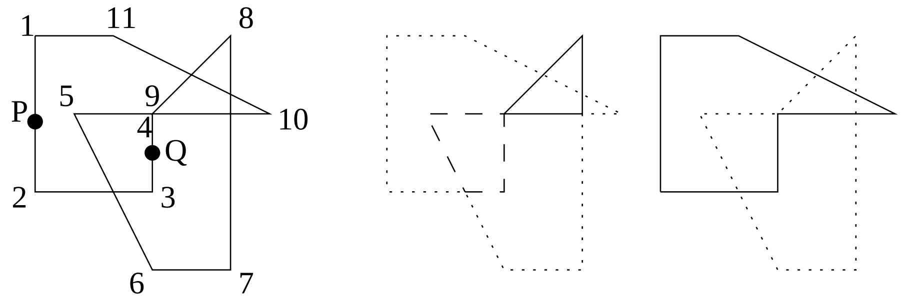


Figure 3.1: *A curve with two different "Umlaufszahlen".*

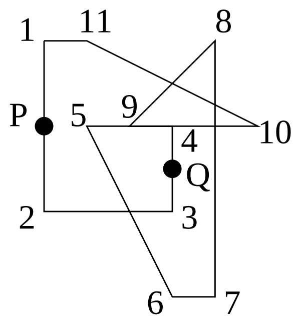


Figure 3.2: *A curve with finite and infinite "Umlaufszahlen".*

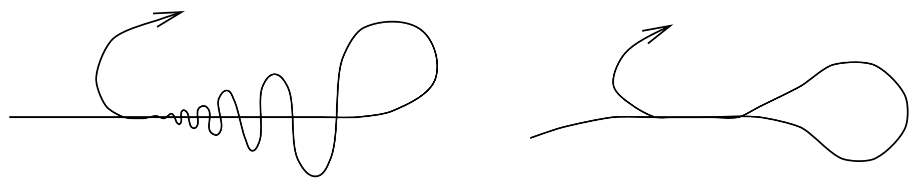


Figure 3.3: *Pieces of curves with no* scs-*factorization.*

point P and "Umlaufszahl" 2 if traversion starts at the point Q. On Figure 3.2 we show a curve with finite "Umlaufszahl" if traversion starts at the point P and infinite "Umlaufszahl" if traversion starts at the point Q. The "Umlaufszahl" or "Umlaufzahl" is used in [14, 25, 31, 35] which should be read with care.

It is possible for a closed curve to have infinite scs-number. On Figure 3.3 are shown two such curves[2]. To give a sufficient condition for a closed curve to possess an scs-factorization we need some notation. Let Γ be a non simple closed curve and let $P \in \Gamma(\mathbb{S}^1)$ be a point of self-intersection of Γ. If the inverse image of the point P under the map Γ, $\Gamma^{-1}(P) \subset \mathbb{S}^1$, consists of m parameter values, then we say the point P has multiplicity m and that the curve Γ has $(m-1)$ self-intersections in the point P. For a closed curve Γ with self-intersection points $P_1, \ldots, P_n$, each of multiplicity m_j, we let $\mathrm{s}(\Gamma) = \sum_{j=1}^{n}(m_j - 1)$ denote the number of self-intersections of Γ.

Theorem 3.4 *Let $\Gamma : \mathbb{S}^1 \to \mathbb{T}$ be a continuous closed curve with only finitely many self-intersections. Then Γ possesses at least one* scs*-factorization and any* scs*-factorization of Γ has order less than or equal to the number of Γ's self-intersections plus one. In particular the simple closed sub-curve number of Γ fulfills* $\mathrm{scsn}(\Gamma) \leq 1 + \mathrm{s}(\Gamma)$.

Proof: If Γ is simple there is nothing to prove. Assume that Γ is not simple but that $\mathrm{s}(\Gamma)$ is finite. Apply an elementary factorization $\Gamma \to S_1 + \Gamma_1$ to Γ. As the multiplicity of the point in which S_1 and Γ_1 are glued together is one less for Γ_1 than for Γ we have that $\mathrm{s}(\Gamma_1) \leq \mathrm{s}(\Gamma) - 1$. Hence, after at most $\mathrm{s}(\Gamma)$ elementary factorizations we obtain an scs-factorization of Γ. $\square$

To give a sufficient condition for a closed curve on the unit 2-sphere to possess an scs-factorization we use the terminology that a closed regular C^1-curve only has transversal self-intersections if no pair of tangents to the curve with the same base point are parallel.

Lemma 3.5 *Let Γ be a closed regular C^1-curve on the unit 2-sphere. If Γ only has transversal self-intersections then* $\mathrm{scsn}(\Gamma)$ *is finite.*

Proof: By transversality and compactness the curve Γ has only finitely many self-intersection points. Since Γ has finite length and Γ is a regular C^1-curve each self-intersection point has finite multiplicity. Now Theorem 3.4 applies. $\square$

Note, that Lemma 3.5 implies that it is a generic property for regular C^1-curves to possess scs-factorizations.

[2] In [42] it is claimed that a *closed spherical curve with continuous geodesic curvature has finite "Umlaufszahl"*. The curve on the right hand side on Figure 3.3 contradicts this statement.

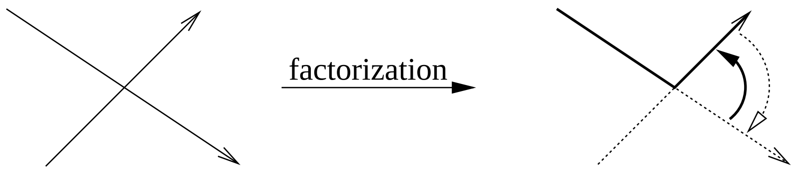


Figure 3.4: *Exterior angles.*

3.3 An index formula on $\mathbb{S}^2$

In this section we give an index of points in the complement of a closed regular C^1-curve on the unit 2-sphere and prove that in case the curve is of type C^2, then the integral over the 2-sphere of this index equals the integrated geodesic curvature of the curve.

Let $\Gamma \to S_1+S_2+\cdots+S_n$ be a simple closed sub-curve factorization of a closed regular C^1-curve on the unit 2-sphere. This scs-factorization is obtained by $(n-1)$ elementary factorizations, each cutting away a simple closed curve defined on an interval, $[a, b]$. We call the point $\Gamma(a) = \Gamma(b)$ a cutting point. If the tangents $\Gamma'(a)$ and $\Gamma'(b)$ are linearly independent we call the cutting point a transversal cutting point. If all cutting points of the scs-factorization are transversal cutting points, then we say that the scs-factorization is a transversal scs-factorization.

Theorem 3.6 *Let Γ be a closed regular C^2-curve on the unit 2-sphere with a transversal* scs-*factorization $\Gamma \to S_1 + S_2 + \cdots + S_n$. Denote the area of the positive resp. negative turned component of the complement of each simple closed sub-curve S_i by $\mu(\Omega_i^+)$ resp. $\mu(\Omega_i^-)$. Then the integral of the geodesic curvature of Γ, $\kappa_g(\Gamma)$, with respect to arc-length satisfies $\kappa_g(\Gamma) = \frac{1}{2}\sum_{i=1}^{n}\left(\mu(\Omega_i^-) - \mu(\Omega_i^+)\right)$.*

Remark 3.7 *If a curve only has transversal self-intersections, then the curve does possess* scs-*factorizations, by Lemma 3.5, and all its* scs-*factorizations are transversal. Note, that the formula given in Theorem 3.6 is true for all transversal* scs-*factorizations of a curve.*

Proof: Let $\Gamma \to S_1 + S_2 + \cdots + S_n$ be a transversal scs-factorization of Γ and let $\mu(\Omega_1^+), \ldots, \mu(\Omega_n^+)$ resp. $\mu(\Omega_1^-), \ldots, \mu(\Omega_n^-)$ be the areas of the positive resp. negative turned components of the complements of these simple closed sub-curves. Let $\alpha_{ij} \in (-\pi, \pi)$ be the exterior angle (see Figure 3.4) between the tangents to S_i at the cutting point between S_i and S_j. If the i'th and the j'th sub-curve do not have a mutual cutting point, or $i = j$, then we set $\alpha_{ij} = \alpha_{ji} = 0$. Note, that for all $1 \le i, j \le n$ the exterior angles fulfil $\alpha_{ij} = -\alpha_{ji}$. Gauss-Bonnet's Theorem for the i'th sub-curve, S_i, gives

$$\int_{S_i} \kappa_{g,\Gamma} d\sigma + \sum_{j=1}^{n} \alpha_{ij} = 2\pi - \mu(\Omega_i).$$

By our scs-factorization of Γ we get

$$\begin{aligned}\int_\Gamma \kappa_{g,\Gamma} d\sigma &= \sum_{i=1}^n \int_{S_i} \kappa_{g,\Gamma} d\sigma = \sum_{i=1}^n \left(2\pi - \mu(\Omega_i^+) - \sum_{j=1}^n \alpha_{ij} \right) \\ &= 2\pi n - \sum_{i=1}^n \mu(\Omega_i^+) - \sum_{i,j=1}^n \alpha_{ij} = 2\pi n - \sum_{i=1}^n \mu(\Omega_i^+).\end{aligned}$$

Let $\tilde{\Gamma}$ denote Γ with reversed orientation. By reversing the orientation of all the simple closed sub-curves in the scs-factorization of Γ we obtain an scs-factorization of $\tilde{\Gamma}$. This gives us

$$-\int_\Gamma \kappa_{g,\Gamma} d\sigma = \int_{\tilde{\Gamma}} \kappa_{g,\tilde{\Gamma}} d\tilde{\sigma} = 2\pi n - \sum_{i=1}^n \mu(\Omega_i^-).$$

Hence,

$$\int_\Gamma \kappa_{g,\Gamma} d\sigma = \frac{1}{2}\left(\int_\Gamma \kappa_{g,\Gamma} d\sigma - \int_{\tilde{\Gamma}} \kappa_{g,\tilde{\Gamma}} d\tilde{\sigma} \right) = \frac{1}{2}\left(\sum_{i=1}^n \mu(\Omega_i^-) - \sum_{i=1}^n \mu(\Omega_i^+) \right).$$

□

One of the ingredients that make the proof of Theorem 3.6 work is that the sum of the exterior angles vanishes. Assume that this is not the case and let $\sum_{i,j=1}^n \alpha_{ij} = a \neq 0$. Using Gauss-Bonnet's Theorem with the first orientation of the curve Γ we get

$$\int_\Gamma \kappa_{g,\Gamma} d\sigma = 2\pi n - \sum_{i=1}^n \mu(\Omega_i^+) - a.$$

With reversed orientation all exterior angles change sign and hence,

$$-\int_\Gamma \kappa_{g,\Gamma} d\sigma = \int_{\tilde{\Gamma}} \kappa_{g,\tilde{\Gamma}} d\tilde{\sigma} = 2\pi n - \sum_{i=1}^n \mu(\Omega_i^-) + a.$$

Subtracting the last equation from the first equation and dividing by two we get

$$\int_0^l \kappa_{g,\Gamma} d\sigma = \frac{1}{2}\sum_{i=1}^n \left(\mu(\Omega_i^-) - \mu(\Omega_i^+) \right) - a.$$

We conclude, that the equation in Theorem 3.6 is valid only if the sum of the exterior angles vanishes. As shown on Figure 3.5 there are non-transversal self-intersections with both exterior angles equal to π. So for a curve with this kind of non-transversal self-intersections the rhs. and the lhs. of the equation in Theorem 3.6 differ by an integral multiple of 2π. If we smoothen the curve on Figure 3.1 and consider the two scs-factorizations of this curve, then the alternating sum of areas, $\frac{1}{2}\sum_{i=1}^n \left(\mu(\Omega_i^-) - \mu(\Omega_i^+) \right)$, for the two

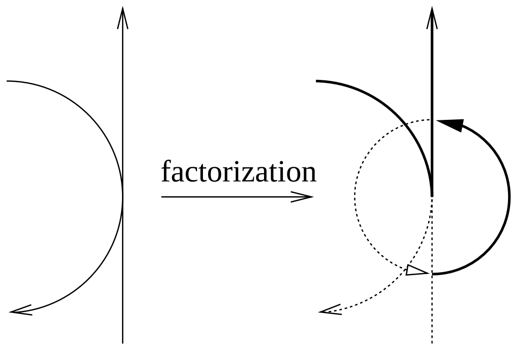


Figure 3.5: *Exterior angles with the same sign.*

scs-factorizations exactly differ by 2π. This is found using a little Linear Algebra remembering that the sum of the seven unknown areas equals 4π.

If instead of the unit 2-sphere we consider any topological 2-sphere we have a theorem similar to Theorem 3.6 where the areas $\mu(\Omega_i^{\pm})$ are exchanged by the integral of Gaussian curvature over the corresponding sets. At first sight this looks like a generalization of Gauss-Bonnet's theorem to non-simple closed curves. But given a closed curve Γ on the unit 2-sphere there is a topological 2-sphere, M, in $\mathbb{R}^3$ and a simple closed curve γ on M such that the image of the surface normal to M along γ equals Γ. Note, that it is necessary that M is a topological 2-sphere since we have to use Gauss-Bonnet's theorem in both orientations of the curve γ, i.e., both components of the complement of γ on M have to be disks[3]. Now the normal image of M gives the formula in Theorem 3.6.

We now prove that in the alternating sum of areas in Theorem 3.6 each connected component of the complement is counted a half integer number of times independent of the scs-factorization. To do this we need some notation. Let $\alpha : [0, l] \to \mathbb{R}^2$ be a plane, continuous closed curve. Recall, that the index or winding number of the plane curve α relative to a point p_0 is a map $\mathrm{Index}(\alpha, p_0) : \mathbb{R}^2 \backslash \alpha([0, l]) \to \mathbb{Z}$ defined on the complement of the curve α, into the integers. By continuity, Index is constant on each connected component of the complement of α and it counts the number of times the plane curve α wraps around each connected component. If a plane curve $\alpha : [0, l] \to \mathbb{R}^2$ is a closed regular C^1-curve then its rotation index, $\mathrm{Index}_R(\alpha)$, is the number of complete turns given by the tangent vector field along the curve. The index and the rotation index of plane curves can be found in eg. [7] pp. 392-393.

[3]Such a surface M can be constructed as follows: We can assume that Γ only has transversal double points and that in a neighbourhood of each double point the curve Γ lies on two great circles. In this neighbourhood we choose a piece of the cylinder with rulings orthogonal to each great circle to lie on our surface M. Lifting one of the great circles fixing the rulings of the cylinder preserves the normal image. Hence, we have a surface M and a simple curve, γ, on M such that the surface normal along γ equal the prescribed curve Γ on the unit 2-sphere. By choosing over- and under- crossings such that γ is unknotted γ bounds a disk (Seifert surface) on M. By reversing the orientation of Γ, and thus also on γ, we also have that the other complement of γ on M is a disk. Hence, M is a topological 2-sphere and γ is a simple closed curve on M such that the image of the surface normal to M along γ equals Γ.

The following theorem (Theorem 3.9) is implicitly used in [5] but first formulated in [32]. This theorem gives a spherical analogy to the index of plane curves. To state this theorem let $-P$ and P be a pair of antipodal points on the unit 2-sphere and denote the stereographic projection from the unit 2-sphere onto the tangent plane of $\mathbb{S}^2$ at P, $T_P\mathbb{S}^2$, by $\Pi_P : \mathbb{S}^2\backslash\{-P\} \to T_P\mathbb{S}^2$.

Definition 3.8 *Let $\Gamma : [0, l] \to \mathbb{S}^2$ be a closed regular curve of type C^1 on the unit 2-sphere. Denote the complement of $\Gamma([0, l])$ by Ω. Let $-P \in \Omega$ and let P be its antipode. Define the map $\mathrm{Ind}_{\Gamma,-P} : \Omega\backslash\{-P\} \to \mathbb{Z}/2$ by*

$$\mathrm{Ind}_{\Gamma,-P}(Q) = \frac{1}{2}\,\mathrm{Index}_R(\Pi_P(\Gamma)) - \mathrm{Index}(\Pi_P(\Gamma), \Pi_P(Q)), \quad Q \in \Omega\backslash\{-P\}.$$

The condition $-P \in \Omega$ in Definition 3.8 ensures that the stereographic projection of the curve Γ is a closed curve. One could define maps from $\Omega\backslash\{-P\}$ using any expression in the rotation index and the winding number, but the linear combination used in Definition 3.8 is, up to a multiplicative constant, the only linear combination giving

Theorem 3.9 *The map $\mathrm{Ind}_{\Gamma,-P} : \Omega\backslash\{-P\} \to \mathbb{Z}/2$ defined in Definition 3.8 is independent of the point $-P \in \Omega$ used to define it. Hereby, we have a well-defined map $\mathrm{Ind}_\Gamma : \Omega \to \mathbb{Z}/2$ from the complement of any closed regular curve of type C^1 on the unit 2-sphere into $\mathbb{Z}/2$.*

A direct proof of Theorem 3.9 can be found in [32] pp. 26-29. Here Theorem 3.9 will follow from Lemma 3.10 which gives a reformulation of the map Ind_Γ.

Lemma 3.10 *Let Γ be a closed regular C^1-curve on the unit 2-sphere. With notation as in Theorem 3.6, we for Q in the complement of Γ have that $\mathrm{Ind}_\Gamma(Q) = \frac{1}{2}(\sharp\{i|Q \in \Omega_i^-\} - \sharp\{i|Q \in \Omega_i^+\})$ for all transversal* scs-*factorizations of* Γ.

Proof: [of Lemma 3.10 and Theorem 3.9] Let Γ be a closed regular C^1-curve on the unit 2-sphere and let $\Gamma = S_1 + S_2 + \cdots + S_n$ be a transversal scs-factorization of Γ. Denote the positive resp. negative turned component of the complement of S_i by Ω_i^+ resp. Ω_i^-. Consider a stereographic projection of the unit 2-sphere such that the image of Γ is closed under this projection. See Definition 3.8.

To simplify notation the stereographic projection of a set, denoted by a capital letter, will be denoted by the corresponding small letter. Hence, q is the projection of the point Q, $\gamma \to s_1 + s_2 + \cdots + s_n$ is the projection of the curve $\Gamma \to S_1 + S_2 + \cdots + S_n$, and $\omega_i^\pm$ is the projection of the set $\Omega_i^\pm$. As Γ and γ are homeomorphic their scs-factorizations are in one-one correspondence. Furthermore, let bc_i be the bounded component of s_i's complement and let ubc_i be the unbounded component of s_i's complement. As each s_i is simple

$$\mathrm{Index}(q, s_i) = \begin{cases} \nu_i, & if \quad q \in bc_i \\ 0, & if \quad q \in ubc_i \end{cases},$$

where $\nu_i = 1$ if s_i runs in the positive direction and $\nu_i = -1$ if s_i runs in the negative direction. Let, in analogue to the proof of Theorem 3.6, α_{ij}, $1 \leq i, j \leq n$, be the jump of the tangent vectors to the sub-curve s_i at the cutting point between the sub-curve s_i and the sub-curve s_j. If we let $\mathrm{Index}_R(\gamma)|s_i$ denote the contribution to $\mathrm{Index}_R(\gamma)$ coming from s_i, then by Hopf's Umlaufsatz we have

$$\mathrm{Index}_R(\gamma)|s_i + \frac{1}{2\pi}\sum_{j=1}^{n}\alpha_{ij} = \nu_i.$$

For the contribution to the map $\mathrm{Ind}_{\Gamma,-P}$ from Definition 3.8 we have

$$\begin{aligned}\mathrm{Ind}_{\Gamma,-P}(Q)|s_i &= \frac{1}{2}\,\mathrm{Index}_R(\gamma)|s_i - \mathrm{Index}(q, s_i)\\ &= \frac{1}{2}\nu_i - \frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} - \left\{\begin{array}{lll}\nu_i, & if & q \in bc_i\\ 0, & if & q \in ubc_i\end{array}\right.\\ &= -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} + \frac{1}{2}\nu_i\left\{\begin{array}{lll}-1, & if & q \in bc_i\\ +1, & if & q \in ubc_i\end{array}\right. .\end{aligned}$$

In case ω_i^+ is the bounded component of s_i's complement $\nu_i = +1$, as Ω_i^+ is defined to be the positive turned component of S_i's complement. Hereby,

$$\begin{aligned}\mathrm{Ind}_{\Gamma,-P}(Q)|s_i &= -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} + \frac{1}{2}\left\{\begin{array}{lll}-1, & if & q \in \omega_i^+\\ +1, & if & q \in \omega_i^-\end{array}\right.\\ &= -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} + \frac{1}{2}\left\{\begin{array}{lll}-1, & if & Q \in \Omega_i^+\\ +1, & if & Q \in \Omega_i^-\end{array}\right. .\end{aligned}$$

In case ω_i^+ is the unbounded component of s_i's complement $\nu_i = -1$. Hereby,

$$\begin{aligned}\mathrm{Ind}_{\Gamma,-P}(Q)|s_i &= -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} - \frac{1}{2}\left\{\begin{array}{lll}-1, & if & q \in \omega_i^-\\ +1, & if & q \in \omega_i^+\end{array}\right.\\ &= -\frac{1}{4\pi}\sum_{j=1}^{n}\alpha_{ij} + \frac{1}{2}\left\{\begin{array}{lll}-1, & if & Q \in \Omega_i^+\\ +1, & if & Q \in \Omega_i^-\end{array}\right. .\end{aligned}$$

The following calculation completes the proof of Lemma 3.10.

$$\begin{aligned}\mathrm{Ind}_{\Gamma,-P}(Q) &= \sum_{i=1}^{n}\mathrm{Ind}_{\Gamma,-P}(Q)|s_i\\ &= -\frac{1}{4\pi}\sum_{i,j=1}^{n}\alpha_{ij} + \frac{1}{2}\left(\sharp\left\{i|Q \in \Omega_i^+\right\} - \sharp\left\{i|Q \in \Omega_i^-\right\}\right)\\ &= \frac{1}{2}\left(\sharp\left\{i|Q \in \Omega_i^+\right\} - \sharp\left\{i|Q \in \Omega_i^-\right\}\right)\end{aligned}$$

The right hand side of the above equation is clearly independent of the stereographic projection used to define the left hand side, $\mathrm{Ind}_{\Gamma,-P}(Q)$. Hence, the map Ind_Γ is well-defined as a map from the complement of a regular closed C^1-curve into the half integers. This proves Theorem 3.9. $\square$

Corollary 3.11 (of Lemma 3.10) *All transversal* scs*-factorizations of a regular C^1-curve on the unit* 2*-sphere have either odd or even degrees.*

Proof: Given a regular C^1-curve on the 2-sphere the map Ind_Γ either takes integer values or values equal to one half plus integers[4]. If Ind_Γ (takes integer values/takes values equal to one half plus integers) then Lemma 3.10 implies that each transversal scs-factorizations of the curve has (even/odd) degree. $\square$

Combining Theorem 3.6 and Lemma 3.10 we get the main result of this chapter.

Theorem 3.12 *Let Γ be a closed regular C^2-curve on the unit* 2*-sphere and let* Ind_Γ *be as in Theorem 3.9. Then the integrated geodesic curvature of Γ, $\kappa_g(\Gamma)$, fulfills*

$$\kappa_g(\Gamma) = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_\Gamma(Q)dA.$$

Proof: By Theorem 3.6, Remark 3.7 , and Lemma 3.10 the desired formula is true for all regular C^2-curves with only transversal self-intersections. As the right hand side of the equation is well-defined for all regular closed C^2-curves the formula is true for all regular closed C^2-curves by continuity. $\square$

Remark 3.13 (Kroneckers Drehziffer) *Let Γ be a closed curve on the unit* 2*-sphere. Let $Q \in \mathbb{S}^2$ be a point such that neither Q nor its antipodal point $-Q$ lie on Γ. Now the stereographic projection, $\Pi_Q(\Gamma)$, of Γ into $T_Q\mathbb{S}^2$ is a closed curve avoiding the origin,* $\mathbf{0}$*, of $T_Q\mathbb{S}^2$. Hereby, $\Pi_Q(\Gamma)$ has a well-defined winding number with respect to the origin,* $\mathrm{Index}(\mathbf{0}, \Pi_Q(\Gamma))$*. As the author reads [4], this winding number is called Kroneckers Drehziffer, $k(Q,\Gamma)$. In [4] p. 83 it is stated that the integral of continuous geodesic curvature of Γ, $\kappa_g(\Gamma)$, fulfil*

$$2\kappa_g(\Gamma) = -\int_{Q\in\mathbb{S}^2} k(Q,\Gamma)dA.$$

As $\mathrm{Index}(\mathbf{0}, \Pi_Q(\Gamma)) = -\,\mathrm{Index}(\mathbf{0}, \Pi_{-Q}(\Gamma))$ *for all pairs of antipodal points, Q and $-Q$, not touching Γ the sphere integral $\int_{\mathbb{S}^2} k(Q,\Gamma)dA$ equals zero for all closed spherical curves. There appears to be a mistake in [4].*

[4] It is easily checked that Ind_Γ takes values equal to one half plus integers for all regular C^1-curves Γ in the equators regular homotopy class and that Ind_Γ takes integer values for all regular C^1-curves Γ in the other (the double-covered equators) regular homotopy class. Confer [45].

In [10] p. 53 the Kroneckers Drehziffer integral formula is mentioned as the dual (in the sense of spherical dual curves) of the Crofton formula for length of spherical curves. If we add (or subtract) the index, Ind_Γ, *of a spherical curve* Γ *and the index,* $\mathrm{Ind}_{-\Gamma}$, *of its antipodal curve* $-\Gamma$ *(according to the orientation chosen on* $-\Gamma$*), then we obtain an index of* Γ *and its antipodal curve with the property that Kroneckers Drehziffer is claimed to have in [10]. It could be interesting to check if this antipodal curve pair weight is the correct dual of the Crofton formula for length of spherical curves.*

3.4 Definition of integrated geodesic curvature

The map Ind, given by Definition 3.8, is defined on closed regular C^1-curves on the unit 2-sphere. We thus give

Definition 3.14 *Let* Γ *be a closed regular* C^1*-curve on the unit* 2*-sphere then the integrated geodesic curvature of* Γ, $\kappa_g(\Gamma)$, *is defined by*

$$\kappa_g(\Gamma) = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_\Gamma(Q)dA.$$

Consider a continuous closed spherical curve possessing more than one scs-factorization. As the two components of the complement of a simple closed sub-curve, Ω^- and Ω^+, are open sets they are Lebesgue measurable. Hence, for a fixed scs-factorization the real number $\frac{1}{2}\sum_{i=1}^n \left(\mu(\Omega_i^-) - \mu(\Omega_i^+)\right)$ is well-defined. By lack of transversality we have to take this number modulo 2π. But it is unknown to the author *if integrated geodesic curvature can be defined, modulo* 2π, *on closed continuous* scs-*factorizeable curves on the unit* 2*-sphere using the expression* $\frac{1}{2}\sum_{i=1}^n \left(\mu(\Omega_i^-) - \mu(\Omega_i^+)\right)$.

3.5 Integrated geodesic curvature of closed indicatrices

In this section we apply Theorem 3.12 to closed spherical curves given as the tangent indicatrix, principal normal indicatrix, binormal indicatrix, or Darboux indicatrix of a space curve. Hereby we obtain a new and short proof of a formula for integrated torsion of space curves with closed tangent indicatrix due to C. Chicone and N.J. Kalton, [5], and we generalize a classical theorem by Jacobi.

Theorem 3.15 *Let* $\Gamma_{\mathbf{t}} : [0,l] \to \mathbb{S}^2$ *be a regular* C^2*-curve on the unit* 2*-sphere. Then the integral of torsion for any space curve* C*, with* $\Gamma_{\mathbf{t}}$ *as tangent indicatrix, equals the integrated geodesic curvature of* $\Gamma_{\mathbf{t}}$[5]*. Furthermore, if* $\Gamma_{\mathbf{t}}$ *is closed then*

$$\tau(C) = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_{\Gamma_{\mathbf{t}}}(Q)dA.$$

[5] This partial result is also stated in [24] and [48].

Remark 3.16 *In analogy with Definition 3.14 we have a natural definition of integrated torsion of a regular C^2-space curve with non-vanishing curvature and closed tangent indicatrix given by the equation in Theorem 3.15.*

Proof: Let $\Gamma_{\mathbf{t}} : [0, l] \to \mathbb{S}^2$ be a closed regular C^2-curve on the unit 2-sphere parametrized by arc-length σ. Any space curve with $\Gamma_{\mathbf{t}}$ as spherical tangent indicatrix can be written as

$$\mathbf{r}(s) - \mathbf{r}(0) = \int_0^s \Gamma_{\mathbf{t}}(\sigma(t))dt, \quad s \in [0, L],$$

where L is the length of this space curve, s is its arc-length, and $\sigma : [0, L] \to [0, l]$ is a non-decreasing C^1-map given by $\sigma = \sigma(s)$.

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \Gamma_{\mathbf{t}}(\sigma(s))$$

$$\mathbf{t}' = \frac{d\mathbf{t}}{ds} = \frac{d\Gamma_{\mathbf{t}}}{d\sigma}(\sigma(s))\frac{d\sigma}{ds}(s) = \kappa_{\mathbf{r}}(s)\mathbf{n}(s)$$

Hence, $\frac{d\sigma}{ds}(s) = \kappa_{\mathbf{r}}(s)$ is the curvature of the space curve at the point $\mathbf{r}(s)$ and $\mathbf{n}(s) = \frac{d\Gamma_{\mathbf{t}}}{d\sigma}(\sigma(s))$. Hereby,

$$\mathbf{n}' = \frac{d\mathbf{n}}{ds} = \frac{d^2\Gamma_{\mathbf{t}}}{d\sigma^2}(\sigma(s))\frac{d\sigma}{ds}(s),$$

giving torsion of the space curve as

$$\tau_{\mathbf{r}}(s) = \mathbf{n}' \cdot \mathbf{b} = \mathbf{n}' \cdot (\Gamma_{\mathbf{t}} \times \frac{d\Gamma_{\mathbf{t}}}{d\sigma})(\sigma(s)) = \frac{d\sigma}{ds}(s)\left[\Gamma_{\mathbf{t}}\frac{d\Gamma_{\mathbf{t}}}{d\sigma}\frac{d^2\Gamma_{\mathbf{t}}}{d\sigma^2}\right](\sigma(s)).$$

Here $[\cdot]$ is the triple scalar product in $\mathbb{R}^3$. As $\kappa_{g,\Gamma_{\mathbf{t}}}(\sigma) = \left[\Gamma_{\mathbf{t}}\frac{d\Gamma_{\mathbf{t}}}{d\sigma}\frac{d^2\Gamma_{\mathbf{t}}}{d\sigma^2}\right](\sigma)$ is the geodesic curvature[6] of $\Gamma_{\mathbf{t}}$ we obtain

$$\begin{aligned}
\int_0^L \tau_{\mathbf{r}}(s)ds &= \int_0^L \left[\Gamma_{\mathbf{t}}\frac{d\Gamma_{\mathbf{t}}}{d\sigma}\frac{d^2\Gamma_{\mathbf{t}}}{d\sigma^2}\right](\sigma(s))\frac{d\sigma}{ds}(s)ds \\
&= \int_0^l \left[\Gamma_{\mathbf{t}}\frac{d\Gamma_{\mathbf{t}}}{d\sigma}\frac{d^2\Gamma_{\mathbf{t}}}{d\sigma^2}\right](\sigma)d\sigma \\
&= \int_0^l \kappa_{g,\Gamma_{\mathbf{t}}}d\sigma = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_{\Gamma_{\mathbf{t}}}(Q)dA,
\end{aligned}$$

where the last equality follows by Theorem 3.12 in case $\Gamma_{\mathbf{t}}$ is closed. $\square$

As previously mentioned, the formula for integrated torsion given in Theorem 3.15, in case of a closed tangent indicatrix, is due to C. Chicone and N.J. Kalton. Their proof

[6] Thus we have derived at the equation $\tau_{\mathbf{r}}(s) = \kappa_{\mathbf{r}}(s)\kappa_{g,\Gamma_{\mathbf{t}}}(\sigma(s))$ that we found in Section 2.4.

of this formula can be found in [5] and in [32] pp. 18-36. This proof is by induction on the number of connected components of the complement of the tangent indicatrix. By homotopying the stereographic projection of the tangent indicatrix and inserting needles to change its Index they use Green's Formula to transform the torsion integral such that the index-formula from Definition 3.8 is recognizable. Observing that the integral over the unit 2-sphere appearing in the Chicone-Kalton formula depends only on the closed tangent indicatrix and not on the space curve itself, lead to the wish of finding a proof of this formula reflecting this fact. Having found such a proof we now give similar formulas for closed principal normal and binormal indicatrices of space curves.

Theorem 3.17 *Let C be a regular curve in $\mathbb{R}^3$ of type C^4 with non-vanishing curvature and closed Frenet frame. Denote the principal normal indicatrix and the binormal indicatrix by $\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{b}} : [0, l] \to \mathbb{S}^2$ respectively. Then*

$$\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{n}}}(Q) dA = 0.$$

If furthermore C has non-vanishing torsion then the total curvature of C fulfills

$$\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{b}}}(Q) dA = \kappa(C).$$

Remark 3.18 *As the lengths of the curves $\Gamma_{\mathbf{t}}, \Gamma_{\mathbf{b}} : [0, l] \to S^2$ are $\kappa(C)$ and $|\tau|(C)$, respectively this theorem gives an "almost duality" between total curvature and total torsion for space curves with closed Frenet-frames. In fact, we have the following identities*

$$\begin{array}{ll} |\Gamma_{\mathbf{t}}| = \kappa(C) & \displaystyle\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{t}}}(Q) dA = \tau(C) \\ |\Gamma_{\mathbf{n}}| = \omega(C) & \displaystyle\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{n}}}(Q) dA = 0 \\ |\Gamma_{\mathbf{b}}| = |\tau|(C) & \displaystyle\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{b}}}(Q) dA = \kappa(C). \end{array}$$

Proof: Let the curve C be as in the theorem and let $\mathbf{r} : [0, l] \to \mathbb{R}^3$ be an arc-length parametrization of C. Let $\mathbf{t_r}(s)$, $\mathbf{n_r}(s)$, $\mathbf{b_r}(s)$, $\kappa_{\mathbf{r}}(s)$, and $\tau_{\mathbf{r}}(s)$ denote the tangent vector, the principal normal vector, the binormal vector, the curvature, and the torsion of the curve C at the point $\mathbf{r}(s)$. Define a curve $\mathbf{x} : [0, l] \to \mathbb{R}^3$ by

$$\mathbf{x}(s) = \int_0^s \mathbf{n_r}(s) ds, \quad \text{for } s \in [0, l].$$

Note, that $\mathbf{x}$ is an arc-length parametrization of a regular curve of type C^3. In order to calculate the torsion, $\tau_{\mathbf{x}}(s)$, of the curve parametrized by $\mathbf{x}$ we find

$$\begin{aligned}
\mathbf{x}' &= \mathbf{n_r} \\
\mathbf{x}'' &= -\kappa_{\mathbf{r}}\mathbf{t_r} + \tau_{\mathbf{r}}\mathbf{b_r} \quad (\neq \mathbf{0} \Rightarrow \kappa_{\mathbf{x}}(s) > 0) \\
\mathbf{x}''' &= -\kappa_{\mathbf{r}}'\mathbf{t_r} - \kappa_{\mathbf{r}}^2\mathbf{n_r} + \tau_{\mathbf{r}}'\mathbf{b_r} - \tau_{\mathbf{r}}^2\mathbf{n_r} \\
[\mathbf{x}'\mathbf{x}''\mathbf{x}'''] &= \begin{vmatrix} 0 & -\kappa_{\mathbf{r}} & -\kappa_{\mathbf{r}}' \\ 1 & 0 & -\kappa_{\mathbf{r}}^2 - \tau_{\mathbf{r}}^2 \\ 0 & \tau_{\mathbf{r}} & \tau_{\mathbf{r}}' \end{vmatrix} = \tau_{\mathbf{r}}'\kappa_{\mathbf{r}} - \kappa_{\mathbf{r}}'\tau_{\mathbf{r}}.
\end{aligned}$$

Hence,

$$\tau_{\mathbf{x}} = \frac{\tau_{\mathbf{r}}'\kappa_{\mathbf{r}} - \kappa_{\mathbf{r}}'\tau_{\mathbf{r}}}{\kappa_{\mathbf{r}}^2 + \tau_{\mathbf{r}}^2} = \frac{\kappa_{\mathbf{r}}^2 \frac{d}{ds}\left(\frac{\tau_{\mathbf{r}}}{\kappa_{\mathbf{r}}}\right)}{\kappa_{\mathbf{r}}^2 + \tau_{\mathbf{r}}^2} = \frac{d}{ds}\left(\mathrm{Arctan}\left(\frac{\tau_{\mathbf{r}}}{\kappa_{\mathbf{r}}}\right)\right).$$

Using Theorem 3.15 we as $\kappa_{\mathbf{x}} > 0$ get

$$\begin{aligned}
\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{n_r}}}(Q)dA &= \int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{t_x}}}(Q)dA \\
= \int_0^l \tau_{\mathbf{x}}(s)ds &= \left[\mathrm{Arctan}\left(\frac{\tau_{\mathbf{r}}}{\kappa_{\mathbf{r}}}\right)\right]_0^l = 0.
\end{aligned}$$

This proves the first part of the theorem. To prove the last part of the theorem let the curve $\mathbf{y} : [0, l] \to \mathbb{R}^3$ be given by

$$\mathbf{y}(s) = \int_0^s \mathbf{b_r}(s)ds, \quad \text{for } s \in [0, l].$$

Note, that $\mathbf{y}$ is an arc-length parametrization of a regular curve of type C^3. In order to calculate the torsion, $\tau_{\mathbf{y}}(s)$, of the curve parametrized by $\mathbf{y}$ we find

$$\begin{aligned}
\mathbf{y}' &= \mathbf{b_r} \\
\mathbf{y}'' &= -\tau_{\mathbf{r}}\mathbf{n_r} \quad (\neq \mathbf{0} \Rightarrow \kappa_{\mathbf{y}} = |\tau_{\mathbf{r}}| > 0) \\
\mathbf{y}''' &= -\tau_{\mathbf{r}}'\mathbf{n_r} + \kappa_{\mathbf{r}}\tau_{\mathbf{r}}\mathbf{t_r} - \tau_{\mathbf{r}}^2\mathbf{b_r} \\
[\mathbf{y}'\mathbf{y}''\mathbf{y}'''] &= \begin{vmatrix} 0 & 0 & \kappa_{\mathbf{r}}\tau_{\mathbf{r}} \\ 0 & -\tau_{\mathbf{r}} & -\tau_{\mathbf{r}}' \\ 1 & 0 & \tau_{\mathbf{r}}^2 \end{vmatrix} = \tau_{\mathbf{r}}^2\kappa_{\mathbf{r}}.
\end{aligned}$$

Hence,

$$\tau_{\mathbf{y}}(s) = \frac{\tau_{\mathbf{r}}^2\kappa_{\mathbf{r}}}{\tau_{\mathbf{r}}^2} = \kappa_{\mathbf{r}}(s),$$

for all $s \in [0, l]$, which together with Theorem 3.15 imply the last statement in the theorem as

$$\int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{b_r}}}(Q)dA = \int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{t_y}}}(Q)dA$$
$$= \int_0^l \tau_{\mathbf{y}}(s)ds = \int_0^l \kappa_{\mathbf{r}}(s)ds = \kappa(C).$$

□

Remark 3.19 *Instead of assuming non-vanishing curvature in Theorem 3.17 we now use the Fenchel Assumption. See Section 2.2. Denote (as in [42]) the index of the plane curve given by* $(\kappa(s), \tau(s))$ *with respect to the origin,* $(0,0)$*, by the nutation,* $\nu(C)$*, of the curve* C*, then the first equation in Theorem 3.17 is replaced by*

$$\int_{\Gamma_{\mathbf{n}}} \kappa_{g,\Gamma_{\mathbf{n}}} d\sigma = \int_{Q\in S^2} \mathrm{Ind}_{\Gamma_{\mathbf{n}}}(Q)dA = \int_C \frac{d}{ds}\left(\mathrm{Arctan}\left(\frac{\tau}{\kappa}\right)\right) ds = 2\pi\nu(C).$$

Combining the first equation in Theorem 3.17 with Lemma 3.10 we obtain a generalization of a theorem by Jacobi (1842) that states: *A simple closed principal normal indicatrix of a regular* C^4*-curve with non-vanishing curvature bisects the unit* 2*-sphere*[7].

Theorem 3.20 *Let* $\Gamma_{\mathbf{n}}$ *be the closed principal normal indicatrix of a regular (not necessarily closed)* C^4*-curve with non-vanishing curvature. If* $\Gamma_{\mathbf{n}}$ *possesses a transversal* scs*-factorization then the sum of the areas of the positive turned complements of the sub-curves equals the sum of the areas of the negative turned complements of the sub-curves.*

Proof: Under the assumptions taken in the theorem we have that

$$0 = \int_{\Gamma_{\mathbf{n}}} \kappa_{g,\Gamma_{\mathbf{n}}} d\sigma = \frac{1}{2}\sum_{i=1}^{n}\left(\mu(\Omega_i^-) - \mu(\Omega_i^+)\right),$$

where the first equality is given in the proof of Theorem 3.17 and the last equality uses Theorem 3.6. □

Another spinoff is the classical

Corollary 3.21 *Let* $C : \mathbb{S}^1 \to \mathbb{R}^3$ *be a closed regular* C^3*-curve lying on a sphere of radius* r*. Then the integrated torsion of* C *is zero. Or equivalently: A tangent indicatrix of a closed spherical regular* C^3*-curve has integrated geodesic curvature zero.*

[7] In [42] there is another generalization of this theorem, but that generalization uses the previous mentioned “Umlaufszahl” which not is well-defined. Moreover, the reference [42] is inconsistent. It gives an example with non zero nutation but assumes non-vanishing curvature. See also the review by S.B. Jackson [Math. Rev. Vol. 8 (1947) p. 226] on [42]. The remarks on [42] given here also counts for [14]

Proof: It is sufficient to prove the theorem in case of the unit 2-sphere. Thus let C be a closed regular C^3-curve on the unit 2-sphere and let $\tilde{C}$ be a space curve with non-vanishing curvature which has C as tangent indicatrix. See eg. the proof of Theorem 3.17. Now the tangent indicatrix of C, $\Gamma_{\mathbf{t},C}$ prescribe the same curve as the principal normal indicatrix of $\tilde{C}$, $\Gamma_{\mathbf{n},\tilde{C}}$. Using Lemma 3.15 and Theorem 3.17 we get $\tau(C) = \kappa_g(\Gamma_{\mathbf{t},C}) = \kappa_g(\Gamma_{\mathbf{n},\tilde{C}}) = 0$. □

It is noteworthy that the generalization of Jacobi's theorem and the well-known fact that regular closed spherical curves have integrated torsion equal to zero in fact are equivalent. Again we note that the generalization of Jacobi's theorem, and hereby Jacobi's theorem in particular, is not to be considered as a "space curve theorem" and certainly not as a "closed space curve theorem", as it is implied by the fact that the principal normal indicatrix is the tangent indicatrix of a closed spherical curve. This simple geometric observation is the basis of Chapter 4, which is exclusively devoted to the analysis of principal normal indicatrices of closed space curves.

The method used in the proof of Theorem 3.17 provides a wealth of integral formulas as follows. Let C be a regular space curve of type at least C^3 with non-vanishing curvature and closed Frenet frame and let X be a unit vector field along the curve C such that X is closed relative to C's Frenet frame. In coordinates that is - if C is given by $\mathbf{r} : [0, l] \to \mathbb{R}^3$ then X given by $s \mapsto \alpha(s)\mathbf{t}(s) + \beta(s)\mathbf{n}(s) + \gamma(s)\mathbf{b}(s)$ has to be a closed regular curve of type C^1 regarded as a curve on the unit 2-sphere. Hence, (α, β, γ) must describe a closed curve[8] on the unit 2-sphere. Now, integration of the geodesic curvature of X gives a new integral formula using Theorem 3.15.

In order to obtain interesting integral formulas the vector field along the curve must have geometric meaning. An example of such a vector field is the Darboux vector field $D(s) = \frac{\tau(s)\mathbf{t}(s)+\kappa(s)\mathbf{b}(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}$. This vector field is the direction of $\mathbf{n} \times \mathbf{n}'$ and is the point of instant rotation of the Frenet frame. Furthermore, it gives the direction of the rulings on the surface known as the rectifying developable of the curve. This ruled surface is the ruled surface with zero Gaussian curvature on which the curve is a geodesic curve. See e.g. [33]. Using this vector field and a straightforward calculation we get

Corollary 3.22 (of Theorem 3.15) *Let C be a regular curve in $\mathbb{R}^3$ of type C^4 with non-vanishing curvature and closed Frenet frame and let D be the Darboux vector field along the curve C. Considering D as a closed curve on $\mathbb{S}^2$ we have*

$$\int_{Q \in S^2} \mathrm{Ind}_D(Q)\,dA = \omega(C) = |\Gamma_{\mathbf{n}}|.$$

[8] In the proof of Theorem 3.17 these "curves" were the points $(0, 1, 0)$ and $(0, 0, 1)$.

3.6 The Tennis Ball Theorem and the Four Vertex Theorem

The Tennis Ball Theorem [1] p. 53 states: *A smooth simple closed spherical curve dividing the sphere into two parts of equal areas has at least four inflection points (points with zero geodesic curvature).* It is natural to note that a non-simple curve on S^2 that bisects S^2 in the sense of Theorem 3.20, need only have two inflection points. An example is a curve of the shape of the figure eight on S^2. Here we draw a connection between the Tennis Tall Theorem and the Four Vertex Theorem for C^3 closed convex simple space curves which restricted to spherical closed curves states: *Any simple closed spherical C^3-curve has at least four vertices (points with zero torsion)*[9], see [12].

Lemma 3.23 *Let Γ be a regular closed spherical C^3-curve and let $\Gamma_{\mathbf{t}}$ be its tangent indicatrix. If the number of vertices $V(\Gamma)$ and the number of inflection points $I(\Gamma)$ of Γ both are finite then $V(\Gamma) \geq I(\Gamma)$ and $V(\Gamma) = I(\Gamma_{\mathbf{t}})$ where $I(\Gamma_{\mathbf{t}})$ is the number of inflection points of $\Gamma_{\mathbf{t}}$.*

Proof: Let Γ be as in the lemma. We can assume that Γ lies on the unit 2-sphere. As the curvature, κ, and the geodesic curvature, κ_g, of Γ fulfil $1 + \kappa_g^2 = \kappa^2$ the curvature has global minima precisely in the inflection points of Γ. By the equation $\kappa\kappa_g\tau = \kappa'$, which is easily derived from [50] equation (6) p. 365, all other local extremas of the curvature of Γ lie in vertices of Γ. Assuming that $V(\Gamma)$ and $I(\Gamma)$ both are finite Rolles theorem gives $V(\Gamma) \geq I(\Gamma)$.

In the proof of Theorem 3.15 we found $\tau = \kappa\kappa_{g,\Gamma_{\mathbf{t}}}$, where $\kappa_{g,\Gamma_{\mathbf{t}}}$ is the geodesic curvature of the tangent indicatrix of Γ, $\Gamma_{\mathbf{t}}$. As the spherical curve Γ has non-vanishing curvature we conclude that $V(\Gamma) = I(\Gamma_{\mathbf{t}})$. $\square$

The Tennis Ball Theorem, the "spherical" Four Vertex Theorem, and their connecting Lemma 3.23 give

Theorem 3.24 *Let Γ be a regular closed spherical C^3-curve and let $\Gamma_{\mathbf{t}}$ be its tangent indicatrix. If Γ or $\Gamma_{\mathbf{t}}$ is simple or Γ is the iterated tangent indicatrix of a simple spherical curve, then Γ has at least four vertices and $\Gamma_{\mathbf{t}}$ has at least four inflection points.*

Remark 3.25 *A curve, Γ, fulfilling the assumption in the Tennis Ball Theorem has integrated geodesic curvature zero, see e.g. Theorem 3.6. By Theorem 4.3 Γ is the tangent indicatrix of each member of an open one parameter family of closed (not necessarily simple) smooth regular curves on the unit 2-sphere, if and only if Γ contains no sub arc with integrated geodesic curvature π. Moreover, if there is a longest curve within this family, Γ^{-1} - say, then Γ^{-1} has integrated geodesic curvature zero, by Lemma 4.11. Now this remark may be iterated.*

[9] The vertices of a spherical curve are the stereographic projections of the vertices (stationary points of the curvature) of the corresponding simple planar curve.

Proof: Let Γ_0 be a sufficiently smooth closed spherical curve and let $\Gamma_1, \Gamma_2, \ldots$ be its iterated tangent indicatrices. By Lemma 3.23 we have

$$I(\Gamma_0) \leq V(\Gamma_0) = I(\Gamma_1) \leq V(\Gamma_1) = I(\Gamma_2) \leq \ldots.$$

If Γ_n, $n \geq 0$, is simple then $V(\Gamma_n) \geq 4$, by the Four Vertex Theorem. Hence, $V(\Gamma_i) \geq 4$ for $i \geq n$ and $I(\Gamma_j) \geq 4$ for $j \geq n+1$. If Γ_n, $n \geq 1$, is simple then by Corollary 3.21 and Theorem 3.6 Γ_n bisects the unit 2-sphere. Hence, $I(\Gamma_n) \geq 4$ by the Tennis Ball Theorem and $V(\Gamma_i) \geq 4$ for $i \geq n-1$ and $I(\Gamma_j) \geq 4$ for $j \geq n$. $\square$

3.7 Topological bounds on geodesic curvature

In this section we consider closed curves on the 2-sphere with *only transversal self-intersections*. The (integral) formulas for integrated geodesic curvature of a spherical curve Γ presented here,

$$\int_\Gamma \kappa_g(\sigma)d\sigma = \int_{Q\in\mathbb{S}^2} \mathrm{Ind}_\Gamma(Q)dA = \frac{1}{2}\sum_{i=1}^{n}\left(\mu(\Omega_i^-) - \mu(\Omega_i^+)\right),$$

give some topological bonds on integrated geodesic curvature. Let max(Ind_Γ) denote the maximal value of Ind_Γ on the complement of Γ and let min(Ind_Γ) denote the minimal value of Ind_Γ. Recall that scsn(Γ) is the simple closed sub-curve number of Γ and finally that s(Γ) is the number of self-intersections of Γ as defined in Section 3.2. As each component of Γ's complement has area less than 4π we get the inequalities

$$-2\pi(\mathrm{s}(\Gamma)+1) \leq -2\pi\ \mathrm{scsn}(\Gamma) \leq 4\pi\ \min(\mathrm{Ind}_\Gamma) < \textstyle\int_\Gamma \kappa_g(\sigma)d\sigma$$

$$\textstyle\int_\Gamma \kappa_g(\sigma)d\sigma < 4\pi\ \max(\mathrm{Ind}_\Gamma) \leq 2\pi\ \mathrm{scsn}(\Gamma) \leq 2\pi(\mathrm{s}(\Gamma)+1).$$

By Theorem 3.17 the total curvature of a regular C^4-curve, C, with non-vanishing curvature and non-vanishing torsion equals the integrated geodesic curvature of its binormal indicatrix, $\Gamma_{\mathbf{b}}$. This gives the inequalities

$$\kappa(C) = \int_{\Gamma_{\mathbf{b}}} \kappa_g(\sigma)d\sigma < 4\pi\ \max(\mathrm{Ind}_{\Gamma_{\mathbf{b}}}) \leq 2\pi\ \mathrm{scsn}(\Gamma_{\mathbf{b}}) \leq 2\pi(\mathrm{s}(\Gamma_{\mathbf{b}})+1).$$

In the following we use some results on total curvature of closed space curves that all can be found in [26]. The first result is due to W. Fenchel who in 1929 proved that a closed space curve has total curvature $\geq 2\pi$, where equality holds if and only if the curve is planar and convex. As we have assumed non-vanishing torsion we have a strict inequality. The other result is due to J.W. Milnor and states that if we let K be a knotted knot type and define the curvature of a knot type $\kappa(K)$ as the greatest lower bound of the total curvature of its representatives, then $\kappa(K) = 2\pi\mu$, where μ is an integer ≥ 2

(the crookedness of the knot type defined as the minimal number of local maxima in any direction of any representative of the knot type K – thus the crookedness equals the bridge number of the knot type). This greatest lower bound is never attained for knotted space curves. By the inequalities

$$2\pi\mu < \kappa(C) < 2\pi \,\mathrm{scsn}(\Gamma_{\mathbf{b}}) \leq 2\pi(\mathrm{s}(\Gamma_{\mathbf{b}}) + 1),$$

where $\mu = 1$ corresponds to unknotted space curves, we get

Theorem 3.26 *Let C be a regular C^3 representative of a knot type K with crookedness $\mu(K)$. If C has both non-vanishing curvature and torsion then the binormal indicatrix of C, $\Gamma_{\mathbf{b}}$, has simple closed sub-curve number* $\mathrm{scsn}(\Gamma_{\mathbf{b}}) \geq \mu(K) + 1$ *and Γ has at least μ self-intersections (in the sense of Section 3.2).*

Example 3.27 *The standard shadow of the trefoil knot has three self-intersections but it can not be obtained as a stereographic projection of the binormal indicatrix of a knotted space curve - since this shadow has simple closed sub-curve number equal to two.*

3.8 Knot diagrams and SCS-factorization

Recall the construction of a Seifert surface from a knot diagram. Firstly the knot diagram is “factorized” into a number of disjoint simple closed curves, the so-called Seifert circles. Each Seifert circle bounds a disc and these disjoined discs are glued together by half twisted bands given by the crossings in which the “factorizations” have taken place. The constructed Seifert surface is an embedded orientable surface with the knot as its only boundary curve.

Let KD be a knot diagram and let $KD \to S_1 + \ldots S_n$ be an scs-factorization of the shadow of this knot diagram. Using the Seifert construction on an scs-factorization of a knot diagram we also get an orientable surface, with the knot as its only boundary curve. In the following we call such a surface an scs-surface. The simple closed sub-curves in an scs-factorization may intersect each other. Hence, an scs-surface may, and generally will, have self-intersections - but each disc is embedded.

Theorem 3.28 *All* SCS-*surfaces are immersed discs.*

Proof: Let KD be a knot diagram of a knot K and let $KD \to S_1 + \ldots S_n$ be an scs-factorization of the shadow of this knot diagram. By the Seifert construction the scs-surface has only one boundary curve, the knot, and the scs-surface is orientable. The genus, g, of a surface constructed by the Seifert construction can be found in any textbook on knots and $2g$ equals the number of half twisted bands minus the number of discs plus one. The Seifert construction from an scs-factorization $KD \to S_1 + \ldots S_n$ replaces the n simple closed sub-curves by n discs and the $(n-1)$ cutting points by $(n-1)$ half twisted bands. Hence, the genus of the scs-surface S is $g(S) = \frac{1}{2}((n-1) - n + 1) = 0$ and the scs-surface is topologically a disc. $\square$

Start with an scs-surface of a knot and deform the surface until it is an embedding. Now the boundary of the embedded surface is unknotted by Theorem 3.28. Hence, by removing the self-intersections of the scs-surface we unknot the boundary curve. On the other hand, change a knot diagram, KD, to a knot diagram of the unknot, UD, by a number of crossing changes. The two knot diagrams KD and UD possess the same scs-factorizations. If KD possesses an scs-factorization such that these changes of crossings do not take place in cutting points, then the self-intersections of the corresponding scs-surface are removed. Therefore, scs-surfaces are intimately connected with the unknotting of knots. Here, we improve a standard upper bound on the unknotting number.

Theorem 3.29 *Let KD be a knot diagram of a knot K and let $KD \to S_1 + \ldots S_n$ be an* scs*-factorization of order n of the shadow (the projection of a knot into the plane of the knot diagram) of this knot diagram. If $c(KD)$ is the number of crossings in KD and $u(K)$ is the unknotting number of the knot K then $u(K) \leq \frac{1}{2}(c(KD) - n + 1)$.*

Proof: Let K, $KD \to S_1 + \ldots S_n$, and $u(K)$ be as in the theorem and let S be the scs-surface defined by the scs-factorization. Recall, that the scs-factorization is obtained by $(n-1)$ elementary factorizations $KD \to S_1 + \Gamma_1 \to S_1 + S_2 + \Gamma_2 \to \cdots \to S_1 + S_2 + \cdots + S_n$. Consider the elementary factorization $KD \to S_1 + \Gamma_1$. Let m_1 denote the number of crossings between S_1 and Γ_1 not counting the cutting point between S_1 and Γ_1. As S_1 and Γ_1 are closed curves and they only have transversal intersections m_1 is even. By changing at most half of the m_1 crossings we can bring S_1 to lie entirely above Γ_1 or entirely below Γ_1. Hence, a disc spanned by S_1 need not intersect Γ_1 after at most $\frac{m_1}{2}$ crossing changes in the knot diagram KD.

Doing these changes of crossings for each elementary factorization we can unknot the diagram KD by changing at most half of the crossings not used as cutting points. Hence, the knot K can be unknotted by use of at most $(c(KD) - (n-1))$ changes. $\square$

The orders of scs-factorizations of a knot diagram are generally changed when the knot diagram is changed by Reidemeister moves. It is therefore natural to define a knot invariant by attaching to each knot the minimal order of all scs-factorizations of any knot diagram of the knot. This gives a measure of complexity of the knot type - but due to the quite surprising Theorem 3.30 this measure only detects knottedness.

Theorem 3.30 *The unknot is the only knot with a knot diagram (a simple curve) that can be factorized into one simple closed curve. Any knotted knot has a knot diagram that can be factorized into two simple closed curves.*

Proof: The first part of the theorem is obvious. To prove the last part let KD be a knot diagram with an scs-factorization $KD \to S_1 + \cdots + S_n$ of order $n \geq 3$ and let S be a

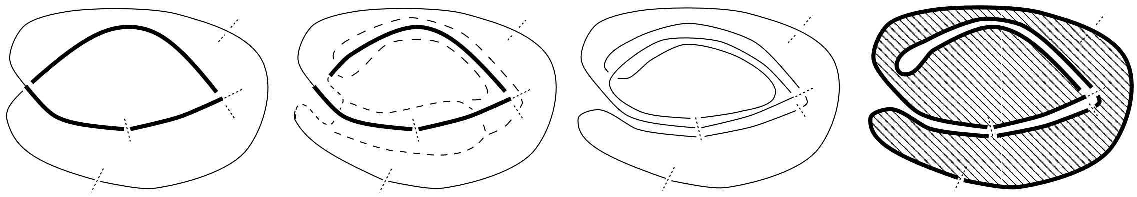


Figure 3.6: *Left: Two neighbouring disjoined simple closed sub-curves. Right: One simple closed sub-curve. And in between an isotopy connecting the two outer knot diagrams.*

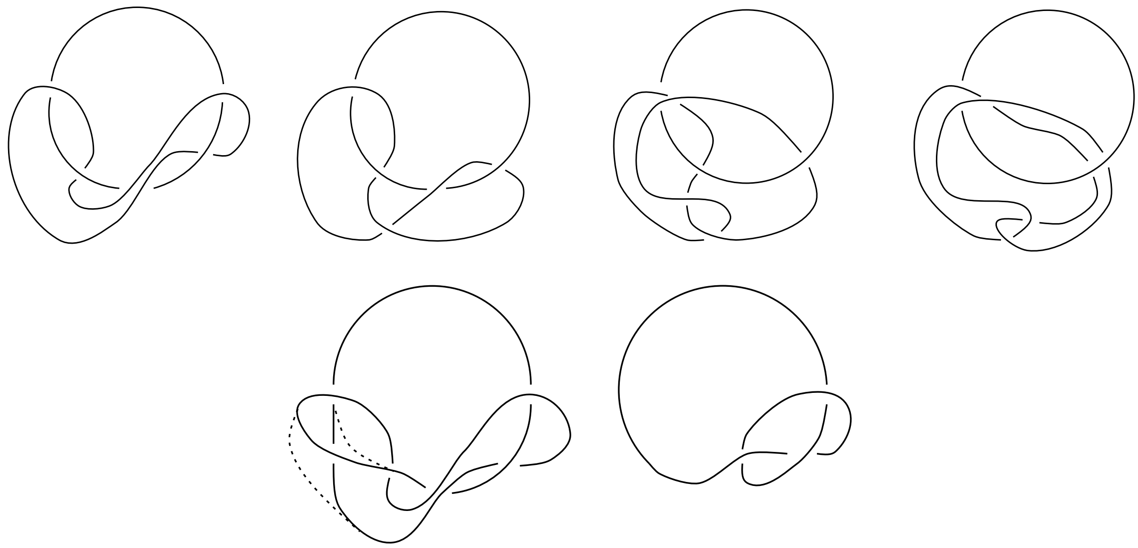


Figure 3.7: *On top an isotopy of the Whitehead link. Below two knot diagrams of the left-handed Trefoil knot.*

corresponding scs-surface. The surface S consists of n discs, $D_1, \ldots, D_n$, each of which is embedded and these discs are connected by $(n-1)$ half-twisted bands. Hence, we may assume that there is only one half-twisted band attached to D_1 connecting D_1 and D_2 - say. If D_1 and D_2 intersects then this intersection can be pushed into $D_3, \ldots, D_n$ without changing the knot type of the boundary of S. The part of the surface S given by D_1 and D_2 and their connecting band is now embedded allowing us to consider it as one embedded disc. We can now flatten out the $(n-1)$-disc surface and obtain a knot diagram with an scs-factorization of order $(n-1)$. See Figure 3.6 . Hence, by changing the knot diagram we can reduce the order of scs-factorizations until order two is reached. ☐

Remark 3.31 (On Theorem 3.29) *On could ask if: Given a knot diagram with an* scs-*factorization of order two and the linking number between the two sub-curves is zero, is this necessary a knot diagram of the unknot? This is a special case of a question if one can improve the upper bound given as* "the number of crossings between S_1 and Γ_1 not counting the cutting point between S_1 and Γ_1" *in the proof of Theorem 3.29 by the natural*

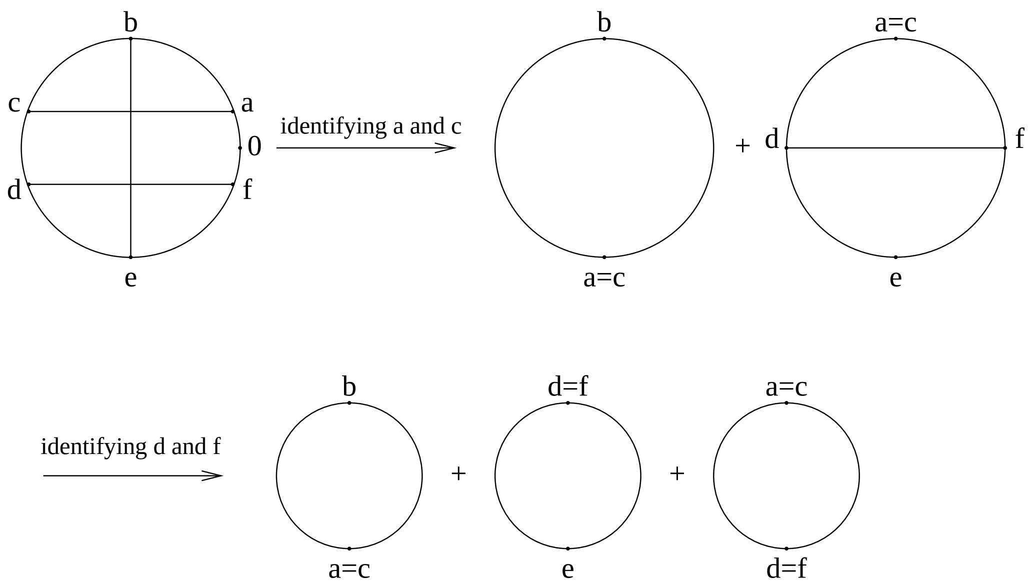


Figure 3.8: The chord diagrams of the first scs-factorization of the curve shown on Figure 3.1.

linking number between the two curves?

On Figure 3.7 at the top is shown an isotopy of the Whitehead link, which is known for the property that even though the linking number between its two components is zero they can not split. The knot diagram to the outmost left of this isotopy has furthermore the property that each of its components is projected to a simple closed curve. Thus the non standard knot diagram of the left-handed Trefoil knot shown on Figure 3.7 to the bottom left, answer the first and hence both of the above questions is to the negative.

3.9 How to find all scs-factorizations of a closed curve

In order to get the best possible upper bound on the unknotting number of all knots with a given shadow, one has to determine the highest possible order of all scs-factorizations of this planar curve. Both to show how to determine this number and for the completeness of our treatment of scs-factorizations we now give a method for finding all scs-factorizations of a given closed curve in a topological space which has self-intersections of order at most two.

Let $\gamma : \mathbb{S}^1 \to \mathbb{T}$ be a closed curve with at most finitely many double points and no points of higher multiplicity. Picture $\mathbb{S}^1$ as the unit circle in the two-plane and draw a straight line segment (chord) between each pair of parameter values that corresponds to a self-intersection of $\Gamma(\mathbb{S}^1)$. This graph is called the chord diagram of Γ, is denoted by $CD(\Gamma)$, and is considered up to orientation preserving reparametrization of $\mathbb{S}^1$. On the top left on Figure 3.8 is shown the chord diagram of the curve on Figure 3.1.

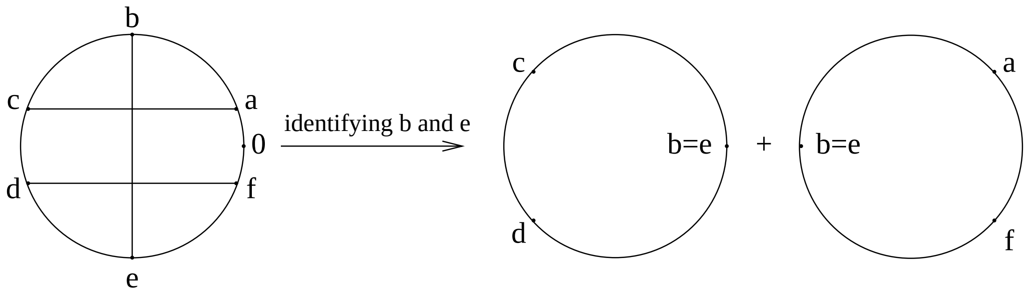


Figure 3.9: Chord diagrams of the second scs-factorization of the curve shown on Figure 3.1.

Pick an arbitrary chord C in $CD(\Gamma)$. The chord C cuts $\mathbb{S}^1$ into two closed intervals I_1 and I_2. Each of these intervals is mapped to a simple closed curve by Γ if and only if no chord, other that C, combine points belonging to the same interval. Assume this is true for the interval I_1 - say. We may and do apply an elementary factorization to $\Gamma = \Gamma_0 \rightarrow S_1 + \Gamma_1$ at the point corresponding to the endpoints of I_1. Hereby the chord diagram $CD(\Gamma_1)$ of the remainder of Γ has exactly the chords from $CD(\Gamma)$ that starts and ends in I_2 except for the chord C. Seen on the level of chord diagrams the elementary factorization "along" the chord C removes C and all chords that is intersected by C. Hence, given the chord diagram of a closed curve, we have an algorithm giving all scs-factorizations of this curve.

As an example, the two scs-factorizations on Figure 3.1 are show at the chord diagram level on Figure 3.8 and Figure 3.9. Picture each elementary factorization, at the chord diagram level, as the unit disc getting squeezed along a chord. In this picture an scs-factorization transforms the original chord diagram disc to an "animal made of a balloon" and we get a visualization of the fact that a scs-surface is an immersed disc (Theorem 3.28).

Chapter 4

Principal normal indicatrices of closed space curves[1]

4.1 Introduction

A theorem due to J. Weiner [49], which is proven curve theoretically by L. Jizhi and W. Youning in [24] and by B. Solomon in [45], implies that a principal normal indicatrix of a closed space curve with non-vanishing curvature has integrated geodesic curvature zero and contains no sub arc with integrated geodesic curvature π. This corollary is not observed in any other reference! We prove that the inverse problem always has solutions under the Fenchel Assumption, i.e., if one allows zero and negative curvature of space curves, and we explain why this not is true if non-vanishing curvature is required. This answers affirmatively an open question posed by W. Fenchel in 1950, [10], under the above assumptions but in general this question is found to be answered in the negative[2]. Recall that under the Fenchel assumption (Section 2.2) the new principal normal N sweeps out a regular C^2 curve on the unit 2-sphere. We denote arc length of the space curve by s and arc length of the principal normal indicatrix by t. As s and t are diffeomorphic, since curvature and torsion not are allowed to vanish simultaneously, we use t as parameter and we denote differentiation with respect to t by a dot, e.g., $\dot{N}$. Furthermore both T and B are piecewise regular spherical curves with tangents that are parallel with N in respective points. This calls for

Definition 4.1 *Let X and Y be two curves on the unit 2-sphere both parametrized by the arc length t of Y. If X is piecewise regular then X is said to have Y as projective tangent image if $\dot{X}(t)$ is parallel with $Y(t)$ for all t.*

[1] With the exception of Example 4.10, all results in this Chapter can be found in [39].

[2] The article [48] may also contain an answer to this question.

4.2 Necessary conditions.

Our first necessary condition is given by Remark 3.19 and it states: *"Let N be a principal normal indicatrix of a closed space curve fulfilling the Fenchel condition. Then the integral of the geodesic curvature of N equals $2\pi\nu$ for some integer ν"*.

Again the integer ν (nutation) is the number of times the plane curve given by $s \mapsto (k, \tau)$ wraps around $(0, 0)$. In case of a closed space curve with non-vanishing curvature $\nu = 0$ and Gauss-Bonnet's theorem yields Jacobi's theorem. The following Proposition has Remark 3.19 as a corollary.

Proposition 4.2 *Let $N : [0, L] \to \mathbb{S}^2$ be a regular C^2-curve on the unit 2-sphere parametrized by arc length t. Then all curves on the unit 2-sphere with N as projective tangent image are given by*

$$T_a(t) = -\cos\left(\int_0^t \kappa_g d\tilde{t} + a\right)\dot{N}(t) + \sin\left(\int_0^t \kappa_g d\tilde{t} + a\right) N(t) \times \dot{N}(t)$$

where $a \in \mathbb{S}^1$, such that, $\cos\left(\int_0^t \kappa_g d\tilde{t} + a\right)$ *is not zero on any interval*[3].

Furthermore, if N is closed then all the curves T_a are closed if and only if $\int_0^L \kappa_g d\tilde{t} = 2\pi\nu$ for some integer ν.

Proof: If a curve T on the unit 2-sphere has N as projective tangent image then $\dot{T}$ is parallel with N. As $T(t)$ is a unit vector for all t, $\dot{T}$ and T are orthogonal. Hence, T and N are orthogonal whenever $\dot{T}$ is non-zero. As $\dot{T}$ may not be zero on an interval we conclude by continuity that T and N are always orthogonal. Hereby we can identify T with an unit vector field along $N \subset \mathbb{S}^2$ and write T as

$$T(t) = -\cos(f(t))\,\dot{N}(t) + \sin(f(t))\,N(t) \times \dot{N}(t)$$

for some function $f = f(t)$. Using that $\ddot{N} = -N + \kappa_g N \times \dot{N}$, where κ_g is the geodesic curvature of N, we get

$$\begin{aligned}
\dot{T} &= -\cos(f)(-N + \kappa_g N \times \dot{N}) + \sin(f)\dot{f}\dot{N} \\
&\quad + \sin(f)N \times (-N + \kappa_g N \times \dot{N}) + \cos(f)\dot{f}N \times \dot{N}. \\
&= \cos(f)N + \sin(f)(\dot{f} - \kappa_g)\dot{N} + \cos(f)(\dot{f} - \kappa_g)N \times \dot{N}.
\end{aligned}$$

Again $\dot{T}$ is parallel with N (in fact T is a parallel unit vector field along N in the sense of parallelism) if and only if $f(t) = \int_0^t \kappa_g d\tilde{t} + a$ for some $a \in \mathbb{S}^1$. In order for T to have N as projective tangent image $\dot{T}(t) = \cos\left(\int_0^t \kappa_g d\tilde{t} + a\right) N(t)$ may not be zero on an interval. This proves the first part of the theorem and the reminder is trivial. □

[3] This condition ensures that T_a not is constant on an interval. The necessity of excluding this possibility, that occurs if and only if N contains a piece of a great circle, is overlooked in [10].

Let N be a regular C^2-curve on the unit 2-sphere and let T_a be one of the spherical curves having N as projective tangent image from Proposition 4.2. By construction, any space curve with T_a as tangent indicatrix has N as one of the two possible choices of its principal normal indicatrix. This explains the chosen notation. Aiming for finding necessary and sufficient conditions for a spherical curve to be the principal normal indicatrix of a closed space curve we from now on only consider the case where both the curves N and T from Prop. 4.2 are closed. As a corollary of Prop. 4.2 we get

Theorem 4.3 *Let N be a closed regular C^2-curve on the unit 2-sphere. Then N is the ordinary tangent image of a regular and closed curve on the unit 2-sphere if and only if it has integrated geodesic curvature zero and contains no sub arc with integrated geodesic curvature π.*

Theorem 4.3 is due to J. Weiner [49]. The proof given in [45] of this theorem is model for the proof of Proposition 4.2, but considers only the regular case.

Proof: Let $N : \mathbb{R}/L\mathbb{R} \to \mathbb{S}^2$ be a closed regular C^2-curve on the unit 2-sphere. All curves on the unit 2-sphere, T_a, with N as projective tangent image are given by Prop. 4.2. From the proof of Prop. 4.2 it follows that T_a is regular if and only if $\cos\left(\int_0^t \kappa_g d\tilde{t} + a\right) \neq 0$ for all $t \in [0, L]$. This is possible if and only if $\left|\int_I \kappa_g dt\right| < \pi$ for all intervals $I \subset \mathbb{R}/L\mathbb{R}$. Furthermore, T_a is closed if and only if $\int_0^L \kappa_g dt = 2\pi\nu$ for some integer ν. As $|\int_0^L \kappa_g dt| < \pi$ we conclude that $\nu = 0$. Under these two integral conditions on the geodesic curvature of N there exists an $a \in \mathbb{S}^1$ such that $\cos\left(\int_0^t \kappa_g d\tilde{t} + a\right) > 0$ for all $t \in [0, L]$ and the theorem follows. $\square$

Using the Fenchel Assumption in stead of the "usual" assumption of non-vanishing curvature of space curves the principal normal indicatrix is still regular but the tangent indicatrix is in general not a regular curve. In the following we need a regularity assumption (positive regularity index) that lies in between these two assumptions.

Definition 4.4 *Let $\mathbf{r} : [0, l] \to \mathbb{R}^3$ be a (not necessarily closed) space curve fulfilling the Fenchel Assumption. By the regularity index of $\mathbf{r}$, $I_{\mathbb{R}^3}(\mathbf{r})$, we mean the length of the interval of $a \in \mathbb{S}^1$ such that $\cos(a)k(s) - \sin(a)\tau(s) > 0$ for all $s \in [0, l]$.*

A perhaps more geometric way of stating Definition 4.4 is: Let α be the minimal angle of a cone in $\mathbb{R}^3$, with the origin as cone point, containing the curve given by $(k(s), \tau(s))$. If $\alpha < \pi$ then $\pi - \alpha$ is the regularity index of the space curve $\mathbf{r}$, otherwise the regularity index of $\mathbf{r}$ is said to be zero. Hence, a space curve with non-vanishing curvature has positive regularity index.

Definition 4.5 *Let $N : [0, L] \to \mathbb{S}^2$ be a C^2-curve on the unit 2-sphere. Let M be the maximal value of $\left|\int_I \kappa_g dt\right|$ for all intervals $I \subset [0, L]$. The regularity index of N, $I_{\mathbb{S}^2}(N)$, is given by $\pi - M$ if $M < \pi$ and zero otherwise.*

Proposition 4.6 *If* $\mathbf{r}$ *is a space curve fulfilling the Fenchel condition and* N *is a principal normal indicatrix of* $\mathbf{r}$*, then their respective regularity indices are equal, i.e.,* $I_{\mathbb{R}^3}(\mathbf{r}) = I_{\mathbb{S}^2}(N)$.

Denote by $\{T_a\}_{a\in\mathbb{S}^1}$ *the family of curves on the unit* 2*-sphere having* N *as projective tangent image. Then the set of the family parameter values* a *for which* T_a *is a regular curve having* N *as ordinary tangent image is an open interval of length* $I_{\mathbb{S}^2}(N)$.

Proof: Let $\mathbf{r}$ and N be as in the theorem. If T and B denote the tangents and binormals of $\mathbf{r}$, consider for all $b \in \mathbb{S}^1$ the curve on the unit 2-sphere given by $C_b(s) = \cos(b)T(s) + \sin(b)B(s)$. If this curve has a projective tangent image then it can be chosen as N. By Frenet's Formulas the curve C_b is regular and has N as ordinary tangent image if and only if $\cos(b)k(s) - \sin(b)\tau(s) > 0$ for all s.

In Prop. 4.2 all curves with N as projective tangent image are given. It is obvious that there is an a_0 such that for all $b = a + a_0$ the two curves C_b and T_a are equal (their parameters are diffeomorphic). From the proof of this Proposition we know that the curve T_a is regular and has N as ordinary tangent image if and only if $\pi/2 < \int_0^t \kappa_g dt + a < \pi/2$ for all t and the theorem follows. $\square$

An immediate consequence of Theorem 4.3 and Proposition 4.6 is

Corollary 4.7 *A principal normal indicatrix of a closed space curve of type* C^4 *with positive regularity index (especially with non-vanishing curvature) has integrated geodesic curvature zero and contains no sub arc with integrated geodesic curvature* π*, i.e., it has positive regularity index.*

In the next section we prove the main result of this chapter, namely, that the inverse of Corollary 4.7 is true.

4.3 Sufficient conditions

To obtain sufficient conditions for a closed curve on the unit 2-sphere to be the principal normal indicatrix of a closed space curve we use the following theorem due to Werner Fenchel [9] as our main tool.

Theorem 4.8 *Let* Γ *be a closed rectifiable curve on the unit* 2*-sphere. Then* Γ *is the tangent indicatrix of a closed space curve if and only if*

$\mathcal{A}$: Γ *is a great circle in case* Γ *is planar*

$\mathcal{B}$: *the origin of* $\mathbb{R}^3$ *is an interior point of the convex hull of* Γ *in case* Γ *is not planar.*

It is straight forward to check that the above condition $\mathcal{B}$ is equivalent to $\mathcal{B}'$: Γ is not contained in a closed hemisphere in case Γ is not planar.

The article [10] closes with an open question that we can formulate as: *Is there in each family of curves from Prop. 4.2 a curve whose convex hull contains the origin of $\mathbb{R}^3$ in its interior?* An affirmative answer to this open question then makes the necessary condition in Remark 3.19 also a sufficient condition for a closed curve on the unit 2-sphere to be the principal normal indicatrix of a closed space curve. We answer this question affirmatively in case the curve has positive regularity index and we show, by giving examples, that otherwise the general answer is in the negative.

Consider an arbitrary curve T_a from Prop. 4.2. Denote by $P(a)$ the set of poles of closed hemispheres containing T_a, i.e.,

$$\begin{aligned} P(a) &= \left\{ p \in \mathbb{S}^2 | p \cdot T_a(t) \geq 0 \;\; \forall \;\; t \in [0, L] \right\} \\ &= \bigcap_{t \in [0,L]} \left\{ p \in \mathbb{S}^2 | p \cdot T_a(t) \geq 0 \right\}. \end{aligned}$$

Note, that $P(a)$ is a closed and convex set in $\mathbb{S}^2$.

We need some analysis of the map $T(t, a) = T_a(t)$. For notational reasons we write $\Phi(t) = \int_0^t \kappa_g d\tilde{t}$. Consider N as a tangential vector field along the map $t \mapsto T_a(t)$. If T_a is piecewise regular then N is a projective tangent image of T_a. Orthogonal to this vector field along the map $t \mapsto T_a(t)$ we have the vector field $\frac{\partial}{\partial a} T(t, a) = \sin(\Phi + a)\dot{N}(t) + \cos(\Phi + a) N \times \dot{N} = T_a(t) \times N(t) = T(t, a + \pi/2)$.

Rewriting $T(t, a)$ as

$$\begin{aligned} T(t, a) &= -\cos(\Phi + a)\dot{N}(t) + \sin(\Phi + a) N \times \dot{N} \\ &= \cos(a) \left(-\cos(\Phi)\dot{N}(t) + \sin(\Phi) N \times \dot{N} \right) \\ &\quad + \sin(a) \left(\sin(\Phi)\dot{N}(t) + \cos(\Phi) N \times \dot{N} \right) \\ &= \cos(a) T(t, 0) + \sin(a) T(t, \pi/2) \end{aligned}$$

we see that for a fixed t the map $a \mapsto T(t, a)$ prescribe a great circle traversed with unit speed. As $\left\{ T(t, a), N(t), T(t, a + \pi/2) = \frac{\partial}{\partial a} T(t, a) \right\}$ is an orthonormal basis for $\mathbb{R}^3$ for all (t, a), the great circle through $T(t, a + \pi/2)$ with $N(t)$ as tangent divides $\mathbb{S}^2$ into points with positive resp. negative inner product with $T(t, a)$. See Figure 4.1. As $\frac{\partial}{\partial a} T(t, a + \pi/2) = T(t, a + \pi) = -T(t, a)$ the vector $\frac{\partial}{\partial a} T(t, a + \pi/2)$ points into the component of the complement of the great circle through $T_{a+\pi/2}$ with tangent $N(t)$ having negative inner product with $T_a(t)$. We can thus rewrite $P(a)$ as

$$P(a) = \bigcap_{t \in [0,L]} \left\{ p \in \mathbb{S}^2 | p \cdot \frac{\partial}{\partial a} T(t, a + \pi/2) \leq 0 \right\}.$$

If $T_{a+\pi/2}(t)$ is a regular point on the curve $T_{a+\pi/2}$ then $N(t)$ is a tangent to $T_{a+\pi/2}$ in $T_{a+\pi/2}(t)$. Thus if T_a lies on a closed hemisphere each point $p \in P(a)$ for all t lies either on the tangent great circle to $T_{a+\pi/2}$ in $T_{a+\pi/2}(t)$ or in the positive component of the complement of this great circle.

Figure 4.1: *On the figure eight curve T_a, we have drawn a part of its tangent great circle through a point, $T_a(t)$. On the piecewise regular curve $T_{a+\pi/2}$ we have drawn the tangent great circle through its corresponding point $T_{a+\pi/2}(t)$. The smaller parts of great circles are pieces of the tangent great circles to $T_{a'}(t)$ for some $a < a' < a + \pi/2$ are included to indicate the a-flow.*

On Figure 4.1 it is obviously not possible for a point on $\mathbb{S}^2$ to lie on one side of all the tangent great circles to the figure eight curve, denoted by T_a. We can therefor conclude that $T_{a-\pi/2}$, and thereby its antipodal curve, $T_{a+\pi/2}$, does not lie on a closed hemisphere. Which also is clear from this figure. With this observation we are ready for our main theorem.

Theorem 4.9 *Let N be a regular closed C^2-curve on the unit 2-sphere with integrated geodesic curvature zero. If N has positive regularity index then N is a principal normal indicatrix of a closed space curve.*

Proof: Let N be a regular closed C^2-curve on the unit 2-sphere having positive regularity index and integrated geodesic curvature zero. By Theorem 4.3 the curve N is the tangent image of a closed regular curve on the unit 2-sphere given by T_a for some a (cf. Prop. 4.2). If T_a is not contained in a closed hemisphere then N is the principal normal indicatrix of a closed space curve (with non-vanishing curvature) by Theorem 4.8 $\mathcal{B}'$.

Assume that T_a is contained in a closed hemisphere. If $T_{a-\pi/2}$ is not contained in a closed hemisphere, then by continuity this is also true for all a' in a neighbourhood of $a - \pi/2$ in $\mathbb{S}^1$. In this neighbourhood of $a - \pi/2$ there is an a' such that $T_{a'}$ is piecewise regular. By Theorem 4.8 the curve N is a principal normal indicatrix of a closed space curve.

We are now left with the non-trivial case that both T_a and $T_{a-\pi/2}$ are contained in open hemispheres. Assuming this we have a pole $p \in P(a-\pi/2)$ of a closed hemisphere containing $T_{a-\pi/2}$. As

$$P(a-\pi/2) = \bigcap_{t\in[0,L]} \left\{ p \in \mathbb{S}^2 | p \cdot \frac{\partial}{\partial a} T(t,a) \leq 0 \right\}$$

the unit vector field $\frac{\partial}{\partial a}T(t,a)$ along the curve T_a always points away from all points in $P(a-\pi/2)$.

Let $D(a)$ be the closed disc with circular boundary on $\mathbb{S}^2$ of smallest spherical radius $\leq \pi/2$ containing T_a. Then $\partial D(a)$ and T_a has contact of first order in all their common points. By orthogonality of $\dot{T}_a \neq \mathbf{0}$ and $\frac{\partial}{\partial a}T(t,a)$ the vectors $\frac{\partial}{\partial a}T(t,a)$ are orthogonal to $\partial D(a)$ in all points of $\partial D(a) \cap T_a$.

Claim: Either these vectors all point into the interior of $D(a)$ or they all point away from $D(a)$.

Proof of claim: The tangent great circle to T_a at $T_a(t)$ divides $\mathbb{S}^2$ into points with positive resp. negative inner product with $T_{a-\pi/2}(t)$ and hence, into points with negative resp. positive inner product with $T_{a+\pi/2}(t) = -T_{a-\pi/2}(t)$. Hereby $P(a-\pi/2)$ and $P(a+\pi/2)$ each is contained in the closure of a connected component of the complement of T_a.

By our assumption T_a is a regular closed curve. Start traversing T_a from an arbitrary point, $T_a(t_0)$, until the first time a pre-traversed point, $T_a(t_1) = T_a(t_2)$, is reached. This gives a simple closed curve from $T_a(t_1)$ to $T_a(t_2)$ that is regular in all points except form $T_a(t_1) = T_a(t_2)$. The vector field $\frac{\partial}{\partial a}T(t,a)$ is orthogonal to $\frac{\partial}{\partial t}T(t,a) \neq \mathbf{0}$. Hence, for all $t \in (t_1, t_2)$ the vector field $\frac{\partial}{\partial a}T(t,a)$ points into only one of the two connected components of the complement of our simple closed curve. The set $P(a+\pi/2)$ is therefore contained in the closure of the component of the complement of our simple closed curve into which the vector field $\frac{\partial}{\partial a}T(t,a)$ points and $P(a-\pi/2)$ is contained in the closure of the other component. We conclude that one of the two sets $P(a-\pi/2)$ and $P(a+\pi/2)$ is contained in $D(A)$.

If $P(a-\pi/2)$ is contained in $D(A)$ the vectors $\frac{\partial}{\partial a}T(t,a)$ in all points of $\partial D(a) \cap T_a$ point away from $D(a)$ and if $P(a+\pi/2)$ is contained in $D(A)$ the vectors $\frac{\partial}{\partial a}T(t,a)$ in all points of $\partial D(a) \cap T_a$ point into the interior of $D(a)$. This proves the claim.

Continuing this proof of Theorem 4.9 we first consider the case that all the vectors $\frac{\partial}{\partial a}T(t,a)$ in all points of $\partial D(a) \cap T_a$ point away from the interior of $D(a)$. As $D(a)$ has minimal radius of all discs containing T_a, the intersection $\partial D(a) \cap T_a$ either contains two points dividing $\partial D(a)$ into halves or $\partial D(a) \cap T_a$ contains at least three points such that any open interval of $\partial D(a)$ of length half the length of $\partial D(a)$ contains at least one of these three points.

Increasing the family parameter a all points initially on $\partial D(a) \cap T_a$ move away from $D(a)$ on great circles orthogonally to $\partial D(a)$ with unit speed. Hence, the radius of $D(a)$ is increased by Δa when a is increased by $\Delta a \geq 0$ at least until this radius becomes $\pi/2$.

Let $\pi/2 > \Delta a \geq 0$ be chosen such that the radius of $D(a + \Delta a)$ equals $\pi/2$, that is $D(a + \Delta a)$ is a closed hemisphere. Assume that $T_{a+\Delta a}$ is planar, i.e., $T_{a+\Delta a} \subset \partial D(a + \Delta a)$. By the assumption that N is a regular closed curve $T_{a+\Delta a}$ traverses the great circle $\partial D(a + \Delta a)$ a finite number of times without changing direction. Hence, $T_{a+\Delta a}$ is the tangent image of a closed planar curve with non-vanishing curvature. From now on we assume that $T_{a+\Delta a}$ is not planar.

Case I: We first consider the case that $\partial D(a) \cap T_a$ contains three points, $T(t_i, a)$, $i = 1, 2, 3$, such that any open interval of $\partial D(a)$ with half the length of $\partial D(a)$ contains at least one of these three points. These three points flow along great circles orthogonal to $\partial D(a)$ with unit speed for increasing family parameter a. Hence, the 2-simplex with $T(t_i, a + \Delta a)$, $i = 1, 2, 3$, as vertices contains the origin of $\mathbb{R}^3$ in its interior. As $T_{a+\Delta a}$ is not planar there is a point $T_{a+\Delta a}(t_0)$ lying in the interior of the hemisphere $D(a + \Delta a)$. By continuity of the map $T(t, a)$ the point $T(t_0, a + \Delta a + \varepsilon)$ still lies in the interior of $D(a + \Delta a)$ for sufficiently small $\varepsilon > 0$. So for a sufficiently small $\varepsilon > 0$ the 3-simplex ,S, with $T(t_i, a + \Delta a + \varepsilon)$, $i = 0, 1, 2, 3$, as vertices contains the origin of $\mathbb{R}^3$ in the interior of its convex hull. As the 3-simplex S is contained in the convex hull of the curve $T_{a+\Delta a+\varepsilon}$ we conclude by Theorem 4.8 that N is a principal normal indicatrix of a closed space curve.

Case II: Finally we consider the case that $\partial D(a) \cap T_a$ contains two points $T(t_i, a)$, $i = 0, 1$, dividing $\partial D(a)$ into halves. Again we have an $\pi/2 > \Delta a \geq 0$ such that the radius of $D(a + \Delta a)$ equals $\pi/2$, i.e., $D(a + \Delta a)$ is a closed hemisphere.

When t converges towards t_0 the two intersection points between the great circles $T(t, \mathbb{S}^1)$ and $T(t_0, \mathbb{S}^1)$, converge to the pole of $D(a)$ and its antipodal point. Hence, by choosing t sufficiently close to t_0, and we do so in the following, these two intersections lie in a given neighbourhood of the pole of $D(a)$ and its antipodal set.

Let P be the plane in $\mathbb{R}^3$ through the pole of $D(a)$ and the two points $T(t_i, a)$, $i = 0, 1$. The plane P divides $D(a)$ into two halves. As T_a is regular and intersects the plane P orthogonally in $T_a(t_0)$ we can choose t_2 and t_3 arbitrarily close to t_0 such that $T(t_2, a)$ and $T(t_3, a)$ lies on each side of P. By the above argument we by choosing t_2 and t_3 sufficiently close to t_0 can ensure that the a-flow of the points $T(t_i, a)$, $i = 2, 3$, first crosses the plane P close to the pole of $D(a)$'s antipodal point.

If $T(t, a) \in \partial D(a)$ for t in an interval containing t_0 in its interior we can pick t_2 and t_3 on each side of t_0 such that the points $T(t_i, a)$, $i = 1, 2, 3$, fulfil the condition in Case I. Hence, we may assume that either $T(t_2, a)$ or $T(t_3, a)$ lies in the interior of $D(a)$.

By the a-flow of the points $T(t_i, a)$, $i = 0, 1, 2, 3$, the origin of $\mathbb{R}^3$ lies in the middle of the edge between $T(t_0, a + \Delta a)$ and $T(t_1, a + \Delta a)$ in the 3-simplex S in $\mathbb{R}^3$ given by its vertices $T(t_i, a + \Delta a)$, $i = 0, 1, 2, 3$. Since either $T(t_2, a + \Delta a)$ or $T(t_3, a + \Delta a)$ lie in the interior of $D(a + \Delta a)$ the convex hull of the 3-simplex S is not planar. Whence, for every sufficiently small $\varepsilon > 0$ the 3-simplex with vertices $T(t_i, a+\Delta a+\varepsilon)$, $i = 0, 1, 2, 3$ contains the origin of $\mathbb{R}^3$ in the interior of its convex hull. Theorem 4.8 now implies that N is a principal normal indicatrix of a closed space curve.

All arguments in Case I and Case II are based on the assumption that all the vectors $\frac{\partial}{\partial a}T(t,a)$ in all points of $\partial D(a) \cap T_a$ point away from the interior of $D(a)$. If instead, according to the Claim, all these vectors point into the interior of $D(a)$ we get an analogous proof - but this time by decreasing the family parameter. $\square$

The following example shows that the assumption of positive regularity index in Theorem 4.9 in general can not be weakened.

Example 4.10 *Let N be a curve with length $< 2\pi$ with shape of a symmetric figure eight on the unit 2-sphere. By symmetry N has integrated geodesic curvature zero. With notation as in the proof of Prop. 4.2 we have that, all curves on the unit 2-sphere with N as projective tangent image has the derivative $\frac{d}{dt}(T_a)(t) = \cos(\phi(t)+a)N(t)$, where t is arc length of N. Hereby, the lengths of these curves all are $< 2\pi$. By Fenchels Lemma, each of these curves is too short to be the tangent indicatrix of a closed space curve. Hence, the curve N is not a principal normal indicatrix of a closed space curve.*

An example of a closed spherical curve with integrated geodesic curvature equal to $2\pi\nu$ for some integer $\nu \neq 0$ that is to short to be a principal normal indicatrix of a closed space curve is as follows[4]. First, construct a curve with the shape of a non-symmetrical figure eight, such that the loop traversed in the negative direction enclose twice the area as the loop traversed in the positive direction and such that both of the two loops forms smooth closed curves that have at least second order contact at their point of intersection. Next, the curve N is given by traversing the positive loop twice and the negative loop ones. If A resp. $2A$ denotes the area enclosed by the positively resp. negatively traversed loop then by the Gauss-Bonnet's Theorem this curve has integrated geodesic curvature $(2\pi - A) + (2\pi - A) + (2\pi - (4\pi - 2A)) = 2\pi$. Finally, a ν-multiple cower (negative ν reverses orientation) has integrated geodesic curvature $2\pi\nu$ and can be constructed to have any given length $< 2\pi$. This calculation is an exercise in using Lemma 3.10 and Theorem 3.12.

4.4 The vanishing of curvature

An immediate consequence of Definition 4.4 is that the regularity index of a closed space curve with either non-vanishing curvature or non-vanishing torsion is greater than zero; and that the regularity index of a closed space curve with both non-vanishing curvature and non-vanishing torsion is greater than $\pi/2$. On the other hand, let N be a spherical curve fulfilling the conditions in Theorem 4.9. Then a closed space curve with N as principal normal indicatrix may have zeros for both its curvature and its torsion. If the

[4] In [39] p. 57 it is claimed that *a regular closed C^2-curve on the unit 2-sphere with integrated geodesic curvature $2\pi\nu$ for some $\nu = \pm 1, \pm 2, \ldots$, has length $> 2\pi$.* This is false! This, I notified the Editor November 5'th 1998, which however was to late for correcting the final version of the article [39].

regularity index of N is greater than $\pi/2$ then a closed space curve $\mathbf{r}$ with N as principal normal indicatrix has either non-vanishing curvature or non-vanishing torsion. This follows from the observation that the tangent indicatrix of $\mathbf{r}$ and the binormal indicatrix of $\mathbf{r}$ by orthogonality can be found in our family of curves, $\{T_a\}_{a\in\mathbb{S}^1}$, as T_a for some $a \in \mathbb{S}^1$ and $T_{a+\pi/2}$. As the regularity index of N is greater that $\pi/2$ it follows from Prop. 4.6 that at least one of these two curves is regular.

Considering space curves with non-vanishing curvature, as most authors do, the possible vanishing of curvature in the solutions given in Theorem 4.9 is annoying. We use the reminder of this chapter to show that the results can not be improved towards solutions with non-vanishing curvature.

Lemma 4.11 *Let T be a closed regular C^3-curve on the unit 2-sphere of length $|T|$ and denote the integral of the geodesic curvature of T by $\kappa_g(T)$. Then any closed regular curve on the unit 2-sphere with the same ordinary tangent indicatrix as T's tangent indicatrix has length less than or equal to $\sqrt{|T|^2 + (\kappa_g(T))^2}$.*

Proof: Let T be as in the lemma and denote its tangent indicatrix by N. From Prop. 4.2 we know that any curve on $\mathbb{S}^2$ with N as ordinary tangent indicatrix can be written as

$$T_a(t) = -\cos(\phi(t) + a)\dot{N}(t) + \sin(\phi(t) + a)N(t) \times \dot{N}(t),$$

where $t \in [0, L]$ is the arc length of N, $\dot{N} = \frac{dN}{dt}$, and $\phi(t) = \int_0^t \kappa_{g,N}(\tilde{t})d\tilde{t}$ fulfills $-\pi/2 < \phi(t) + a < \pi/2$ for all $t \in [0, L]$, where $\kappa_{g,N}$ is the geodesic curvature of the curve N.

As $\frac{dT_a}{dt} = \cos((\phi(t) + a))N(t)$ and $\cos((\phi(t) + a))$ is positive for all $t \in [0, L]$ we find the squared length of the curve T_a as

$$|T_a|^2 = \left(\int_0^L \cos((\phi(t) + a))dt\right)^2 .$$

Let σ denote the arc length of the curve T_a. As T_a is a curve on the unit 2-sphere it has curvature $\geq$ 1. Hence, the map $t \mapsto \sigma(t)$ is a diffeomorphism and we find that $\frac{dt}{d\sigma} = \frac{1}{\cos(\phi)}$.

Next we calculate the geodesic curvature of the curve T_a, κ_g, as a function of the arc length t of the curve N.

$$\frac{d^2T_a}{d\sigma^2} = \frac{dN}{d\sigma} = \frac{dN}{dt}\frac{dt}{d\sigma} = \frac{1}{\cos(\phi)}\dot{N}$$

$$\begin{aligned}\kappa_g(t) &= \left(T_a \times \frac{dT_a}{d\sigma}\right) \cdot \frac{d^2T_a}{d\sigma^2}(t) \\ &= \big(\big(-\cos(\phi(t) + a)\dot{N}(t) \\ &\quad + \sin(\phi(t) + a)N(t) \times \dot{N}(t)\big) \times N(t)\big) \cdot \frac{1}{\cos(\phi)}\dot{N}(t) \\ &= \tan((\phi(t) + a))\end{aligned}$$

This gives the integration formula

$$\int_{T_a} \kappa_g(\sigma) d\sigma = \int_N \kappa_g(\sigma(t)) \frac{d\sigma}{dt} dt = \int_N \sin(\phi(t) + a) dt.$$

Hereby we have that

$$|T_a|^2 + (\kappa_g(T_a))^2 = \left(\int_0^L \cos(\phi(t) + a) dt \right)^2 + \left(\int_0^L \sin(\phi(t) + a) dt \right)^2$$

To prove the lemma it is sufficient to prove that the right hand side of the above equation is independent of the family parameter a.

$$\begin{aligned}
&\frac{d}{da} \left(\left(\int_0^L \cos(\phi(t) + a) dt \right)^2 + \left(\int_0^L \sin(\phi(t) + a) dt \right)^2 \right) \\
&= 2 \Big(- \int_0^L \cos(\phi(t) + a) dt \int_0^L \sin(\phi(t) + a) dt \\
&\qquad + \int_0^L \sin(\phi(t) + a) dt \int_0^L \cos(\phi(t) + a) dt \Big) \\
&= 0
\end{aligned}$$

□

Consider a closed curve T on the unit 2-sphere with the shape of the figure eight and length $< 2\pi$. By symmetry, the integral of T's geodesic curvature is zero. Hence, this curve is, by Lemma 4.11, the longest regular curve on the unit 2-sphere having its tangent indicatrix, N, as ordinary tangent indicatrix. A curve with length $< 2\pi$ is too short to be the tangent indicatrix of a closed space curve (Fenchels Lemma). We conclude that the curve N is not the principal normal indicatrix of a closed space curve with non-vanishing curvature. However, since N is the ordinary tangent indicatrix of T the regularity index of N is positive by Prop. 4.6. Hence, positive regularity index is a necessary but insufficient condition for a closed curve on the unit 2-sphere to be the principal normal indicatrix of a closed space curve with non-vanishing curvature.

Chapter 5

A short note on the Vassiliev invariants[1]

The main result in this chapter is that the Vassiliev invariants are dense in the set of numeric knot invariants if and only if they separate knots. As a motivating example we show how the characterization of the Vassiliev invariants' restriction to the $(2, 2i+1)$ torus knots, given in [47], may be used (bare handed) to prove that Vassiliev invariants are dense under this restriction.

5.1 Introduction

Any numeric invariant, u, of oriented knots in oriented space can be extended to singular knots with m transverse double points (or smashed crossings) via inductive use of the Vassiliev skien relation on Figure 5.1.

Figure 5.1: *The Vassiliev skien relation.*

If one extends invariants linearly to finite sums of knots, one can think of an m-singular knot as an alternating sum of the 2^m knots where the m smashed crossings are replaced by positive or negative crossings. The signs of the knots in this alternating sum is simply the product of the signs of the replacing crossings. This proves consistency of the extension to singular knots via the Vassiliev skien relation. Henceforward all invariants will be considered in this extended sense.

An invariant, u, is said to be of type n if n is the smallest integer such that the extension of u vanishes on all $(n+1)$-singular knots. If an invariant is of type n for some n it is called a Vassiliev invariant or an invariant of finite type.

The Vassiliev invariants have some properties which make one think of polynomials. One is that Vassiliev invariants and polynomials both form graded algebras. Another is the

[1] The results in this Chapter can be found in [36] and especially the main result, Theorem 5.5, is also given in [38].

similarity between the Vassiliev skien relation and partial derivation of polynomials, under which both objects are of finite type. Furthermore, if one restrict a Vassiliev invariant to a twist sequence of knots, then these values of the Vassiliev invariant are indeed described by a polynomial in one variable.

The twist sequence technique will be explained in the next section, where the restrictions of all Vassiliev invariants to the twist sequence of $(2, 2i+1)$ torus knots are explicitly calculated. Moreover, it is proven that the Vassiliev invariants, not only separate $(2, 2i+1)$ torus knots as observed in [47] page 400, but that the Vassiliev invariants are dense in numeric knot invariants restricted to $(2, 2i+1)$ torus knots.

One of the open problems about Vassiliev invariants is (quoting [2] p. 424):

> *Problem 1.1. Is there an analog of Taylor's theorem in our context - can an arbitrary knot invariant be approximated by Vassiliev invariants? Do Vassiliev invariants separate knots?*

In [2] there is no proof (or citation) that the two questions in Problem 1.1. are equivalent. In the authors opinion the following quotation from [3] pp. 282-3 indicates that these two questions not are generally considered equivalent.

> *Setting aside empirical evidence and unsolved problems, we can ask some easy questions which will allow us to sharpen the question of whether Jones or Vassiliev invariants determine knot type. ... This leaves open the question of whether there are sequences of Vassiliev invariants which converge to these[2] invariants.*

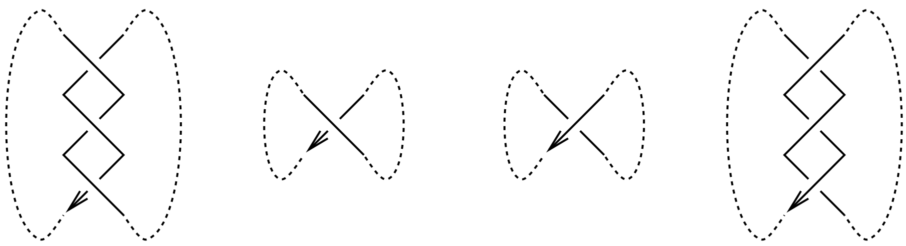
The main result in this paper is that the answers to the two questions in Problem 1.1. are the same. In other words, Vassiliev invariants are dense in knot invariants in the compact-open topology (or arbitrary knot invariant can be approximated by Vassiliev invariants) if and only if Vassiliev invariants separate knots. In the following three sections we take a detour to explicitly calculate the restriction of all Vassiliev invariants to $(2, 2i+1)$ torus knots. Readers familiar with the notion of twist sequences may wish to skip these sections and jump to Section 5.4.

5.2 Vassiliev invariants restricted to $(2, 2i+1)$ torus knots

We now introduce the notion of twist sequences of knots. Let there be given a (singular) knot with two parallel strands. Then a twist move adds two positive crossings between these two strands. The inverse of this twist move simply adds two negative crossings at the same spot.

The two (singular) knots in the right hand side of the Vassiliev skien relation are connected by a twist move. Thus the study of Vassiliev invariants and twist moves on knots are closely related.

[2] The crossing number, the unknotting number, and the braid index which are argued not to be Vassiliev invariants in the part omitted in this quotation.



T_{-2} T_{-1} T_0 T_1

Figure 5.2: *The* $(2, 2i+1)$ *torus knots. For a general twist sequence,* $\{K_i\}_{i=-\infty}^{+\infty}$*, discard the dotted parts of this figure.*

If one continue making twist moves on the same place on a knot then one obtain a twist sequence. An example of a twist sequence is the $(2, 2i+1)$ torus knots shown on Figure 5.2.

Given a sequence of numbers or (singular) knots, $\{b_i\}_{i=-\infty}^{+\infty}$, define its difference sequence $\{\Delta b_i\}_{i=-\infty}^{+\infty}$ by $\Delta b_i = b_{i+1} - b_i$. Consider the difference sequence of a twist sequence of knots. Each $\Delta K_i = K_{i+1} - K_i$ can be seen as K_{i+1} with one of its crossings smashed. With this notation the Vassiliev skien relation takes the form $u(\Delta K_i) = \Delta u(K_i)$. By induction we have $u(\Delta^j K_i) = \Delta^j u(K_i)$, where each $\Delta^j K_i$ has j double points more than K_i.

Let v be a Vassiliev invariant of type n and let $\{K_i\}_{i=-\infty}^{+\infty}$ be a twist sequence of knots. Then by definition $v(\Delta^{n+1} K_i) = 0$ for all i. Hence, the $(n+1)$'st difference sequence of $\{v(K_i)\}_{i=-\infty}^{+\infty}$ vanishes. Therefore the sequence $\{\Delta^n v(K_i)\}_{i=-\infty}^{+\infty}$ is constant, the sequence $\{\Delta^{n-1} v(K_i)\}_{i=-\infty}^{+\infty}$ is a polynomial, at most, of first degree in the variable i, and the sequence $\{v(K_i)\}_{i=-\infty}^{+\infty}$ is, at most, of degree n in i.

On the other hand, given an n degree polynomial, p, considered as the sequence $\{p(i)\}_{i=-\infty}^{+\infty}$, the j'th difference sequence is an $(n-j)$ degree polynomial for each $j = 0, \ldots, n$. Hence, the $(n+1)$'st difference sequence of $\{p(i)\}_{i=-\infty}^{+\infty}$ vanishes. So, if the $(n+1)$'st difference sequence of an arbitrary sequence of numbers is the first to vanish, then this sequence of numbers is described by a polynomial of degree n.

We will now address the problem about which polynomials that come from Vassiliev invariants restricted to twist sequences by studying $(2, 2i+1)$ torus knots. R. Trapp gives in [47] page 400 the following theorem.

Theorem 5.1 (R. Trapp) *The invariant* p *defined on* $\{T_i\}$ *(*$(2, 2i+1)$ *torus knots see Figure 5.2) extends to a finite-type invariant* v *on all knots if and only if the difference sequence of* $\{p(i)\}$ *vanishes for some* $n+1$ *and* $\sum_{j=1}^{n} (-1)^j \Delta^j p(0) = 0$.

The following lemma gives two equivalent and handier conditions for a polynomial on $(2, 2i+1)$ torus knots to extend to a Vassiliev invariant v on all knots.

Lemma 5.2 *Let p be an n'th degree polynomial, i.e.,*

$$p(i) = a_0 + a_1 i + \cdots + a_n i^n.$$

Then the following conditions on p are equivalent.

(1) $\sum_{j=1}^{n}(-1)^j \Delta^j p(0) = 0.$

(2) $\sum_{j=1}^{n}(-1)^j a_j = 0.$

(3) $p(-1) = p(0).$

Proof: That (2) and (3) are equivalent follows directly, since $p(0) = a_0$ and $p(-1) = a_0 + \sum_{j=1}^{n}(-1)^j a_j$.

We now show equivalence of (1) and (2). We use the notation $p_n(x) = x^n$ for $n \geq 1$. As $\sum_{j=1}^{n}(-1)^j \Delta^j p(0)$ is a linear functional on the set of polynomials the equivalence of (1) and (2) will follow, if $\sum_{j=1}^{n}(-1)^j \Delta^j p_n(0) = (-1)^n$. To prove this last equation we have to consider parts of the sequences $p_n(i), \Delta p_n(i), \Delta^2 p_n(i), \ldots, \Delta^n p_n(i)$.

		$i=0$	$i=1$	$i=2$
$p_n(i)$	:	0^n	1^n	2^n
$\Delta p_n(i)$	:	$1^n - 0^n$	$2^n - 1^n$	$3^n - 2^n$
$\Delta^2 p_n(i)$	:	$2^n - 2\cdot 1^n + 0^n$	$3^n - 2\cdot 2^n + 1^n$	$\star$
$\Delta^3 p_n(i)$	:	$3^n - 3\cdot 2^n + 3\cdot 1^n - 0^n$	$\star$	$\star$

This leads to the general formula

$$\Delta^j p_n(0) = (-1)^j \sum_{i=1}^{j}(-1)^i \begin{pmatrix} j \\ i \end{pmatrix} i^n.$$

The following calculation completes the proof.

$$\begin{aligned}
&\textstyle\sum_{j=1}^{n}(-1)^j \Delta^j p_n(0) = \sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^i \begin{pmatrix} j \\ i \end{pmatrix} i^n \\
&= \textstyle\sum_{i=1}^{n}\sum_{j=i}^{n}(-1)^i \begin{pmatrix} j \\ i \end{pmatrix} i^n = \sum_{i=1}^{n}(-1)^i i^n \sum_{j=i}^{n} \begin{pmatrix} j \\ i \end{pmatrix} \\
&= \textstyle\sum_{i=1}^{n}(-1)^i i^n \begin{pmatrix} n+1 \\ i+1 \end{pmatrix}, \text{ by (1.52) in [18]} \\
&= \textstyle\sum_{i=1}^{n}(-1)^i i^n \frac{n+1}{i+1} \begin{pmatrix} n \\ i \end{pmatrix} = (n+1)\sum_{i=1}^{n}(-1)^i \begin{pmatrix} n \\ i \end{pmatrix} \frac{i^n}{i+1} \\
&= (n+1)(-1)^n \frac{1^{n-1}}{\begin{pmatrix} 1+n \\ n \end{pmatrix}}, \text{ by (1.47) in [18]} \\
&= (-1)^n
\end{aligned}$$

$\square$

In [47] pages 403-404 it is proven that the space of Vassiliev invariants is closed under uniform convergence on all knots. Whence, one is led to study pointwise limits of Vassiliev invariants. The following theorem states that any numeric invariant can be obtained as a pointwise limit of Vassiliev invariants restricted to $(2, 2i+1)$ torus knots.

Theorem 5.3 *Restricted to* $(2, 2i+1)$ *torus knots Vassiliev invariants are dense in knot invariants in the compact-open topology.*

Proof: We use the discrete topology on the set of knots. Thus a compact set of knots is just a finite set of knots, and basic open sets of invariants consists of invariants that are "close" on finite sets of knots.

In the twist sequence $\{T_i\}_{i=-\infty}^{+\infty}$ we have that T_{-1} and T_0 both are the unknot and all other knots in this sequence are pairwise different. Hence, a knot invariant restricted to $\{T_i\}_{i=-\infty}^{+\infty}$ is a sequence $\{b_i\}_{i=-\infty}^{+\infty}$ with $b_{-1} = b_0$.

Theorem 5.1 and Lemma 5.2 imply that a Vassiliev invariant of type n on $\{T_i\}_{i=-\infty}^{+\infty}$ is an n-degree polynomial fulfilling $p(-1) = p(0)$. Now, given any compact set of $\{T_i\}_{i=-\infty}^{+\infty}$, choose an integer N such that $\{T_{-N}, \ldots, T_N\}$ contains this compact set. The $(2N+1)$ values $b_{-N}, \ldots, b_N$ determine an, at most, $2N^{\text{th}}$ degree polynomial, p, with the property that $p(-1) = p(0)$. This polynomial now agrees with the original invariant on the compact set. $\square$

Theorem 5.1 and Lemma 5.2 allow us to calculate a basis for the Vassiliev invariants restriction to $(2, 2i+1)$ torus knots. Note, that in this paper the trivial (i.e. constant) knot invariant is considered as a Vassiliev invariant of degree zero. All other Vassiliev invariants in the following theorem are normalized such that they vanish on the unknot.

Theorem 5.4 *The space of Vassiliev invariants restricted to* $(2, 2i+1)$ *torus knots is spanned*[3] *by* $v_0(T_i) = 1$ *and* $v_n(T_i) = (1+i)i^{n-1}$ *for* $n \geq 2$ *and for all* i.

Proof: Lemma 5.2 gives no restriction on the constant maps. Hence, we can choose the normalized trivial Vassiliev invariant as $v_0(T_i) = 1$ for all i. In degree one we have the constraint $-a_1 = 0$, which gives nothing new. In degree two the constraint is $-a_1 + a_2 = 0$ leading to Casson's invariant $v_2(T_i) = i + i^2 = (1+i)i$ for all i. Finally, we see that we can make the general choice $v_n(T_i) = (1+i)i^{n-1}$ for $n \geq 2$ and for all i. $\square$

5.3 The weight systems from $v_0, v_2, v_3, \ldots$

Consider an invariant of degree n evaluated on an n-singular knot. If one change a crossing in the n-singular knot the value of the degree n invariant will not change. This follows

[3] In [47] v_3 is chosen such that $v_3(T_i) = -i + i^3$.

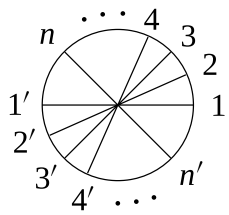


Figure 5.3: *The chord diagram D_n.*

from the fact that the difference between these two n-singular knots is an $(n+1)$-singular knot on which the invariant vanishes. Hence, the pairs of parameter values of the singular points, up to orientation preserving diffeomorphism of the circle, are the only information about the n-singular knot left over. In a chord diagram this is indicated by connecting each pair of parameter values of singular points by a chord. The chord diagram underlying $\Delta^n T_i$ is shown on Figure 5.3.

Denote the weight systems (see [2] page 432) from v_n by w_n. We now assume that the extension of v_n is a degree n invariant. Then

$$w_n(D_n) = v_n(\Delta^n T_0) = \Delta^n v_n(T_0) = (-1)^n \sum_{i=1}^{n} (-1)^i \binom{n}{i} i^n = n! \text{ for } n \geq 2.$$

In the last identity we have used (1.13) in [18]. The “weight system” from v_0 is given by $w_0(\bigcirc) = 1$.

5.4 Separation and denseness are equivalent

Now we drop the restriction to $(2, 2i+1)$ torus knots and return to pointwise limits of Vassiliev invariants on all knots.

Theorem 5.5 *Vassiliev invariants are dense in knot invariants in the compact-open topology if and only if Vassiliev invariants separate knots.*

Proof: If Vassiliev invariants are dense in knot invariants they approximate an invariant that separate knots. Hereby, Vassiliev invariants also separate knots.

At this point it is useful to introduce the canonical basis of the space of knot invariants. Therefore, for any knot K' let $\delta_{K'}$ be the knot invariant defined by

$$\delta_{K'}(K) = \begin{cases} 1, \text{ if } K = K' \\ 0, \text{ else.} \end{cases}$$

As a Vassiliev invariant has polynomial growth on twist sequences of K' the invariant $\delta_{K'}$ is not a Vassiliev invariant.

Given a compact (i.e. finite) set of knots H we construct a Vassiliev invariant v such that $v|_H = \delta_{K'}|_H$. This will prove the Theorem, since any numeric knot invariant I can be written as $I = \sum_{\text{knots } K} I(K)\delta_K$.

Recall, that the constant knot invariant, 1, is a Vassiliev invariant of type zero. If the knot K' is not contained in H, then $\delta_{K'}|_H$ is the zero map which extend to a Vassiliev invariant. If H just consists of the knot K', then $1|_H = \delta_{K'}|_H$. We can w.o.l.g. assume that $H = \{K', K_1, K_2, \ldots, K_m\}$. Let K_i be any knot in H different from K'. Assuming that Vassiliev invariants separate knots, there exist a Vassiliev invariant, u, with the property $u(K_i) \neq u(K')$. The knot invariant v_i defined by

$$v_i(K) = \frac{u(K) - u(K_i)}{u(K') - u(K_i)}$$

has the property $v_i(K') = 1$ and $v_i(K_i) = 0$. The invariants v_i and u are of the same type. Hence, v_i is a Vassiliev invariant. Finally, the knot invariant, v, defined by

$$v = \prod_{i=1}^{m} v_i$$

is a Vassiliev invariant, due to the algebraic structure of Vassiliev invariants, and the invariant v has the property $v|_H = \delta_{K'}|_H$. $\square$

Remark 5.6 *Theorem 5.5 is true restricted to any subset of the set of all knots.*

In [22] it is proven that *either Vassiliev invariant distinguish all oriented knots, or there exists prime, unoriented knots which they do not distinguish*. This combined with Theorem 5.5 sharpens the question about denseness of Vassiliev invariants.

Chapter 6

The Isotopy Classes of Flat Surfaces in euclidean 3-space

6.1 Introduction

By a flat surface in euclidean 3-space we mean an embedded surface with everywhere zero Gaussian curvature (see [46]). The main result of this chapter is

Theorem 6.1 (a) *In* 3*-space, any compact surface with nonempty boundary is isotopic to a flat surface.* (b) *Two such flat surfaces are isotopic through ordinary surfaces if and only if they are isotopic through flat surfaces.*

Theorem 6.1 is the flat analogue of the main theorem in [17] by Herman Gluck and Liu-Hua Pan.

Theorem 6.2 (H. Gluck and L.-H. Pan, [17]) (a) *In* 3*-space, any compact orientable surface with nonempty boundary can be deformed into one with positive curvature.* (b) *Any two such surfaces with positive curvature can be deformed into one another through surfaces of positive curvature if and only if they can be deformed into one another through ordinary surfaces, preserving their natural orientations.*

This chapter ends with an elaboration on the nature of the, to Theorem 6.1 and Theorem 6.2, analogous result concerning compact negative curvature surfaces with nonempty boundary. This elaboration leads to Conjecture 6.8 that, if true, describes the isotopy classes of compact negative curvature surfaces with nonempty boundaries.

It is well known that there do not exist a closed compact flat complete surface embedded in 3-space even though the torus has a flat metric. However, part (a) of Theorem 6.1 ensures that there exists a flat surface in every isotopy class of compact surfaces with nonempty boundary. Hence, the nonempty boundaries allow flat surfaces in 3-space to be arbitrarily knotted and twisted. Part (b) of Theorem 6.1 then ensures that the isotopy classes of flat surfaces are in one-one correspondence with the isotopy classes of ordinary surfaces which have no curvature constraint.

A corollary of Theorem 6.1, Corollary 7.1, gives an akin to Seifert surfaces, namely, any simple closed space curve can be deformed until it bounds a flat orientable compact surface.

We now outline the strategy of the proof of our main theorem. A compact connected surface with non-vanishing boundary deformation contracts to a "topological" spine, that is, to a finite number of simple closed curves in 3-space that all intersect in one common point. Under isotopy of an embedded surface through embeddings a topological spine is mapped to topological spines of all surfaces in the isotopy. We can and do in the following assume that topological spines do not intersect the boundaries of the surfaces on which they lie.

Consider a Möbius strip and an orientable cylinder in 3-space. The topological spines of these two surfaces are both a simple closed curve in 3-space. If these two curves represent the same knot type then the two topological spines are ambient isotopic, but the two surfaces are not isotopic. In order to tell if two surfaces with isotopic spines are isotopic, we to each closed curve in a topological spine attach the number of times the surface "twists" around this closed curve. Considering surfaces of positive curvature, as in [17], the natural choice of "twisting number" is the self-inking number[1] of the considered closed space curves, see eg. [30]. Curves on flat surfaces may have vanishing curvature, and (worse) flat surfaces may be unorientable. Hence, in case of flat surfaces the self-linking number can not be used as "twisting number". However, simple closed curves on embedded surfaces do have a natural twisting number, given in [34], namely

Definition 6.3 *Let γ be a simple closed curve on an embedded surface S in 3-space. Let $N_\varepsilon(\gamma) \subset S$ be a tubular neighbourhood of radius $\varepsilon > 0$ of γ in S. Then the twisting number* $\operatorname{twist}(\gamma, S)$*, for $\varepsilon > 0$ sufficiently small, is given by* $\operatorname{twist}(\gamma, S) = \frac{1}{2}\operatorname{link}(\gamma, \partial N_\varepsilon(\gamma))$.

In the above definition $\operatorname{link}(\gamma, \partial N_\varepsilon(\gamma))$ is the total linking number between γ and the link $\partial N_\varepsilon(\gamma)$, that is, $\operatorname{link}(\gamma, \partial N_\varepsilon(\gamma))$ is the sum of the linking numbers between γ and all (one or two) components of $\partial N_\varepsilon(\gamma)$. We observe that restricted to positive curvature surfaces this twisting number agrees with the self-linking number. As the linking number is invariant under ambient isotopy an immediate consequence of Definition 6.3 is

Proposition 6.4 *The twisting number of a simple closed curve on an embedded surface in 3-space is invariant under isotopy through simple closed curves on the surface and invariant under isotopy of the surface through embedded surfaces in 3-space.*

The general idea of the proof of the main theorem is from a topological spine, with twisting numbers attached, to construct a flat model surface giving the desired twisting numbers. Making this construction sufficiently canonical we get an isotopy of flat model surfaces from an isotopy of a spine, with twisting numbers attached. Finally, the main result of this chapter follows by proving that any compact surface with nonempty boundary is isotopic to a flat model surface, and that if the given surface is flat, then there is

[1]The self-linking number of a given curve is the linking number between the given curve and a curve obtained by slightly pushing the given curve along the principal normals.

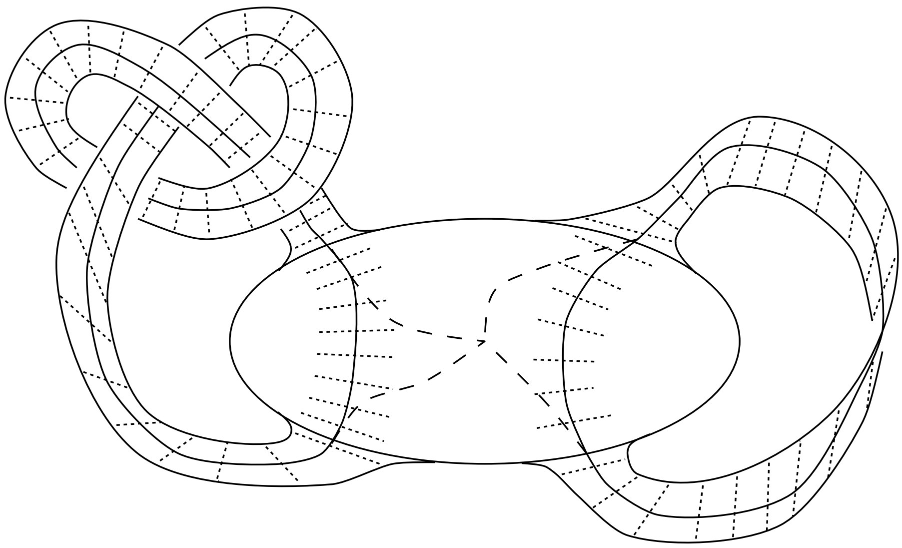


Figure 6.1: *A flat model surface built on a slightly modified spine. The original "topological" spine is indicated by dotted arcs.*

an isotopy through flat surfaces to a flat model surface. A flat model surface is shown on Figure 6.1.

Let p be the point of intersection of the closed curves in a topological spine. Then the flat model surface is planar in a region containing p in its interior. By Prop. 6.4 the twisting numbers are invariant under isotopy through simple closed curves on the surface. Hence, we may choose the curves as we please on the planar region. Each closed curve in the spine is an axis of a flat globally ruled strip coinciding with the planar region such that the closed curve has the desired twisting number with respect to this ruled strip. The planarity of the region containing p makes it possible to glue the strips together to a regular surface. Hence, a crucial step of our proof is to construct globally ruled flat strips sufficiently canonical. We do this in the following section.

6.2 Flat closed strips in 3-space

By a closed strip in 3-space we mean an embedding of the Möbius strip or the orientable cylinder, both with boundary, into 3-space. In [34] it is proven that a closed strip in 3-space is, up to ambient isotopy, given by the knot type and the twisting number of a simple closed curve traversing the strip once. See also [23], in which is proven that, except for the $\pm 1/2$-twisted Möbius strips, the isotopy class of a closed strip is uniquely given by the knot/(oriented link, in case of orientable closed strips) type of its boundary.

Proposition 6.5 *Let γ be a simple closed space curve. Assume the curvature of γ vanishes only on a finite set of intervals and points and assume the torsion of γ vanishes whenever curvature vanishes.*

Then for any half integer, t, the curve γ is an axis of a flat ruled surface S_t, such that, $\mathrm{twist}(\gamma, S_t) = t$ *if and only if γ both possesses points with positive and negative torsion. Furthermore, the rulings can, except for a neighbourhood of a point with positive torsion and a neighbourhood of a point with negative torsion, be chosen orthogonal to the curve.*

Remark 6.6 *From the proof of Prop. 6.5 given below it follows that on an interval of a curve where orthogonal rulings are chosen, these rulings are uniquely given when one ruling in one point is specified. Hereby, Prop. 6.5 gives a, for our purpose, sufficiently canonical construction of flat globally ruled closed strips. For related results see [5].*

Proof: Let $\mathbf{r} : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ be a closed space curve parametrized by arc length, s, and let V be a choice of trivialization of the normal bundle, i.e., a closed unit normal vector field, V, along $\mathbf{r}$, such that $\mathrm{link}(\mathbf{r}, \mathbf{r} + \varepsilon V) = 0$ for $\varepsilon > 0$ sufficiently small. If T denotes the unit tangent vectors to $\mathbf{r}$ and we set $U = T \times V$ then $\{T, V, U\}$ is an orthonormal basis for $\mathbb{R}^3$ for each point on $\mathbf{r}$. By orthogonality we have Frenet like equations

$$\begin{array}{rcrrr} T' & = & & aV & +bU \\ V' & = & -aT & & +cU \\ U' & = & -bT & -cV & \end{array},$$

where primes indicate differentiation with respect to s and $a, b, c : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}$ are periodic functions.

A ruled surface with $\mathbf{r}$ as axis is given by $f(s,t) = \mathbf{r}(s) + t\mathbf{q}(s)$ for some vector field $\mathbf{q}$ along $\mathbf{r}$. This surface is regular if

$$\mathbf{0} \neq \frac{\partial f(s,t)}{\partial s} \times \frac{\partial f(s,t)}{\partial t} = (T(s) + t\mathbf{q}'(s)) \times \mathbf{q}(s).$$

Along $\mathbf{r}$, that is for $t = 0$, we find that $\frac{\partial f(s,t)}{\partial s} \times \frac{\partial f(s,t)}{\partial t} = T(s) \times \mathbf{q}(s)$. By continuity and compactness this ruled surface is regular in a neighbourhood of $\mathbf{r}$ if $\mathbf{q}$ is never parallel with the tangents to $\mathbf{r}$. Hereby we can write $\mathbf{q}$ as

$$\mathbf{q} = \alpha T + \cos\theta V + \sin\theta U$$

where $\alpha, \theta : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}$ are cylinder coordinates. Using these coordinates the ruled surface closes up if and only if there exists an integer p such that $\theta(L) - \theta(0) = \pi p$ and $\alpha(L) = (-1)^p \alpha(0)$. As $\mathrm{link}(\mathbf{r}, \mathbf{r} + \varepsilon V) = 0$, for $\varepsilon > 0$ sufficiently small, our axis has twisting number $\mathrm{twist}(\mathbf{r}, f) = p/2$ on such a ruled surface.

A ruled surface is flat if and only if $0 = \left[T \;\; \mathbf{q} \;\; \mathbf{q}'\right] = T \cdot (\mathbf{q} \times \mathbf{q}')$.

$$\begin{aligned}\mathbf{q}' &= \alpha' T + \alpha a V + \alpha b U - \theta' \sin\theta V - a\cos\theta T + c\cos\theta U \\ &\quad + \theta' \cos\theta U - b\sin\theta T - c\sin\theta V\end{aligned}$$

$$\begin{aligned} 0 &= \left[T \;\; \mathbf{q} \;\; \mathbf{q}'\right] \\ &= \begin{vmatrix} 1 & \star & \star \\ 0 & \cos\theta & \alpha a - \theta' \sin\theta - c\sin\theta \\ 0 & \sin\theta & \alpha b + \theta' \cos\theta + c\cos\theta \end{vmatrix} \\ &= \alpha b \cos\theta + \theta' + c - \alpha a \sin\theta \\ &\Updownarrow \\ \theta' &= \alpha(a\sin\theta - b\cos\theta) - c \end{aligned}$$

On an interval where $\mathbf{r}$ has positive curvature, κ, we can rewrite $T' \neq \mathbf{0}$ setting $a = \kappa\cos\phi$ and $b = \kappa\sin\phi$. This gives

$$\theta' = \alpha\kappa\,(\sin\theta\cos\phi - \cos\theta\sin\phi) - c = \alpha\kappa\sin(\theta - \phi) - c. \tag{6.1}$$

From the famous formula: *link equals twist plus writhe* (as the curvature of $\mathbf{r}$ may have zeros see eg. [27]) we see that

$$Tw(\mathbf{r}, V) = \frac{1}{2\pi}\int_0^L V(s)' \cdot U(s) ds = \frac{1}{2\pi}\int_0^L c(s) ds = -Wr(\mathbf{r})$$

as we have chosen the linking number between $\mathbf{r}$ and $\mathbf{r} + \varepsilon V$ to be zero. For $\alpha \equiv 0$ and θ constant Equation (6.1) yields, that the ruled surface given by $\mathbf{r}(s) + t(\cos(\theta)V(s) + \sin(\theta)U(s))$ is flat if and only if $c \equiv 0$. Let $\tilde{V}$ be the unique unit normal vector field along $\mathbf{r}$ such that $\tilde{V}(0) = V(0)$ and $\tilde{V}$ has $c \equiv 0$. Note, that in general $\tilde{V}$ does not close up since $\tilde{V}$ is twisted $Tw(\mathbf{r}, V)$ less than V. Using this new (non-closed) frame $\tilde{V}$ and setting $\tilde{U} = T \times \tilde{V}$ we obtain $\mathbf{q} = \alpha T + \cos\theta V + \sin\theta U = \alpha T + \cos\tilde{\theta}\tilde{V} + \sin\tilde{\theta}\tilde{U}$ and

$$\tilde{\theta}' = \alpha\kappa\sin\left(\tilde{\theta} - \tilde{\phi}\right). \tag{6.2}$$

The necessary and sufficient conditions for the flat surface to close up and to have twisting number twist$(\mathbf{r}, f) = p/2$ for a given $p \in \mathbb{Z}$ is that

$$\tilde{\theta}(L) - \tilde{\theta}(0) = p\pi - Wr(\mathbf{r}) \quad \text{and} \quad \alpha(L) = (-1)^p \alpha(0). \tag{6.3}$$

We need to describe the frame $\tilde{V}$ using the Frenet Apparatus. For this let I be an interval where $\mathbf{r}$ has positive curvature. By orthogonality we may write $\tilde{V}$ as $\tilde{V} = \cos v N + \sin v B$ and the Frenet equations give

$$\tilde{V}' = -v' \sin v N + \cos v\,(-\kappa T + \tau B) + v' \cos v B - \tau \sin v N.$$

The ruled surface with $\mathbf{r}$ as axis and $\tilde{V}$ as rulings is flat if and only if

$$\begin{aligned} 0 = c &= \begin{bmatrix} T & \tilde{V} & \tilde{V}' \end{bmatrix} \\ &= \begin{vmatrix} 1 & \star & \star \\ 0 & \cos v & -v' \sin v - \tau \sin v \\ 0 & \sin v & \tau \cos v + v' \cos v \end{vmatrix} \\ &= \tau + v' \end{aligned}$$

Hence, on the interval, I, where $\mathbf{r}$ has positive curvature the vector field $\tilde{V}$ is given by

$$\tilde{V}(s) = \cos\left(-\int_{s_0}^{s} \tau(s)ds + k_I\right) N(s) + \sin\left(-\int_{s_0}^{s} \tau(s)ds + k_I\right) B(s) \tag{6.4}$$

for some constant k_I. On a straight segment of $\mathbf{r}$ (where $\mathbf{r}$ has zero curvature) a similar calculation shows that the vector field $\tilde{V}$ is constant on this segment. From the equations

$$\begin{aligned} T' &= \kappa N \\ &= \kappa \cos\tilde{\phi}\tilde{V} + \kappa \sin\tilde{\phi}\tilde{U} \\ &= \kappa \cos\tilde{\phi} \left(\cos\left(-\int \tau(s)ds + k_I\right) N(s) + \sin\left(-\int \tau(s)ds + k_I\right) B(s)\right) \\ &\quad \kappa \sin\tilde{\phi} \left(-\sin\left(-\int \tau(s)ds + k_I\right) N(s) + \cos\left(-\int \tau(s)ds + k_I\right) B(s)\right) \end{aligned}$$

we deduce that

$$\begin{aligned} 1 &= \cos\tilde{\phi} \cos\left(-\int \tau(s)ds + k_I\right) - \sin\tilde{\phi} \sin\left(-\int \tau(s)ds + k_I\right) \\ &= \cos\left(\tilde{\phi} - \int \tau(s)ds + k_I\right). \end{aligned}$$

Hence,

$$\tilde{\phi}(s) = \int_{s_0}^{s} \tau(s)ds - k_I \quad \text{modolo} \quad 2\pi.$$

Thus we can rewrite Equation (6.2) as

$$\tilde{\theta}'(s) = \alpha(s)\kappa(s) \sin\left(\tilde{\theta}(s) - \int_{s_0}^{s} \tau(s)ds + k_I\right) \tag{6.5}$$

on an interval, I, where $\kappa > 0$. If $\kappa = 0$ then $\tilde{\theta}' = 0$ and $\tilde{V}' = \tilde{U}' = \mathbf{0}$.

If $\tau > 0$ on some interval then, by a proper choice of α, $\tilde{\theta}$ can decrease arbitrarily much on this interval and if $\tau < 0$ on an interval then $\tilde{\theta}$ can increase arbitrarily much on this interval. If $\tau(s_+) > 0$ and $\tau(s_-) < 0$, then there are intervals I_+ and I_- not containing 0 such that τ has constant sign on each interval. Choosing $\alpha = 0$ on $[0, L] \setminus (I_+ \cup I_-)$

the angle $\tilde{\theta}$ is constant on $[0, L] \setminus (I_+ \cup I_-)$. By controlling α on I_+ and I_- the desired difference, see (6.3), $\tilde{\theta}(L) - \tilde{\theta}(0) = p\pi - Wr(\mathbf{r})$ can be obtained for any $p \in \mathbb{Z}$ and $\alpha(L) = (-1)^p \alpha(0) = 0$ is trivially fulfilled. On the other hand if the torsion of $\mathbf{r}$ do not take both signs then the obtainable twisting numbers are, as a consequence of Equation (6.5), bounded either from above or from below. $\square$

Consider an isotopy between two closed curves such that all curves in this isotopy possess points with both positive and negative torsion. Prop. 6.5 ensures that for any given twisting number each curve in this isotopy is an axis of a globally ruled closed strip giving the desired twisting number. As we shall see below, we can indeed for any twisting number construct a strip isotopy from such a curve isotopy. This construction is the corner stone in the proof of the following lemma. As mentioned in the introduction this is a crucial lemma for the proof of Theorem 6.1, and it was posed as a question to the author by Herman Gluck.

Lemma 6.7 *Two flat closed strips in* 3*-space are isotopic through ordinary closed strips in* 3*-space if and only if they are isotopic through flat closed strips.*

Proof: Let $H(s, t, u)$ be an isotopy between two flat closed strips $H_0 = H(\cdot, \cdot, 0)$ and $H_1 = H(\cdot, \cdot, 1)$ and let $a_i \subset H_i$ be a topological spine of H_i, $i = 0, 1$.

Claim: We may assume that

- $\mathcal{A}$: A neighbourhood of a_i on H_i can be parametrized as a globally ruled flat surface with a_i as axis.
- $\mathcal{B}$: The space curves a_0 and a_1 both have non-vanishing curvature.
- $\mathcal{C}$: The self-linking number of a_0 and a_1 are equal.
- $\mathcal{D}$: The torsion of both a_0 and a_1 take both signs.

Proof of claim: Part $\mathcal{A}$: By the characterization of flat surfaces given in [46] the surfaces H_0 and H_1 are piecewise ruled surfaces. That is, on a compact subset of H_0 resp. H_1, namely the closure of the set of points, called parabolic points, where one of the principal curvatures is non-zero, unique rulings are given by the zero principal curvature directions. We call this the ruled regions. The remainder of H_0 resp. H_1 consists of planar points, i.e., points with both principal curvatures equal to zero, and this is indeed a union of planar regions. These planar regions can in a neighbourhood of a curve be parametrized as a ruled surface with this curve as axis. The only restriction is that the rulings must be chosen in the considered plane and they may not cross the tangents to the curve in order to get a regular surface containing the curve.

The ruled regions of H_0 and H_1 are compact sets. Hence, the spines a_0, and a_1, can be chosen such that a_0 and a_1 has transversal intersections with the rulings except in a finite number of points. On Figure 6.2 is shown why non-transversal intersections in general

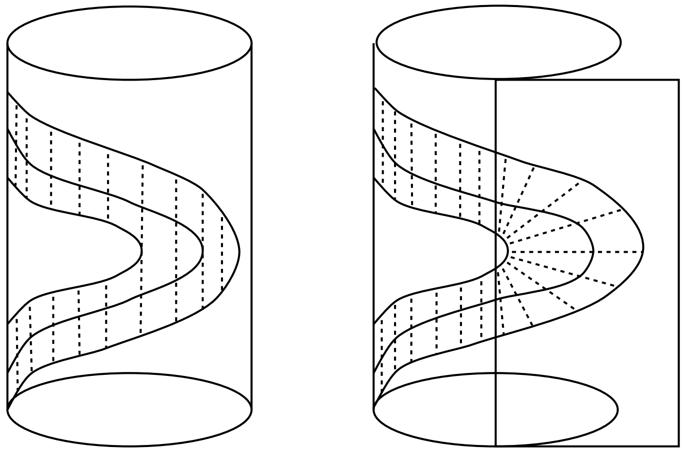


Figure 6.2: *Left, an unavoidable non transversal intersection between curve and rulings. Right, a local deformation of the surface (pulling the label off the bottle), such that, the deformed surface locally can be parametrized as a ruled surface.*

can not be avoided on a given surface and it is indicated how we by a slight isotopy avoid this problem. We can and do for convenience assume that these non transversal intersections do not lie on the boundary of the ruled regions. Furthermore, from now on we consider only neighbourhoods of our curves, on the surfaces they lie on.

For each non transversal intersection point q consider the unique curve, c, on the considered surface, H_0 or H_1, such that the curve c is orthogonal to the rulings and goes through $q = c(0)$. Let V be a vector field along c giving the directions of the rulings of the surface. In a neighbourhood of q our surface is given by $f(s,t) = c(s) + tV(s)$, $-\varepsilon \leq s \leq \varepsilon$. The curve a_i is locally given by $f(g(t),t)$, where $g(0) = 0$, $g'(0) = 0$, and we may assume that $g''(0) < 0$. We may also assume that q is a parabolic point, i.e., one of the principal curvatures $k \neq 0$. By the choice of the curve c, its curvature is bigger than or equals $|k|$ at the point q. Consider the plane P through q orthogonal to the ruling through q. The projection of c onto P, $\pi(c)$, has curvature $|k|$ at q. Hence, by a partition of unity a neighbourhood of q on c can be isotoped to $\pi(c)$ preserving non-vanishing curvature. Hereby, the torsion vanishes on this neighbourhood. Within the plane P we now make $\pi(c)$ straight in a smaller neighbourhood of q. Using the rulings given by $V_u(-\varepsilon) = V(-\varepsilon)$, V_u is orthogonal to c_u, and the ruled surface given by $f_u(s,t) = c_u(s) + tV_u(s)$ is flat, we get an isotopy as constructed in the proof of Proposition 6.5 of H_0 or H_1 that makes H_0 or H_1 planar in a region containing q in its interior. By compactness of H_0 resp. H_1 this isotopy can be assumed to go through embeddings.

On the isotoped part of our surface, now given by $f_1(s,t) = c_1(s) + tV_1(s)$, the curve given by $t \mapsto f_1(g(t),t)$ lies in a plane for t in a neighbourhood of zero. Note, that since the parametrization of this curve is fixed with respect to axes and rulings no further non transversal intersections with rulings has occurred in our constructed isotopy. We

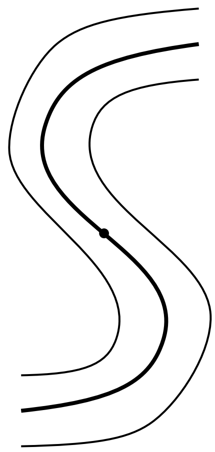


Figure 6.3: *An unavoidable vanishing of curvature of spines on a planar region of a flat surface.*

are free to chose rulings in the constructed planar region. Hence, we can parametrize a neighbourhood of the curve given by $t \mapsto f_1(g(t), t)$ on the isotoped surface as a ruled surface with this curve as axis.

By such local isotopies that only concern neighbourhoods of the non transversal intersections between a_0 and a_1 and the unique rulings of H_0 resp. H_1 we obtain curves $\tilde{a}_0$ resp. $\tilde{a}_1$ on $\widetilde{H_0}$ resp. $\widetilde{H_1}$, such that, a neighbourhood of $\tilde{a}_i$ on $\widetilde{H_i}$, $i = 0, 1$, can be parametrized as a (globally) ruled surface with $\tilde{a}_i$ as axis. Note, that on a flat globally ruled Möbius strip the rulings are only projectively well-defined globally. This proves part $\mathcal{A}$.

To prove part $\mathcal{B}$, we recall that the curvature of a_i as space curve, κ, the geodesic curvature, κ_g, and the normal curvature, κ_n, fulfil the equation $\kappa^2 = \kappa_g^2 + \kappa_n^2$. As a_i, assuming $\mathcal{A}$, is transversal to the rulings of H_i the normal curvature $\kappa_n(s)$ is zero if and only if $a_i(s)$ is a planar point. Hence, the curvature of a_i vanishes if and only if a_i has zero geodesic curvature in a planar point of H_i.

First, consider the case that a_i has zero curvature in $a_i(s)$ and that $a_i(s)$ is a limit point of parabolic points (points with one principal curvature different from zero) of H_i. By a local isotopy of a_i on H_i through axes of H_i the zero of a_i's geodesic curvature can be moved to a parabolic point giving a_i non-vanishing curvature at $a_i(s)$.

Otherwise, a_i has zero curvature in $a_i(s)$ and a neighbourhood of $a_i(s)$ on H_i consists only of planar points. See Figure 6.3. Let R denote the planar region of H_i containing $a_i(s)$ and let P be the plane containing R. Deforming a_i on R we may assume that the geodesic curvature of a_i vanishes only in a finite number points. Let $a_i(s)$ denote one of these points. Fixing the projection of a_i onto the plane P and lifting a neighbourhood of $a_i(s)$ on H_i through cylinder surfaces makes the normal curvature of the deformed a_i at $a_i(s)$ non-zero. As the projection of the deformed surface into the plane P only has vanishing curvature at the point $a_i(s)$ this finally gives a_i non-vanishing curvature everywhere. This proves part $\mathcal{B}$.

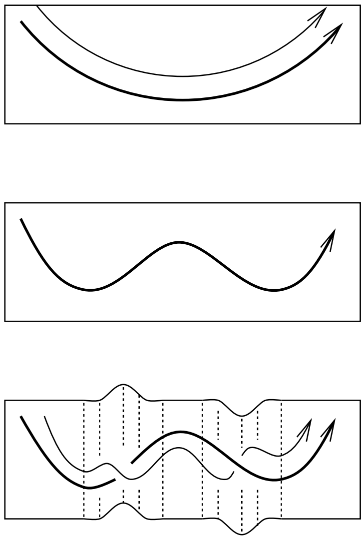


Figure 6.4: *On the top a planar space curve (the fat curve) together with a curve pushed off along its principal normals. At the bottom a flat surface obtained by an obvious isotopy of the above plane through flat surfaces. The (fat) curve on the flat surface has positive curvature and the (thin) curve is pushed off along its principal normals. The two crossings between these curves are positive. Hence, the self-linking number of the fat curve is increased by one.*

Part $\mathcal{C}$: Using an isotopy as constructed in $\mathcal{A}$ we may assume that H_0 has a planar region R and by $\mathcal{B}$ we may assume that a_i, $i = 0, 1$, has non-vanishing curvature. Hereby the self-linking numbers of a_0 and a_1 are defined, but they need not be equal. To prove claim $\mathcal{C}$ we give a deformation of H_0 to $\widetilde{H_0}$ through flat surfaces, that is the identity outside R, such that a spine $\tilde{a}_0$, with non-vanishing curvature, of $\widetilde{H_0}$ has the same self-linking number as a_1.

On Figure 6.4 is shown how to increase the self-linking number by one. By interchanging up and down on this figure the self-linking number is decreased by one instead. Hence, by inserting a finite number of "bumps" on the planar region R any given self-linking number can be obtained. This proves part $\mathcal{C}$.

Part $\mathcal{D}$: By an isotopy, eg. as indicated on Figure 6.4, we may assume that H_i do not lie in a plane. Hence, there is a region of H_i in which one of the principal curvatures, k, is non-zero. By Prop. 2 in [46] p. 279 the geodesic torsion, i.e., the torsion of a geodesic curve with unit tangent $\mathbf{x}$ (see Prop 3 in [46] p. 281), is $\tau_g(\mathbf{x}) = k \sin\theta \cos\theta$, where θ is the angle between $\mathbf{x}$ and the principal direction with zero principal curvature. As the principal curvature $k \neq 0$ in the considered region an axis can be isotoped to contain a geodesic segment with positive torsion and a geodesic segment with negative torsion. This proves the part $\mathcal{D}$ and hereby the proof of the **claim** is complete.

By part $\mathcal{B}$ and $\mathcal{C}$ of the **claim** there exists an isotopy between the two axes through positive curvature curves. This fact is due to H. Gluck and L.-H. Pan [16]. Denote such an positive curvature isotopy between the axes a_i of H_i, $i = 0, 1$, by $a(s, u) : \mathbb{S}^1 \times [0, 1] \to \mathbb{R}^3$. On each curve a_u consider the osculating plane P_u to the point $a_u(0)$. The plane P_u is spanned by the tangent and principal normal to a_u at $a_u(0)$. By compactness there is a common ε-ball around $a_u(0)$ in P_u such that the planar projection of a_u to P_u is regular and has non-vanishing curvature for all $u \in [0, 1]$. Fixing these planar projections we isotope all curves to be planar within an $\epsilon/2$-ball of $a_u(0)$ preserving non-vanishing curvature.

For $\varepsilon/2 > \delta > 0$ there is a family A_u^δ, $u \in [0, 1]$, of affine transformations of 3-space[2], such that, $A_u^\delta(a(-\delta, u)) = (0, 0, 0)$ and $A_u^\delta(a(\delta, u)) = (1, 0, 0)$ for all $u \in [0, 1]$ and such that $A_u^\delta(P_u)$ is the xy-plane. By choosing $\delta > 0$ sufficiently small we by compactness can assume that the x-axis can be used as parameter of the pieces of the curves $s \mapsto A_u^\delta(a(s, u))$ lying in between $(0, 0, 0)$ and $(1, 0, 0)$. Hence, locally the curve isotopy is given by $[0, 1] \times [0, 1] \ni (x, u) \mapsto (x,\ f_u(x), 0)$. We now isotope all f_u to be identical for $x \in [1/3, 2/3]$ and to give positive curvature on a slightly larger interval. Hereby, we may introduce points with zero curvature – but as they occur only on planar pieces of our curve isotopy they do not disturb the construction of rulings that still remains. See Proposition 6.5. The constructed planar curve-piece that is identical for all curves in the isotopy is now rolled onto a cylinder (as on Figure 6.2 reed from the right hand side to the left hand side). The resulting space curve-piece then has both positive and negative torsion which even may be chosen constant on two sub-intervals using circular Helices. We refer to this common segment as the $\pm\tau$-segment. We are now in possession of the requisite axes. Hence, we need only to specify their rulings to complete our proof.

Along the curve $s \mapsto A_0^\delta(a(s, 0))$ (or by short hand $A_0^\delta(a(\cdot, 0))$) we have a ruling vector field $q_0(s)$ parametrizing a neighbourhood of $A_0^\delta(a(\cdot, 0))$ on $A_0^\delta(H_0)$. This vector field is given by A_0^δ of the rulings of H_0 along a_0. Specifying one ruling $q_0(s^\star)$ we continuously change the rulings along $A_0^\delta(a(\cdot, 0))$ to be orthogonal to the curve and equal to $q_0(s^\star)$ at $A_0^\delta(a(s^\star, 0))$ except for a segment where $A_0^\delta(a(\cdot, 0))$ has positive torsion and one with negative torsion. The existence of these segments is ensured by part $\mathcal{D}$ of the **claim**. In these two segments we compensate for the twisting of the rulings that occur during this change. Next, under the isotopy inserting the $\pm\tau$-segment we still compensate within these two segments for the twisting of the rulings, caused by keeping the rulings orthogonal to the isotoped segment under this curve isotopy. Fixing the space curve and one ruling outside the $\pm\tau$-segment we now isotope the rulings to be orthogonal to the curve outside the $\pm\tau$-segment while only compensating inside the $\pm\tau$-segment.

During the curve isotopy from $A_0^\delta(a(\cdot, 0))$ to $A_1^\delta(a(\cdot, 1))$ (with the $\pm\tau$-segment inserted on all curves) the rulings outside the $\pm\tau$-segment are given by a choice of one orthogonal ruling in one point of each curve and demanding that the rulings are orthogonal to the curves and that they give a flat surface. Hereby the rulings in the endpoints of the $\pm\tau$-segment vary continuously during the isotopy. This makes it possible to control the rulings on the $\pm\tau$-segment to match the boundary conditions of Equation (6.5) in the proof of Prop. 6.5.

[2]The image of a flat surface under an affine transformation is flat. Furthermore, affine transformations maps rulings to rulings.

By performing the preparations of the rulings on $A_0^\delta(a(\cdot,0))$ and the curve itself "time reversed" on $A_1^\delta(a(\cdot,1))$ we, from the curve isotopy from $A_0^\delta(a(\cdot,0))$ to $A_1^\delta(a(\cdot,1))$, get a surface isotopy through globally ruled flat strips from $A_0^\delta(H_0)$ to $A_1^\delta(H_1)$. Pulling this isotopy back using the affine mappings A_u^δ we finally get our desired isotopy from H_0 to H_1 through globally ruled flat strips. □

6.3 Flat surfaces in 3-space

We are now ready to prove the main theorem of this chapter which implies that the isotopy classes of flat surfaces are in one-one correspondence with the isotopy classes of ordinary surfaces which have no curvature constraint.

Proof: [of Theorem 6.1] We first prove part *(b)*. Let S_u, $u \in [0,1]$, be an isotopy between two flat compact connecteded surfaces S_0 and S_1 with nonempty boundaries. In order to prove part *(b)* we construct an isotopy from S_0 to S_1 through flat surfaces.

Let $s_0 \subset S_0$ be a topological spine of S_0 and denote by $s_u \subset S_u$ the images of this spine under the isotopy. Similarly let $p_0 \in s_0$ be the intersection point of the closed curves in s_0 and let $p_u \in S_u$ be p_0's images under the isotopy. From now on we only consider neighbourhoods of the spines s_u on the surfaces S_u.

Claim: We may assume that each surface S_u is planar in an ε-neighbourhood of p_u for a fixed $\varepsilon > 0$.

Proof of claim: By local isotopies, as constructed in the proof of Lemma 6.7, we can assume that S_0 is planar in a neighbourhood of p_0 and that S_1 is planar in a neighbourhood of p_1. A partition of unity between S_u and the tangent plane to S_u at p_u, $T_{p_u}S_u$, makes S_u planar in an ε-neighbourhood of p_u. By compactness there is a common $\varepsilon > 0$ such that all the surfaces, S_u, in the isotopy locally can be isotoped through embeddings to be planar in an ε-neighbourhood of the images of p_0 on each surface. This proves the **claim**.

A closed simple curve, a, on S_0 has a twisting number twist (a, S_0) that, by Prop 6.4, is invariant under isotopy of a on S_0. Especially we are free to choose the closed curves in the spine s_0 on the planar ε-neighbourhood of p_0 of S_0. Hence, we may consider the spine s_0, and thus also its images s_u, each as a planar ε-disk together with a finite number of simple curves a_u^i starting and ending in pairwise disjoint points on the boundary of this disk. And furthermore, each of these curves has a twisting number attached.

To prove part *(b)* we from each curve isotopy a_u^i, $u \in [0,1]$, induced by the given surface isotopy, construct an isotopy of closed flat strips S_u^i, $u \in [0,1]$, such that the a_u^i's have the required twisting numbers on these flat strips, i.e., for all i and u twist $\left(a_u^i, S_u^i\right) =$ twist $\left(a_0^i, S_0\right)$. Furthermore we construct the S_u^i's such that they coincide with the planar ε-neighbourhoods of p_u on S_u. Smoothing the edges between the closed strips S_u^i and the planar ε-disks gives an isotopy between S_0 and S_1 through flat surfaces.

Restricting S_0 to a neighbourhood of the simple closed curve a_0^i and taking the image of this set under the given isotopy, S_u, $u \in [0, 1]$, we get an isotopy between two flat closed strips, S_0^i and S_1^i. By Lemma 6.7 these two flat closed strips are isotopic through flat closed strips. All, except for two, of the local isotopies (see the proof of Lemma 6.7) concern only the ruled regions of S_0^i and of S_1^i. Hence, they do not change the planar ε-disks of these two surfaces. The remaining local isotopies, that ensure that neither S_0^i nor S_1^i is contained in a plane and that the self-linking numbers of their spines are equal, can be applied anywhere on these surfaces, and can therefore be kept away from their planar ε-disks.

The choice of rulings on the globally ruled flat strips in the isotopy between S_0^i and S_1^i constructed in the proof of Lemma 6.7, is (except for the segments with positive and negative torsion that can be kept away from the ε-disks) always orthogonal to the curves. These rulings are specified by one ruling in on point of each axis and demanding that the corresponding ruled surfaces are flat. To each axis we now specify a ruling in a point lying in the planar ε-disk, such that, this ruling together with the tangent to the axis in this point form an o.n.b. of this plane. By the orthogonality of the rulings the ruled surfaces coincide with the planar ε-disks. This follows from Equation 6.4 as planar curves have zero torsion.

The curves in the spines s_u are pairwise disjoint outside the ε-disks. Hence, compactness ensures that a sufficiently small neighbourhood of the spines in the isotopy of flat surfaces between S_0 and S_1, we now have constructed, in fact is embedded.

By the compactness of S_0 it consists in general of finitely many connected components. The proof given in the connected case carries over to the general case without changes except for a need of an index corresponding to an enumeration of the connected components. The proof of part *(b)* is completed.

To prove part *(a)* it is, as in the proof of part *(b)*, enough to consider the case that S is connected. Let S be a compact connected surface with nonempty boundary and let $s \subset S$ be a topological spine of S. We may assume that S is planar in a neighbourhood, N, of the point in which the curves in the spine s intersect. Again, consider a spine, s, of S as a finite number of closed curves, a_i, entering the planar ε-disk – each with a twisting number attached.

We may assume that each a_i has non-vanishing curvature. The proof of this claim is analogous to the proof of the claim $\mathcal{B}$ in the proof of Lemma 6.7, when noticing that a_i has zero curvature in $a_i(s)$ if and only if the geodesic curvature of a_i is zero in $a_i(s)$ and $a_i'(s)$ is an asymptotic direction.

A neighbourhood of the curve a_i on S is isotopic to a part of the ruled surface with a_i as axis and rulings chosen such that they together with the tangents of a_i form o.n. bases of the tangent planes of S along a_i. As S may be unorientable these rulings are only projectively well-defined. Such an isotopy may be constructed using the normal exponential map and compactness.

On each a_i we insert an $\pm\tau$-segment which is kept away from the planar part N of the surface S. This can be done preserving non-vanishing curvature. During this local curve isotopy we choose orthogonal vectorfields q_i along each a_i, such that, the twisting number between a_i and the ruled surface given by a_i and q_i is fixed. This induces a surface isotopy of the (total) piecewise ruled surface. Each a_i now fulfills the conditions of Proposition 6.5. Hence, there exists a vectorfield V_i along a_i, such that the hereby defined surface is flat and such that the twisting number of a_i with respect to this surface is the same as the twisting number of the surface defined by a_i and q_i. Furthermore, we may chose $V_i = q_i$ on the planar part N of S.

Using the cylinder coordinates as in Proposition 6.5 the vectorfields V_i and q_i are given by $\left(\alpha_{V_i}, \theta_{V_i}\right)$ resp. $\left(0, \theta_{q_i}\right)$. By construction the surface isotopy induced by the vectorfield isotopy:

$$[0, 1] \ni u \mapsto \left(u\alpha_{V_i}, u\theta_{V_i} + (1-u)\theta_{q_i}\right)$$

is the identity on the planar part on S and it makes a neighbourhood of the curve a_i into a flat surface. Doing this for each closed curve in the spine of S completes the proof of part *(a)* and hereby the proof Theorem 6.1. □

6.4 On the isotopy classes of negative curvature surfaces

The isotopy classes of flat surfaces are described by Theorem 6.1 and the isotopy classes of positive curvature surfaces are described by Theorem 6.2. These theorems raise the question: *Is there a result analogous to Theorem 6.1 and Theorem 6.2 concerning the isotopy classes of negative curvature surfaces?* We answer this question in the negative, by pointing out, that the lack of umbilic points on negative curvature surfaces subdivide each isotopy class of surfaces (except for the disc's class) into at least countable infinitely many isotopy classes of negative curvature surfaces.

Let S be a compact surface with non-vanishing boundary and negative curvature. By the negative curvature, the principal directions corresponding to positive resp. negative principal curvature define two smooth globally defined line-fields on S. Assume S is not a disk and let s be a spine on S. As we have seen in this chapter, the isotopy class (with no curvature restriction) of S is determined by the isotopy class of the spine s with twisting numbers attached to each closed curve. Let a_i be a closed curve in the spine s. The rotation of the principal directions relative to the tangents of a_i, when traversing a_i once, defines a half integer valued index. By continuity, this rotational index is independent of deformations of a_i through simple closed curves on S and it is independent of isotopy of S through negative curvature surfaces. Hence, considering isotopy of negative curvature surfaces through negative curvature surfaces, each closed curve in a spine has an index additional to and independent of its twisting number. This causes the claimed subdivision of the isotopy classes of ordinary compact surfaces with nonempty boundary and motivates

Conjecture 6.8 (a) *In* 3*-space, any compact surface with nonempty boundary is isotopic to a negative curvature surface.* (b) *Any two such negative curvature surfaces, S_1 and*

S_2, are isotopic through negative curvature surfaces if and only if there exists an isotopy through ordinary surfaces between S_1 and S_2, such that, for each simple closed curve on S_1, this curve and its image on S_2, under this isotopy, have equal rotational indices with respect to the principal directions on the respective surfaces[3].

The reason why the rotational index does not cause subdivision of isotopy classes in case of flat resp. positive curvature surfaces is that they may have umbilic points (planar regions on flat surfaces) in which all directions are principal. Hence, the above rotational index is generally not well-defined on flat resp. positive curvature surfaces. As shown on Figure 6.2, the isotopies constructed in this chapter use planar regions to unwind the principal directions (rulings), such that, they newer tangentiate the axes. A similar remark counts for the positive curvature model surfaces in [17] where positive curvature strips are pieced together on a spherical surface piece.

Hence, even though the flat model surfaces used here and the model positive curvature surfaces used in [17] easily can be changed into "model negative curvature surfaces", they can only produce surfaces on which the netto rotation of the principal directions with respect to each closed curve in a spine is zero. All other isotopy classes of negative curvature surfaces has to be treaded using other model surfaces or perhaps using entirely different methods.

[3] These indices depend only on the regular homotopy classes of the curves.

Chapter 7

Boundaries of flat surfaces in 3-space

7.1 Introduction

A simple closed curve in 3-space bounds an orientable compact embedded surface, that is, a Seifert surface. By part (a) of Theorem 6.1 this surface is isotopic to a flat surface. The boundary of this flat surface has the same knot type as the given curve. Hence, an immediate consequence of Theorem 6.1 is

Corollary 7.1 *Any simple closed space curve can be deformed until it bounds an orientable compact embedded flat surface.*

Corollary 7.1 leaves open the possibility that any simple closed space curve bounds a flat surface. In Section 7.5 it is proven, by giving an example, that this not is the case. A simple closed space curve may be chosen to bound any given number of distinct flat surfaces[1]. The most obvious example of a curve bounding more that one flat surface is the smooth intersection curve between two flat cylinders in 3-space. Apart from the above mentioned results, the results on boundaries of flat surfaces in this chapter are of three different types.

The first type of results concerns surfaces with non-negative curvature and consists of Theorem 7.2 in Section 7.2 only. By Theorem 7.2 the genus of non-negative curvature surfaces (especially flat surfaces), bounded by a given curve, is bounded by the total curvature of this curve. Applied to the set of torus knots, it follows that it takes high total curvature for a knot to bound a non-negative curvature surface.

The second type of results concerns the number and multiplicities of 3-singular points, i.e., points of vanishing torsion (vertices) or of vanishing curvature, on the boundary of a flat immersed surface. These results are gathered in Section 7.3 and Section 7.4. The main result in Section 7.3 is Theorem 7.4, which implies that the number of 3-singular points on the boundary of a flat immersed surface is greater than or equal to twice the absolute value of the Euler characteristic of the surface. Section 7.4 mainly concern the

[1] The curve given by $(\cos(t), \sin(t), \cos(nt))$, $t \in [0, 2\pi]$ bounds n, for $n = 1, 2$, and $(n+2)$, for $n \geq 3$, distinct flat discs. The two extra discs, in case $n \geq 3$, each contain a planar region

situation that a given simple closed curve bounds two distinct flat discs which is, pointed out to be, intimate with the Four Vertex Theorem.

The last type of results is contained in Section 7.6 and is a set of necessary and, in a slightly weakened sense, sufficient conditions for a simple closed curve to be, what we denote, a generic boundary of a flat immersed surface without planar regions. The solutions are given via explicit parametrizations. These conditions are easily generalized to the problem of links bounding flat surfaces. The general problem of giving necessary and sufficient conditions for a closed space curve to bound a flat surface is listed as an industrial problem in [11].

7.2 Total curvature of a curve bounding an orientable non-negative curvature surface

The following theorem is an immediate corollary of the Gauss-Bonnet's Theorem in case of surfaces with negative Euler characteristic, in case of a disc it uses W. Fenchel's result asserting that the total curvature of a closed space curve is greater that or equal to 2π.

Theorem 7.2 *Let $\gamma = \partial S$ be a simple closed curve bounding an immersed orientable non-negative curvature (especially flat) surface S of genus $\mathrm{g}(S)$ (and Euler characteristic $\chi(S)$) immersed in 3-space. Then the total curvature of γ is bigger than or equal to $2\pi|\,\chi(S)| = 2\pi|2\,\mathrm{g}(S) - 1|$.*

Proof: Let S, $\gamma = \partial S$, $\mathrm{g}(S)$, and $\chi(S)$ be as in the theorem. As a closed space curve, by Fenchels Theorem [8], has total curvature bigger than or equal to 2π there is nothing to prove unless $\chi(S) < -1$. Henceforth, assume that $\chi(S) < -1$. The geodesic curvature of $\gamma \subset S$, κ_g, is numerically smaller than the curvature of γ thought of as a space curve, κ. Hence we have the inequality $\int_\gamma \kappa(s)ds \geq \int_{\gamma\subset S} |\kappa_g(s)|ds$. As S has non-negative curvature, K, the Gauss-Bonnet's Theorem yields $|\int_{\gamma\subset S} \kappa_g(s)ds| = |2\pi\,\chi(S) - \int_S K dA| \geq 2\pi|\,\chi(S)| = 2\pi|1 - 2\,\mathrm{g}(S)|$. □

The following example shows that the lower bound of total curvature of a knot bounding an orientable non-negative curvature surface given in Theorem 7.2 can be arbitrarily much larger than the infimum of curvature needed for the knot to have its knot type.

Example 7.3 *In this example we consider the part of the torus knots, $K(q,r)$, given by $q > r > 1$ which are relatively prime. The bridge number of $K(q,r)$ equals r (Theorem 7.5.3 in [28]). In general the bridge number of a knot type equals the crookedness of the knot type, i.e., $\frac{1}{2\pi}$ times the infimum of the total curvature taken over all curves of the given knot type. See [26]. Hence, the total curvature of a curve with knot type $K(q,r)$ can get arbitrarily close to $2\pi r$. In fact, the total curvature of the standard parametrization of a torus knot lying on a torus of revolution tends to $2\pi r$ when the thickness of the torus tends to zero.*

Now let γ be a knot of type $K(q,r)$ bounding a flat embedded surface S. The minimal genus of any Seifert surface, i.e., of any embedded orientable surface, bounded by the here considered torus knots are $\frac{(q-1)(r-1)}{2}$. See e.g. [28], Theorem 7.5.2. Hence, if one of our torus knots bounds an orientable non-negative curvature surface S, then this knot has total curvature $\geq 2\pi|1-2\,\mathrm{g}(S)| \geq 2\pi|(q-1)(r-1)-1| = 2\pi(qr-q-r)$, by Theorem 7.2. The lower bound on the total curvature of a knot bounding an orientable non-negative curvature surface can thus be arbitrarily much larger than the infimum of curvature needed to have its knot type. In particular torus knots on a thin torus of revolution do in general not bound an embedded oriental surface with non-negative curvature.

7.3 The number of 3-singular points on the boundary of a flat surface

The main result in this section is

Theorem 7.4 *The boundary of a flat and compact (immersed) surface in 3-space with Euler characteristic $\chi(S)$ and p planar regions has at least $2(|\chi(S)|+p)$ 3-singular points.*

Remark 7.5 *We give examples (Example 7.16 and 7.17) showing that Theorem 7.4 is optimal for disks and closed strips in the case $p=0$.*

A zero torsion point on a space curve is called a vertex. Thus in a vertex the first three derivatives of the curve are linear dependent. If the first three derivatives of the curve are linear dependent this is also known as a 3-singular point of the curve, see e.g. [6]. As we assume that our space curves are regular they have either zero curvature or zero torsion in a 3-singular point. Assuming non-vanishing curvature the 3-singular points asserted by Theorem 7.4 are indeed vertices. For a list of 2- and 4-vertex theorems for closed space curves see e.g. [6].

To prove our generalized vertex theorem, Theorem 7.4, some preparations are needed. The first is to introduce a map similar to the integrand in the Gauss integral formula of the self-linking number of a closed space curve. This map is the main subject of investigation in the following.

Proposition 7.6 *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a simple closed regular C^n-curve, $n \geq 3$, in 3-space. Then the map $F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ given by*

$$F_\gamma(s,t) = \begin{cases} \frac{[\gamma'(s)\ \ \gamma(s)-\gamma(t)\ \ \gamma'(t)]}{|\gamma(s)-\gamma(t)|^4}, & \text{for } s \neq t \\ -\frac{1}{12}\left[\gamma'(s)\ \ \gamma''(s)\ \ \gamma'''(s)\right], & \text{for } s = t \end{cases}$$

is of type $C^{(n-3)}$.

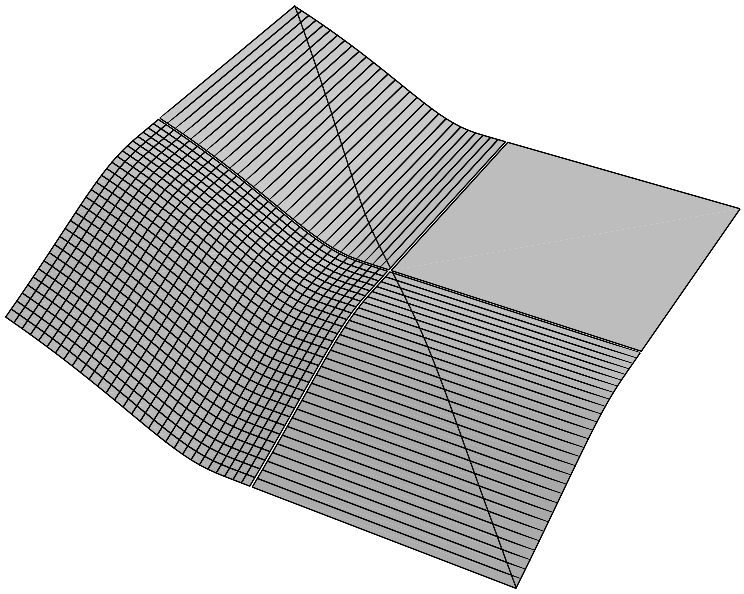


Figure 7.1: *The graph of a C^{∞}-function giving a flat surface corresponding to the north eastern half plane.*

Proof: Note that F_γ is of type $C^{(n-1)}$ except at the diagonal and that F_γ is symmetric (i.e. $F_\gamma(s,t) = F_\gamma(t,s)$ for all $(s,t) \in \mathbb{S}^1 \times \mathbb{S}^1$). With these two observations it is sufficient to consider the limit $t \to s$ for fixed s using a Taylor expansion of γ at $\gamma(s)$. The further details are left to the reader. $\square$

Let $p \in S$ be a parabolic point or a limit point of parabolic points on a flat surface S. In the following S needs only be immersed in 3-space. In case p is parabolic the zero principal direction gives the unique ruling R_p of S through p, see [46] Col. 6 p. 359. If p is an interior flat point on S and p is a limit point of parabolic points then a unique ruling through p is given by continuity, see [46] Col. 7 and 8 p. 361; however a non-vanishing vector field giving the direction of the rulings need not be smooth at p. This is the case on the flat surface shown on Figure 7.2. In case p is not an interior point on S and p is a limit point of parabolic points there may be two distinct rulings of S through p. See Figure 7.1. By a ruling, R_p, of S through a point p on S we only refer to the connected component, of the intersection between the straight line in 3-space, defined by R_p, and S, which contains p. If this set only contains p then R_p is considered as the point p and a direction. Although all results in this section holds for immersed flat surfaces we, mainly for notational reasons, only state and prove them for embedded flat surfaces. Note, that a flat surface has unique rulings through each of its points if and only if the parabolic points of S are dense in S.

Definition 7.7 *Let γ be a curve on the boundary of a flat surface S with boundary. Assume that γ tangentiate a ruling R_p on S in the point $p \in S$. This tangentiation is called*

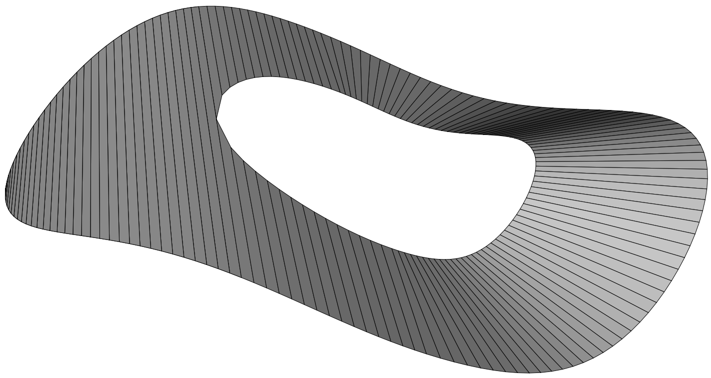


Figure 7.2: *A flat orientable C^4-surface with no global ruling vector field. The surface consists of a cylinder part to the left and a cone part to the right. The line-field defined by the rulings is continuous at the intersection of the two parts of the surface, but not continuous differentiable.*

generic if the plane containing R_p and a surface normal in $p \in S$ strictly supports γ locally. A generic tangentiation is called interior if p is an interior point on R_p and it is called exterior if not (in this case R_p consists of one point).

On the flat surface shown on Figure 7.2 the "outer" boundary curve has a generic exterior tangentiation with the rulings of the surface at the out most left of this figure and the "inner" boundary curve, at its out most left, has a generic interior tangentiation with the rulings of the surface.

Proposition 7.8 *(Bitangent plane) Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a boundary curve of a flat surface S in 3-space. If $\gamma(s)$ and $\gamma(t)$ are two distinct points on γ lying on the same ruling of S, then $F_\gamma(s,t) = 0$.*

Proof: Let γ and S be as in the proposition and let $\gamma(s)$ and $\gamma(t)$ be two distinct points on γ lying on the same ruling, R, of S. The surface normal is constant on the straight line segment R. Hence, all tangent planes of points on R are identical when identified with planes in 3-space. The vectors $\gamma'(s)$, $\gamma'(t)$, and $\gamma(s) - \gamma(t)$ lie in the same plane, thus their triple scalar product vanishes. As $\gamma(s) - \gamma(t) \neq \mathbf{0}$ it follows that $F_\gamma(s,t) = 0$. $\square$

Lemma 7.9 *(Non-generic tangentiation) Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a boundary curve of a flat surface S in 3-space. If γ has a non-generic tangentiation with a ruling of S in $\gamma(s)$ then γ has vanishing curvature at $\gamma(s)$. Especially $\gamma(s)$ is an 3-singular point of γ, i.e., $F_\gamma(s, s) = 0$.*

Proof: Let $I : U \to \mathbb{R}^2$ be an isometry of a neighbourhood of $\gamma(s)$ on S into the plane. The ruling which is tangentiated by γ at $\gamma(s)$ is a geodesic on S and is thus mapped to a straight line segment, L, by I. The plane curve $I(\gamma(\mathbb{S}^1) \cap U)$ tangentiate L and is not locally strictly supported by L (assuming non-generic tangentiation). Hereby, the planar curvature of $I(\gamma(\mathbb{S}^1) \cap U)$ is zero at $I(\gamma(s))$. Hence, the geodesic curvature $\kappa_g(s)$ of γ, being intrinsic, is zero at $\gamma(s)$. As $\gamma'(s)$ is an asymptotic direction on S, also the normal curvature $\kappa_n(s)$ of γ at $\gamma(s)$ is zero. Finally, the curvature $\kappa(s)$ of γ at $\gamma(s)$ is zero as $\kappa^2 = \kappa_g^2 + \kappa_n^2$. □

Lemma 7.10 *(Exterior tangentiation) Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a boundary curve of a flat surface S in 3-space. If γ has an exterior tangentiation with a ruling of S in $\gamma(s)$, then $\gamma(s)$ is an 3-singular point of γ. I.e., $F_\gamma(s, s) = 0$.*

Proof: Let $I : U \to \mathbb{R}^2$ be an isometry of an ε-neighbourhood of $\gamma(s)$ on S into the plane. The planar curve $I(\gamma(\mathbb{S}^1) \cap U)$ is locally supported by its tangent at $I(\gamma(s))$ and we may assume that it has non-vanishing curvature. See the proof of Lemma 7.8. Let $R \subset U$ be a ruling of S contained in U. As R is a geodesic on S it is mapped to a straight line segment by the isometry I. By continuity we by choosing R sufficiently close to $\gamma(s)$ can assume that the angle between $I(R)$ and the tangent line at $I(\gamma(s))$ is small. Hereby R has exactly two intersections with γ, $\gamma(s_1)$ and $\gamma(s_2)$ say, on each side of $\gamma(s)$ in $\gamma(\mathbb{S}^1) \cap U$. Proposition 7.8 now assert that $F_\gamma(s_1, s_2) = 0$. Considering a sequence of rulings on S with $\gamma(s)$ as a limit point we obtain that $F_\gamma(s, s) = 0$, as $F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ is continuous. □

Lemma 7.11 *(Interior tangentiation) Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a boundary curve of a flat surface S in 3-space. If γ has an interior tangentiation with a ruling of S in $\gamma(s)$, then $\gamma(s)$ is an 3-singular point of γ. I.e., $F_\gamma(s, s) = 0$.*

Proof: Consider a chart $\phi : S \supset U \to \mathbb{R}^2_+ = \mathbb{R} \times [0, \infty[$ on S around $\gamma(s)$, such that $\phi(U)$ is contained in the upper half plane, $\gamma(\mathbb{S}^1) \cap U$ is mapped to the x-axis, and $\phi(\gamma(s)) = (0, 0)$. Choose $\delta > 0$ such that $[-\delta, \delta] \times \{0\} \subset \phi(\gamma(\mathbb{S}^1) \cap U)$ and choose a smooth function $f : [-\delta, \delta] \to [0, \infty[$ such that $f(x) = 0$ for $|x| \geq \delta/2$ and $f(0) = 1$. The image of the, by γ tangentiated ruling R, under ϕ, $\phi(R)$, is a curve in $\mathbb{R}^2_+$ that only intersects (and thus tangentiate) the x-axis at the origin. Hence, for each sufficiently small $\varepsilon > 0$ the graph of εf intersects $\phi(R)$ in at least two points, $(x^-(\varepsilon), f(x^-(\varepsilon))$

and $(x^+(\varepsilon),\ f(x^+(\varepsilon))$ where $x^-(\varepsilon) < 0 < x^+(\varepsilon)$ say. Let $\gamma_\varepsilon : [-\delta, \delta] \to U$ denote the curve given as the pullback of the graph of εf under ϕ. Proposition 7.8 asserts that $F_{\gamma_\varepsilon}(x^-(\varepsilon), x^+(\varepsilon)) = 0$ as $\gamma_\varepsilon(x^-(\varepsilon))$ and $\gamma_\varepsilon(x^+(\varepsilon))$ both lie on the ruling R. As x^+, $x^- \to 0$ for $\varepsilon \to 0$ we by continuity obtain that $0 = \lim_{\varepsilon\to 0} F_{\gamma_\varepsilon}(x^-(\varepsilon), x^+(\varepsilon)) = F\gamma_0(0,0) = F_\gamma(s,s)$. $\square$

Lemma 7.12 *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a boundary curve of a flat surface S in 3-space. If $\gamma(s)$ is contained in a planar region of S, then $\gamma(s)$ is an 3-singular point of γ. I.e., $F_\gamma(s,s) = 0$.*

Proof: An example of the considered situation is shown of Figure 7.1. As in the proof of Lemma 7.11 consider a family of curves γ_ε which are pushed into the interior of S locally around $\gamma(s)$. We observe that each γ_ε is planar on an interval I_ε and thus $F_{\gamma_\varepsilon}(I_\varepsilon \times I_\varepsilon) \equiv 0$. By continuity we conclude that $F_\gamma(s,s) = 0$. $\square$

The proof of the generalized vertex theorem, Theorem 7.4, is by counting the number of points in which the boundary of a flat surface S either tangentiates a ruling of S or intersects a planar region of S. For a start the situation where neither of these possibilities occur is treaded.

Proposition 7.13 *Let S be a flat compact connected surface (immersed) in 3-space. If S has unique rulings through each of its points and the intersections between the boundary of S and each ruling of S are transversal, then S is either a Möbius strip or an orientable closed strip in 3-space. In particular S has Euler characteristic zero.*

Proof: Let S be as in the proposition and let R be a ruling of S. We now cut S along R and let $\gamma : [a,b] \to \partial S$ be a regular curve chosen such that $\gamma(s) \in R$ if and only if $s = a$ or $s = b$. For each $s \in [a,b]$ consider the ruling $R_{\gamma(s)}$ of S through $\gamma(s)$. Apart from the point $\gamma(s)$ the ruling $R_{\gamma(s)}$ has, by compactness of S, at least one more intersection with the boundary of S. Denote such an intersection point by $p \in \left(R_{\gamma(s)} \cap \partial S\right) \setminus \{\gamma(s)\}$. By the assumption that the intersections between the boundary of S and each ruling of S are transversal the point p is an endpoint of the ruling $R_{\gamma(s)}$. Thus $R_{\gamma(s)} \cap \partial S$ consists only of p and $\gamma(s)$ and we may write $p = p(s)$ defining a curve $p : [a,b] \to \partial S$ with the same properties as γ.

Consider the continuous map $H : [a,b[\times[0,1] \to S$ given by $H(s,u) = u\gamma(s) + (1-u)p(s)$. For each $s \in [a,b[$ the image of $\{s\} \times [0,1]$ under H is the ruling of S through $\gamma(s)$. As γ is regular and there is a unique ruling through each point of S the map H is injective. The image of $[a,b[\times[0,1]$ under H is both open and closed in S. Thus H is surjective as S is connected. According to the identification of $H(a \times [0,1])$ with $H(b \times [0,1])$ the surface S is either a Möbius strip or an orientable cylinder in 3-space. In any of the two cases the surface S has Euler characteristic zero. $\square$

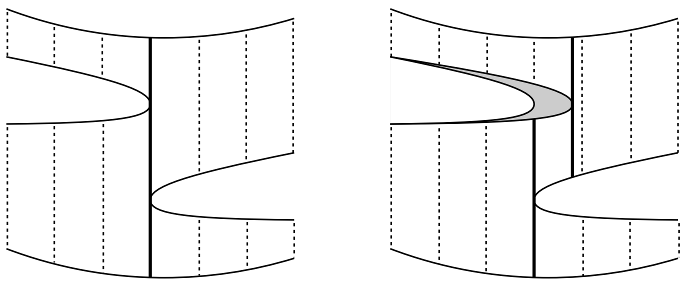


Figure 7.3: *Removing a double tangentiation of a ruling by contracting the surface.*

Theorem 7.14 *Let S be a flat, compact, and connected surface with boundary in 3-space and denote the Euler characteristic of S by $\chi(S)$. If S has unique rulings through each of its points then the boundary of S tangentiate the rulings of S in at least $2|\chi(S)|$ distinct points. Moreover, if the boundary of S has generic tangentiations with the rulings of S only and if it has $E < \infty$ exterior tangentiations and $I < \infty$ interior tangentiations with the rulings of S then $\chi(S) = \frac{1}{2}(E - I)$.*

Proof: In case the boundary of S do not tangentiate the rulings of S the theorem follows by Proposition 7.13. Assume therefore that the boundary of S do tangentiate the rulings of S. A non generic tangentiation can be removed by a local isotopy of the boundary of S through curves on S. Thus we may assume that all tangentiations are generic and that ∂S tangentiate the rulings of S in an only finite number of distinct points. Finally, fixing the number of generic tangentiations, n, we may assume that each tangentiated ruling is tangentiated only once. See Figure 7.3[2].

Let R_i, $i = 1, \ldots, n$, denote the genericly tangentiated rulings of S. For each R_i we by cutting along rulings of S, which lie close to R_i, obtain a closed neighbourhood, N_i, of the tangentiated ruling R_i. By cutting sufficiently close to each R_i we can assume that the closed neighbourhoods N_i, $i = 1, \ldots, n$ are pairwise disjoined.

Let $\tilde{S}$ be the closure of $S \setminus \cap_{i=1}^{n} N_i$ and let C denote one of the finitely many connecteded components of $\tilde{S}$. As the boundary of S only has transversal intersections with the rulings of C the argument used in the proof of Prop. 7.13 shows that C is the bijective image of a rectangle $[a, b] \times [0, 1]$. By the construction of this bijection the restriction of the boundary of S to C is the image of $[a, b] \times \{0\}$ and $[a, b] \times \{1\}$ and the image of $\{a\} \times [0, 1]$ and $\{b\} \times [0, 1]$ are two of the rulings of S along which we have cut S to obtain the neighbourhoods N_i, $i = 1, \ldots, n$. Hence, the closure of $S \setminus \cap_{i=1}^{n} N_i$ is the bijective image of the disjoined union of a finite number of regular four-gons.

Let R be an exterior genericly tangentiated ruling and let N be a neighbourhood of R as constructed above. See Figure 7.4 to the left hand side. Let p denote the point of tangentiation (the fat point on this figure). Our transversality argument shows that $N \setminus \{p\}$

[2]This last assumption is not strictly necessary for the remainder of this proof – but it reduces considerably the number of cases needed to be considered.

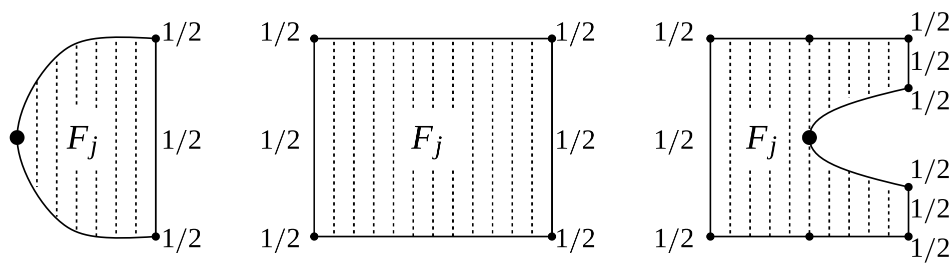


Figure 7.4: *Left, a neighbourhood of a generically tangentiated exterior ruling. Right, a neighbourhood of a generically tangentiated interior ruling. In between a connected component of the reminder of the surface.*

is the bijective image of $]a, b] \times [0, 1]$. As $\{a\} \times [0, 1]$ is mapped to p we by collapsing this edge of $[a, b] \times [0, 1]$ obtain N as the bijective image of a triangle.

Let R be an interior genericly tangentiated ruling and let N be a neighbourhood of R as constructed above. See Figure 7.4 to the right hand side. The set N may be considered as the bijective image of a 9-gon. Our given surface S is hereby pieced into a union of 3, 4, and 9-gon faces and these faces are glued together along distinct rulings of S. Hence, we have obtained a polygonalization of S.

Recall that the Euler characteristic of S is given by $\chi(S) = F - E + V$, where F, E, resp. V is the number of distinct faces, distinct edges, resp. distinct vertices of any polygonalization of S. We want to count the Euler characteristic face-wise and do this

- by counting each of the rulings, which we have cut S along to obtain a polygonalization, as one half edge in each of the two faces in which it is an edge and

- by counting each endpoint of these rulings as one half vertex in each of the two faces in which it is a vertex.

Let F_j, $j \in \{1, \ldots, F\}$, be a face in the polygonalization of S. Denote by χ_j its contribution to the Euler characteristic of S and denote by T_j the number of tangentiation points between the boundary of S and the rulings of S that lie in F_j. The Euler characteristic of S, $\chi(S)$, satisfies $\chi(S) = \sum_{j=1}^{F} \chi_j$ and the total number of tangentiation points, $T(S)$, satisfies $T(S) = \sum_{j=1}^{F} T_j$. The last equality follows as each tangentiation point only is contained in the face, which is chosen as a neighbourhood of this point.

On Figure 7.4 is shown a 3, 4, and 9-gon face together with the weights of their edges and vertices that differ from one. A triangle face contains one tangentiation point and its contribution to the Euler characteristic of S is $1-(1+1+1/2)+(1+1/2+1/2) = 1/2$. A 4-gon face contains zero tangentiation point and its contribution to the Euler characteristic of S is $1-(1+1+1/2+1/2)+(4\cdot(1/2)) = 0$. A 9-gon face contains one tangentiation point and its contribution to the Euler characteristic of S is $1 - (6 \cdot 1 + 3 \cdot (1/2)) + (3 \cdot 1 + 6 \cdot (1/2)) = -1/2$. Hence, for each face F_j, $j = 1, \ldots, F$, we have the inequality $T_j \geq 2|\chi_j|$. The theorem now follows by summation of these face-wise inequalities, since we have

$$T(S) = \sum_{j=1}^{F} T_j = 2\sum_{j=1}^{F} \left|\chi_j\right| \geq 2\left|\sum_{j=1}^{F} \chi_j\right| = 2\left|\chi(S)\right|.$$

□

We now prove the main theorem, Theorem 7.4, of this section.

Proof: [of Theorem 7.4.] We may assume that the boundary has only finitely many 3-singular points. A planar region on a flat compact surface intersects the boundary of the surface. By Lemma 7.12 each point of intersection between a planar region and boundary is an 3-singular point on the boundary. Assuming that the boundary has only finitely many 3-singular points the surface S has only finitely many planar regions and its boundary intersects each planar region in an at most finite number of points. Denote the finite number of planar regions on S by p.

Claim: *Let P be a planar region of S which has n intersections with the boundary of S. Then there exists an isotopy of S, which is the identity outside a neighbourhood of P, such that, the deformed part of S has unique rulings through each point and the boundary of the deformed part has $(n-2)$ tangentiations with these rulings.*

By the **claim** the given flat surface S is isotopic to a flat surface $\tilde{S}$ which has unique rulings through each of its points. The boundary of the surface $\tilde{S}$ tangentiate the rulings of $\tilde{S}$ in at least $2|\chi(\tilde{S})| = 2|\chi(S)|$ distinct points. Each of these points is an 3-singular point by the lemmas 7.9, 7.10, and 7.11. By the **claim** we, for each planar region on the original surface S, can count two 3-singular points more on the boundary of S than on the boundary of $\tilde{S}$. Hence, the boundary of S has at least $2\,(|\chi(S)| + p)$ 3-singular points.

Proof of the claim: Let P be a planar region on S which has n intersections with the boundary of S. The boundary of P consists of n rulings, R_i, $i = 1, \dots, n$, of S connecting these n intersection points. This follows by [46] Col. 8 p. 361. Hence, P is a planar (convex) n-gon. Let q be an interior point of P and consider for each of the rulings R_i a curve $c_i \subset S$ starting at q and leaving P orthogonally through R_i. We may assume that the curves c_i, $i = 1, \dots, n$, only intersect in the point q. Hence, these curves gives a local spine of the surface S in a neighbourhood of P. Contract the surface on a neighbourhood of P such that the boundary is close to this spine inside P. See Figure 6.1, where the dotted arcs give a picture of the local spine. A neighbourhood of q can be isotoped to a part of a cylinder surface, such that, the number of tangentiations between boundary and rulings is $(n-2)$. The remaining planar part of P consists of neighbourhoods of each c_i. These strips can be deformed to become globally ruled e.g. using cones to pull pieces of them out of the plane. This proves the claim and hereby the theorem. □

We conclude this section with a discussion of the optimality of Theorem 7.4.

Theorem 7.15 *In each isotopy class of flat, compact, and connecteded surfaces, with Euler characteristic $\chi(S)$, there exists a surface S with unique rulings through each of its points such that the boundary of this surface tangentiate its rulings exactly $2|\chi(S)|$ times. I.e., the inequality in Theorem 7.14 is an equality.*

Proof: The case that S is a disc is trivial, see e.g. Figure 7.5. Let S be a flat, compact, and connected surface in 3-space and let $\chi(S) \leq 0$ denote the Euler characteristic of S. By the isotopies of flat surfaces in 3-space constructed in Section 6.3 the given surface S is isotopic to a flat model surface, M. See Figure 6.1. The surface M is build from one planar disk and $(1 - \chi(S))$ strips each starting and ending at this disk. The boundary of M thus intersects the planar disc in $2(1 - \chi(S))$ segments. Furthermore, each strip is given by an axis and a vectorfield along this axis such that if the boundary of the strip is chosen close to and parallel with this axis then it does not tangentiate the rulings. Using an isotopy as constructed in the **Claim** in the proof of Theorem 7.4 we get a flat surface with unique rulings through each of its points and the boundary of this surface has exactly $2(1 - \chi(S)) - 2 = 2|\chi(S)|$ generic interior tangentiations with its rulings. □

Knowing that the lower bound on the number of 3-singular points descented from ruling-tangentiations and intersections with planar regions is optimal it is, e.g. in spirit of the Four Vertex Theorem, natural to ask if the total number of 3-singular points is larger than twice the absolute value of the Euler characteristic of the flat surface? Below, we give examples showing that the answer to this question is in the negative in case of disks and closed strips. Furthermore, combining these two examples it is possible to construct optimal examples with any negative Euler characteristic – but optimal example within each isotopy class seems out of reach due to the boundedness of obtainable twisting numbers caused by using axes (spines) with non-vanishing torsion. Cf. Proposition 6.5.

Example 7.16 *The closed curve on Figure 7.5 bounds a flat cylinder surface. This curve is partly made by two pieces of circular Helices with constant curvature $1/2$ and constant torsion $1/2$ resp. $-1/2$. The projection, into the plane of this paper, of the two curve-pieces joining these helixcial pieces are tenth degree polynomials chosen such that the space curve has non-vanishing C^1-curvature and C^1-torsion that only changes sign twice. The curvature and torsion of one of the joining pieces are shown of Figure 7.6.*

Example 7.17 *Let $\mathbf{r} : \mathbb{S}^1 \to \mathbb{R}^3$ be a closed space curve with non-vanishing curvature and non-vanishing torsion. A family of such curves is given in [6]. The twisting numbers of flat closed strips on which $\mathbf{r}$ is an axis is bound either from above or from below, depending on the sign of the torsion of $\mathbf{r}$. This is the cornerstone in the proof of Proposition 6.5. Newer the less, $\mathbf{r}$ is both an axis of a Möbius strip and of an orientable cylinder. In both cases the flat closed strip S possesses a parametrization of the form $(s, t) \mapsto \mathbf{r}(s) + t\mathbf{q}(s)$ for some unit vectorfield $\mathbf{q}$ along $\mathbf{r}$. Choosing the boundary of S as $s \mapsto \mathbf{r}(s) \pm \varepsilon\mathbf{q}(s)$ compactness ensures that it has non-vanishing curvature and non-vanishing torsion simply by choosing $\varepsilon > 0$ sufficiently small.*

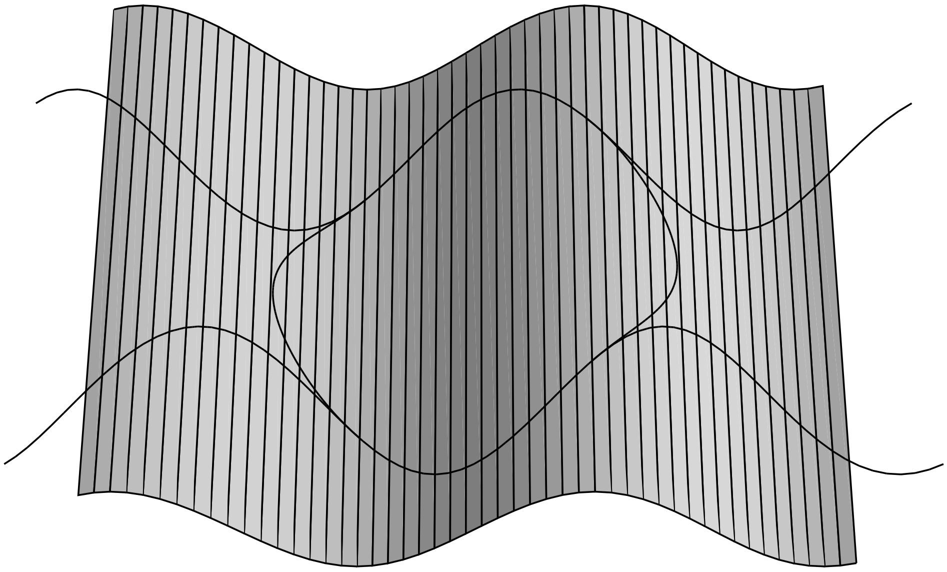


Figure 7.5: *Two circular Helices and a closed curve on a flat cylinder surface.*

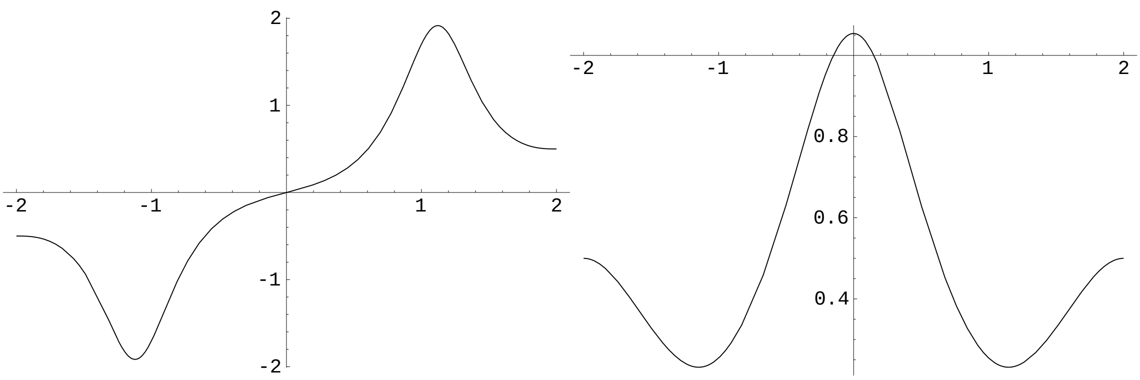


Figure 7.6: *Torsion, to the left hand side, and curvature, to the right hand side of one of the curve pieces that join the circular Helices on Figure 7.5.*

7.4 On the Four Vertex Theorem

In this section the results on the number of 3-singular points on the boundary of a flat surface obtained in Section 7.3 are related to the Four Vertex Theorem and to a generalization of the Four Vertex Theorem given in [40]. The Four Vertex Theorem was conjectured by P. Scherk in 1936 and was first proven in full generality in 1994 by V.D. Sedykh [44]. To point out this relation it is necessary to elaborate on the number, and multiplicities, of 3-singular points on a closed space curve bounding two destinct flat discs.

Definition 7.18 *Let $\gamma : I \to \mathbb{R}^3$ be a space curve of type at least three and define $D_\gamma : I \to \mathbb{R}^3$ by $D_\gamma(s) = [\gamma'(s)\, \gamma''(s)\, \gamma'''(s)]$. We say that a 3-singular point $\gamma(s^\star)$ of γ has multiplicity m if the first $(m-1)$ derivatives of D_γ vanish for $s = s^\star$.*

The map D_γ (D for determinant and for diagonal of $F\gamma$) from Definition 7.18 is periodic if the curve γ is closed. Hence, if the sum of the multiplicities of the 3-singular point of a closed curve is finite then it is even.

Theorem 7.19 *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a closed space curve of type at least four. If γ bounds two immersed discs D_1 and D_2 such that the interior of $D_1 \cap D_2$ is empty in D_1 (and in D_2), then γ has at least two 3-singular points and the sum of the multiplicities of γ's 3-singular points is at least four.*

Proof: Let γ, D_1, and D_2 be as in the theorem. We may and do assume that γ has only finitely many 3-singular points. If D_i, $i = 1, 2$, contains a planar region then γ has at least four 3-singular points by Theorem 7.4. Hence, we may assume that both D_1 and D_2 has unique rulings through each of their points. As D_i, $i = 1, 2$, is a disc the number of exterior tangentiations minus the number of interior tangentiations between γ and the rulings of D_i equals two, by Theorem 7.14. Thus we may assume that there is an one parameter family of rulings of D_i that starts and ends in two points q_i^1 and q_i^2 of exterior tangentiation. The pairs q_1^1 and q_1^2 resp. q_2^1 and q_2^2 are pairs of distinct points on γ. Hence, γ has at least two 3-singular points.

In case the set $\{q_1^1, q_1^2, q_2^1, q_2^2\}$ consists of four distinct points the theorem follows. The two remaining cases, where $\{q_1^1, q_1^2, q_2^1, q_2^2\}$ consists of two resp. three distinct points, follow from the assertion: "*If γ has external tangentiation with a ruling of both D_1 and D_2 in $q_1^1 = q_2^1$ - say, then $q_1^1 = q_2^1$ is a 3-singular point of multiplicity at least two*".

To prove the assertion note, that the map F_γ is C^1 as γ is assumed to be C^4 (see Proposition 7.6). Let $q_1^1 = q_2^1 = \gamma(s)$, then $-12F_\gamma(s,s) = D_\gamma(s) = 0$. By the assumption that the interior of $D_1 \cap D_2$ is empty in D_1 and D_2 there are two curves, that not are equal on any interval, in the plane through (s,s) on which F_γ is identical equal to zero. Hence, (s,s) is a stationary point of F_γ by the implicit function theorem. In particular $\frac{d}{ds}(-12F\gamma(s,s)) = \frac{dD_\gamma}{ds}(s) = 0$ and the 3-singular point $\gamma(s)$ has multiplicity at least two. $\square$

The following two examples illustrate Theorem 7.19 and show that it is optimal.

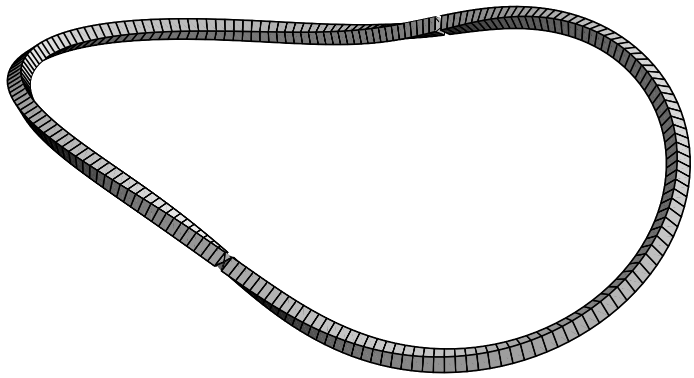


Figure 7.7: *A space curve given by* $r(t) = (3/2\cos(t) + 1/2\cos(2t), 3/2\sin(t) + 1/2\sin(2t), 2\sin(t))$ *shown as a tube with edges pushed off along principal and bi- normals. The curve has two points of zero curvature in which its principal normals and hereby also its binormals have opposite limits. The tube inflects in both of these points.*

Example 7.20 *The curve shown on Figure 7.7 has two* 3*-singular points in which its curvature vanishes. Hence, they both have multiplicity at least two. From Figure 7.8 we get a picture of how there are two families of rulings of two distinct flat discs, spanned by this curve. Both of these families starts and ends in the two* 3*-singular points on the curve.*

Example 7.21 *The curve shown on Figure 7.9 has three* 3*-singular points. One of these is has multiplicity two and is a zero curvature point. The other two* 3*-singular points have multiplicity one and are points of zero torsion. From Figure 7.10 we get a picture of how there are two families of rulings of two distinct flat discs, spanned by this curve. Both of these families starts in the zero curvature point on the curve and they end in distinct zero torsion points.*

We now discus the relation between the Four Vertex Theorem [44] and its generalization in [40] and Theorem 7.19. Recall that the Four Vertex Theorem states that *a simple convex closed* C^3 *space curve* (i.e., a space curve contained in the boundary of its convex hull) *with non-vanishing curvature has at least four vertices*. The generalization of the Four Vertex Theorem in [40] is

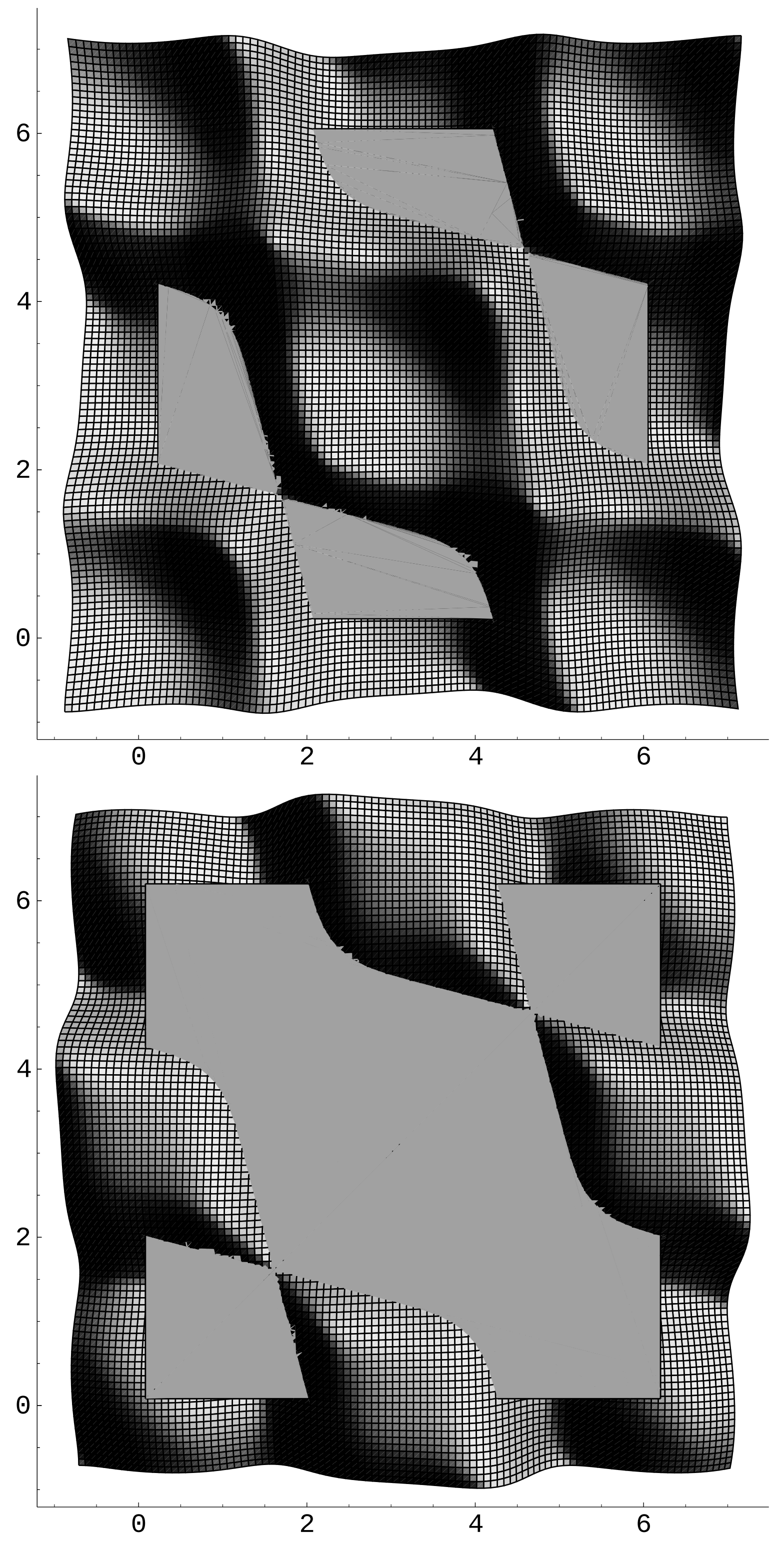


Figure 7.8: *Top and bottom view of the intersection between the graph of $F_{\mathbf{r}}$, with* **r** *as on Figure 7.7, and the zero plane.*

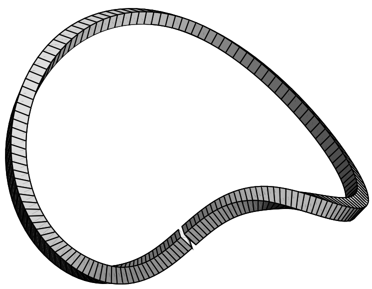


Figure 7.9: *A space curve given by* $r(t) = (\sin(t), \sin(t)(1 - \cos(t)), (1 - \cos(t))^3)$ *shown as a tube with edges pushed off along principal and bi- normals. This curve is studied in [43] and is in [33] shown to be one of the simplest trigonometric polynomial space curves who's rectifying developable is a flat and analytic Möbius Strip. For this it is necessary that it has one point of zero curvature which causes an inflection of its principal and bi- normals. This is the points in which the tube inflects*

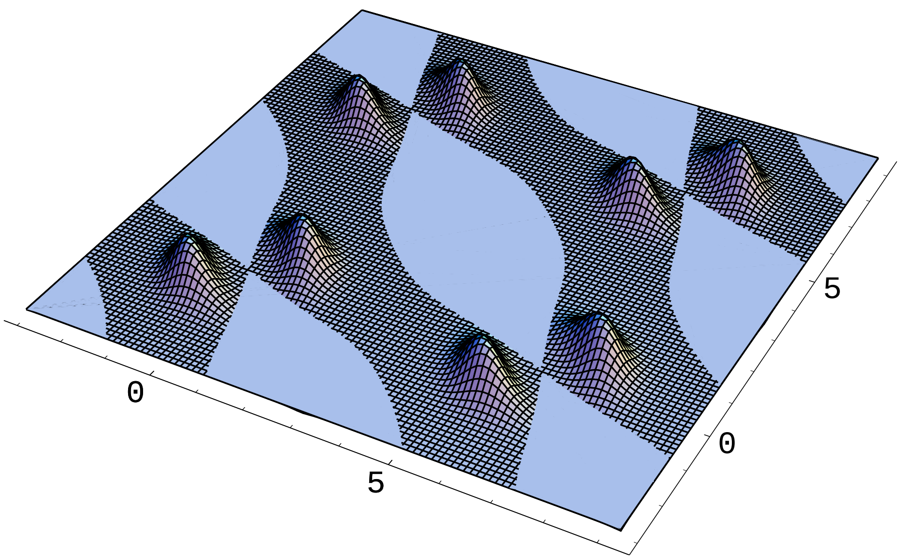


Figure 7.10: *A view of the intersection between the graph of* $F_{\mathbf{r}}$*, with* $\mathbf{r}$ *as on Figure 7.9, and the zero plane.*

Theorem 7.22 (M.C. Romero Fuster and V.D. Sedykh, [40]) *Given any C^3 closed simple convex space curve, its total numbers S, K, and V of singular points, zero curvature points and vertices satisfy the inequality:*

$$3S + 2K + V \geq 4.$$

Observing that the vertices of a curve γ are zeros of D_γ with multiplicity at least 1, whereas the zero curvature and singular points are zeros of D_γ of multiplicities at least 2 and 3 respectively they in [40] assert

Corollary 7.23 (M.C. Romero Fuster and V.D. Sedykh, [40]) *The number of zeros of the function $D_\gamma(t)$ associated to any C^3 closed simple convex space curve, γ, is at least 2, and the sum of their multiplicities is at least 4.*

A connection between Corollary 7.23 and Theorem 7.19 is that the boundary of the convex hull of a strictly convex closed space curve consists of two flat $C^{1,1}$-surfaces (see Theorem 1.2.2. in [13]). A closed space curve γ is said to be strictly convex if through each point p on the curve there is a support plane such that $\gamma \setminus \{p\}$ lies strictly on one side of this plane. Hence, Theorem 7.19 implies Corollary 7.23 restricted to strictly convex curves and is itself free of the convexity condition.

An 3-singular point, with multiplicity two, on a space curve need not be a point of zero curvature[3]. Hence, Theorem 7.19 does not imply the generalization of the Four Vertex Theorem, Theorem 7.22, restricted to regular strictly convex curves. However, computer experiments motivate Conjecture 7.24 which, if true, establishes this implication and thereby frees the generalized Four Vertex Theorem from the convexity condition.

Conjecture 7.24 *If a curve, γ, bounds two distinct discs and $\gamma(s)$ is a point of exterior tangentiation with respect to both discs (as considered in the proof of Theorem 7.19) then γ has zero curvature at $\gamma(s)$.*

Remark 7.25 (On Conjecture 7.24) *By the assumption in Conjecture 7.24 there are two distinct curves through (s, s) on which F_γ is zero. See the figures 7.8 and 7.10. Computer experiments suggests that if $\gamma(s)$ is a vertex of multiplicity two but not a zero curvature point then $F_\gamma(s, s)$ is an isolated zero of F_γ. A possible approach to Conjecture 7.24, in case of analytic space curves, is to analyse arbitrarily high degree Taylor expansions of F_γ around (s, s) in order to conclude that if $\gamma(s)$ is a vertex of multiplicity at least two and $\gamma(s)$ not is a zero curvature point, then all derivatives of γ are contained in the span of the first two. By analyticity, the curve γ is planar spanning only one flat (planar) disc. However this approach seems a lot of work and if successful then not very illuminating.*

[3]The curve given by $\gamma(t) = \left(t, t^2, t^5\right)$ has non-vanishing curvature and $D_\gamma(t) = 120t^2$.

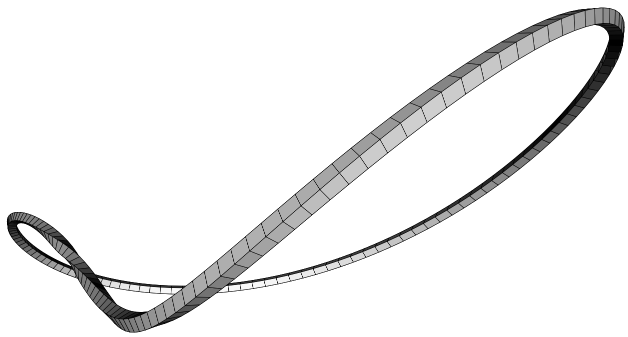


Figure 7.11: *A space curve given by* $r(t) = (3/2\cos(t) + 1/2\cos(2t), 3/2\sin(t) + 1/2\sin(2t), 2\sin(t))$ *shown as a tube with edges pushed off along principal and bi- normals.*

7.5 A closed curve which does not bound a flat compact surface

To prove that there do exist simple closed space curves which do not bound any compact flat surface we use the following corollary of Proposition 7.8 and the lemmas 7.9, 7.10, 7.11, and 7.12.

Corollary 7.26 *Let* $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ *be a simple closed space curve bounding an immersed flat compact surface and let* $F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ *be the map from Prop. 7.6. Then for each* $s \in \mathbb{S}^1$ *there exists an* $t \in \mathbb{S}^1$*, such that,* $F_\gamma(s,t) = 0$*.*

Proof: Let $\gamma(s)$ be a point on the boundary curve, γ, of a flat surface S. If $\gamma(s)$ is a point in which γ tangentiate a ruling of S then $F_\gamma(s,s)$ is zero by Lemma 7.9, 7.10, or 7.11. If there is a ruling of S through $\gamma(s)$ and this intersection is transversal, then, by compactness of the surface, this ruling has at least one more intersection with γ apart form $\gamma(s)$. Denote such an intersection point by $\gamma(t)$ for some $t \neq s$. Proposition 7.8 asserts that $F(s,t)$ is zero. Finally, if there is not a ruling of S through $\gamma(s)$ then S and hereby also γ is planar in a neighbourhood of $\gamma(s)$. Lemma 7.12 then asserts that $F_\gamma(s,s) = 0$.

□

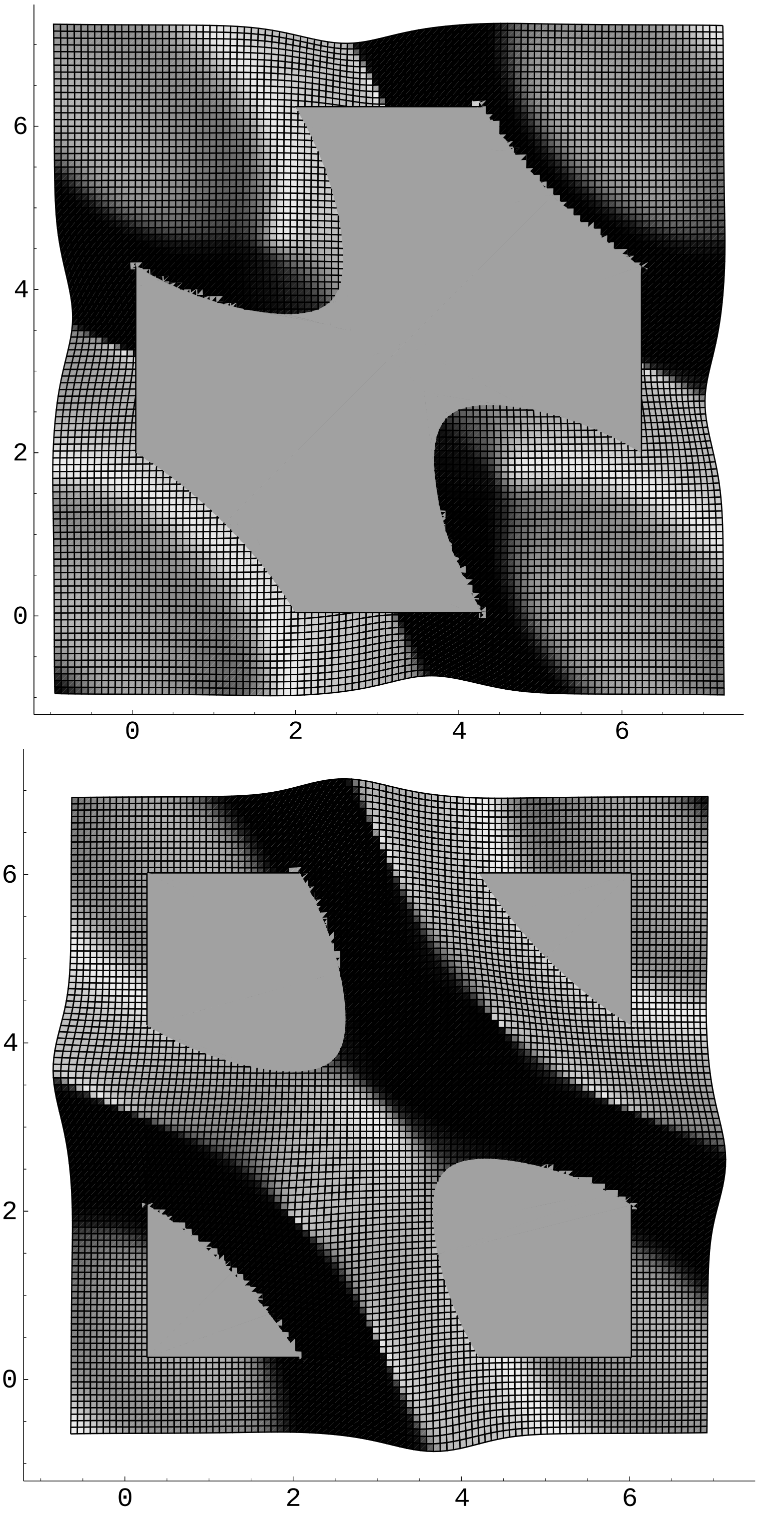


Figure 7.12: *Top and bottom view of the intersection between the graph of $F_{\mathbf{r}}$, with* **r** *as on Figure 7.11, and the zero plane.*

Example 7.27 *In this example we consider a curve, shown on Figure 7.11, on a torus found by S.I.R. Costa*[4] *[6]. The curve has non-vanishing curvature, its torsion changes sign twice (see Figure 7.12), and it possesses the parametrization* $r(t) = (3/2\cos(t) + 1/2\cos(2t), 3/2\sin(t) + 1/2\sin(2t), 2\sin(t))$, *where* $t \in [0, 2\pi]$. *On Figure 7.12 the intersection between the surface given by* $(s,t) \mapsto (s,t,F_\gamma(s,t))$ *and the* st*-plane is shown. From this picture and Corollary 7.26 it follows that this curve does not bound a flat compact immersed surface of any topology since, for all* s *in a neighbourhood of* π *and all* $t \in [0, 2\pi]$ *we have* $F_\gamma(s,t) \neq 0$.

Example 7.27 proves the main result of this section.

Theorem 7.28 *There exist simple closed curves in* 3*-space (with two vertices) that do not bound any flat compact immersed surface.*

7.6 Generic boundaries of flat surfaces

In this section we consider flat surfaces with unique rulings through each of their points only. Hereby the rulings of a given flat surface can be used to parametrize it. A parametrization with rulings (asymptotic curves) as one of the two families of parameter curves is smooth on the open and dense set of parabolic points, but needs not be continuous differentiable at planar points. See eg. Figure 7.2. However, a parametrization by rulings defines surface normals on the open and dense set of flat points that extends continuously to planar points. We thus say that a compact surface with nonempty boundary and with well-defined surface normals is flat if its Gauss image has measure zero on the unit 2-sphere. The main result of this section is a set of necessary and (sufficient in a weakened sense) conditions for a knot to be, what we will call, a generic boundary of a flat surface with unique rulings through each of its points. The general problem of giving necessary and sufficient conditions for a closed space curve to bound a flat surface is listed as an industrial problem in [11].

Remark 7.29 *Considering the* C^1 *Whitney topology on the set of compact flat immersions with non-vanishing boundary, it is not generic for a flat surface to have unique rulings through each of its points. A planar region as shown on Figure 6.3 has, however, a flat surface with unique rulings arbitrarily* C^1 *close to it - but e.g. the planar disc on a flat model surface can not be removed by a* C^1 *perturbation. The reason for this is that the contraction of the planar disc to a neighbourhood to a "local spine" used in the proof of the claim in the proof of Theorem 7.4 is unavoidable and can not be performed even arbitrarily* C^0 *small.*

Considering flat surfaces as compact subsets of 3*-space there is, however, always a flat surface with unique rulings arbitrarily close to a given flat surface.*

[4]For visual reasons the third coordinate is multiplied by a constant. Note, that flatness of surfaces is affine invariant.

Arbitrarily C^1-close to the boundary of a flat surface with unique rulings there is a new contracted boundary with non-vanishing curvature[5]. The proof of this assertion is analogous to the proof of Lemma 6.7 Part $\mathcal{B}$. By Lemma 7.9 this new boundary has only generic tangentiations with the rulings of S and we may also assume that no ruling of S is tangentiated by the boundary of S in more than one point. Hence, it is a generic condition for the boundary of a flat surface with unique rulings to fulfill these two conditions.

Definition 7.30 *Let S be a flat compact surface with non-vanishing boundary immersed in* 3*-space. If S has unique rulings through each of its points, then S is said to have generic boundary if its boundary has non-vanishing curvature (only generic tangentiations with the rulings of S) and no ruling of S is tangentiated by the boundary of S in more than one point.*

A geometric outline of the following is: Given a flat compact surface with generic boundary, cut this surface along the at most finitely many genericly tangentiated rulings. The surface hereby falls into a finite number of faces, such that, the restriction of the original surfaces boundary to each face has only transversal intersections with the rulings of the face. Thus a face possesses a parametrization, as constructed in the proof of Proposition 7.13, of the form $]a, b[\times[0, 1] \ni (s, u) \mapsto u\gamma(s) + (1 - u)\mu(t(s))$ where $\gamma(]a, b[)$ and $\mu(t(]a, b[))$ are open arcs on the boundary on S, such that the u-parameter curves are the rulings of the face and the map $t = t(s)$ defines a homeomorphism between these two arcs. The flatness and regularity of a face together with the transversality of boundary-ruling intersections then give conditions on the two arcs and their combining homeomorphism $t = t(s)$. (With these conditions fulfilled for a given face one could say that one brick in a puzzle is given.) The remaining conditions ensure that all faces fit together to one surface. (Defining a solved puzzle.) For notational reasons, only the case of one boundary curve is considered. The results, are easily generalized to links bounding flat immersed surfaces.

Let $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ be a generic boundary of a flat surface S with unique rulings through each of its points. Let ET and IT denote the sets of points on γ in which it has generic exterior resp. generic interior tangentiations with the rulings of S. Assuming that γ is a generic boundary the sets $ET = \{\gamma(e_1), \ldots, \gamma(e_{\#ET})\}$ and $IT = \{\gamma(i_1), \ldots, \gamma(i_{\#IT})\}$ are finite disjoined sets of pairwise disjoined points. By Theorem 7.14 the Euler characteristic of S is given by $\chi(S) = \frac{1}{2}(\#ET - \#IT)$. Furthermore, the sets ET and IT are subsets of the set $\gamma\left(D_\gamma^{-1}(0)\right)$ of γ's vertices by Lemma 7.10 resp. by Lemma 7.11.

Consider a point of interior tangentiation $\gamma(i_n)$. The ruling of S through $\gamma(i_n)$ is a segment of the tangent line to γ at $\gamma(i_n)$ and it contains $\gamma(i_n)$ in its interior. This ruling is tangentiated only once by γ, as γ is a generic boundary. Hence, this ruling has two unique transversal intersections with γ - one, $\gamma(i_n^+)$, in the forward direction of the

[5]Unlike curves bounding compact positive curvature surfaces, who's self-linking numbers are zero (see [17]), the self-linking numbers of curves with non-vanishing curvature bounding compact flat surfaces are unrestricted. This assertion follows directly from the proof of Lemma 6.7 part $\mathcal{C}$.

tangent line of γ at $\gamma(i_n)$ and one, $\gamma(i_n^-)$, in the backward direction. Each of the two sets $IT^{\pm} = \{\gamma(i_1^{\pm}), \dots, \gamma(i_{\#IT}^{\pm})\}$ consists of pairwise distinct points and the two sets are disjoined. This proves the following Proposition.

Proposition 7.31 (Partition) *Let the curve γ be the generic boundary of a flat immersed compact surface in 3-space and denote by $V = \gamma\left(D_\gamma^{-1}(0)\right)$ the set of vertices of γ. Then V contains two disjoined finite (possible empty) sets $ET = \{\gamma(i_1), \dots, \gamma(i_{\#ET})\}$ and $IT = \{\gamma(e_1), \dots, \gamma(e_{\#IT})\}$, such that the forward resp. backward tangent half line to γ at $\gamma(i_n)$, $n = 1, \dots, \#IT$, each has at least one transversal intersection, denoted by $\gamma(i_n^+)$ resp. $\gamma(i_n^-)$, with γ. Moreover $\gamma(i_n^+), \gamma(i_n^-) \notin ET \cup IT$ and the $i_n^{\pm}$ are distinct. (The curve is said to fulfill the partition property, Property $\mathcal{P}$.)*

Denote by $I_1, \dots, I_N$ the $\#ET + 3\#IT = N$ open intervals given by $(\mathbb{R}/L\mathbb{Z}) \setminus \gamma^{-1}\left(ET \cup IT \cup IT^+ \cup IT^-\right)$. These are the intervals corresponding to the arcs into which γ is cut when the surface S is cut along the tangentiated rulings. Consider one of the $N/2$ faces, F, of S minus the tangentiated rulings. By the uniqueness of the rulings of S, and by the transversality of their intersections with the boundary of S restricted to the face F, they define a homeomorphism $\phi_{jl} : I_j \leftrightarrow I_l$ (as constructed in the proof of Proposition 7.13) between the j'th and the l'th interval - say. The face F possesses the parametrization $H_{jl} : I_j \times [0,1] \to \mathbb{R}^3$ given by $H_{jl}(s,u) = u\gamma(s) + (1-u)\gamma\left(\phi_{jl}(s)\right)$.

The surface S is flat. Hence, by Proposition 7.8 $F_\gamma\left(s, \phi_{jl}(s)\right) \equiv 0$ for all $s \in I_j$ and for all $j = 1, \dots, N$. Furthermore, since S is not planar on a region it follows that for each $s \in I_j$ the map F_γ is not identically equal to zero on any neighbourhood of $(s, \phi_{jl}(s))$.

The homeomorphism ϕ_{jl} is implicitly given by the equation $F_\gamma(s,t) = 0$ or equivalently by the equation

$$G(s,t) = \left[\gamma'(s)\ \gamma(s) - \gamma(t)\ \gamma'(t)\right] = 0.$$

Consider the two partial derivatives $\frac{\partial G(s,t)}{\partial s} = \left[\gamma''(s)\ \gamma(s) - \gamma(t)\ \gamma'(t)\right]$ and $\frac{\partial G(s,t)}{\partial t} = \left[\gamma'(s)\ \gamma(s) - \gamma(t)\ \gamma''(t)\right]$. If $\frac{\partial G(s,t)}{\partial s} = 0$ then $\gamma''(s)$ lies in the span of $(\gamma(s) - \gamma(t))$ and $\gamma'(t)$, i.e., in the tangent plane of the surface S at $\gamma(s)$. Hence, the normal curvature in the direction of $\gamma'(s)$ is zero. Furthermore, the straight line (ruling) from $\gamma(s)$ to $\gamma(t)$ is an asymptotic curve on S. As $(\gamma(s) - \gamma(t))$ and $\gamma'(s)$ both are asymptotic directions in the tangent plane of the surface S at $\gamma(s)$, and the Gaussian curvature of S is zero at $\gamma(s) \in S$, the point $\gamma(s)$ is a flat point on S. By Corollary 7 in [46] p. 631 all points on the ruling from $\gamma(s)$ to $\gamma(t)$ are flat points on S. Hence, $\gamma(t)$ is also a flat point on S and thus also $\frac{\partial G(s,t)}{\partial t} = 0$ concluding that (s,t) is a singular point of G. From this observation it follows that

> If $\left(s, \phi_{jl}(s)\right)$ is a regular point of G, then ϕ_{jl} is a diffeomorphism between neighbourhoods of s and $\phi_{jl}(s)$. In this case the corresponding points on the surface S given by $H_{jl}(s,u)$ are all parabolic points on S.

If $\left(s, \phi_{jl}(s)\right)$ is a singular point of G, then the corresponding ruling of S consists of flat points.

This proves the following proposition.

Proposition 7.32 (Flatness) *Let the curve γ be the generic boundary of a flat immersed compact surface in 3-space. Consider the $\#ET + 3\#IT = N$ open intervals, $I_1, \ldots, I_N$, given by $(\mathbb{R}/L\mathbb{Z}) \setminus \left(\gamma^{-1}\left(ET \cup IT \cup IT^+ \cup IT^-\right)\right)$. Then for each I_j there exists an I_l and a homeomorphism $\phi_{jl} : I_j \to I_l$, such that, $F_\gamma\left(s, \phi_{jl}(s)\right) = 0$ for all $s \in I_j$. In case both ET and IT are empty there exist an orientation preserving homeomorphism $\phi : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}/L\mathbb{Z}$, such that, $F_\gamma\left(s, \phi(s)\right) = 0$ for all $s \in \mathbb{R}/L\mathbb{Z}$. If $\left(s, \phi_{jl}(s)\right)$ (or $(s, \phi(s))$) is a regular point of the map F_γ, then ϕ_{jl} (or ϕ) is a diffeomorphism between neighbourhoods of s and $\phi_{jl}(s)$ (or $\phi(s)$). Furthermore, F_γ is not identically equal to zero on any neighbourhood of $\left(s, \phi_{jl}(s)\right)$ (or $(s, \phi(s))$) for any $s \in I_j$ (or $s \in \mathbb{R}/L\mathbb{Z}$). (The curve γ is said to fulfill the flatness property, Property $\mathcal{F}$.)*

The arc $\gamma(I_j)$ is transversal to all rulings of the face F. I.e.,

$$\gamma'(s) \times \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right) \neq \mathbf{0},$$

for all $s \in I_j$ and for all $j = 1, \ldots, N$. We say that γ fulfills the transversality property, Property $\mathcal{T}$.

Regularity of S requires that, whenever ϕ_{jl} is continuously differentiable, then

$$\begin{aligned}\mathbf{0} \neq\ & \frac{\partial H_{jl}(s,u)}{\partial s} \times \frac{\partial H_{jl}(s,u)}{\partial u} \\ =\ & \left(u\gamma'(s) + (1-u)\phi_{jl}'(s)\gamma\left(\phi_{jl}(s)\right)\right) \times \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right)\end{aligned}$$

for all $u \in [0,1]$. The vector $\frac{\partial H_{jl}(s,u)}{\partial s} \times \frac{\partial H_{jl}(s,u)}{\partial u}$ is parallel with the surface normal in $H_{jl}(s,u)$. As the surface normal is constant along the ruling of S given by $H_{jl}\left(\{s\} \times [0,1]\right)$ the above condition is equivalent to the condition

$$0 < \left(\frac{\partial H_{jl}(s,0)}{\partial s} \times \frac{\partial H_{jl}(s,0)}{\partial u}\right) \cdot \left(\frac{\partial H_{jl}(s,1)}{\partial s} \times \frac{\partial H_{jl}(s,1)}{\partial u}\right),$$

which reduces to

$$\begin{aligned}0 < \phi_{jl}'(s) & \left(\gamma'(s) \cdot \gamma'\left(\phi_{jl}(s)\right) \left|\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right|^2\right. \\ & \left. - \left[\gamma'(s) \cdot \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right)\right]\left[\gamma'\left(\phi_{jl}(s)\right) \cdot \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right)\right]\right).\end{aligned}$$

We say that γ fulfills the regularity property, Property $\mathcal{R}$.

We now give the conditions for the $N/2$ faces to fit together to a surface.

Proposition 7.33 (Faces fit together) *Let the curve γ be the generic boundary of a flat immersed compact surface in 3-space. Let eg. $[e_n, \star]$ resp. $[\star, e_n]$ denote the closure of the one of intervals $I_1, \dots, I_N$ that starts resp. ends in e_n and let the pairing of the intervals $I_1, \dots, I_N$ by the homeomorphisms ϕ_{jl} be indicated by $I_j \overset{\text{pre.}}{\leftrightarrow} I_l$ resp. $I_j \overset{\text{rev.}}{\leftrightarrow} I_l$ when orientation preserving resp. orientation reversing. In case of an external tangentiation then $[e_n, \star] \overset{\text{rev.}}{\leftrightarrow} [\star, e_n]$. In case of an internal tangentiation point, $\gamma(i_n)$, the conditions are given by the following tabular, where $\mathbf{n}(i_n)$ is the principal normal to γ at $\gamma(i_n)$.*

	$\gamma'(i_n^+) \cdot \mathbf{n}(i_n) > 0$	$\gamma'(i_n^+) \cdot \mathbf{n}(i_n) < 0$
$\gamma'(i_n^-) \cdot \mathbf{n}(i_n) > 0$	$[i_n, \star] \overset{\text{pre.}}{\leftrightarrow} [i_n^+, \star]$ $[\star, i_n] \overset{\text{rev.}}{\leftrightarrow} [i_n^-, \star]$ $[\star, i_n^+] \overset{\text{pre.}}{\leftrightarrow} [\star, i_n^-]$	$[i_n, \star] \overset{\text{rev.}}{\leftrightarrow} [\star, i_n^+]$ $[\star, i_n] \overset{\text{rev.}}{\leftrightarrow} [i_n^-, \star]$ $[i_n^+, \star] \overset{\text{rev.}}{\leftrightarrow} [\star, i_n^-]$
$\gamma'(i_n^-) \cdot \mathbf{n}(i_n) < 0$	$[i_n, \star] \overset{\text{pre.}}{\leftrightarrow} [i_n^+, \star]$ $[\star, i_n] \overset{\text{pre.}}{\leftrightarrow} [\star, i_n^-]$ $[\star, i_n^+] \overset{\text{rev..}}{\leftrightarrow} [i_n^-, \star]$	$[i_n, \star] \overset{\text{rev.}}{\leftrightarrow} [\star, i_n^+]$ $[\star, i_n] \overset{\text{pre.}}{\leftrightarrow} [\star, i_n^-]$ $[i_n^+, \star] \overset{\text{pre.}}{\leftrightarrow} [i_n^-, \star]$

(We say that γ fulfills Property $\mathcal{FIT}$.)

Proof: Note, that at a point of interior tangentiation, $\gamma(i_n)$, the principal normal to the boundary $\mathbf{n}(i_n)$ is tangent to the surface and points orthogonal away from the surface. The conditions are easily checked using Figure 7.4. $\square$

Remark 7.34 (On Proposition 7.33) *In case of an orientable flat immersed compact surface all the homeomorphisms ϕ_{jl}, $j = 1, \dots, \#ET + 3\#IT$, are orientation reversing.*

Gathering the three previous propositions we get

Theorem 7.35 *A simple closed regular C^4-curve which is the generic boundary of a flat immersed compact surface in 3-space fulfills property $\mathcal{P}$, $\mathcal{F}$, $\mathcal{T}$, $\mathcal{R}$, and $\mathcal{FIT}$.*

Let $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ be a simple closed C^4 space curve with non-vanishing curvature fulfilling the properties $\mathcal{P}$, $\mathcal{F}$, $\mathcal{T}$, $\mathcal{R}$, and $\mathcal{FIT}$ and let

$$S_\gamma = \left\{ H_{jl}(s, u) | s \in \overline{I_j},\;\; u \in [0, 1],\;\; j = 1, \dots, \#ET + 3\#IT \right\}$$

denote the from γ constructed "surface". Furthermore, let $S_\gamma^\circ = S_\gamma \setminus \gamma\,(\mathbb{R}/L\mathbb{Z})$ denote the constructed "surface" minus its "boundary", $\gamma\,(\mathbb{R}/L\mathbb{Z})$. The remainder of this chapter is used to prove that there exists a C^1-parametrization of S_γ°. Thus S_γ° is an immersed C^1-surface in 3-space with γ as "boundary". The question if S_γ is an immersed C^1-surface with boundary, i.e., if S_γ is extendible, is not an issue of elaboration here. Note however, that if $\left(s, \phi_{jl}(s)\right)$ is a regular point of F_γ, then the parametrization given by H_{jl} is locally smooth and extends beyond γ simply by extension of rulings until the regularity property might be violated. Hence, on the dense subset of parabolic points on S_γ minus the points of tangentiation it is locally possible to extend S_γ to a larger immersed (flat) surface.

The proof of existence of a C^1-parametrization of S_γ° is based on a result by Herman Gluck, Theorem 4.1. in [15][6]. To state this result let M be a subset of $\mathbb{R}^3$ and x_0 a point of M and let P be a plane in $\mathbb{R}^3$ through x_0. Then P is said to be a tangent plane to M at x_0 if

$$\lim_{x \to x_0} \frac{d_{\mathbb{R}^3}(x, P)}{d_{\mathbb{R}^3}(x, x_0)} = 0,$$

for $x \in M \setminus \{x_0\}$, where $d_{\mathbb{R}^3}$ is the euclidean metric on $\mathbb{R}^3$. Let $G_{n,2}$ denote the Grassman manifold of 2-dimensional linear subspaces (i.e., 2-planes through the origin) in $\mathbb{R}^n$. If P is a 2-plane in $\mathbb{R}^n$ then P_0 will denote the 2-plane through the origin of $\mathbb{R}^n$ which is parallel to P.

Theorem 7.36 (H. Gluck, [15]) *Let M be a two-dimensional C^0 manifold in $\mathbb{R}^n$. Then M is a C^1 manifold in $\mathbb{R}^n$ if and only if*

(1) *M has a tangent plane $P(x)$ at each point $x \in M$;*

(2) *the map $M \to G_{n,2}$ sending $x \to P_0(x)$ is continuous;*

(3) *for each $x \in M$, the orthogonal projection π_x of $\mathbb{R}^n$ onto $P(x)$ is one-one (and therefore a homeomorphism into) on some neighbourhood U of x in M.*

Definition 7.37 *A compact C^1-surface with nonempty boundary is said to be flat if its Gauss image has Lebesgue measure zero.*

Theorem 7.38 *Let $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ be a simple closed C^4 space curve with non-vanishing curvature fulfilling the properties $\mathcal{P}$, $\mathcal{F}$, $\mathcal{T}$, $\mathcal{R}$, and $\mathcal{FIT}$. Then S_γ° is the image of a flat (in the sense of Definition 7.37) C^1-immersion.*

Proof: Let γ be a simple closed C^4 space curve fulfilling the properties $\mathcal{P}$, $\mathcal{F}$, $\mathcal{T}$, $\mathcal{R}$, and $\mathcal{FIT}$. By $\mathcal{F}$ there is a homeomorphism $\phi_{jl} : I_j \leftrightarrow I_l$ between two open intervals I_j and I_l such that $F_\gamma\left(s, \phi_{jl}(s)\right) = 0$ for all $s \in I_j$. Let $\left(s, \phi_{jl}(s)\right)$ be a regular point of the map F_γ, then ϕ_{jl} locally is a diffeomorphism. Within such an s-neighbourhood consider the, by construction, flat regular surface given by the smooth map $H_{jl}(s, u) = u\gamma(s) + (1-u)\gamma\left(\phi_{jl}(s)\right)$. Let $\nu(s, u)$ denote a surface normal which is independent of u by the flatness of the surface. A choice of $\nu(s, u) = \nu(s)$ is:

$$\nu(s) = \frac{\gamma'(s) \times \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right)}{\left|\gamma'(s) \times \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right)\right|}$$

Note, that by the transversality property, ν is well-defined on all of $H_{jl}\left(I_j \times [0, 1]\right)$. We now show that $\nu(s)$ is continuously differentiable along all of $\gamma\left(I_j\right)$. Restricted to an

[6] A proof of this theorem is however not given in this reference.

open subset of I_j where ϕ_{jl} is continuously differentiable (i.e., to a neighbourhood of a s-value for which $\left(s, \phi_{jl}(s)\right)$ is a regular point of F_γ) we calculate $\nu'(s)$ with respect to the basis $\left\{\gamma(s) - \gamma\left(\phi_{jl}(s)\right), \gamma'(s), \nu(s)\right\}$ of $\mathbb{R}^3$. Differentiation of the equation $0 \equiv \nu(s) \cdot \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right)$ yields

$$\begin{aligned} 0 &= \nu'(s) \cdot \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right) + \nu(s) \cdot \left(\gamma'(s) - \frac{d\phi_{jl}(s)}{dt}\gamma'\left(\phi_{jl}(s)\right)\right) \\ &= \nu'(s) \cdot \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right) \end{aligned}$$

Differentiation of the equation $0 \equiv \nu(s) \cdot \gamma'(s)$ yields $0 = \nu'(s) \cdot \gamma'(s) + \nu(s) \cdot \gamma''(s)$. And finally differentiation of the equation $1 \equiv \nu(s) \cdot \nu(s)$ yields $0 = \nu'(s) \cdot \nu(s)$. Thus $\nu'(s)$ can be written of the form

$$\begin{aligned} \nu'(s) &= -\left(\nu(s) \cdot \gamma''(s)\right)\gamma'(s) \\ &= \frac{\left[\gamma''(s)\ \gamma(s) - \gamma\left(\phi_{jl}(s)\right)\ \gamma'(s)\right]}{\left|\gamma'(s) \times \left(\gamma(s) - \gamma\left(\phi_{jl}(s)\right)\right)\right|}. \end{aligned}$$

Whenever $\left(s, \phi_{jl}(s)\right)$ is a singular point of F_γ, then $\left[\gamma''(s)\ \gamma(s) - \gamma\left(\phi_{jl}(s)\right)\ \gamma'(s)\right] = 0$. Hence, ν is C^1 along γ on the open arc $\gamma\left(I_j\right)$. By the expressions of $\nu(s)$ and $\nu'(s)$ it follows immediately that $\nu(s)$ and $\nu'(s)$ extends continuously across the endpoints of the intervals I_j, $j = 1, \ldots, \#ET + 3\#IT$, corresponding to the points IT^+ and IT^-. Hereby $\nu(s)$ and $\nu'(s)$ also extends continuously across the points of interior tangentiation, IT. Finally, we check the limits of ν at points of exterior tangentiation. For this let $\gamma(0) \in ET$ be a point in ET and consider the Taylor expansion of γ:

$$\gamma(s) = \gamma(0) + \gamma'(0)s + \frac{1}{2}\gamma''(0)s^2 + \frac{1}{6}\gamma'''(o)s^3 + \mathbf{o}(s^3),$$

where $\frac{\mathbf{o}(s^3)}{s^3} \to \mathbf{0}$ for $s \to 0$. In the following $s > 0 > t$ and $s \to 0$ if and only $t = \phi_{jl}(s) \to 0$.

$$\begin{aligned} \gamma(s)' \times (\gamma(s) - \gamma(t)) &= \left(\gamma'(0) + \gamma''(0)s + \gamma'''(0)s^2 + \mathbf{o}(s^2)\right) \\ &\quad \times \left(\gamma'(0)(s-t) + \frac{1}{2}\gamma''(0)(s^2 - t^2) + \mathbf{o}(s^2) + \mathbf{o}(t^2)\right) \\ &= \gamma'(0) \times \gamma''(0)\left(\frac{1}{2}s^2 - \frac{1}{2}t^2 - s^2 + st\right) + \mathbf{o}(s^2) + \mathbf{o}(t^2) \\ &= -\frac{1}{2}\gamma'(0) \times \gamma''(0)\,(s-t)^2 + \mathbf{o}(s^2) + \mathbf{o}(t^2) \end{aligned}$$

As $s > 0 > t$ it follows that $\left|\frac{\mathbf{o}(s^2)}{(s-t)^2}\right| \leq \left|\frac{\mathbf{o}(s^2)}{s^2}\right| \to 0$ for $s, t \to 0$ and $\left|\frac{\mathbf{o}(t^2)}{(s-t)^2}\right| \leq \left|\frac{\mathbf{o}(t^2)}{t^2}\right| \to 0$ for $s, t \to 0$. The limits of $\nu(s)$ are thus parallel with the binormals of γ at points of exterior tangentiation and the derivative of $\nu(s)$ vanishes at these points.

The from $\mathcal{P}$, $\mathcal{F}$, $\mathcal{T}$, and $\mathcal{R}$ constructed "faces" fit together by $\mathcal{FIT}$. The resulting subset in $\mathbb{R}^3$, S°_γ, has a well-defined normal at each point. And a (projectively) double cover of the "Gauss-image" is given by $\nu\,(\mathbb{R}/L\mathbb{Z})$. The projective "Gauss-image" hereby equals the image of a non-regular C^1 closed curve on the unit 2-sphere. Hence, if the above constructed subset of $\mathbb{R}^3$, S°_γ, is the image of an C^1 immersion, then it is an immersed flat surface in 3-space in the sense of Definition 7.37.

In order to apply Theorem 7.36 note that S°_γ is a two-dimensional C^0 immersed manifold in $\mathbb{R}^3$, with a well-defined continuous "normal" vector field ν. Hence, condition (2) of Theorem 7.36 is fulfilled and there is a candidate to a tangent plane at each point of S°_γ. It remains to check that for each $x \in S^\circ_\gamma$ the plane through x with normal $\nu(x)$ is a tangent plane and that the orthogonal projection to this plane is locally one-one.

First consider an open "face" of S°_γ given by H_{jl} restricted to the open domain $I_j\times]0,1[$. Again, if $s^\star \in I_j$ and $\left(s^\star, \phi_{jl}(s^\star)\right)$ is a regular point of F_γ, there is a neighbourhood $N_{s^\star}$ of $s^\star$ in I_j such that $H_{jl}|N_{s^\star}\times]0,1[$ is a smooth regular parametrization of a flat smooth surface. It remains to check the conditions (1) and (3) of Theorem 7.36 when $\left(s^\star, \phi_{jl}(s^\star)\right)$ is a singular point of F_γ. As previously mentioned, the plane P with normal $\nu(s^\star)$ and through $\gamma(s^\star)$ is the osculating plane of γ both at $\gamma(s^\star)$ and at $\gamma\left(\phi_{jl}(s^\star)\right)$. By property $\mathcal{T}$, γ is transversal to the line of $\mathbb{R}^3$ through $\gamma(s^\star)$ and $\gamma\left(\phi_{jl}(s^\star)\right)$ at both of these points. By this it is obvious that if $N_{s^\star}$ is a neighbourhood of $s^\star$ in I_j, then the projection of $H_{jl}\left(N_{s^\star}\times]0,1[\right)$ onto P is one-one, for $N_{s^\star}$ sufficiently small. Hence, condition (3) of Theorem 7.36 is fulfilled by the considered open "face".

To show condition (1) of Theorem 7.36 on an open "face" of S°_γ, choose a coordinatesystem of $\mathbb{R}^3$, such that, the xy-plane is the osculating plane of γ at $\gamma(s^\star)$ and at $\gamma\left(\phi_{jl}(s^\star)\right)$ and, such that, both of the points $\gamma(s^\star)$ and $\gamma\left(\phi_{jl}(s^\star)\right)$ lie on the y-axis. Reparametrize the two neighbourhoods of γ around $\gamma(s^\star)$ and $\gamma\left(\phi_{jl}(s^\star)\right)$ as $\xi(s) = \gamma\left(s - s^\star\right)$ and $\mu(t) = \gamma\left(t - \phi_{jl}(s^\star)\right)$ for $|s|$ and $|t|$ both within neighbourhoods of zero. As ξ (resp. μ) is transversal to the y-axis, and the xy-plane is the osculating plane of ξ (resp. μ) at $\xi(0)$ (resp. $\mu(0)$), it follows that $\xi'(0)\cdot(1,0,0) \neq 0$ (resp. $\mu'(0)\cdot(1,0,0) \neq 0$). The projection of ξ (resp. μ) onto the x-axis defines thus locally a diffeomorphism by $x(s) = \xi(s)\cdot(1,0,0)$ (resp. $x(t) = \mu(t)\cdot(1,0,0)$). Restricting x, s, and t to neighbourhoods of zero, there exist constants $0 < c_1 \leq c_2 \leq 1$ and $0 < c_3 \leq c_4 \leq 1$ such that $c_1|x| \leq |s| \leq c_2|x|$ and $c_3|x| \leq |t| \leq c_4|x|$. The considered neighbourhood on S°_γ has an one-one projection on to the xy-plane and may therefor be written of the form

$$(x, y, h(x, y)) = u\xi(s) + (1-u)\mu(t),$$

where $u = u(x,y)$, $s = s(x,y)$, $t = t(x,y)$, and s and t are homeomorphic via the flatness property of γ. Moreover the hight function, h, may be written of the form

$$\begin{aligned} h(x,y) &= (u\xi(s) + (1-u)\mu(t))\cdot(0,0,1) \\ &= u\left(as^3 + o(s^3)\right) + (1-u)\left(bt^3 + o(t^3)\right), \end{aligned}$$

as the xy-plane is the osculating plane of the two curve pieces at their intersection with the y-axis.

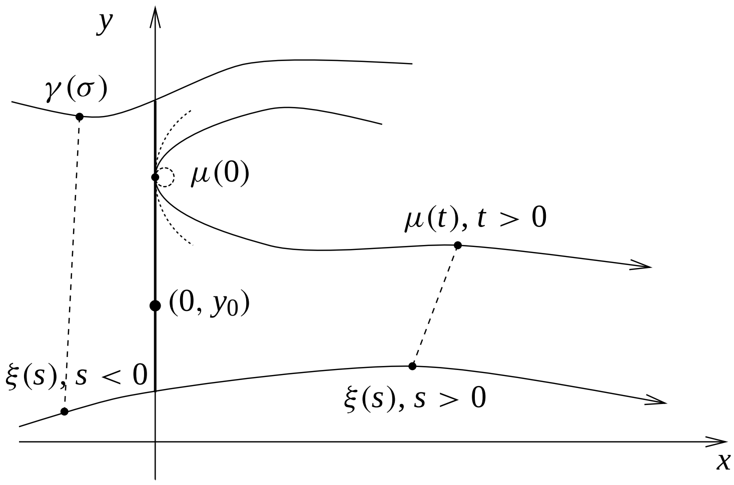


Figure 7.13: *Three faces glued together along a segment of the y-axis.*

Let $(0, y_0, 0)$ be a point on the y-axis in between $\xi(0)$ and $\mu(0)$ and let $(x, y, 0) \neq (0, y_0, 0)$ be a neighbouring point in the xy-plane. In case $x = 0$ the distance from $(x, y, h(x, y))$ to the xy-plane is zero. Hence, when proving that the xy-plane is a tangent plane to a point of S_γ° on the y-axis we may, and do in the following, assume that $x \neq 0$.

$$\begin{aligned}
&\frac{d_{\mathbb{R}^3}\left((x, y, h(x, y)), xy\text{–plane}\right)}{d_{\mathbb{R}^3}\left((x, y, h(x, y)), (0, y_0, h(0, y_0))\right)}\\
&\leq \frac{|h(x, y)|}{|x|}\\
&= \frac{|u\left(as^3 + o(s^3)\right) + (1-u)\left(bt^3 + o(t^3)\right)|}{|x|}\\
&\leq \frac{|uas^3|}{|x|} + \frac{|uo(s^3)|}{|x|} + \frac{|(1-u)bt^3|}{|x|} + \frac{|(1-u)o(t^3)|}{|x|}\\
&\leq \frac{|uac_2^3x^3|}{|x|} + \frac{|uc_2o(s^3)|}{|s|} + \frac{|(1-u)bc_4^3x^3|}{|x|} + \frac{|(1-u)c_4o(t^3)|}{|t|}
\end{aligned}$$

The last inequality follows from the inequalities $c_1|x| \leq |s| \leq c_2|x|$ and $c_3|x| \leq |t| \leq c_4|x|$ noticing that the lower bounds yield $s \neq 0$ and $t \neq 0$ as $x \neq 0$. Both s and $t \to 0$ for $x \to 0$, whereby both $\frac{|uc_2o(s^3)|}{|s|}$ and $\frac{|(1-u)c_4o(t^3)|}{|t|} \to 0$ for $x \to 0$. The xy-plane is thus a tangent plane to S_γ° at each of their points of intersection. Each of the open faces of S_γ° is hereby the image of a C^1-immersion by Theorem 7.36.

Finally, consider the tangent plane candidates on the straight line segment on which the open faces of S_γ° are glued together. See Figure 7.13 that introduces the notation used in the following.

The transversality of the intersection of ξ with the y-axis ensures that the projection of S_γ° onto the xy-plane is one-one in a neighbourhood of the point $(0, y_0)$. Again, ξ

is transversal to the y-axis and tangentiate the xy-plane, whereby there exist constants $0 < c_1 \leq c_2 \leq 1$ such that $c_1|x| \leq |s| \leq c_2|x|$. Moreover, the hight of ξ above the xy-plane is at least a second degree polynomial in s. The xy-plane is the osculating plane of μ at $\mu(0)$. The curve μ tangentiate the y-axis at $\mu(0)$ and has non-vanishing curvature. Hence, there (as indicated on Figure 7.13) exists constants $0 < c_3 \leq c_4$ such that $c_3 t^2 \leq |x| \leq c_4 t^2$. This gives the estimate $h(x, y) = u\left(as^2 + o(s^2)\right) + (1-u)\left(bt^3 + o(t^3)\right)$ of the height function when restricted to the $(x > 0)$-half-plane. Hence, for $x > 0$ it follows that

$$\begin{aligned}\varepsilon_{(0,y_0)}(x, y) &= \frac{d_{\mathbb{R}^3}\left((x, y, h(x, y)), xy\text{-plane}\right)}{d_{\mathbb{R}^3}\left((x, y, h(x, y)), (0, y_0, h(0, y_0))\right)}\\ &\leq \frac{|h(x, y)|}{|x|}\\ &\leq \frac{|uac_2^2x^2|}{|x|} + \frac{|uc_2o(s^2)|}{|s|} + \frac{\left|(1-u)b\left(\frac{|x|}{c_3}\right)^{\frac{3}{2}}\right|}{|x|} + \frac{|(1-u)o(t^3)|}{c_3t^2},\end{aligned}$$

which by similar arguments as above tends to zero for $0 < x \to 0$. Again for $(x, y) = (0, y) \neq (0, y_0)$ the function $\varepsilon_{(0,y_0)}$ is identically equal to zero. On the $(x < 0)$-half-plane there are two possibilities; either the y-axis corresponds to a regular or to a singular point of the map F_γ. In both cases the previous considerations of an open "face" of S_γ° show that $\varepsilon_{(0,y_0)}(x, y)$ also tends to zero for $0 > x \to 0$. Altogether the map $\varepsilon_{(0,y_0)}(x, y)$ is continuous at $(0, y_0)$ with value zero and the xy-plane is a tangent plane of S_γ° along the considered line.

The continuous manifold S_γ° thus fulfill the three conditions of Theorem 7.36 and is therefor an C^1-immersion, which is flat in the sense of Definition 7.37. □

Theorem 7.35 and Theorem 7.38 have natural generalizations to links being generic boundaries of flat surfaces with unique rulings through each of their points. Considering links instead of knots, one basically has to exchange the map F_γ with a set of analogous maps, one for each pair of curves in a link. In case of knots, there are either local parametrizations, H_{jl}, of the flat surface given by the finite number of intervals I_j or there is one global parametrization given by the whole circle in case of a Möbius strip who's boundary is transversal to all its rulings, see Proposition 7.13. Considering links there may be a mixture of intervals and circles defining the parametrizations.

Bibliography

[1] V.I. Arnold. *Topological Invariants of Plane Curves and Caustics*, volume 5 of *University Lecture Series*. AMS. Providence, 1994.

[2] D. Bar-Natan. On the vassiliev knot invariants. *Topology*, 34(2):423–472, 1995.

[3] Joan S. Birman. New points of view in knot theory. *Bull. Amer. Math. Soc. (N.S.)*, 28(2):253–287, 1993.

[4] W. Blaschke. *Integralgeometrie, Systeme von Diff. Gl.* Chelsea Publishing Company New York, 1949.

[5] C. Chicone and N.J. Kalton. Flat embeddings of the Möbius strip in $\mathbb{R}^3$. Department of Mathematics, University of Missouri, Columbia, MO 65211., 1984.

[6] S.I.R. Costa. On closed twisted curves. *Proc. Amr. Math. Soc.*, 109(1):205–214, 1990.

[7] M.P. do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Inc, first edition, 1976.

[8] W. Fenchel. Über Krümmung und Windung geschlossener Raumkurve. *Math. Ann*, 101:238–252, 1929.

[9] W. Fenchel. Geschlossene Raumkurven mit vorgeschriebenem Tangentenbild. *Jber. Deutschen Math. Verein.*, 39:183–185, 1930.

[10] W. Fenchel. On the differential geometry of closed space curves. *Bull. Amer. Math.*, 57:44–54, 1950.

[11] Avner Friedman. *Mathematics in industrial problems. Part 10*. Springer-Verlag, New York, 1998.

[12] M.C. Romero Fuster and V.D. Sedykh. On the number of singularities, zero curvature points and vertices of a simple convex space curve. *Journal of Geometry*, 52(1):168–172, 1995.

[13] Mohammad Ghomi. *Strictly Convex Submanifolds And Hyper surfaces Of Positive Curvature*. PhD thesis, The Johns Hopkins University, Baltimore, Maryland, 1998. Available at http://www.math.jhu.edu/ ghomi/.

[14] Oswald Giering. Verallgemeinerung eines Satzes von Jacobi und Scherrer. *Math. Z.*, 122(4):321–326, 1971.

[15] Herman Gluck. Geometric characterization of differentiable manifolds in Euclidean space. In *Topology Seminar (Wisconsin, 1965)*, pages 197–209. Ann. of Math. Studies, No. 60, Princeton Univ. Press, Princeton, N.J., 1966.

[16] Herman Gluck and Liu-Hua Pan. Knot theory in the presence of curvature, I. Preprint, 1994.

[17] Herman Gluck and Liu-Hua Pan. Embedding and knotting of positive curvature surfaces in 3-space. *Topology*, 37(4):851–873, 1998.

[18] Henry W. Gould. *Combinatorial identities*. Henry W. Gould, Morgantown, W.Va., 1972. A standardized set of tables listing 500 binomial coefficient summations.

[19] Johann Hartl. Über direkt integrierbare Kurven und strenge Böschungskurven im $\mathbb{R}^n$ sowie geschlossene verallgemeinerte Hyperkreise. *Manuscripta Math.*, 42(2-3):273–284, 1983.

[20] Johann Hartl. Zur integration der Frénet-Gleichungen. In *Proceedings of the 3rd Congress of Geometry (Thessaloniki, 1991)*, pages 187–193, Thessaloniki, 1992. Aristotle Univ. Thessaloniki.

[21] Cheng Chung Hwang. A differential-geometric criterion for a space curve to be closed. *Proc. Amer. Math. Soc.*, 83(2):357–361, 1981.

[22] Greg Kuperberg. Detecting knot invertibility. *J. Knot Theory Ramifications*, 5(2):173–181, 1996.

[23] R. H. Kyle. Embeddings of Möbius bands in 3-dimensional space. *Proc. Roy. Irish Acad. Sect. A.*, 57:131–136, 1955.

[24] Jizhi Liu and Youning Wang. On Weiner's theorem for tangent indicatrices of regular closed spherical curves. *Beijing Shifan Daxue Xuebao*, 32(1):36–39, 1996.

[25] C. Lübbert. Symmetrieeigenschaften ebener periodischer Bewegungsvorgänge. *Z. Angew. Math. Mech.*, 52:311–314, 1972.

[26] J.W. Milnor. On the total curvature of knots. *Annals of Mathematics*, 52(2):248–257, 1950.

[27] H.K. Moffatt and R.L. Ricca. Helicity and the Călugăreanu invariant. *Proc. R. Soc. Lond. A*, 439:411–429, 1992.

[28] K. Murasugi. *Knot Theory and Its Applications*. Birkhäuser Boston, 1996.

[29] Yu. A. Nikolaevskiĭ. On a problem of W. Fenchel. *Mat. Zametki*, 56(5):87–97, 158, 1994. translation in Math. Notes **56**, no. 5-6, 1158–1164 (1994).

[30] William Pohl. The self-linking number of a closed space curve. *J. of Math. and Mechanics*, 17:975–985, 1968.

[31] Helmut Pottmann. Bemerkungen zum Satz von Jacobi und Scherrer über das Hauptnormalenbild einer Raumkurve. *Rad Jugoslav. Akad. Znan. Umjet.*, (435):27–33, 1988.

[32] T. Randrup and P. Røgen. How to twist a knot. Master's thesis, Mathematical Institute, Technical University of Denmark, 1995. (Avaliable at http://www.mat.dtu.dk/publications/masterthesis-1995-02.html).

[33] T. Randrup and P. Røgen. Sides of the Möbius strip. *Arch. Math.*, 66:511–521, 1996.

[34] T. Randrup and P. Røgen. How to twist a knot. *Arch. Math.*, 68:252–264, 1997.

[35] Eckehard Röding. Über meromorphe Funktionen auf endlichen Riemannschen Flächen vom Betrag Eins auf den Randlinien. *Math. Nachr.*, 78:309–318, 1977.

[36] P. Røgen. On denseness of Vassiliev invariants. Preprint: Inst. of Math., Tech. Univ. of Denmark, MAT-REPORT No. 1995-24:1–8.

[37] P. Røgen. Gauss Bonnet's formula and closed Frenet frames. *Geom. Dedicata*, 73(3):295–315, December 1998.

[38] P. Røgen. On density of Vassiliev invariants. *J. Knot Theory Ramifications*, 8(2):1–4, March 1999. (To appear).

[39] P. Røgen. Principal normal indicatrices of closed space curves. *Geom. Dedicata*, 74(1):49–60, January 1999.

[40] M. C. Romero Fuster and V. D. Sedykh. On the number of singularities, zero curvature points and vertices of a simple convex space curve. *J. Geom.*, 52(1-2):168–172, 1995.

[41] Ştefania Ruscior. Réseaux à torsions géodésiques égales et constantes. *Bul. Inst. Politehn. Iaşi (N.S.)*, 2:1–7, 1956.

[42] W. Scherrer. Über das Hauptnormalenbild einer Raumkurve. *Comment. Math. Helv.*, 19:115–133, 1947.

[43] Gideon Schwarz. A pretender to the title "canonical Moebius strip". *Pacific J. Math.*, 143(1):195–200, 1990.

[44] V. D. Sedykh. Four vertices of a convex space curve. *Bull. London Math. Soc.*, 26(2):177–180, 1994.

[45] B. Solomon. Tantrices of spherical curves. *Amer. Math. Monthly*, 1:30–39, 1996.

[46] M. Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 3. Publish or Perish, Wilmington, second edition, 1979.

[47] Rolland Trapp. Twist sequences and Vassiliev invariants. *J. Knot Theory Ramifications*, 3(3):391–405, 1994. Random knotting and linking (Vancouver, BC, 1993).

[48] You Ning Wang and Ji Zhi Liu. Closed spherical curves and the Jacobi theorem. *Acta Math. Sinica*, 40(2):202–206, 1997. (Chinese).

[49] J. Weiner. Flat tori in $\mathbb{S}^3$ and their Gauss maps. *Proc. London Math. Soc.*, 62:54–76, 1991.

[50] Y. Ch. Wong. A global formulation of the condition for a curve to lie in a sphere. *Monatsh. Math.*, 67:363–365, 1963.

Index