A Laplacian on metric measure spaces

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Abstract

We introduce a Laplacian on a class of metric measure spaces via a direct pointwise mean value definition. Fundamental properties of this Laplacian, such as its symmetry as an operator on functions satisfying a Neumann or Dirichlet condition, are established.

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1 Introduction

The purpose of this paper is to introduce a Laplacian in the setting of metric measure spaces. While harmonic functions have been studied on doubling metric measure spaces satisfying a Poincaré inequality by several authors, see e.g. [6], [1] and [8], these functions are usually defined indirectly as minimizers of a certain energy integral. In contrast to this, we will give a specific, pointwise definition of the Laplacian.

While this Laplacian can be defined in a very general setting, we need to restrict ourselves to metric measure spaces, which are sufficiently locally homogeneous in order to establish some of the usual, basic properties, like e.g. symmetry of the Laplacian as a linear operator on functions satisfying a Neumann or Dirichlet condition. We will also establish a version of Green's formulas for sets which has, in a suitable sense, codimension 1 boundary.

In the final section we will show that on a Riemannian manifold, the Laplacian introduced in this paper, is a constant multiple of the usual Laplacian associated to the Riemannian metric.

The main results are summarized in Theorem 33 and Theorem 41.

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Prerequisites

Let (X, d, μ) be a metric measure space, with μ a Borel measure on X. We will use the following standard notation:

$$B(p,r) = \{q \in X |\, d(p,q) < r\}$$

is the open ball of radius r centered at p, and for a locally integrable function

$$f_{B(p,r)} := \fint_{B(p,r)} f(q)\mu(q) := \frac{1}{\mu(B(p,r))} \int_{B(p,r)} f(q)\mu(q)$$

is the mean value of f over $B(p,r)$. A function $f : X \to \mathbb{R}$ is *Lipschitz* if

$$|f(p) - f(q)| \leq L d(p,q) \tag{1}$$

for some $L \geq 0$ and all $p, q \in X$. We will use $\mathrm{LIP}\, f$ to denote the infimum over all real numbers L such that (1) holds. $\mathrm{LIP}(X)$ will denote the space of all Lipschitz functions on X.

For $f \in \mathrm{LIP}(X)$ we define, c.f. [5], the local Lipschitz constant of f at $p \in X$ as:

$$\mathrm{Lip}\, f(p) := \limsup_{q \to p, q \neq p} \frac{|f(p) - f(q)|}{d(p,q)},$$

which is interpreted as 0 if p is an isolated point.

Poincaré Inequalities:

We will assume that an inequality of the following type holds for any Lipschitz function f:

$$\sup_{q \in B(p,r)} |f(p) - f(q)| \leq C_P r \{\fint_{B(p,\tau r)} (\mathrm{Lip}\, f(q))^s \mu(q)\}^{\frac{1}{s}}, \tag{2}$$

where $B(p,r)$ is an arbitrary ball of sufficiently small radius $r \leq r_P$ and $s, C_P > 0\,, \tau \geq 1$ are constants.

Recall that a Borel measure μ on X is said to be doubling with doubling constant $C_d \geq 1$, if

$$\mu\big(B(p,2r)\big) \leq C_d \mu\big(B(p,r)\big) \tag{3}$$

for all $p \in X$ and all $r \geq 0$. μ is said to be locally doubling if (3) holds for balls of radius bounded from above $r \leq r_d$.

An s'-Poincaré inequality, is an inequality of the form:

$$\fint_{B(p,r)} |f(q) - f_{B(p,r)}|\mu(q) \leq C'_P r \{\fint_{B(p,r)} (\mathrm{Lip}\, f(q))^{s'} \mu(q)\}^{\frac{1}{s'}}, \tag{4}$$

for any Lipschitz function f, constants $C'_P, s' > 0$ and an arbitrary ball $B(p,r)$. If (4) only holds on balls with radius bounded from above, we call (4) a *local* Poincaré inequality.

In certain cases a local Poincaré inequality will guarantee (2) with $\tau = 1$ for some sufficiently large s, when μ is locally doubling. This is the case if every ball is a so-called John domain, e.g. when X is a proper length space. See [4] chapter 9.

If a Poincaré inequality holds globally, i.e. for any radius $r > 0$, and μ is doubling, we get (2) with $\tau = 5$ and some sufficiently large s without further assumptions on the geometry of balls, see [4] Theorem 5.1.

Thus in both cases we have:

$$\sup_{q\in B(p,r)} |f(p) - f(q)| \leq C_P r \{ \fint_{B(p,5r)} (\mathrm{Lip}\, f(q))^s \mu(q) \}^{\frac{1}{s}}, \tag{5}$$

for r sufficiently small, since when μ is (locally) doubling,

$$\{ \fint_{B(p,r)} (\mathrm{Lip}\, f(q))^s \mu(q) \}^{\frac{1}{s}} \leq C \{ \fint_{B(p,5r)} (\mathrm{Lip}\, f(q))^s \mu(q) \}^{\frac{1}{s}},$$

where $C > 0$ depends on the doubling constant only.

Definition of the Laplacian:
We are now ready to define the object, that we will be studying.

Definition 6. Let (X, d, μ) be a proper metric measure space, with μ a Radon measure on X s.t. $\mu(B(p, r)) > 0$ for every ball $B(p, r)$ of radius $r > 0$. For a function $f : X \to \mathbb{R}$ and $p \in X$ define

$$\overline{\Delta}_\mu f(p) := \limsup_{r\to 0, r>0} \left\{ \frac{2}{r^2} \fint_{B(p,r)} \big(f(q) - f(p)\big)\, \mu(q) \right\} \in \mathbb{R}^* \tag{7}$$

and

$$\underline{\Delta}_\mu f(p) := \liminf_{r\to 0, r>0} \left\{ \frac{2}{r^2} \fint_{B(p,r)} \big(f(q) - f(p)\big)\, \mu(q) \right\} \in \mathbb{R}^*, \tag{8}$$

where $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the extended real line.

If $\underline{\Delta}_\mu f(p) = \overline{\Delta}_\mu f(p) \in \mathbb{R}^*$, we denote this common limit by $\Delta_\mu f(p)$, and define it to be the μ-Laplacian of f at p.

Thus if the Laplacian of f exists, and is finite, at $p \in X$, we can write

$$f_{B(p,r)} = f(p) + \frac{1}{2}\Delta_\mu f(p) r^2 + \epsilon_p(r) r^2, \tag{9}$$

where $\epsilon_p(r)$ is some function such that $\epsilon_p(r) \to 0$ for $r \to 0$.

The requirement that μ has support everywhere, $\mu(B(p, r)) > 0$ for $r > 0$, is just a convenience to avoid discussing mean values over sets of measure zero.

The definition of the Laplacian involves both the metric and the measure, and unless there is some kind of compatibility between these objects, one cannot expect the Laplacian to have nice properties similar to those of e.g. the Laplacian on a Riemannian manifold. In general, if the Laplacian with respect to μ behaves nicely, this might not be the case for the Laplacian with respect to $f\mu$, even when f is a Lipschitz function.

The compatibility condition between the measure and the metric, that we shall be using here, is the following:

Definition 10. A locally doubling measure μ is said to be *compatible* if there are constants $r_P, r_h > 0$ and $C_h \geq 0$, s.t.

1. Inequality (5) is satisfied for sufficiently small radii $r \leq r_P$.
2. There is a set $\mathcal{E} \subset X$ of measure zero, $\mu(\mathcal{E}) = 0$, s.t. for $p \in \mathcal{H} := X \setminus \mathcal{E}$ we have:
$$\lim_{r\to 0} \frac{1}{r} \fint_{B(p,r)} |\frac{\mu((B(p,r))}{\mu(B(q,r))} - 1|\mu(q) = 0, \tag{11}$$
3. And finally
$$\frac{1}{r} \fint_{B(p,r)} |\frac{\mu((B(p,r))}{\mu(B(q,r))} - 1|\mu(q) \leq C_h \tag{12}$$
for $d(p, \mathcal{E}) \geq 2r$ and $r \leq r_h$. Here we set $d(p, \mathcal{E}) := \infty$ if $\mathcal{E} = \emptyset$.

Example 13.

- On a Riemannian manifold with boundary having Ricci curvature bounded from below the standard volume measure (i.e. n-dimensional Hausdorff measure) is compatible with exceptional set $\mathcal{E} = \partial M$. This follows since on a Riemannian manifold (without boundary) of dimension n, we have $\mu\big(B(p,r)\big) = c_n r^n + O(r^{n+2})$ for $r \to 0$, with a specific bound on the error in terms of curvature. See (some of the) details below. Also M is locally doubling and satisfies a local Poincaré inequality, c.f. [4] chapter 10.
- On a 1-dimensional simplicial complex, i.e. a weighted graph with edges, the 1-dimensional Hausdorff measure $\mathcal{H}^1$ is compatible, if the degree of the vertices (0-simplex) is bounded from above; (5) is easily verified, also $\mathcal{H}^1$ is doubling. The exceptional set is seen to consist of the vertices (0-simplices) of degree different from 2.
- Clearly any homogeneous measure, $\mu(B(p,r)) = \mu(B(q,r))$, $\forall p, q \in X$, is compatible with exceptional set $\mathcal{E} = \emptyset$, if μ is doubling and satisfies (5).

2 The Laplacian as a symmetric operator

In this section we will study the situation, where the Laplacian acts as a symmetric operator on a class of functions.

Notation 14. To shorten proofs, we will sometimes use the notation:

$$[f, g]_+ := \frac{2}{r^2} \int_X \{\fint_{B(p.r)} f(p)\big(g(q) - g(p)\big)\mu(q)\}\mu(p),$$

and $[f,g]_-$ for the same term, with the order of integration "reversed"

$$[f,g]_- := \frac{2}{r^2}|\int_X \{\fint_{B(q.r)} f(p)\big(g(q)-g(p)\big)\mu(p)\}\mu(q),$$

Lemma 15. *Let μ be compatible, let f,g be Lipschitzs functions and assume that the support of $f\operatorname{Lip} g$, $\operatorname{supp}(f\operatorname{Lip} g)$, is compact. If either*

1. $\operatorname{Lip} g(p) \leq Cd(p,\mathcal{E})$, *for some $C \geq 0$. Or*
2. $f(p) = 0$ *for $p \in \mathcal{E}$,*

then

$$\int_X \frac{1}{r^2}\{\fint_{B(p,r)} f(p)\big(g(q)-g(p)\big)\mu(q)\}\mu(p) = \frac{1}{2}[f,g]_+$$

is convergent as $r \to 0$ iff

$$\int_X \frac{1}{r^2}\{\fint_{B(q,r)} f(p)\big(g(q)-g(p)\big)\mu(p)\}\mu(q) = \frac{1}{2}[f,g]_-$$

is convergent, in which case the limits equal.

Proof.

$$\begin{aligned}
&|\frac{1}{2}[f,g]_- - \frac{1}{2}[f,g]_+| = \\
&\frac{1}{r^2}|\int_X\int_X \frac{\mu(B(p,r))}{\mu(B(q,r))}\frac{\delta_r(p,q)}{\mu(B(p,r))} f(p)\big(g(q)-g(p)\big)\mu(p)\mu(q) \\
&\qquad - \int_X\int_X \frac{\delta_r(p,q)}{\mu(B(p,r))} f(p)\big(g(q)-g(p)\big)\mu(q)\mu(p)| \\
&= \frac{1}{r^2}|\int_X\int_X (\frac{\mu(B(p,r))}{\mu(B(q,r))}-1)\frac{\delta_r(p,q)}{\mu(B(p,r))} f(p)\big(g(q)-g(p)\big)\mu(q)\mu(p)| \\
&\leq \int_X |f(p)|\{\frac{1}{r}\fint_{B(p,r)} |\frac{\mu(B(p,r))}{\mu(B(q,r))}-1|\mu(q)\}\{\frac{1}{r}\sup_{q\in B(p,r)} |g(q)-g(p)|\}\mu(p) \\
&\leq \int_X |f(p)|\{\frac{1}{r}\fint_{B(p,r)} |\frac{\mu(B(p,r))}{\mu(B(q,r))}-1|\mu(q)\}C_P\{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s\mu(q)\}^{\frac{1}{s}}\mu(p)
\end{aligned} \tag{16}$$

Fubini is used to interchange the order of integration in the first term of the second line, while the Poincaré inequality is applied in the final step; we have also used the notation:

$$\delta_r(p,q) := \begin{cases} 1 \text{ if } d(p,q) < r \\ 0 \text{ otherwise} \end{cases},$$

for $r > 0$.

Since μ is locally doubling, we always have $\mu((B(p,r)) \leq C_d \mu(B(q,r))$ when $d(p,q) \leq r$ (and r is sufficiently small).Thus

$$\frac{1}{r} \fint_{B(p,r)} |\frac{\mu((B(p,r))}{\mu(B(q,r))} - 1|\mu(q) \leq \frac{C_d + 1}{r}. \tag{17}$$

If now $d(p, \mathcal{E}) \leq 5r$, then clearly by the triangle inequality $d(q, \mathcal{E}) \leq 10r$ on $B(p, 5r)$, thus in case 1 holds:

$$\{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \leq \{\fint_{B(p,5r)} Cd(q,\mathcal{E})^s \mu(q)\}^{\frac{1}{s}} \leq 10Cr.$$

Hence in this case the integrand

$$|f(p)|\{\frac{1}{r} \fint_{B(p,r)} |\frac{\mu(B(p,r))}{\mu(B(q,r))} - 1|\mu(q)\} C_P \{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \tag{18}$$

is dominated by $C_1|f(p)|$ with support on a $5r$-neighborhood of $\operatorname{supp}(f \operatorname{Lip} g)$, which is compact since X is proper.

In case $d(p, \mathcal{E}) > 5r$ we use

$$\{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \leq \operatorname{LIP} g$$

while (for r sufficiently small) by (12)

$$\frac{1}{r} \fint_{B(p,r)} |\frac{\mu(B(p,r))}{\mu(B(q,r))} - 1|\mu(q) \leq C_h.$$

Thus the integrand (18) is dominated by $C_2|f(p)|$, with support on a $5r$-neighborhood of $\operatorname{supp}(f \operatorname{Lip} g)$.

Hence, in any case, for sufficiently small r, under the condition 1 the integrand is bounded by the compactly supported L^1-function $C_3|f(p)|$ for some $C_3 > 0$. Then since the integrand is converging pointwise to 0 a.e., we can use dominated convergence to conclude that the integral vanishes in the limit $r \to 0$.

If instead 2 holds, we use $|f(p)| \leq \operatorname{LIP} f d(p, \mathcal{E})$ and (17) in a similar fashion to get, that the integrand is bounded by a constant in a $2r$-neighborhood of $\mathcal{E}$, and by $C_3|f(p|$ outside this neighborhood, to get the same conclusion. □

Notice that since $[1, f]_+ = -[1, f]_-$, we have

$$\int_X \Delta_\mu f \, \mu = \frac{1}{2} \lim_{r \to 0}([1, f]_+ - [1, f]_-) = \lim_{r \to 0}[1, f]_+ \tag{19}$$

when we have the dominated convergence condition, c.f. Definition 28. Thus using Lemma 15:

$$\int_X \Delta_\mu f \, \mu = 0.$$

Lemma 20. *Suppose that f is Lipschitz, $|\overline{\Delta}_\mu f(p)| < \infty$ and $|\underline{\Delta}_\mu f(p)| < \infty$, then*

$$\liminf_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q) - f(p)\big)^2 \mu(q) = \liminf_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q) - f_{B(p,r)}\big)^2 \mu(q)$$

and

$$\limsup_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q) - f(p)\big)^2 \mu(q)$$
$$= \limsup_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q) - f_{B(p,r)}\big)^2 \mu(q) \le \operatorname{Lip} f(p)^2,$$

Proof. We have

$$(f(p) - f(q))^2 = \big(f(p) - f_{B(p,r)} - (f(q) - f_{B(p,r)})\big)^2$$
$$= \big(f(p) - f_{B(p,r)}\big)^2 - 2\big(f(p) - f_{B(p,r)}\big)\big(f(q) - f_{B(p,r)}\big) + \big(f(q) - f_{B(p,r)}\big)^2$$

thus

$$\frac{1}{r^2} \fint_{B(p,r)} \big(f(p) - f(q)\big)^2 \mu(q) =$$
$$\frac{1}{r^2}\big(f(p) - f_{B(p,r)}\big)^2 + \frac{1}{r^2} \fint_{B(p,r)} (f(q) - f_{B(p,r)})^2 \mu(q).$$

Clearly, under the hypothesis above, $\lim_{r\to 0} \frac{1}{r^2}\big(f(p) - f_{B(p,r)}\big)^2 = 0$, so we get

$$\liminf_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(p) - f(q)\big)^2 = \liminf_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} (f(q) - f_{B(p,r)})^2 \mu(q),$$

and similarly for lim sup. By definition of $\operatorname{Lip} f(p)$, for any $\epsilon > 0$:

$$(f(q) - f(p))^2 \le (\operatorname{Lip} f(p) + \epsilon)^2 r^2,$$

when $d(q,p) \le r$ and r is sufficiently small. So clearly

$$\limsup_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} (f(q) - f(p))^2 \mu(q) \le (\operatorname{Lip} f(p))^2. \tag{21}$$

□

The estimate (21) can be a quite rough. For a smooth function f on a Riemannian n-manifold M^n, one can show that:

$$\lim_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} (f(q) - f(p))^2 \mu(q) = \frac{n}{n+2} \fint_{\mathbb{S}^{n-1}} \big(df_p(v)\big)^2 dv$$
$$= \frac{1}{n+2} \| \operatorname{grad} f(p) \|^2, \tag{22}$$

where $\mathbb{S}^{n-1}$ is the unit sphere in T_pM. Compare to Proposition 52 below.

Lemma 23. *Assume that μ is compatible, and f is a compactly supported Lipschitz function with $\Delta_\mu f \in L^2(X,\mu)$ satifying either condition 1 or 2 of Lemma 15 and furthermore*

SC $\frac{1}{r^2} ⨍_{B(p,r)} \big(f(p) - f(q)\big)^2 \mu(q)$ *is convergent as $r \to 0$ for almost all $p \in X$.*

DC $r^{-2}|f(p) - f_{B(p,r)}| \leq \rho(p)$ *for some $\rho \in L^2(X,\mu)$ and $r \leq r_c$ sufficiently small.*

Then

$$\int_X \left\{ \lim_{r\to 0} \frac{1}{r^2} ⨍_{B(p,r)} \big(f(q) - f(p)\big)^2 \mu(q) \right\} \mu(p) = \big\langle -\Delta_\mu f, f \big\rangle \tag{24}$$

Proof. First of all, since $\lim_{r\to 0} \frac{1}{r^2} ⨍_{B(p,r)} \big(f(q) - f(p)\big)^2 \mu(q) \leq C \operatorname{Lip} f(p)^2 \leq C \operatorname{LIP} f$ on the support of f, we get by dominated convergence:

$$\int_X \lim_{r\to 0} \frac{1}{r^2} ⨍_{B(p,r)} \big(f(q)-f(p)\big)^2 \mu(q)\mu(p) = \lim_{r\to 0} \int_X \frac{1}{r^2} ⨍_{B(p,r)} \big(f(q)-f(p)\big)^2 \mu(q)\mu(p).$$

Then since $(f(p) - f(q))^2 = f(p)(f(p) - f(q)) + f(q)(f(p) - f(q))$, we can continue the equation as:

$$\begin{aligned} = \lim_{r\to 0} \int_X \frac{1}{r^2} ⨍_{B(p,r)} f(p)\big(f(p) - f(q)\big)\mu(q)\mu(p) \\ + \lim_{r\to 0} \int_X \frac{1}{r^2} ⨍_{B(p,r)} f(q)\big(f(q) - f(p)\big)\mu(q)\mu(p). \end{aligned} \tag{25}$$

We then use Lemma 15 to interchange the order of integration in the second term, which is convergent since the first term is:

$$-\lim_{r\to 0} \frac{1}{2}[f,f]_- = -\lim_{r\to 0} \frac{1}{2}[f,f]_+ = -\frac{1}{2}\langle f, \Delta_\mu f\rangle,$$

where the limit is moved back inside the first integrals using dominated convergence, condition DC. □

Lemma 26. *If μ is a doubling measure, then there is a constant $C > 0$ depending only on the doubling constant for μ, such that for a Lipschitz function f with $|\overline{\Delta}_\mu f(p)| < \infty$ and $|\underline{\Delta}_\mu f(p)| < \infty$ almost everywhere, we have*

$$\limsup_{r\to 0} \frac{1}{r^2} ⨍_{B(p,r)} \big(f(q) - f(p)\big)^2 \mu(q) \geq \frac{1}{C}\big(\operatorname{Lip} f(p)\big)^2, \tag{27}$$

for almost all $p \in X$.

Proof. By Proposition 4.3.3 in [5], we have for a Lipschitz function f:

$$\limsup_{r\to 0} \frac{1}{r} \fint_{B(p,r)} |f(q) - f_B(p,r)|\mu(q) \geq \frac{1}{C} \operatorname{Lip} f(p),$$

for almost all $p \in X$ and a constant C depending only on the doubling constant for μ. But then by Lemma 20

$$\limsup_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q)-f(p)\big)^2 \mu(q) = \limsup_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q)-f_{B(p,r)}\big)^2 \mu(q) \geq$$

$$\limsup_{r\to 0} \left(\frac{1}{r} \fint_{B(p,r)} |f(q) - f_{B(p,r)}|\mu(q) \right)^2 \geq \frac{1}{C^2} \big(\operatorname{Lip} f(p)\big)^2$$

□

In view of the previous lemmas we combine properties to define a class of *admissible* functions on which the Laplacian acts nicely.

Definition 28. Define $\mathcal{A}$ to be the Lipschitz functions $f \in \mathrm{LIP}(X)$, such that

CS supp(f) is compact.

L2 $\Delta_\mu f$ is well defined a.e., and gives a function in $L^2(X, \mu)$.

DC Dominated Convergence: There is an $r_c(f) > 0$ and a $\rho(f) \in L^2(X, \mu)$, depending on f , such that $r^{-2}|f(p) - f_{B(p,r)}| \leq \rho(p)$ for $r \leq r_c$.

SC $\frac{1}{r^2} \fint_{B(p,r)} \big(f(p) - f(q)\big)^2 \mu(q)$ is convergent as $r \to 0$ for almost all $p \in X$.

Now define the following two subclasses of $\mathcal{A}$:

- Let $\mathcal{A}_N$ be the functions in $\mathcal{A}$ satisfying also the condition that f has critical points on the exceptional set $\mathcal{E}$ in the sense of Lemma 15: $\operatorname{Lip} f(p) \leq C_f d(p, \mathcal{E})$, for some $C_f \geq 0$ depending on f.
- Let $\mathcal{A}_D$ be the functions in $\mathcal{A}$ satisfying the condition $f(p) = 0$ for $p \in \mathcal{E}$.

A simple calculation reveals that condition SC above is equivalent to $\Delta_\mu(f^2)$ being well defined a.e., given that $\Delta_\mu f$ is well defined almost everywhere.

As defined $\mathcal{A}$ need not be a vector space, however in interesting cases $\mathcal{A}$ should contain a large, possibly dense, vector space as a subset. It is not hard to see, that if f, g and $f + g$ are all in $\mathcal{A}$, then $\Delta_\mu(fg)$ is well defined almost everywhere, and defines a function in $L^1(X, \mu)$. The functions in $\mathcal{A}$ should be thought of as a ”very nice” class of functions, and in general one would be interested in a suitably defined completion of these functions.

The condition defining $\mathcal{A}_N$, having critical points on the exceptional set, should be thought of as a rough generalization of a Neumann condition, while $\mathcal{A}_D$ is simply the functions in $\mathcal{A}$ satisfying also a Dirichlet type condition.

Proposition 29. *If μ is a compatible measure, then Δ_μ is a symmetric operator on $\mathcal{A}_N$ as well as on $\mathcal{A}_D$:*

$$\left\langle \Delta_\mu f, g \right\rangle = \left\langle f, \Delta_\mu g \right\rangle, f, g \in \mathcal{A}_*, \tag{30}$$

where $\mathcal{A}_*$ is either $\mathcal{A}_N$ or $\mathcal{A}_D$.

Proof. Given $f, g \in \mathcal{A}$ we have

$$\begin{aligned}
&\left\langle g, \Delta_\mu f \right\rangle - \left\langle f, \Delta_\mu g \right\rangle = \\
&\int_X g(p) \lim_{r\to 0} \frac{2}{r^2} \fint_{B(p,r)} (f(q) - f(p)\mu(q)\mu(p) \\
&\qquad - \int_X f(p) \lim_{r\to 0} \frac{2}{r^2} \fint_{B(p,r)} (g(q) - g(p)\mu(q)\mu(p) = \\
&\lim_{r\to 0} \int_X \frac{2}{r^2} \fint_{B(p,r)} \big(g(p)f(q) - g(p)f(p)\big)\mu(q)\mu(p) \\
&\qquad - \lim_{r\to 0} \int_X \frac{2}{r^2} \fint_{B(p,r)} \big(f(p)g(q) - f(p)g(p)\big)\mu(q)\mu(p) =,
\end{aligned} \tag{31}$$

where we have used dominated convergence to move the limits. Using Lemma 15 to interchange the order of integration in the first term, and then interchanging p and q, we can continue (31) as:

$$= \lim_{r\to 0} \int_X \frac{2}{r^2} \fint_{B(p,r)} \big(f(p)g(p) - f(q)g(q)\big)\mu(q)\mu(p). \tag{32}$$

The product fg is Lipschitz (since $\operatorname{supp}(fg)$ is compact) with local Lipschitz constant $\operatorname{Lip} fg \leq |f| \operatorname{Lip} g + |g| \operatorname{Lip} f$, and thus with critical points on $\mathcal{E}$. Using Notation 14, we can write the term (32) as:

$$-\frac{1}{2} \lim_{r\to 0} ([1, fg]_+ - [1, fg]_-),$$

hence the term vanishes by Lemma 15. □

We can now combine these results into the following theorem:

Theorem 33. *Let (X, d, μ) be a proper, metric measure space, with μ a compatible measure. Then $-\Delta_\mu$ is a nonnegative, symmetric operator on any vector space contained in either $\mathcal{A}_N$ or $\mathcal{A}_D$. Furthermore there is a constant $C \geq 1$ depending only on the doubling constant for μ s.t. for $f \in \mathcal{A}_N \cup \mathcal{A}_D$*

$$\frac{1}{C} | \operatorname{Lip} f\|_2^2 \leq \left\langle -\Delta_\mu f, f \right\rangle \leq \| \operatorname{Lip} f\|_2^2 \tag{34}$$

If it is the case, as it will be in many interesting examples, that the set of admissible functions $\mathcal{A}$ is dense in $L^2(X, \mu)$, then we can consider the self-adjoint Friedrichs extension of the Laplacian acting on functions with a Dirichlet condition, and thus as usual the spectral theory of Δ_μ as an unbounded operator on $L^2(X, \mu)$. We will not go further into this in the present paper though.

It is an interesting question, when (34) can be replaced by

$$\big\langle -\Delta_\mu f, f\big\rangle = C' \|\operatorname{Lip} f\|_2^2. \tag{35}$$

This is the case e.g. on a Riemannian manifold, compare to Proposition 52 below. It remains to be investigated, when this property is preserved under measured Gromov-Hausdorff convergence.

3 Green's Formulas

It is possible in a quite general setting also to establish some weak versions of the classical Green's formulas, when the exceptional set is in some sense of codimension 1. We will use assumptions that make the proofs fairly easy. These could most likely be weakened considerably.

Definition 36. We will say that the exceptional set $\mathcal{E}$ has *essential codimension* 1, if there is a measure ν with support on $\mathcal{E}$, a constant $C > 0$ and for each $p \in \mathcal{E}$ a curve $\gamma_p : [0,1] \to X$ paramatrized by arclength with $\gamma_p(0) = p$, s.t. for every compactly supported continuous function $f \geq 0$:

$$\int_{B(\mathcal{E},r)} f(p)\mu(p) \leq C \int_{\mathcal{E}} \int_0^r f(\gamma_p(t))dt\,\nu(p), \tag{37}$$

where $B(\mathcal{E}, r)$ is a r-neighborhood of $\mathcal{E}$ and $r \leq r_0(f)$ is sufficiently small.

Lemma 38. *Let f be continuous, g Lipschitz and suppose that $f \operatorname{Lip} g$ has compact support. Furthermore, assume that $\mathcal{E}$ has essential codimension 1 and that μ is a compatible measure. Then*

$$\begin{aligned}\limsup_{r\to 0} \frac{1}{r^2}\Big| \int_X \Big\{ ⨍_{B(q,r)} f(p)\big(g(q) - g(p)\big)\mu(p)\Big\}\mu(q) \\ - \int_X \Big\{ ⨍_{B(p,r)} f(p)\big(g(q) - g(p)\big)\mu(q)\Big\}\mu(p)\Big| \\ \leq C' \int_{\mathcal{E}} |f(p)| \operatorname{Lip} g(p)\,\nu(p),\end{aligned}$$

for some constant $C' \geq 0$.

Proof. We continue as in (16) until we arrive at the the last line. Then

$$
\begin{aligned}
&\frac{1}{2r^2}|[f,g]_- - [f,g]_+| \\
&\le \int_X |f(p)|\{\frac{1}{r}\fint_{B(p,r)} |\frac{\mu(B(p,r))}{\mu(B(q,r))} - 1|\mu(q)\} C_P \{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \mu(p) \\
&= \int_{B(\mathcal{E},2r)} |f(p)|\{\frac{1}{r}\fint_{B(p,r)} |\frac{\mu(B(p,r))}{\mu(B(q,r))} - 1|\mu(q)\} C_P \{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \mu(p) \\
&\quad + \int_{X\setminus B(\mathcal{E},2r)} |f(p)|\{\frac{1}{r}\fint_{B(p,r)} |\frac{\mu(B(p,r))}{\mu(B(q,r))} - 1|\mu(q)\} C_P \{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \mu(p) \\
&\le C \int_{\mathcal{E}} \int_0^{2r} |f(\gamma_q(t))| \frac{C_1}{r} C_P \{\fint_{B(\gamma_p(t),5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} dt\, \nu(p) \\
&\qquad + \int_{X\setminus B(\mathcal{E},2r)} |f(p)| C_h C_P \{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \mu(p),
\end{aligned}
$$

assuming that r is sufficiently small so that (12) is satisfied. Since μ is doubling we have

$$
\{\fint_{B(\gamma_p(t),5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \le C_2 \{\fint_{B(p,7r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}},
$$

for some C_2 depending only on the doubling constant. So we can continue the calculation as:

$$
\begin{aligned}
&\le C C_1 C_2 C_P \int_{\mathcal{E}} \{\fint_{B(p,7r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \frac{1}{r} \int_0^{2r} |f(\gamma_p(t))| dt\, \nu(p) \\
&\quad + \int_{X\setminus B(\mathcal{E},2r)} |f(p)| C_h C_P \{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \mu(p) \\
&\le C_3 \int_{\mathcal{E}} \{\fint_{B(p,7r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \frac{1}{r} \int_0^{2r} |f(\gamma_p(t))| dt\, \nu(p) \\
&\quad + C_4 \int_X |f(p)| \{\fint_{B(p,5r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \mu(p)
\end{aligned}
$$

By Lebesgue's differentiation theorem (and dominated convergence) the limit of the first term is:

$$
\begin{aligned}
C_3 \int_{\mathcal{E}} \{\fint_{B(p,7r)} \operatorname{Lip} g(q)^s \mu(q)\}^{\frac{1}{s}} \frac{1}{r} \int_0^{2r} |f(\gamma_p(t))| dt\, \nu(p) \to \\
2C_3 \int_{\mathcal{E}} |f(p)| \operatorname{Lip} g(p) \nu(p), \quad (39)
\end{aligned}
$$

while the integrand of the second term is dominated by

$$
|f(p)| \operatorname{LIP} g,
$$

with support on a $5r$-neighborhood of $\operatorname{supp}(f \operatorname{Lip} g)$, which is compact. Hence, since the integrand is converging pointwise to 0, we get by dominated convergence, that the limit of the second term is 0. $\square$

As in (19) we have

$$\int_X \Delta_\mu f \,\mu = \frac{1}{2} \lim_{r\to 0}([1,f]_+ - [1,f]_-),$$

under the dominated convergence condition in Definition 28. Thus under the hypotheses of the Lemma above:

$$|\int_X \Delta_\mu f \,\mu| \le \frac{C}{2} \int_{\mathcal{E}} \operatorname{Lip} f \,\nu. \tag{40}$$

This also follows from Green's formulas:

Theorem 41 (Green's Formulas)**.** *Suppose that μ is a compatible measure, and that $\mathcal{E}$ has essential codimension 1. Then there are constants $C_1, C_2 \ge 0$ s.t. for f, g Lipschitz functions satisfying conditions CS, L2 and DC of Definition 28.*

$$|\langle \Delta_\mu f, g\rangle - \langle f, \Delta_\mu g\rangle| \le C_1 \int_{\mathcal{E}} (|f| \operatorname{Lip} g + |g| \operatorname{Lip} f)\,\nu \tag{42}$$

$$\big|\, C_2 \|\operatorname{Lip} f\|_2^2 + \langle f, \Delta_\mu f\rangle\big| \le \frac{1}{2} C_1 \int_{\mathcal{E}} |f| \operatorname{Lip} f \,\nu \tag{43}$$

where we for (43) *also require condition SC of Definition 28. Furthermore if $\Delta_\mu(fg)$ is defined a.e., we have:*

$$|\langle f, \Delta_\mu g\rangle| \le \langle \operatorname{Lip} g, \operatorname{Lip} f\rangle + \frac{1}{2} C_1 \int_{\mathcal{E}} |f| \operatorname{Lip} g \,\nu, \tag{44}$$

Proof. We use notation as above, 14. Calculating as in Proposition 29 we get:

$$\begin{aligned}
&|\langle \Delta_\mu f, g\rangle - \langle f, \Delta_\mu g\rangle| \\
&= \lim_{r\to 0} \big|[g,f]_+ - [f,g]_+\big| = \lim_{r\to 0} \big|[g,f]_+ - [g,f]_- - ([f,g]_+ - [g,f]_-)\big| \\
&\le \limsup_{r\to 0} \big|[g,f]_+ - [g,f]_-\big| + \limsup_{r\to 0} \big|[f,g]_+ - [g,f]_-\big| \\
&= \limsup_{r\to 0} \big|[g,f]_+ - [g,f]_-\big| + \limsup_{r\to 0} \frac{2}{r^2} \big| \int_X \fint_{B(p.r)} \big(g(q)f(q) - g(p)f(p)\big)\mu(q)\mu(p)\big| \\
&= \limsup_{r\to 0} \big|[g,f]_+ - [g,f]_-\big| + \frac{1}{2} \limsup_{r\to 0} \big|[1,fg]_+ - [1,fg]_-\big| \\
&\le C \int_{\mathcal{E}} |g| \operatorname{Lip} f \,\nu + \frac{1}{2} C \int_{\mathcal{E}} (|g| \operatorname{Lip} f + |f| \operatorname{Lip} g)\,\nu,
\end{aligned}$$

which proves (42). Now for (43). Following the proof of Lemma 23, we get (25) rewritten as:

$$\int_X \Big\{ \lim_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q) - f(p)\big)^2 \mu(q) \Big\} \mu(p) + \frac{1}{2} \langle \Delta_\mu f, f\rangle = -\frac{1}{2} \lim_{r\to 0} [f,g]_-$$

and thus

$$\int_X \left\{ \lim_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q)-f(p)\big)^2 \mu(q) \right\} \mu(p) + \frac{1}{2}(\langle \Delta_\mu f, f\rangle + \lim_{r\to 0}[f,f]_+$$

$$= \int_X \left\{ \lim_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q)-f(p)\big)^2 \mu(q) \right\} \mu(p) + \langle \Delta_\mu f, f\rangle$$

$$= \frac{1}{2}\lim_{r\to 0}([f,f]_+ - [f,f]_-)$$

and then the result follows by Lemma 38 and Lemma 26.

Finally, we deal with (44). By Hölder's inequality:

$$\limsup_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} (f(q)-f(p))(g(q)-g(p))\mu(q) \leq$$

$$\limsup_{r\to 0}\{\frac{1}{r^2} \fint_{B(p,r)} (f(q)-f(p))^2\mu(q)\}^{\frac{1}{2}} \limsup_{r\to 0}\{\frac{1}{r^2} \fint_{B(p,r)} (g(q)-g(p))^2\mu(q)\}^{\frac{1}{2}}$$

$$\leq \operatorname{Lip} f \operatorname{Lip} g, \quad (45)$$

c.f. Lemma 23. Also

$$(f(q)-f(p))(g(q)-g(p)) = \big(f(q)g(q)-f(p)g(p)\big)-f(p)\big(g(q)-g(p)\big)-g(p)\big(f(q)-f(p)\big)$$

thus $\frac{1}{r^2}\fint_{B(p,r)}(f(q)-f(p))(g(q)-g(p))\mu(q)$ is convergent as $r\to 0$ iff $\Delta_\mu(fg)$ is well defined at p. Then since

$$(f(q)-f(p))(g(q)-g(p)) = f(q)(g(q)-g(p)) - f(p)(g(q)-g(p))$$

we get by dominated convergence:

$$\int_X \lim_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} (f(q)-f(p))(g(q)-g(p))\mu(q)\mu(p) =$$

$$-\lim_{r\to 0}\frac{1}{2}[f,g]_- - \frac{1}{2}\langle f, \Delta_\mu g\rangle = \lim_{r\to 0}\frac{1}{2}([f,g]_+ - [f,g]_-) - \langle f, \Delta_\mu g\rangle,$$

thus (44) follows by Lemma 38 and (45). □

4 Comparison with Riemannian manifolds

We will end the analysis of the "mean value" Laplacian in this paper, by showing, that it is in fact proportional to the usual Laplacian in the setting of Riemannian manifolds. Refer to [7] or [2] for background on Riemannian geometry.

Let (M,g) be a n-dimensional Riemannian manifold and let $f: M\to\mathbb{R}$ be a smooth function. The Riemannian Laplacian at $p\in M$ of f is usually defined by

$$\Delta_R f(p) := \operatorname{trace}(H_p^f),$$

where H_p^f is the Hessian of f at p, and the trace is taken with respect to the Riemannian metric. Another way of getting the same is:

$$\Delta_R f(p) = \frac{n}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} H_p^f(v,v)dv, \tag{46}$$

where ω_{n-1} is the volume of the unit $n-1$-sphere in T_pM.

On an n-dimensional Riemannian manifold the Riemannian volume measure conincides with n-dimensional Hausdorff measure, which we will denote by $\mathcal{H}^n$, while $n-1$-dimensional Hausdorff measure on codimension 1 sets, will be written $\mathcal{H}^{n-1}$.

Let $S(p,r) = \partial B(p,r) = \{q \in M |\, d(p,q) = r\}$ denote the sphere of radius r centered at p.

Normal and polar coordinates: We get normal coordinates $(x_1, x_2, \dots, x_n)$ around $p \in M$, by identifying (T_pM, g_p) with $\mathbb{R}^n$ via a linear isometry and using the exponential map, $\exp_p : U \to M$, on a neighbourhood U of $0_p \in T_pM$. In normal coordinates we have for the matrix of the (pulled back) metric:

$$g_{ij} = \delta_{ij} + O(r^2), \tag{47}$$

where $r = \sqrt{g_p(x,x)} = d(p, \exp_p(x))$. As usual we can introduce *polar coordinates* on T_pM by writing $w \in T_pM$ as $w = rv$, with $v \in \mathbb{S}^{n-1} \subset T_pM$. In the usual frame on T_pM associated to polar coordinates $\partial_r, \partial_{\theta_1}, \dots, \partial_{\theta_{n-1}}$, with $g_p(\partial_{\theta_\alpha}, \partial_{\theta_\beta}) = \delta_{\alpha\beta} r^2$, the pulled back metric is:

$$g = dr^2 + g_{\alpha\beta} d\theta^\alpha d\theta^\beta, \quad \alpha, \beta \in \{1, \dots, n-1\},$$

where $dr, d\theta^\alpha$, $\alpha = 1, 2, \dots, n-1$ is the dual frame. From (47) we get, by the transformation between polar and cartesian coordinates:

$$g_{\alpha\beta} = \delta_{\alpha\beta} r^2 + O(r^4)$$

Hence the *the volume density* in polar coordinates is

$$\lambda(r,v) = \sqrt{|\{g_{\alpha\beta}\}|} = \sqrt{r^{2(n-1)} + O(r^{2n})} = r^{n-1}\sqrt{1 + O(r^2)} = r^{n-1} + O(r^{n+1}),$$

and therefore

$$\mathcal{H}^{n-1}(S(p,r)) = \int_{\mathbb{S}^{n-1}} \lambda(r,v)dv = \omega_{n-1} r^{n-1} + O(r^{n+1})$$

Lemma 48. *On the Riemannian manifold M^n pick μ to be Riemannian volume measure, then for a smooth function f:*

$$\Delta_R f(p) = \lim_{r \to 0} \frac{2n}{r^2} \fint_{S(p,r)} \big(f(q) - f(p)\big) \mathcal{H}^{n-1}(q) \tag{49}$$

Proof. Given the smooth function f, we can expand it in polar coordinates around p as

$$f(r,v) = f(p) + r df_p(v) + \frac{1}{2} r^2 H_p^f(v,v) + o(r^2).$$

Define $S_r := \mathcal{H}^{n-1}(S(p,r))$. Then:

$$\begin{aligned}
\lim_{r\to 0} \frac{2n}{r^2} \fint_{S(p,r)} \big(f(q) - f(p)\big)\mathcal{H}^{n-1}(q) = \\
\lim_{r\to 0} \frac{2n}{r^2} \big(\frac{1}{S_r} \int_{\mathbb{S}^{n-1}} f(r,v)\lambda(r,v)dv - f(p)\big) = \\
\lim_{r\to 0} \big(\frac{2n}{rS_r} \int_{\mathbb{S}^{n-1}} df_p(v)\lambda(r,v)dv + \frac{n}{S_r} \int_{\mathbb{S}^{n-1}} H_p^f(v,v)\lambda(r,v)dv\big) + 0 \\
= \lim_{r\to 0} \big(\frac{2n}{rS_r} \int_{\mathbb{S}^{n-1}} df_p(v)(r^{n-1} + O(r^{n+1}))dv + \frac{n}{S_r} \int_{\mathbb{S}^{n-1}} H_p^f(v,v)(r^{n-1} + O(r^{n+1}))dv \\
= \lim_{r\to 0} \big(\frac{2nr^{n-1}}{rS_r} \int_{\mathbb{S}^{n-1}} df_p(v)dv + \frac{O(r^{n+1})}{rS_r} + \frac{nr^{n-1}}{S_r} \int_{\mathbb{S}^{n-1}} H_p^f(v,v)dv + \frac{O(r^{n+1})}{S_r}\big) \\
= \frac{n}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} H_p^f(v,v)dv = \Delta_R f(p)
\end{aligned}$$

since $\int_{\mathbb{S}^{n-1}} df_p(v)dv = 0$. □

Lemma 50. *For a smooth function $f : M^n \to \mathbb{R}$ we have:*

$$\lim_{r\to 0} \frac{1}{r^2} \fint_{S(p,r)} \big(f(q) - f(p)\big)\mathcal{H}^{n-1}(q) = \frac{n+2}{n} \lim_{r\to 0} \frac{1}{r^2} \fint_{B(p,r)} \big(f(q) - f(p)\big)\mathcal{H}^n(q) \tag{51}$$

Proof. Given $p \in M$ define $S_r := \mathcal{H}^{n-1}\big(S(p,r)\big)$ and $B_r := \mathcal{H}^n\big(B(p,r)\big)$. Then S_r and B_r are smooth for $r > 0$ close to 0 and (see above) $\frac{d}{dr}B_r = S_r = \omega_{n-1}r^{n-1} + O(n+1)$, thus $B_r = \frac{1}{n}\omega_{n-1}r^n + O(r^{n+2})$.

We can then write

$$\fint_{B(p,r)} (f(p) - f(q)\mathcal{H}^n(q) = \frac{1}{B_r} \int_0^r \{\int_{S(p,r)} (f(p) - f(q)\mathcal{H}^{n-1}(q)\}dr.$$

From the previous lemma it follows that $\int_{S(p,r)}(f(p) - f(q)\mathcal{H}^{n-1}(q) = ar^{n+1} + o(r^{n+1})$. Thus we see, that the comparison of the limits (51) will follow if

$$\frac{f}{g} = br^2 + o(r^2) \implies \frac{F}{G} = \frac{n}{n+2} br^2 + o(r^2),$$

when $g = r^{n-1} + o(r^{n-1})$ and$f = br^{n+1} + o(r^{n+1})$ and F, G are antiderivatives of f, g respectively. However this is an easy calculation; details are left to the reader. □

From these two lemmas it then follows:

Proposition 52. *Let $f : M^n \to \mathbb{R}$ be a smooth function, then*

$$\Delta_\mu f(p) = \frac{1}{n+2}\Delta_R f(p),$$

when we pick μ to be Riemannian volume measure.

Concluding Remarks

Proposition 52 suggests that for a sufficiently nice metric space, the natural Laplacian associated to the metric should be:

$$(d+2)\Delta_\mu, \tag{53}$$

where μ is taken to be (local) Hausdorff measure and d is the (local) Hausdorff dimension. We claim, that in fact this is the pointwise form of the Laplacian introduced in [3] on Gromov-Hausdorff limits of Riemannain manifolds with Ricci-curvature bounded from below.

We think that the utility of the simple direct definition of the Laplacian given here, has been demonstrated. Using the definition the calculus of the Laplacian is reduced to simple algebraic manipulations, always keeping convergence questions in mind, of course.

Finally, having a Laplacian on a metric measure space opens up for defining various curvature concepts, via some of the many connections between these entities in Riemannian geometry, c.f. [7]. The usefulness of such an approach remains to be investigated further.

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