Sobolev spaces and variational problems

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Abstract

In this short note we shall introduce some important concepts for the theoretical and numerical treatment of partial differential equations. The writing stile is of necessity compact, and references for further information are given.

1 Dual spaces

Let A be a nonempty subset of the real numbers $\mathbb{R}$. $s \in \mathbb{R}$ is a majorant of A if $x \leq s$ for every $x \in A$; and $s \in \mathbb{R}$ is a minorant of A if $x \geq s$ for every $x \in A$. A has a smallest majorant, which is denoted by the supremum of A, or $\sup A$; and A has a largest minorant, which is denoted by the infimum of A, or $\inf A$.

Let $(X, \|\cdot\|)$ be a Banach space. A linear functional on X is mapping $L : X \mapsto \mathbb{R}$ such that

$$L(ax + by) = aL(x) + bL(y) \;\forall x, y \in X,\; \forall a, b \in \mathbb{C}.$$

A linear functional on X is bounded if there exists a finite, positive constant C such that

$$\forall x \in X: \; |L(x)| \leq C\|x\|.$$

The *dual space of* X, denoted by X', is the space of all bounded linear functionals on X. The dual space can be equipped with the norm

$$\|L\|_{X'} \equiv \sup\{|L(u)| : \; \|u\|_X \leq 1\}$$

with respect to which X' is a Banach space. A bounded linear functional on a Banach space X is often written in the form

$$L(u) \equiv \langle L, u\rangle_{X' \times X}, \quad u \in X,$$

one says that $\langle \cdot, \cdot \rangle_{X' \times X}$ is the *duality pairing* between X and its dual X'.

Now, let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Fix some $y \in X$ and consider the linear functional

$$X \ni x \mapsto \langle y, x \rangle \in \mathbb{R} \tag{1}$$

The Cauchy-Schwartz inequality ensures that this linear functional is bounded:

$$\forall x \in X : \quad |\langle y, x \rangle| \leq \|y\| \|x\|$$

the norm of the functional is $\|y\|$. One can prove that *all* bounded linear functionals on X arise in this way. That is, for any bounded linear functional $L \in X'$ there exists a unique element $y \in X$ such that L has the form

$$L(x) = \langle y, x \rangle \ \forall x \in X.$$

Furthermore, $\|L\|_{X'} = \|y\|$. This is the *Riesz representation theorem.* Because of this theorem, one often identifies a Hilbert space with its dual, writing $X = X'$.

For more information on these subjects, see [3].

2 Sobolev spaces

We begin by defining the domains on which we shall consider partial differential equations.

Definition 2.1 We say that the domain $\Omega \subset \mathbb{R}^n$ is *Lipschitz* if every point on the boundary $\Gamma = \delta\Omega$ has a neigborhood N such that, after a rotation and translation, $\Gamma \cap N$ is described by the equation

$$x_n = \phi(x_1, \ldots x_{n-1})$$

where ϕ is uniformly Lipschitz continuous, i.e. there exists a finite constant C such that

$$\forall x, y \in \mathbb{R}^{n-1} : \ |\phi(x) - \phi(y)| \leq C|x - y|$$

Moreover, $\Omega \cap N$ is on one side of $\Gamma \cap N$.

Convex domains with corners are Lipschitz, whereas domains with slits or cusps are not Lipschitz.

Let $C_0^\infty(\Omega)$ be the set of compactly supported C^∞ functions mapping Ω into $\mathbb{C}$. We denote by $L^1_{\mathrm{loc}}(\Omega)$ the measurable functions $f : \Omega \mapsto \mathbb{C}$ such that $\int_K |f(x)| dx < \infty$ for every compact subset K of Ω.

We say that a function f has a *weak derivative of order* $l \in \mathbb{N}$, if there exists a $g \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega g(x)\phi(x)dx = (-1)^l \int_\Omega f(x)\phi^{(l)}(x)dx \quad \forall \phi \in C_0^\infty(\Omega),$$

where $\phi^{(l)}(x)$ is the l-th order derivative of ϕ. If the above equation holds, we write $D_w^l f = g$.

To avoid cumbersome notation when dealing with partial differential operators, we introduce the *multi-index notation.* If $\Omega \subset \mathbb{R}^n$, we define

$$\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_n)$$
$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

and for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ we define

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n \alpha^n$$

and the partial differential operator ∂^α acting on a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ as

$$\partial^\alpha f(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

We call $|\alpha|$ the *order* of the partial differential operator.

DEFINITION 2.2 Let k be a non-negative integer. The Sobolev space $H^k(\Omega)$ is defined as the space of $L_2(\Omega)$ functions such that their (weak) derivatives $\partial^\alpha f \in L_2(\Omega)$ for every α with $|\alpha| \leq k$; and we equip this space with the inner product resp. norm

$$\langle f, g \rangle = \sum_{|\alpha| \leq k} \langle \partial^\alpha f, \partial^\alpha g \rangle$$
$$\|f\| = \sqrt{\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L_2(\Omega)}^2}$$

It can be proved that the $H^k(\Omega)$ spaces are Hilbert spaces for the above inner product.

3 Traces

Assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain.

Until now, we have defined the L_p spaces as the space of functions $f : \Omega \mapsto \mathbb{C}$ for which

$$\int_\Omega |f(x)|^p dx < \infty.$$

equipped with the norm

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

In order for this to define a norm, we need the property

$$\|f\|_{L_p(\Omega)} = 0 \Rightarrow f = 0. \tag{2}$$

But, if $\|f\|_{L_p(\Omega)} = 0$, we can only conclude that $f = 0$ *almost everywhere* in Ω, hence f is not necessarily the zero function.

This problem is solved by introducing an *equivalence relation* $\sim$ on $L_p(\Omega)$, by

$$f \sim g \Leftrightarrow f = g \text{ almost everywhere }.$$

This is indeed an equivalence relation, i.e.

$$f \sim f, \quad f \sim g \Leftrightarrow g \sim f, \quad f \sim g \wedge g \sim h \Rightarrow f \sim h$$

for any $f, g, h \in L_p(\Omega)$. Now $L_p(\Omega)$ is redefined as the space of *equivalence classes* of functions whose p'th power is integrable, subject to the above equivalence relation. Hence the elements in $L_p(\Omega)$ are classes of functions being equal almost everywhere. Performing this redefinition, the property (2) holds.

For more details on the L_p spaces, see [4].

Suppose we have a domain $\Omega \subset \mathbb{R}^n$ with a boundary $\delta\Omega$. Due to the definition of $L_p(\Omega)$ above, it does not make sense to speak of values of elements of $L_p(\Omega)$ on the boundary $\delta\Omega$, since the boundary is a set of measure zero, with the n-dimensional volume as measure. Hence one must be careful when interpreting boundary conditions for differential equations.

For $f \in C_0^\infty(\Omega)$, we define γf, the *trace* of f, as the function on $\delta\Omega$ whose values are those of f on the boundary $\delta\Omega$. For each positive integer k, One can equip the space of traces of $C_0^\infty(\Omega)$ functions with a norm:

$$\|\gamma f\| \equiv \inf\{\|g\|_{H^k(\Omega)} : g \in C_0^\infty(\Omega), \gamma f = \gamma g\}$$

One can extend the trace operator from $C_0^\infty(\Omega)$ in such a way that the trace operator γ becomes a bounded operator from $H^k(\Omega)$ onto a space $H^{k-\frac{1}{2}}(\delta\Omega)$ on which one has the norm

$$\|\gamma f\| \equiv \inf\{\|g\|_{H^k(\Omega)} : g \in H^k(\Omega), \gamma f = \gamma g\}$$

The space $H^{k-\frac{1}{2}}(\delta\Omega)$ is called *the space of traces of* H^k *functions.*

We define the spaces $H_0^k(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the $H^k(\Omega)$ norm. It turns out that these spaces have an equivalent characterization using the traces: The space $H_0^k(\Omega)$ is the space of $H^k(\Omega)$ functions satisfying

$$\gamma u = \gamma \frac{\partial u}{\partial n} = \ldots = \gamma \frac{\partial^{k-1} u}{\partial n^{k-1}} = 0.$$

We shall also need the Sobolev spaces for negative integers. These are defined as

$$H^k(\Omega) \equiv (H_0^{-k}(\Omega))'$$

For more information on Sobolev spaces and their duals, see [1], [2].

4 Variational formulation of boundary value problems

4.1 Classical and weak solutions to the Dirichlet problem

DEFINITION 4.1 Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $f \in C_b(\Omega)$, the vector space of bounded, continuous functions. Let $a_{\sigma\gamma} : \Omega \mapsto \mathbb{R}$ for $|\sigma|, |\gamma| \leq k$ be sufficiently regular functions. We say that a function

$$u \in C_b^{2k}(\Omega) \cap C_b^{2k-1}(\overline{\Omega})$$

is a classical solution of the Dirichlet problem if

$$\sum_{0 \leq |\sigma|, |\gamma| \leq k} (-1)^{|\sigma|} \partial^\sigma (a_{\sigma\gamma}(\mathbf{x}) \partial^\gamma u) = f \quad \text{ on } \Omega \tag{3}$$

$$\partial^\alpha u = 0 \quad \text{ for } |\alpha| \leq k-1 \text{ on the boundary } \delta\Omega. \tag{4}$$

This is the concept of solution which is well known from 01246 Applied Mathematics. However, we shall work with a more general concept of solution than the classical one.

DEFINITION 4.2 Define the bilinear form

$$a(u,v) \equiv \sum_{0 \leq |\sigma|, |\gamma| \leq k} (-1)^{|\sigma|} \langle (a_{\sigma\gamma}(\mathbf{x})) \partial^\gamma u, \partial^\sigma v \rangle_{L_2(\Omega)}$$

Let $f \in H_0^{-k}(\Omega)$ be given. A function $u \in H_0^k(\Omega)$ is a *weak solution* of the Dirichlet problem if

$$a(u,v) = \langle f, v \rangle \quad \forall v \in H_0^k(\Omega)$$

We note that a classical solution is also a weak solution, but the opposite is not in general the case.

4.2 General variational problems

Boundary value problems are often formulated in variational form as

$$\text{Find } u \in V \text{ such that } a(u,v) = F(v) \quad \forall v \in V, \tag{5}$$

where $F \in V'$. V is called the approximation space, or the variational space.

Let us introduce the following terminology:

DEFINITION 4.3 Let $\mathcal{H}$ be a Hilbert space, let V be a subspace of $\mathcal{H}$, and let $a(\cdot,\cdot)$ be a bilinear form on $\mathcal{H}$.

- $a(\cdot,\cdot)$ is *symmetric* if $a(u,v) = a(v,u) \quad \forall u, v \in \mathcal{H}$

- $a(\cdot,\cdot)$ is *bounded on* $\mathcal{H}$ if there exists a constant $C < \infty$ such that

$$|a(u,v)| \leq C \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \quad \forall u,v \in \mathcal{H}$$

- $a(\cdot,\cdot)$ is *coercive* on $V \subset \mathcal{H}$ if there exists a constant $C < \infty$ such that

$$\|u\|_{\mathcal{H}}^2 \leq Ca(u,u), \quad \forall u \in V$$

LEMMA 4.4 Let V be a closed subspace of a Hilbert space $\mathcal{H}$, and let $a(\cdot,\cdot)$ be a symmetric, bilinear form being continuous on $\mathcal{H}$ and coercive on V. Then $(V, a(\cdot,\cdot))$ is a Hilbert space.

PROOF Take $u \in V$. If $u = 0$, then $a(u,u) = 0$ by boundedness. Conversely, if $a(u,u) = 0$, then $u = 0$ by coercivity. Then $a(\cdot,\cdot)$ is an inner product on V. Let u_n be a Cauchy sequence in $(V, a(\cdot,\cdot))$. By coercivity, u_n is also a Cauchy sequence in the norm on $\mathcal{H}$, hence convergent to $u \in \mathcal{H}$. Since V is closed in $\|\cdot\|_{\mathcal{H}}$, $u \in V$. By boundedness,

$$a(u_n - u, u_n - u) \leq C \|u_n - u\|^2$$

hence $u_n \to u$ in $a(\cdot,\cdot)$, proving that $(V, a(\cdot,\cdot))$ is complete. □

When $a(\cdot,\cdot)$ is an inner product on V, it is frequently called the *energy inner product*. The corresponding norm $\|\cdot\|_a^2 = a(\cdot,\cdot)$ is called the *energy norm*.

On the existence and uniqueness of solutions to variational problems, we have the following theorem:

THEOREM 4.5 Assume that V is a closed subspace of a Hilbert space $\mathcal{H}$, and that $a(\cdot,\cdot)$ is symmetric, continuous on $\mathcal{H}$ and coercive on V. Assume furthermore that $F \in V'$. Then the variational problem

$$\text{Find } u \in V \text{ so that } a(u,v) = F(v), \quad \forall v \in V \tag{6}$$

has a unique solution $u \in V$.

PROOF Since $(V, a(\cdot,\cdot))$ is a Hilbert space, and $F \in V'$, there exists by the Riesz representation theorem a unique $u \in V$, so that F has the form $a(u,v) = F(v)$ for all $v \in V$. □

We note that the theorem also holds if $a(\cdot,\cdot)$ is nonsymmetric. This is the *Lax-Milgram theorem*, see [2], [1].

4.3 The Galerkin method

When solving the variational problem (6) numerically, one constructs a finite dimensional subspace $V_n \subset V$, in which one has a basis $\{x_1, \ldots, x_n\}$. Then, the restriction of (6) to V_n is considered:

$$\text{Find } u \in V_n \text{ so that } a(u,v) = F(v), \quad \forall v \in V_n. \tag{7}$$

One approximates the solution u by expanding in the basis for V_n

$$u \simeq \sum_{i=1}^{n} c_i x_i. \tag{8}$$

The solution to (7) is (8) with the coefficients $\{c_i\}_{i=1,\ldots,n}$ being the solution to the linear system of equations

$$\sum_{i=1} a(x_i, x_j) c_i = \langle f, x_j \rangle \quad j = 1, \ldots, n \tag{9}$$

This approach is known as the *Galerkin method* for solving (6). The matrix with entries $\{a(x_i, x_j)\}_{i,j=1,\ldots,n}$ is called the stiffness matrix. For more information, see [2].

4.4 Concluding remarks

The Dirichlet problem is characterized by homogeneous boundary conditions, that can be incorporated into the approximation space, such as $V = H_0^1(\Omega)$. There are various strategies for handling more general boundary conditions, these are commonly referred to as penalty, and/or Lagrange multiplier techniques. Here, the theory of traces of functions in Sobolev spaces play an important role. We shall hear more about this later in the course.

References

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[4] W. Rudin. *Real and Complex Analysis.* McGraw-Hill, 3rd edition, 1994.