Characterizing the Sphere by Mean Distance

Simon L. Kokkendorff*
Department of Mathematics
National University of Ireland, Maynooth
e-mail: Simon.Kokkendorff@maths.may.ie

Abstract

We discuss the metric invariants *extent, rendezvous number and mean distance* of a compact metric space X. The main result of this paper is Theorem 4 stating that the round sphere $\mathbb{S}_1^n$ of constant curvature 1 has maximal mean distance among Riemannian n-manifolds with Ricci curvature $Ric \geq n-1$, and that such a manifold is diffeomorphic to a sphere if the mean distance is close to $\frac{\pi}{2}$.

Keywords: Metric invariants, extent, rendezvous number, mean distance, potential, curvature.

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Introduction

Throughout this paper X will denote a compact metric space with metric d. We always assume that X contains at least 2 points. In the first section we describe a measure theoretic setup, which allows us to introduce and treat metric invariants such as the extent, rendezvous number and mean distance of a compact metric space in a unified way.

In the second section we apply some of the machinery to the case where X is a homogeneous space. Finally in the third section we specialize to compact

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Riemannian manifolds and show that the round sphere of curvature 1 has maximal mean distance among Riemannian manifolds with $Ric \geq n-1$. Theorem 4 is the main result of this paper.

The Associated Quadratic Form

For background on measures and integration we refer to [10]. The set of distributions, signed measures or Radon charges (as they are called in [10]) is the $\mathbb{R}$-span of all Radon measures on X. We shall use the notation $\mathfrak{M}(X)$ to denote the set of Radon charges. Any Radon charge can be decomposed as $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive Radon measures. We can choose the decomposition such that μ^+ and μ^- are concentrated on disjoint subsets:

$$Y^+ \cap Y^- = \emptyset,\ \mu^+(X \setminus Y^+) = 0 \text{ and } \mu^-(X \setminus Y^-) = 0, \tag{1}$$

c.f. [10] 6.5.7. The norm or total absolute mass of μ is $\|\mu\| = |\mu|(X)$, where $|\mu|$ denotes the measure $\mu^+ + \mu^-$. With this norm $\mathfrak{M}(X)$ is a Banach space isometrically isomorphic to $C(X)^*$, the dual of the space of continuous functions with the uniform norm, cf. Proposition 6.5.9 in [10].

$\mathfrak{M}(X)$ is equipped with the w^*-topology (the weak topology):

$$\mu_n \to \mu \text{ iff } \int_X f\,\mu_n \to \int_X f\,\mu \text{ for all } f \in C(X)$$

The subset of $\mathfrak{M}(X)$ consisting of positive measures, i.e. $\mu = |\mu|$, will be denoted $\mathfrak{M}(X)^+$, while the subset of *probability measures* is defined as

$$\mathcal{P}(X) := \{\mu \in \mathfrak{M}(X)^+ |\, \mu(X) = 1\}$$

The *support* of a distribution $\mu \in \mathfrak{M}(X)$ is the *minimal closed* subset $Y \subseteq X$ such that $|\mu|(X \setminus Y) = 0$.

We will denote the Dirac point measure with support $p \in X$ by $\delta_p \in \mathcal{P}(X)$.

Two important facts: $\mathcal{P}(X)$ is w^*-compact, c.f. 2.5.2, 2.5.7 in [10], and the finitely supported or *atomic* distributions form a w^*-dense subset; this second statement follows from the Krein-Millman Theorem, [10] 2.5.8.

Given two distributions $\mu, \nu \in \mathfrak{M}(X)$, $\mu \otimes \nu$ denotes the product distribution on $X \times X$. We have a symmetric, bilinear form on $\mathfrak{M}(X)$ associated to the metric d:

Definition 1. Define a symmetric bilinear form $I : \mathfrak{M}(X) \times \mathfrak{M}(X) \to \mathbb{R}$ as

$$I(\mu,\nu) := \int\limits_{X\times X} d(\cdot,\cdot) \quad \mu \otimes \nu := \int\limits_{X\times X} d(\cdot,\cdot) \quad \mu^+ \otimes \nu^+ + \int\limits_{X\times X} d(\cdot,\cdot) \quad \mu^- \otimes \nu^- \\ - \int\limits_{X\times X} d(\cdot,\cdot) \quad \mu^+ \otimes \nu^- - \int\limits_{X\times X} d(\cdot,\cdot) \quad \mu^- \otimes \nu^+ \quad (2)$$

The corresponding quadratic form is denoted shorthand by $I(\mu) := I(\mu,\mu)$.

There are interesting connections between the geometry of X and algebraic features of the associated quadratic form, see e.g. [9]. Using instead a kernel of the form $f(d(x,y))$ in the definition above, where $f : \mathbb{R} \to \mathbb{R}$ is some appropriate modification function, would give a range of other interesting quadratic forms. Here we shall stick to the distance kernel in itself and mainly consider the restriction of the form to the set of probability measures $\mathcal{P}(X)$.

Definition 2. For a distribution $\mu \in \mathfrak{M}(X)$ we define the associated potential of μ as

$$\mathbf{p}_\mu(p) := \int_X d(p,q)\mu(q) = \int_X d(p,q)\mu^+(q) - \int_X d(p,q)\mu^-(q) \quad (3)$$

For a probability measure $\mu \in \mathcal{P}(X)$ we call $\mathbf{p}_\mu$ the mean distance function and use the notation $\mathbf{p}_\mu = \mathrm{md}_\mu$.

Thus by Fubini's Theorem, c.f. [10], and linear algebra we have for $\mu, \nu \in \mathfrak{M}(X)$

$$I(\mu,\nu) = \int_X \mathbf{p}_\mu \nu = \int_X \mathbf{p}_\nu \mu \quad (4)$$

Note that $\mathrm{md}_{\delta_p}(q) = d(p,q)$.

The following basic facts are easy to prove (this is carried out in [9]).

Lemma 1. *The following maps are w^*-continuous:*

$$\Psi : \mathfrak{M}(X) \to C(X),\ \mu \mapsto \mathbf{p}_\mu$$

$$\mathcal{E} : \mathfrak{M}(X) \to \mathbb{R},\ \mu \mapsto I(\mu)$$

Metric Invariants

We shall associate two numbers to a general compact metric space X. The first is the *extent*:

Definition 3. The *extent* of X is defined as

$$\operatorname{xt}(X) := \sup_{\mu \in \mathcal{P}(X)} I(\mu) \tag{5}$$

The extent is thus the *maximal mean distance of* X with respect to probability measures on X. The concept was introduced and studied by Grove and Markvorsen in [6]. The equivalence of the definition used there and the one given above follows from the w^*-density of finitely supported measures, c.f. [9].

By w^*-continuity of the map $\mu \mapsto I(\mu)$ and w^*-compactness of $\mathcal{P}(X)$ we immediately have:

Theorem 1. *There is a probability measure $\mu \in \mathcal{P}(X)$ s.t. $I(\mu) = \operatorname{xt}(X)$*

Thus the sup in the definition can be replaced by a max. It is not difficult to see, c.f. [6] or [9], that in general

$$\frac{1}{2} \operatorname{diam} X \leq \operatorname{xt}(X) < \operatorname{diam}(X) \tag{6}$$

The connection between extent and curvature is investigated in [6]. From the work there it follows:

Theorem 2. *Let X be an n-dimensional, $n \geq 2$, Alexandrov space with curv$(X) \geq 1$. Then*

$$\operatorname{xt}(X) \leq \operatorname{xt}(\mathbb{S}^n_1) = \frac{\pi}{2}, \tag{7}$$

where $\mathbb{S}^n_1$ is the n-dimensional sphere of constant curvature 1. Moreover, equality holds if and only if X is isometric to a spherical suspension $\Sigma_1 E$, where E is an $(n-1)$-dimensional Alexandrov space with $\operatorname{diam}(E) \leq \pi$ *and curv$(E) \geq 1$ if $n > 2$.*

The next magic number is the *rendezvous number*, which was introduced for a connected and compact metric space by Gross in [7] and later generalized by Thomassen to any compact metric space in [13]. The concept also appears in the paper [1].

The theorem below is simply an extension of the results in [13], where formulations are via finite subsets, to the setting of probability measures. The result could be formulated more generally, it is not necessary here that d is a metric, in fact a continuous symmetric function on $X \times X$ would do.

Theorem 3. *Let X be a compact metric space, then*

$$\min_{\mu \in \mathcal{P}(X)} \max_{p \in X} \mathrm{md}_\mu(p) = \max_{\mu \in \mathcal{P}(X)} \min_{p \in X} \mathrm{md}_\mu(p) \tag{8}$$

Consequently there is a unique number $\mathrm{rv}(X) > 0$ *such that for any* $\nu \in \mathcal{P}(X)$*, there exists points* $p, q \in X$ *satisfying* $\mathrm{md}_\nu(p) \geq \mathrm{rv}(X)$ *and* $\mathrm{md}_\nu(q) \leq \mathrm{rv}(X)$.

Proof. First note that by weak compactness and continuity it makes sense to use $\min, \max$ rather than $\inf, \sup$. Equation (8) follows from Theorem 2.3 in [13] by w^*-density of finitely supported measures in $\mathcal{P}(X)$ and w^*-continuity of the mappings involved.

The existence and uniqueness of the number $\mathrm{rv}(X)$ follows from (8) since we would have

$$\mathrm{rv}(X) \geq \max_{\mu \in \mathcal{P}(X)} \min_{p \in X} \mathrm{md}_\mu(p)) \text{ and } \mathrm{rv}(X) \leq \min_{\mu \in \mathcal{P}(X)} \max_{p \in X} \mathrm{md}_\mu(p)$$

$\square$

Definition 4. The rendezvous number of a compact metric space is the unique number $\mathrm{rv}(X)$ with properties listed in Theorem 3.

Note that if X is *connected*, then for every $\mu \in \mathcal{P}(X)$ there will be a $p \in X$ such that $\mathrm{md}_\mu(p) = \mathrm{rv}(X)$. A distribution $\mu \in \mathcal{P}(X)$ such that

$$\max_{p \in X} \mathrm{md}_\mu(p) = \min_{\mu \in \mathcal{P}(X)} \max_{p \in X} \mathrm{md}_\mu(p)$$

will be called a min max-distribution; similarly we will talk about max min-distributions. Such distributions exist by weak compactness and continuity.

Recall that the *radius* and *excess* of X are defined as:

$$\mathrm{rad}(X) := \min_{p\in X}\max_{q\in X} d(p,q) \text{ and } \mathrm{exc}(X) := \min_{p,q\in X}\max_{r\in X} \big(d(p,r)+d(r,q)-d(p,q)\big)$$

The following is a list of a few basic properties of the rendezvous number:

Proposition 1.

1. $\mathrm{rv}(X) \leq \mathrm{xt}(X)$
2. $\frac{1}{2}\,\mathrm{diam}(X) \leq \mathrm{rv}(X) \leq \mathrm{rad}(X)$
3. $\mathrm{exc}(X) = 0 \implies \mathrm{rv}(X) = \frac{1}{2}\,\mathrm{diam}(X)$

Proof. 1. is clear since $\mathrm{xt}(X) = \sup_\mu \int_X \mathrm{md}_\mu(\cdot)\mu \geq \min_{p\in X}\mathrm{md}_\mu(p)$.

The first inequality of the second statement follows by considering the distribution $\mu = \frac{1}{2}(\delta_p+\delta_q)$, where p, q realize $\mathrm{diam}(X)$. The second inequality follows by considering the distribution δ_p, where p realizes $\mathrm{rad}(X)$, hence $\max_{q\in X}\mathrm{md}_{\delta_p}(q) = \mathrm{rad}(X)$.

Finally assume that p, q realize $\mathrm{exc}(X) = 0$. Then p, q are antipodal and $d(p,q) = \mathrm{diam}(X)$. This means that the potential of $\frac{1}{2}(\delta_p + \delta_q)$ is constant and equal to $\frac{1}{2}\,\mathrm{diam}(X)$. □

Mean Distance We will often assume that we have a preferred associated measure $\mu_0 \in \mathcal{P}(X)$ on X, which is a *normalized volume measure* in the sense that $\mu_0(B(p,r)) > 0$ for all $p \in X$ and $r > 0$, where

$$B(p,r) := \{q \in X | d(p,q) < r\}$$

is the open ball of radius r and center p.

For a Riemannian manifold we take $\mu_0 = \frac{1}{\mathrm{vol}(M)}\mathrm{vol}(\cdot)$, where vol denotes Riemannian volume. In general we could take μ to be normalized Hausdorff measure, whenever the Hausdorff measure of the appropriate dimension is finite and positive. E.g. for a finite space, μ_0 would then be normalized counting measure.

Definition 5. Let X be a compact metric space with an associated normalized volume measure $\mu_0 \in \mathcal{P}(X)$, then the *mean distance* of X is defined to be

$$\mathrm{md}(X) := I(\mu_0) = \int_{X\times X} d(\cdot,\cdot)\mu_0 \otimes \mu_0 \tag{9}$$

Whenever we consider mean distance with respect to the standard measure μ_0, we drop the subscript notation, so $\mathrm{md} := \mathrm{md}_{\mu_0}$ is the mean distance function of the standard measure.

Homogeneous Spaces

The isometry group of X, $\mathrm{Isom}(X)$, acts on $\mathfrak{M}(X)$ by pull back $\mu \mapsto \mu \circ \sigma$ and the process of taking potentials commutes with the corresponding pull back action on $C(X)$:

$$\mathbf{p}_\mu \circ \sigma = \mathbf{p}_{\mu\circ\sigma} \tag{10}$$

As usual we will call X *homogeneous* if the isometry group acts transitively on X. Then we can identify X with the quotient $\mathrm{Isom}(X)/\,\mathrm{Isom}(X)_p$, where $\mathrm{Isom}(X)_p := \{\sigma \in \mathrm{Isom}(X) |\, \sigma(p) = p\}$ is the isotropy group of some $p \in X$.

Let $\mathrm{Con}(X)$ denote the set of probability measures with constant potential $\mathrm{Con}(X) := \Psi^{-1}(\mathbb{R}) \cap \mathcal{P}(X)$. We can think of a distribution μ with constant potential as a critical point of the "energy" functional $I(\mu)$ under mass preserving variations, since $I(\mu, \nu) = 0$ for any ν s.t. $\nu(X) = 0$. A probability measure which is preserved by the action of $\mathrm{Isom}(X)$ on a homogeneous space is clearly such a critical measure with constant potential, hence:

Proposition 2. *On a homogeneous space X with a normalized volume measure μ_0 which is preserved by the action of* $\mathrm{Isom}(X)$ *we have:*

$$\mathrm{rv}(X) = \mathrm{md}(X)$$

Thus for a homogeneous Riemannian manifold X we have $\mathrm{rv}(X) = \mathrm{md}(X)$. In general however both $\mathrm{rv}(X) < \mathrm{md}(X)$ and the opposite inequality occurs.

The following is an adaptation of Theorem 13 in [1]:

Proposition 3. *On a homogeneous space X any* $\min\max$ *or* $\max\min$ *distribution $\mu \in \mathcal{P}(X)$ has constant potential,* $\mathrm{md}_\mu = \mathrm{rv}(X)$.

Proof. We prove the $\max\min$ case, the other is similar. For each $\max\min$-distribution μ, define $C_\mu := \{p \in X |\ \mathrm{md}_\mu(p) = \mathrm{rv}(X)\}$. Then C_μ is closed. For a finite number of $\max\min$-distributions we have $\bigcap_{i=1}^n C_{\mu_i} \neq \emptyset$. For if not the distribution $\nu = \frac{1}{n}\sum_{i=1}^n \mu_i \in \mathcal{P}(X)$ would have a larger minimum than $\mathrm{rv}(X)$, which is impossible. Hence by compactness of X the intersection $\bigcap C_\mu := C$ of all such C_μ is nonempty. Now by homogeneity of X every $\min\max$-potential md_μ must be constant. If not, pick a point $p \in C$, a point q s.t. $\mathrm{md}_\mu(q) > \mathrm{rv}(X)$ and an isometry with $\sigma(q) = p$. Then the $\max\min$-distribution $\mu\circ\sigma$ would not have minimal potential at p, a contradiction. $\square$

We trivially have in general $\mathrm{md}(X) \leq \mathrm{xt}(X)$. What happens if we actually have equality? We shall need the following lemma, see Theorem 2 in [1] (c.f. also Theorem 13 p. 82 in [9]).

Lemma 2. *If $I(\mu) = \mathrm{xt}(X)$ for $\mu \in \mathcal{P}(X)$, then for all p in the support of μ we have:*

$$\mathrm{md}_\mu(p) = \sup_{q \in X} \big(\mathrm{md}_\mu(q)\big) = \mathrm{xt}(X) \tag{11}$$

Proposition 4. $\mathrm{md}(X) = \mathrm{xt}(X)$ *implies that the mean distance function of μ_0 is constant, hence* $\mathrm{rv}(X) = \mathrm{md}(X) = \mathrm{xt}(X)$*. Furthermore this implies* $\{\mu \in \mathcal{P}(X) | I(\mu) = \mathrm{xt}(X)\} = \mathrm{Con}(X)$ *if X is homogeneous.*

Proof. By the requirement that μ_0 is a normalized volume measure the support of μ_0 is the whole of X. But then by Lemma 2 above, the mean distance function is constant, hence $\mathrm{xt}(X) = \mathrm{md}(X) = \mathrm{rv}(X)$. Then clearly any probability measure with constant potential (which has to be $\mathrm{rv}(X)$) realizes $\mathrm{xt}(X)$. On the other hand any distribution realizing $\mathrm{xt}(X)$ would be a min max-distribution by Lemma 2 and thus have constant potential if X is homogeneous by Proposition 3. □

Proposition 5. *For the round sphere $\mathbb{S}^n_1$ we have:*

$$\mathrm{rv}(\mathbb{S}^n_1) = \mathrm{md}(\mathbb{S}^n_1) = \mathrm{xt}(\mathbb{S}^n_1) = \frac{\pi}{2}, \tag{12}$$

and a distribution $\mu \in \mathcal{P}(\mathbb{S}^n_1)$ realizes $\mathrm{xt}(\mathbb{S}^n_1)$ *iff* $\mu \in \mathrm{Con}(\mathbb{S}^n_1)$.

Proof. We will first show that $\mathrm{md}(\mathbb{S}^n_1) = \frac{\pi}{2}$, then the proposition above applies. More generally we will show that an antipodally invariant measure $\mu \in \mathcal{P}(\mathbb{S}^n_1)$ has constant potential $\frac{\pi}{2}$ and thus realizes $\mathrm{xt}(\mathbb{S}^n_1)$. Here we say that μ is antipodally invariant if $\mu \circ \sigma_A = \mu$, where σ_A is the antipodal isometry.

Hence assume that $\mu \circ \sigma_A = \mu$. For any mean distance function md_ν we have $\mathrm{md}_\nu(p) + \mathrm{md}_\nu(\sigma_A(p)) = \pi$ as is easily seen (e.g. by integrating wrt. $\delta_p + \delta_{\sigma_A(p)}$, which has constant potential π). But since $\mathrm{md}_\mu(p) = \mathrm{md}_\mu(\sigma_A(p))$ by antipodal invariance of μ, the claim follows. □

We claim that in fact any measure with constant potential must be antipodally invariant. Hence for the *extremal boundary* of the convex set $\mathrm{Con}(\mathbb{S}^n_1)$, we have

$$\partial_C(\mathrm{Con}(\mathbb{S}^n_1)) \underset{homeo}{\cong} \mathbb{R}P^n, \tag{13}$$

since the extremal boundary of antipodally invariant probability measures consists of the measures $\frac{1}{2}(\delta_p + \delta_{\sigma_A(p)})$, $p \in \mathbb{S}^n_1$. This will be discussed further in a forthcoming paper...

The situation $\mathrm{md}(X) = \mathrm{xt}(X)$ seems to be quite special, indeed it is tempting to conjecture:

Conjecture: A compact Riemannian manifold X of dimension n with $\mathrm{md}(X) = \mathrm{xt}(X)$ is isometric to a sphere of constant curvature $\mathbb{S}^n_\kappa$.

To support the conjecture we note that it can easily be shown that X must be of *negative type* and thus at least simply connected, c.f. [8] or [9].

Curvature and Mean Distance of Riemannian Manifolds

To connect mean distance with curvature we trivially have, since $\mathrm{md}(X) \leq \mathrm{xt}(X)$ and $\mathrm{md}(\mathbb{S}^n_1) = \mathrm{xt}(\mathbb{S}^n_1) = \frac{\pi}{2}$, a corollary to Theorem 2:

Corollary 1. *Let X be an n-dimensional, $n \geq 2$, Alexandrov space with curv$(X) \geq 1$, equipped with a normalized volume measure $\mu_0 \in \mathcal{P}(X)$. Then*

$$\mathrm{md}(X) \leq \mathrm{md}(\mathbb{S}^n_1) = \frac{\pi}{2} \tag{14}$$

This could also be shown directly by a Toponogov comparison argument, which would give a comparison on integrands. In the Riemannian category it seems to be possible to weaken the curvature condition while still maintaining the maximality of $\mathrm{md}(\mathbb{S}^n_1)$.

Mean distance measures the way X "spreads out" and this is controlled by Ricci curvature. Note however that the maximality of $\mathrm{xt}(\mathbb{S}^n_1)$ does *not* hold, when we weaken the curvature condition to $Ric \geq n-1$. This is shown by the examples of manifolds with large Ricci curvature and large diameter by Anderson and Otsu, c.f. [12] 9.1.8 and 9.1.9.

First we list some equivalent ways of measuring the effectiveness of the way X spreads out, when X is an n-dimensional Riemannian manifold with

$Ric(X) \geq n-1$. The first three are due to Colding, c.f. [3] [4] , and the last to Petersen [11].

1. $\mathrm{vol}(X)$ is close to $\mathrm{vol}(\mathbb{S}^n_1)$
2. $\mathrm{rad}(X)$ is close to π
3. X is Gromov-Hausdorff close to $\mathbb{S}^n_1$
4. $\lambda_{n+1}(X)$ close to $\lambda_{n+1}(\mathbb{S}^n_1) = n$, where λ_{n+1} is the $(n+1)$'th eigenvalue of the laplacian.

By a theorem of Cheeger and Colding, [2], these statements imply that X is diffeomorphic to $\mathbb{S}^n$. The goal is to add

$$\mathrm{md}(X) \text{ close to } \tfrac{\pi}{2}$$

to the list above.

Theorem 4. *Let X be an n-dimensional, $n \geq 2$, Riemannian manifold with $Ric(X) \geq n-1$. Then*

1. $\mathrm{md}(p) \leq \frac{\pi}{2}$ *for any $p \in X$, and if* $\mathrm{md}(p) = \frac{\pi}{2}$ *for some $p \in X$ then $X = \mathbb{S}^n_1$*
2. *Hence* $\mathrm{md}(X) \leq \frac{\pi}{2}$ *and* $\mathrm{md}(X) = \frac{\pi}{2}$ *iff $X = \mathbb{S}^n_1$.*
3. *Furthermore there is an $\epsilon(n) > 0$ s.t.* $\mathrm{md}(X) \geq \frac{\pi}{2} - \epsilon$ *implies that X is diffeomorphic to $\mathbb{S}^n$.*

Proof. Notation will be as in [12], we use polar coordinates around a point $p \in X$ and $\lambda(r,\theta)$ denotes the Riemannian volume density. By Myers Theorem, [12] Theorem 9.1.2, $\mathrm{diam}(X) \leq \pi$. We can extend the volume density to a ball of radius π by taking it to be 0 outside the segment domain $\mathrm{seg}_p \subset T_pX$. Hence

$$\mathrm{md}(p) = \frac{1}{V}\int_{\mathrm{seg}_p} r\lambda(r,\theta)dr \wedge d\theta = \int_0^\pi r\,\mathrm{vol}(S_r)dr, \tag{15}$$

where $S_r = \partial B(p,r)$ is the sphere of radius r around p and $V = \mathrm{vol}(X)$. Continuing by partial integration we obtain:

$$\mathrm{md}(p) = \frac{1}{V}\big(\pi V - \int_0^\pi \mathrm{vol}(B(p,r))dr\big) = \pi - \int_0^\pi \frac{\mathrm{vol}(B(p,r))}{V}dr \tag{16}$$

But by relative volume comparison, [12] Lemma 9.1.6, we have:

$$\frac{\mathrm{vol}(\mathbb{B}(r))}{\mathrm{vol}(\mathbb{S}_1^n)} \leq \frac{\mathrm{vol}(B(p,r))}{\mathrm{vol}(X)},\ r \in [0,\pi], \tag{17}$$

where $\mathbb{B}(r)$ denotes a ball of radius r in $\mathbb{S}_1^n$. Thus the comparison for $\mathrm{md}(p)$ follows since the mean distance function on $\mathbb{S}_1^n$ is constant $\frac{\pi}{2}$. If we have equality $\mathrm{md}(p) = \frac{\pi}{2}$, we must also have equality in the comparison (17) for all $r \in [0,\pi]$. Thus $\mathrm{diam}(X) = \pi$ and X is isometric to $\mathbb{S}_1^n$, c.f. [12] 9.1.4.

The second statement 2. clearly follows from 1.

For the third statement, consider the function $D(p) := \max_q d(p,q)$. Then $\mathrm{rad}(X) = \min D(p)$ and $\mathrm{diam}(X) = \max D(p)$. Suppose that $D(p) < \pi - \epsilon$, for some $\epsilon > 0$. Then the right hand side of (17) will achieve its maximum value 1 before $\pi - \epsilon$. We thus get an estimate $\mathrm{md}(p) < \frac{\pi}{2} - \delta(\epsilon, n)$, with

$$\delta(\epsilon, n) = \int_{\pi-\epsilon}^{\pi} (1 - \frac{\mathrm{vol}(\mathbb{B}(r))}{\mathrm{vol}(\mathbb{S}_1^n)})dr$$

So suppose that $\mathrm{rad}(X) < \pi - \epsilon$ and that $p \in X$ realizes $\mathrm{rad}(X)$. Then since D is clearly 1-Lipschitz, we have that $D(q) < \pi - \frac{\epsilon}{2}$ on $B(p, \frac{\epsilon}{2})$. We then have an estimate $\mathrm{md}(q) < \frac{\pi}{2} - \delta(\frac{\epsilon}{2}, n)$ on $B(p, \frac{\epsilon}{2})$ and thus

$$\mathrm{md}(q) < \frac{\pi}{2} - \delta(\frac{\epsilon}{2}, n)\mathbf{1}_{B(p,\frac{\epsilon}{2})} \text{ on } X$$

But this then implies that $\mathrm{md}(X) < \frac{\pi}{2} - f_p(\frac{\epsilon}{2}, n)$, where

$$f_p(\epsilon, n) = \frac{\mathrm{vol}(B(p,\epsilon))}{\mathrm{vol}(X)}\delta(\epsilon, n) \geq \frac{\mathrm{vol}(\mathbb{B}(\epsilon))}{\mathrm{vol}(\mathbb{S}_1^n)}\delta(\epsilon, n) > 0$$

Thus mean distance close to $\frac{\pi}{2}$ implies that $\mathrm{rad}(X)$ is close to π and hence that X is diffeomorphic to $\mathbb{S}^n$ by the work of Colding and Cheeger, [2]. □

The proof shows that mean distance close to $\frac{\pi}{2}$ implies that radius is close to π. But to add mean distance close to $\frac{\pi}{2}$ to the list of equivalent conditions above, we need also one of the other directions. This direction can be seen in many ways. The most useful is perhaps to note that from Colding's work on volume convergence when there is a lower bound on Ricci-curvature, [5], it follows:

Theorem 5. *Let $\{X_n\}$ be a sequence of Riemannian manifolds of dimension m s.t. $X_n \to X$, with respect to Gromov-Hausdorff distance, where X is an m-dimensional Riemannian manifold. If there is a uniform lower bound $Ric(X_n) \geq k \in \mathbb{R}$, then* $\mathrm{md}(X_n) \to \mathrm{md}(X)$.

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