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Conformal boundaries of warped products

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Abstract:

In this note we prove a result on how to determine the conformal boundary of a type of warped product of two length spaces in terms of the individual conformal boundaries. In the situation, that we treat, the warping and conformal distortion functions are functions of distance to a base point. The result is applied to produce examples of CAT(0)-spaces, where the conformal and ideal boundaries differ in interesting ways.

Key words: Length space, conformal boundary, ideal boundary, warped product, CAT(0)-space.

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1 Background

There are various notions of boundaries at infinity of a metric space. In this paper we will discuss and compare two of these.

The ideal boundary is a classical concept usually defined as a set of equivalence classes of paths converging to infinity. The precise definition will be given shortly. Conformal boundaries are defined via conformal distortions of the metric space, and thus depends on the choice of distortion function. In certain classes of spaces, e.g. Gromov hyperbolic spaces, there is a range of canonical choices of distortions, which produce homeomorphic boundaries, c.f. [3]. These turn out also to be homeomorphic to the ideal boundary with a canonically defined topology, see [5]. In the class of CAT(0)-spaces however, there is no canonical choice of conformal distortion, and the ideal and conformal boundary will typically be different.

The purpose of this note is to prove a general result on how to determine the conformal boundary of a warped product, and then apply this to produce CAT(0)-spaces, where the two types of boundaries differ in interesting ways.

1.1 Length Spaces and conformal distortions

We refer to [4] and [5] for more details on the concepts we briefly introduce below.

A length space is a metric space (X, d), where the distance between two points is given as the infimum of lengths of rectifiable curves connecting points:

$$d(p,q) = \inf_{\gamma \in \Gamma(p,q)} \mathfrak{L}(\gamma),$$

where $\Gamma(p,q)$ denotes the set of rectifiable paths having p and q as endpoints. Given a continuous function $\rho : X \to (0,\infty)$, we can define the conformally distorted length metric:

$$\sigma_\rho(p,q) := \inf_{\gamma \in \Gamma(p,q)} \int_\gamma \rho \, ds,$$

see e.g. [3] for more details. One then defines the conformal boundary of X with respect to the distortion ρ as

$$\partial_\rho X := \overline{X}_\rho \setminus X,$$

where $\overline{X}_\rho$ denotes the metric completion of X with respect to the length metric σ_ρ.

In this note we focus attention on the case, where the distortion is a function of the distance to a base point $o \in X$:

$$\rho(p) = g\big(d(o,p)\big),$$

and $g : [0,\infty) \to (0,\infty)$ is a continuous function. If (X, d) has an infinitely long geodesic (see below) and $g \in L^1([0,\infty))$, then we always have $\partial_g X \neq \emptyset$. Here $\partial_g X$ is used as a short notation for the boundary with respect to a conformal distortion of the type described above, even though the boundary could depend also on the choice of base point. See [5] for more details on conformal boundaries of this type.

1.2 Ideal Boundaries

A K-rough geodesic $\gamma : I \to X$ is a path, such that for all $s < t \in I$, we have

$$d(\gamma(s), \gamma(t)) \geq \mathfrak{L}\big(\gamma([s,t])\big) - K, \tag{1.1}$$

for some $K \geq 0$. Here we do do not distinguish notationally between paths and their images. If (1.1) holds with $K = 0$, γ is called a geodesic. A length space is called *geodesic* if any two points can be joined by a geodesic. Any *proper* (i.e. where closed balls are compact) length space is geodesic, c.f. [4].

A K-rough ray γ in X is an infinitely long K-rough geodesic with one endpoint, i.e. when parametrized by arclength it can be defined on $[0, \infty)$. A *ray* is then a 0-rough ray, that is, an infinitely long geodesic with one endpoint.

Two paths γ_1, γ_2 are said to be *asymptotic* if the Hausdorff distance between their images is finite $d_{\mathcal{H}}(\gamma_1, \gamma_2) < \infty$. This defines an equivalence relation $\sim$ on the set of paths. One easily checks that for two rough rays γ_1, γ_2 parametrized by arclength, we have $\gamma_1 \sim \gamma_2$ iff

$$\sup_{t>0} d(\gamma_1(t), \gamma_2(t)) < \infty.$$

We define the *ideal boundary* $\partial_I X$ to be the set of equivalence classes of (geodesic) rays. It will prove convenient to work with the slightly more general notion of a *rough ideal boundary*, $\underset{RI}{\partial} X$, defined to be the set of equivalence classes of rough rays. Clearly, we have $\partial_I X \subseteq \underset{RI}{\partial} X$. If X is Gromov hyperbolic then we have $\partial_I X = \underset{RI}{\partial} X$, however it is possible to have a strict inclusion:

Example 1.2. In general for an Hadamard manifold M^n, i.e. a complete, simply connected Riemannian manifold of nonpositive curvature, it is well known that $\partial_I M$ is homeomorphic to $\mathbb{S}^{n-1}$, when equipped with a canonically defined topology, c.f. [4]. Even in the simplest case $M = \mathbb{R}^2$, we have $\partial_I \mathbb{R}^2 \subset \underset{RI}{\partial} \mathbb{R}^2$. Given a ray γ (a half-line going to infinity) it is possible to construct a rough ray, which "zig-zags" around γ, in a way such that the two paths are not asymptotic according to the definition above.

CAT(0)-spaces

A geodesic triangle T in a metric space X is the union on of three geodesics, $\gamma_1 \in \Gamma(a,b)\,, \gamma_2 \in \Gamma(b,c)\,, \gamma_3 \in \Gamma(c,a)$. A geodesic triangle T is said to satisfy the CAT(0)-inequality if it is at least as *slim* as a comparison triangle with the same side lengths $\tilde{T} \subset \mathbb{R}^2$. See e.g. [4] for the precise definition. A CAT(0)-space is then a geodesic space in which all geodesic triangles satisfy the CAT(0)-condition.

1.3 Warped Products

We will define the warped product of two length spaces as in [1] and [2], see also [6]:

Let B and F be length spaces, and let $f : B \to (0, \infty)$ be a continuous function. Then define the following length structure on $B \times F$:

$$\mathfrak{L}_f(\gamma) = \int_\gamma \sqrt{v_B^2(t) + f^2(\gamma_B(t))v_F(t)^2}dt, \tag{1.3}$$

where $\gamma = (\gamma_B, \gamma_F)$ and v_B, v_F are the speeds of γ_B, γ_F respectively, which are defined almost everywhere. We will write

$$d_f(p,q) = \inf_{\gamma \in \Gamma(p,q)} \mathfrak{L}_f(\gamma)$$

for the length metric induced from the length structure $\mathfrak{L}_f$. The *warped product* $B \times_f F$ is then $B \times F$ equipped with the metric d_f.

We use standard terminology as for Riemannian warped products, c.f. [7]. B is called the base and F is called the fiber. Subsets of the form $B \times \{q\}$, $q \in F$, are called *leaves*, while the subsets $\{p\} \times F$, $p \in B$ are called *fibers*. A curve of the form $t \mapsto (\alpha(t), q)$ is called *horizontal*, while $t \mapsto (p, \beta(t))$ is called a *vertical* curve.

From the definition of the warped product metric, it is clear that the projection onto the base coordinate $\pi_B : B \times_f F \to B$ is 1-Lipschitz,

$$d_B(\pi_B(p), \pi_B(q)) \leq d(p,q).$$

It is also evident that the leaves $B \times \{q\}$, $q \in F$ are all isometric to B. Furthermore, if $f : B \to (0, \infty)$ has a global minimum at $p \in B$, one easily checks that the fiber $\{p\} \times F$ is isometric to F with metric scaled by $f(p)$.

Lemma 1.4. *The identity map $B \times_f F \to B \times F$ is a homeomorphism when $B \times F$ is equipped with the product topology. If B and F are complete, then so is $B \times_f F$. Thus if B and F are proper, so is $B \times_f F$.*

Proof. Given $p, q \in B \times F$, we have

$$d_f(p,q) \leq f(\pi_B(p))d_F(\pi_F(p), \pi_F(q)) + d_B(\pi_B(p), \pi_B(q)), \tag{1.5}$$

by first moving in the fiber through p and then in the leaf through q. This shows that the identity is continuous from $B \times F$ to $B \times_f F$.

Now choose a ball $\mathcal{B} \subset B$ of radius r around $\pi_B(p)$ s.t. $f(x) \geq \frac{1}{2} f(\pi_B(p))$ for $x \in \mathcal{B}$. Given $\epsilon > 0$, choose $0 < \delta < \min\{\epsilon, f(\pi_B(p))\frac{\epsilon}{2}, r\}$. Then if $d_f(p,q) < \delta < r$ we can ensure that for an almost minimizing curve $\gamma = (\gamma_B, \gamma_F)$ connecting p and q, we have $\gamma_B \subset \mathcal{B}$, since $\mathfrak{L}_B(\gamma_B) \leq \mathfrak{L}_f(\gamma)$. Thus

$$\mathfrak{L}_f(\gamma) \geq (\min_\gamma f)\mathfrak{L}_F(\gamma_F) \geq \frac{1}{2} f(\pi_B(p))\mathfrak{L}_F(\gamma_F) \geq \frac{1}{2} f(\pi_B(p))d_F(\pi_F(p), \pi_F(q))$$

and we conclude that $d_f(p,q) \geq \frac{1}{2} f(\pi_B(p)) d_F(\pi_F(p), \pi_F(q))$,
hence $d_F(\pi_F(p), \pi_F(q)) < \epsilon$ and $d_B(\pi_B(p), \pi_B(q)) < d_f(p,q) < \epsilon$, which shows that the identity $B \times_f F \to B \times F$ is continuous.

Since the projection π_B is 1-Lipschitz, we get a Cauchy sequence $\{\pi_B(p_n)\} \subset B$, when $\{p_n\}$ is Cauchy in $B \times_f F$. An easy variation of the argument given above shows that also $\{\pi_F(p_n)\} \subset F$ is Cauchy, and thus the d_f-Cauchy sequence $\{p_n\}$ is convergent in the product topology, hence also in $B \times_f F$.

Finally, if B and F are both proper, hence complete, then $B \times_f F$ is complete. $B \times_f X$ is also locally compact since the product topology is, hence $B \times_f F$ is proper by the Hopf-Rinow Theorem, c.f. [4].

□

2 Boundaries of warped products

We will from now on consider warping functions, which are functions of the distance to a point. We assume that a base point $o_B \in B$ is choosen, and for $p \in B$ introduce the notation $|p| := d_B(o_B, p)$. For $f : [0, \infty) \to (0, \infty)$ a continuous function, we consider warping functions of the form

$$f \circ |\cdot| : B \to (0, \infty)$$

and also use the shorthand notation $B \times_f F$ for warped products with this type of composite warping functions. The meaning should be clear from context.

Lemma 2.1. *Let $f : [0, \infty) \to [1, \infty)$ be a strictly increasing continuous function with $f(0) = 1$, satisfying:*

$$\frac{1}{f} \in L^1([0,\infty)) \text{ and } \sup_{t>0} f(t) \int_t^\infty \frac{1}{f(s)} ds < \infty \tag{2.2}$$

Then the rough ideal boundary of $X = B \times_f F$ can be identified as a set with:

$$\underset{RI}{\partial} X = (\underset{RI}{\partial} B \times F) \cup \underset{RI}{\partial} F \tag{2.3}$$

Proof. First of all since all leaves $B \times \{q\}$ are isometric to B, and likewise because the fiber $\{o_1\} \times F$ is isometric to F it is clear that $\underset{RI}{\partial} B \times F \cup \underset{RI}{\partial} F \subseteq \underset{RI}{\partial} X$. So we need to show that any rough ray $\gamma = (\gamma_B, \gamma_F)$ in X is asymptotic to a either a horizontal or a vertical rough ray.

Assume that $\gamma = (\gamma_B, \gamma_F)$ is a rough ray parametrized by arclength, so $v_F \leq \frac{1}{f(|\gamma_B|)}$ a.e.. Then there is a $K \geq 0$ s.t. for all $t \geq 0$ we have:

$$\begin{aligned} t - K &\leq d\big((o_B, \gamma_F(0)), \gamma(t)\big) \\ &\leq d\big((o_B, \gamma_F(0), (o_B, \gamma_F(t)\big) + d\big(o_B, \gamma_F(t)), \gamma(t)\big) \\ &\leq \mathfrak{L}_F(\gamma_{F|[0,t]}) + |\gamma_B(t)| \\ &\leq \int_0^t \frac{1}{f(|\gamma_B(s)|)} ds + |\gamma_B(t)| \end{aligned} \tag{2.4}$$

by first moving in the fiber $\{o_B\} \times F$, then in the base and applying the triangle inequality.

Assume now that $|\gamma_B(t)|$ is bounded, hence γ_F and thus F must be unbounded. It immediately follows from (2.4) that γ_F must be a rough ray, so that γ is asymptotic to the vertical ray $t \mapsto (o_B, \gamma_F(t))$. We will show a little more, namely that $\lim_{t\to\infty} \gamma_B(t) = o_B$.

We have:

$$\int_0^t \frac{1}{f(|\gamma_B(s)|)} ds \geq \mathfrak{L}_F(\gamma_{F|[0,t]}) \geq t - K_0$$

for some $K_0 > 0$. If $\gamma_B(t) \not\to o_B$ for $t \to \infty$ then there is some $\epsilon > 0$ s.t. there exists arbitrarily large t with $|\gamma_B(t)| > \epsilon$. But then $I := \{t \geq 0 \,|\, |\gamma_B(t)| > \frac{\epsilon}{2}\}$ has infinite measure, since the speed of γ_B is bounded by 1. So

$$t - \int_0^t \frac{1}{f(|\gamma_B(s)|)} ds = \int_0^t (1 - \frac{1}{f(|\gamma_B(s)|)}) ds \geq \mu(I \cap [0,t])(1 - \frac{1}{f(\frac{\epsilon}{2})}) \to \infty \quad (2.5)$$

Hence we conclude that $\lim_{t\to\infty} \gamma_B(t) = o_B$.

Now assume that $|\gamma_B(t)|$ is unbounded, so the set $J := \{t > 0 \,|\, |\gamma_B(t)| > 2c_1\}$ has infinite measure for any $c_1 > 0$ and as in (2.5) we conclude that $\frac{1}{f(c_1)} t - \mathfrak{L}_F(\gamma_{F|[0,t]}) \to \infty$. Since by (2.4) $|\gamma_B(t)| \geq t - K - \mathfrak{L}_F(\gamma_{F|[0,t]})$, we get $|\gamma_B(t)| - c_2 t \to \infty$ for any $c_2 \in [0,1)$. However by a change of parameters this implies that $\mathfrak{L}_F(\gamma_F) \leq \int_0^\infty \frac{1}{f(|\gamma_B(t)|)} dt$ is finite, and thus that $|\gamma_B(t)| > t - K_2$ for some $K_2 > 0$, i.e. γ_B is a rough ray in B.

Since the length $\mathfrak{L}_F(\gamma_F) \leq \int_0^\infty \frac{1}{f(|\gamma_B(t)|)} dt$ is finite, $\gamma_F(t)$ is convergent to some point $p \in F$. Thus γ is asymptotic to the horizontal rough ray $t \mapsto (\gamma_B(t), p)$, since by moving vertically:

$$\sup_{t>0} d\big(\gamma(t), (\gamma_B(t), p)\big) \leq \sup_{t>0} f(|\gamma_B(t)|) \int_t^\infty \frac{1}{f(|\gamma_B(s)|)} ds < \infty, \quad (2.6)$$

where we use the condition on f, (2.2), and the fact that $t - K_3 \leq |\gamma_B(t)| \leq t + K_3$, for some $K_3 > 0$ and all $t \geq 0$. □

Let's return to the conformal boundary $\partial_g X$, defined with respect to a distortion function $g : [0,\infty) \to (0,\infty)$ and a choice of base point $o \in X$. There is a map

$$J_X : \underset{RI}{\partial} X \to \partial_g X,$$

by choosing sequences going to infinity along each rough ray, e.g. $x_n = \gamma(n)\,, n \in \mathbb{N}$. We will call g k-quasidecreasing if

$$g(t) \leq k g(s)$$

for all $0 \leq s \leq t$ and some $k > 0$. We will need the following result whose proof will only be sketched.

Lemma 2.7. *If X is a proper unbounded length space and $g \in L^1([0,\infty))$ is quasidecreasing, then $J_X : \underset{RI}{\partial} X \to \partial_g X$ is well defined and surjective.*

Proof. That J_X is well defined, i.e. that it does not depend on the choice of representative of an equivalence class of rough rays, neither on the choice of sequence $\{\gamma(t_n)\}$ for $\mathbb{N} \ni t_n \to \infty$, follows from the fact that we must have $\gamma(t) \to 0$ for $t \to 0$. So for any two sequences $x = \{x_n\}, y = \{y_n\}$ choosen along equivalent rays, we have $\sigma_g(x_n, x_m) \to 0, \sigma_g(y_n, y_m) \to 0$ and $\sigma(x_n, y_m) \to 0$ for $n, m \to \infty$, where σ_g is the conformally distorted metric. Thus x and y will define equivalent σ_g-Cauchy sequences converging to some point in $\partial_g X$.

That J_X is surjective, in fact already as a map from $\partial_I X \subseteq \underset{RI}{\partial} X$, is seen using the argument in the proof of Theorem 2.2 (b) of [5]: In short, given a σ_g-Cauchy sequence $\{x_n\} \subset X$ converging to a point in $\partial_g X$, we must have $d(o, x_n) \to \infty$. In the proof of Theorem 2.2 (b) a geodesic ray γ with $\gamma(0) = o$ is constructed such that $\delta(g(k), y_n) \leq 1$, where $y = \{y_n\}$ is a subsequence of $\{x_n\}$. Clearly $J_X(\gamma)$ is then equivalent to y, since $g(t) \to 0$ for $t \to \infty$. Furthermore a subsequence of a σ_g-Cauchy sequence is equivalent to the original sequence. □

Now we fix also a base point $o_F \in F$ and thus a base point $o_X := (o_B, o_F) \in X = B \times_f F$. We use the notation $|p|_B := d_B(o_B, p), |q|_F := d_F(o_F, q), |r|_X := d_X(o_X, r)$ for points $p \in B, q \in F, r \in X$. σ_B, σ_F and σ_X will denote the conformally distorted metrics of B, F and X with respect to the choosen base points and a fixed distortion function g.

Theorem 2.8. *Let B and F be pointed, proper length spaces, with B unbounded. Let furthermore $f : [0,\infty) \to [1,\infty)$ be a warping function satisfying the requirements of Lemma 2.1 and $g : [0,\infty) \to (0,\infty)$ be a distortion function satisfying the requirements of Lemma 2.7 and furthermore:*

$$g(a+b) \geq k_1 g(a) g(b) \text{ and } f(t)g(t) \geq k_2, \tag{2.9}$$

for some positive constants k_1, k_2 and all $a, b, t \geq 0$. Then the conformal boundary of $X = B \times_f F$ is homeomorphic to the gluing of $\partial_g B \times \overline{F}$ onto $\partial_g F$ along $\partial_g B \times \partial_g F$, using the projection map. If $\partial_g F = \emptyset$, the gluing is simply $\partial_g B \times F$.

Proof. For each leaf the embedding $B \hookrightarrow B \times \{q\} \subseteq X$ is isometric, and $|(p,q)|_X \geq |p|_B$, hence $g(|(p,q)|_X) \leq k_0 g(|p|_B)$. Thus clearly any σ_B-Cauchy sequence $\{p_n\} \subset B$ is also a σ_X-Cauchy sequence in the leaf $B \times \{q\}$. Hence we have a map

$$\psi_B : \partial_g B \times F \to \partial_g X$$

Likewise the embedding of the standard fiber $F \hookrightarrow \{o_B\} \times F \subset X$ is isometric, with $|(o_B, q)|_X = |q|_F$, so we also have a map

$$\psi_F : \partial_g F \to \partial_g X$$

Now define a map $\psi : \partial_g B \times \overline{F} \to \partial_g X$ by

$$\psi((x,y)) = \begin{cases} \psi_B((x,y)) & (x,y) \in \partial_g B \times F \\ \psi_F(y) & (x,y) \in \partial_g B \times \partial_g F \end{cases}$$

Another way of getting the gluing of $\partial_g B \times \overline{F}$ and $\partial_g F$ is by considering the quotient $\partial_g F \times \overline{F} \,/\sim$, where $\partial_g B \times \{y\}$ is collapsed to a point for $y \in \partial_g F$. Since ψ is constant on equivalence classes we may consider it as a map $\psi : \partial_g B \times \overline{F} \,/\sim \to \partial_g X$.

Equip $\partial_g B \times \overline{F}$ with the product topology induced by the conformally distorted metrics σ_B, σ_F and give $\partial_g B \times \overline{F} \,/\sim$ the quotient topology. We will show that ψ is a homeomorphism.

By Lemma 2.7 the conformal distortion map $J_X : \underset{RI}{\partial} X \to \partial_g X$ is surjective and using the description in Lemma 2.1, we see that ψ is surjective: For $x \in \partial_g X$ choose a rough ray γ s.t. $J_X(\gamma) = x$. Then γ is either asymptotic to a horizontal ray (γ_B, q) and x is in the image $\psi_B(\partial_g B \times F)$ or γ is asymptotic to a vertical ray (o_B, γ_F) and x is in the image $\psi_F(\partial_g F)$.

Let $\gamma = (\alpha, \beta) : [t_0, t_1] \to X$ be a path in X. By first moving in the fiber $\{o_B\} \times F$ and then horizontally in a leaf, we have a triangle with side lengths $|\beta(s)|_F, |\alpha(s)|_B, |(\alpha(s), \beta(s))|_X$, so by the triangle inequatlity:

$$|(\alpha(s), \beta(s))|_X \leq |\alpha(s)|_B + |\beta(s)|_F$$

so using that g is quasi-decreasing and condition (2.9) we have

$$g(|(\alpha(s), \beta(s))|_X) \geq k_0 g(|\alpha(s)|_B + |\beta(s)|_F) \geq k_0 k_1\, g(|\beta(s)|_F)\, g(|\alpha(s)|_B) \tag{2.10}$$

Then

$$\int_\gamma g(|\gamma(s)|_X) \sqrt{v_\alpha^2 + f^2(|\alpha(s)|_B) v_\beta^2}\, ds \geq \int_\gamma k_0 k_1 g(|\beta(s)|_F)\, g(|\alpha(s)|_B)\, f(|\alpha(s)|_B)\, v_\beta\, ds \tag{2.11}$$

By assumption there is a positive constant k_2 such that $g(t)f(t) \geq k_2$ for all $t > 0$. Hence

$$\mathfrak{L}_{\sigma_X}(\gamma) \geq k_0 k_1 k_2 \mathfrak{L}_{\sigma_F}(\beta), \tag{2.12}$$

so we conclude that

$$\sigma_X\big((p_1, q_1), (p_2, q_2)\big) \geq K_0 \sigma_F(q_1, q_2), \tag{2.13}$$

$K_0 = k_0 k_1 k_2$. A similar crude analysis shows that

$$\mathfrak{L}_{\sigma_X}(\gamma) \geq k_0 k_1 \min_{t_0 \leq t \leq t_1} (g(|\beta(t)|_F)) \mathfrak{L}_{\sigma_B}(\alpha) \tag{2.14}$$

and we conclude from (2.12)

$$\epsilon := \sigma_X\big((p_1,q_1),(p_2,q_2)\big) \\ \geq K_1 \min\{g(t)\,|\, t\in [|q_1|_F - \frac{\epsilon}{K_0}, |q_2|_F + \frac{\epsilon}{K_0}]\}\sigma_B(p_1,p_2), \quad (2.15)$$

assuming that $|q_1|_F \leq |q_2|_F$. By continuity (2.13) extends to $\partial_g X$, i.e. to Cauchy sequences of the form (x,y), for $x \in \overline{B}\,, y\in \overline{F}$. The estimate (2.15) extends to Cauchy sequences of the form $(x,q)\,, x\in \partial_g B\,, q\in F$, i.e. to the image $\psi(\partial_g \times F)$.

We then easily deduce that ψ is injective on $\partial_g B \times \overline{F} \,/ \sim$. Because if $\psi\big((x_1,y_1)\big) = \psi\big((x_2,y_2)\big)$ we must have $y_1 = y_2 := y \in \overline{F}$ by (2.13), and for $y \in F$ we see from (2.14) that $\psi\big((x_1,y)\big) = \psi\big((x_2,y)\big)$ iff $x_1 = x_2$.

Using (2.13) and (2.15) we see that ψ^{-1} is continuous on the horizontal sequences $\psi(\partial_g \times F)$. Continuity of ψ^{-1}, in the quotient topology, at a vertical sequence $\psi_F(\partial_g F)$ follows directly from (2.13).

A similar analysis shows that ψ is continuous. However this also follows from the fact that the involved spaces are compact and Hausdorff. □

3 Exotic boundaries of CAT(0)-spaces

In this section we apply the previous results to give examples of boundaries of CAT(0)-spaces. That the ideal and conformal boundaries typically differ in this category is illustrated by the simplest example: $\underset{RI}{\partial}\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^{n-1}$, but $\partial_g \mathbb{R}^n$ is always a single point for $n \geq 2$, and any quasidecreasing distortion function $g \in L^1([0,\infty))$.

The following theorem is the main result in [1]:

Theorem 3.1 (Alexander & Bishop). *If B and F are complete CAT(0) spaces and $f : B \to (0,\infty)$ is convex, then $B \times_f F$ is CAT(0).*

If we furthermore require that the function $f : [0,\infty) \to [1,\infty)$ in the construction of Theorem 2.8 is convex, then since distance functions are convex in CAT(0)-spaces, it follows easily that the composition $f \circ |\cdot||_B$ is convex, and thus by the Theorem of Alexander and Bishop that $B \times_f F$ is CAT(0), when B and F are CAT(0)-spaces.

3.1 Examples

We will give a few examples of Riemannian CAT(0) spaces, i.e. Hadamard manifolds, with interesting boundary, calculated using Theorem 2.8. When B is a Hadamard manifold, we get a smooth convex warping function satisfying the requirements of Lemma 2.1 by choosing $f(t) = \cosh(t)$. If a Hadamard manifold X^n has curvature bounded away from zero, $\kappa \leq \kappa_0 < 0$, and is thus Gromov hyperbolic, then the conformal boundary $\partial_g X$ is homeomorphic to the ideal boundary $\mathbb{S}^{n-1}$, when we choose the distortion such that $g(t) \geq K \exp(-\lambda t)$ for some sufficiently small λ. See [5] for details. Clearly when choosing such a distortion function, we can fulfill the requirements of Theorem 2.8. Another possibility is to choose a so-called Floyd function with only polynomial decay, see [5].

With these choices of f and g we have:

- If $B = \mathbb{R}$ and $F = \mathbb{R}^{n+1}$, then X is CAT(0) and $\partial_g X$ is homeomorphic to $\mathbb{S}^n \sqcup_p \mathbb{S}^n$, a 1-point union of two n-spheres. This follows since $\partial_g \mathbb{R}^{n+1}$ is a single point $\{p\}$ and $\overline{\mathbb{R}^{n+1}}$ is $\mathbb{S}^n$, while $\partial_g \mathbb{R}$ consists of two points.

- If $B = \mathbb{R}^2$ and $F = \mathbb{H}^n$, hyperbolic n-space, then X is CAT(0) and $\partial_g X = \mathbb{B}^n$, the closed unit ball.

Now one sees, that iterating this procedure with some of the ingredients described above, it is possible to construct disturbingly complicated conformal boundaries, even when $\partial_I X$ is $\mathbb{S}^n$.

Let's also give some non-Riemannian examples of CAT(0)-spaces. Here we can take e.g. $f = \cosh(t)$ or $f(t) = \exp(t)$. Again g is some appropriate distortion function and $X = B \times_f F$.

- Let $B = \mathbb{R}$ and let F be a tree T. Then X is CAT(0), with $\partial_g X$ the doubling of T along $\partial_g T$. In this example the ideal and conformal boundaries are the same.

- Let $B = \mathbb{R}^2$ and let F be a tree T. Then X is CAT(0) and $\partial_g X$ is simpy the closure of the tree $\overline{T} = T \cup \partial_g T$.

- Let B be a tree with finitely many branchings, so that $\partial_g B$ is finite. Then $\partial_g X$ is finitely many copies of $\overline{F}$ glued along $\partial_g F$. According to whether F is CAT(0), then so is X.

Final Question: It is possible to construct even more general boundaries by making identifications on the ones constructed above, using shortcuts in the space "inside". Given a compact n-manifold, M, is it then always possible to construct a Riemannian metric on $\mathbb{R}^{n+2}$ such that the conformal boundary as described in this note is homeomorphic to M? If not, what is the smallest possible coodimension relative to the space lying inside?

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