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Symmetric, discrete fractional splines and Gabor systems

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In this paper we consider fractional splines as windows for Gabor frames. We introduce two new types of symmetric, fractional splines in addition to one found by Unser and Blu. For the finite, discrete case we present two families of splines: One is created by sampling and periodizing the continuous splines, and one is a truly finite, discrete construction. We discuss the properties of these splines and their usefulness as windows for Gabor frames and Wilson bases.

Keywords: Fractional splines; Hurwitz zeta function; Gabor systems; Wilson bases.

AMS Subject Classification: 42C15

1. Introduction

Fractional splines are a simple generalization of regular B-splines to fractional orders. They have been developed by Unser and Blu in a series of papers [22,3,4] mostly in the context of wavelets and fractional Brownian motion.

Fractional splines are interesting in relation to Gabor frames, because they provide a smooth parameter family of functions ranging from the rectangular box function, which generates an orthonormal Gabor basis, to the Gaussian function, which is the function with the best time/frequency concentration. The fact that splines converge to the Gaussian as the order grows is shown in [21] along with some considerations of B-splines as Gabor windows for continuous Gabor frames. A result about the non-existence of certain Gabor frames with spline windows was reported in [11].

Fractional splines retain most of the well-known properties of regular B-splines, but they are no longer compactly supported. This makes them less useful for applications where data is continuously produced and analyzed (e.g. processing long music signals). However, in many applications (e.g. image processing) all data are available at the time of processing, and fast algorithms exists for these kinds of Gabor systems, [24,1,20].

A regular B-spline of order n can be defined as n convolutions of the rectangular function. Because a convolution of two functions can be computed by pointwise multiplication of their Fourier transforms, we get the classical expression for the B-spline β^n of order $n = 0, 1, 2, \ldots$ in the Fourier domain:

$$\widehat{\beta^n}(\omega) = \mathrm{sinc}(\omega)^{n+1}, \quad \omega \in \mathbb{R}. \tag{1.1}$$

This definition extends naturally to fractional orders $\alpha > -\frac{1}{2}$:

$$\widehat{\beta_+^\alpha}(\omega) = \mathrm{sinc}(\omega)^{\alpha+1}, \quad \omega \in \mathbb{R}. \tag{1.2}$$

This is a simple shift of the β_+^α spline defined in [22]. This spline is not even around $x = 0$, because is has a complex valued Fourier transform.

Symmetry is an important goal for window construction because symmetric windows are a requirement for Wilson bases [7,5] and the modified discrete cosine transform (MDCT) [17,18]. Both Wilson bases and the MDCT are cosine modulated filter banks closely related to Gabor frames of twice the redundancy. In [3] Blu and Unser's presents a family of symmetric, fractional splines. In this paper, we present two other definitions, yielding splines that generates tighter (better conditioned) Gabor frames than Blu and Unsers definition.

A central property of B-splines is that they form a partition of unity, PU, meaning that the sum of integer translates of a B-spline is a constant function. This property can be important i.e. in image processing, where it is advantageous to be able to represent a large, almost constant area of an image using few coefficients. In relation to local, trigonometric bases, this has been investigated in [14,2].

In [3,22] Unser and Blu have shown methods to generate discrete fractional splines by sampling. In this paper, we present two other methods for producing discrete and finite splines. In Section 4 we present a method to obtain finite, discrete fractional splines by sampling and periodizing their continuous counterparts. In Section 5 we present another method where we transfer the definition of the continuous splines to the finite, discrete setting. Finally, in Section 6 we make numerical comparisons of the different types of splines and study their usefullness as windows for Gabor / Wilson / MDCT frames and bases.

2. Definitions

We define the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ as

$$(\mathcal{F}f)(\omega) = \hat{f}(\omega) = \int_{x\in\mathbb{R}} f(x)\, e^{-2\pi i \omega x} dx, \quad \omega \in \mathbb{R}, \tag{2.1}$$

and the Discrete Fourier Transform (DFT) of $h \in \mathbb{C}^L$ as

$$(\mathcal{F}h)(k) = \hat{h}(k) = \frac{1}{\sqrt{L}} \sum_{l=0}^{L-1} h(k)\, e^{-2\pi i k l / L}, \quad k = 0, \ldots, L-1, \tag{2.2}$$

where i denotes the imaginary unit. The sinc function is given by

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \in \mathbb{R} \setminus \{0\} \\ 1 & x = 0 \end{cases}. \tag{2.3}$$

The sinc function is the Fourier transform of the box function β_+^0, which is explicitly given by

$$\beta_+^0(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2} \\ \frac{1}{2} & \text{if } |x| = \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases} \tag{2.4}$$

Convolution of two functions is given by

$$(f * g)(x) = \int_{y\in\mathbb{R}} f(y)\overline{g(x-y)}dy, \quad x \in \mathbb{R},\ f,g \in L^1(\mathbb{R}) \tag{2.5}$$

$$(f * g)(l) = \sum_{k=0}^{L-1} f(k)\overline{g(l-k)}, \quad l = 0,\dots,L-1,\ f,g \in \mathbb{C}^L. \tag{2.6}$$

The convolution of two functions can be computed in the Fourier domain:

$$\widehat{f * g}(\omega) = \widehat{f}(\omega)\,\widehat{g}(\omega), \quad \omega \in \mathbb{R}, \forall f,g \in L^1(\mathbb{R}), \tag{2.7}$$

$$\widehat{f * g}(k) = \sqrt{L}\widehat{f}(k)\,\widehat{g}(k), \quad k = 0,\dots,L-1, \forall f,g \in \mathbb{C}^L. \tag{2.8}$$

A family of elements $\{e_j\}_{j\in J}$ in a separable Hilbert space $\mathcal{H}$ is called a frame if constants $0 < A \le B < \infty$ exist such that

$$A\|f\|_{\mathcal{H}}^2 \le \sum_{j\in J} \left|\langle f, e_j\rangle_{\mathcal{H}}\right|^2 \le B\|f\|_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H}. \tag{2.9}$$

The constants A and B are called lower and upper frame bounds, respectively.

We define the Wiener space by

$$W(\mathbb{R}) = \left\{ f \,\middle|\, \sum_{n\in\mathbb{Z}} \operatorname*{ess\,sup}_{x\in[0,1]} |f(x+n)| < \infty \right\}. \tag{2.10}$$

The Wiener space is a subspace of $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.

We define Gabor systems for $L^2(\mathbb{R})$ and $\mathbb{C}^L$ by

Definition 2.1. *A Gabor system (g, α, β) for $L^2(\mathbb{R})$ with $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$ is given by*

$$g_{m,n}(x) = e^{2\pi i m\beta x} g(x - na), \quad , m,n \in \mathbb{Z}. \tag{2.11}$$

Definition 2.2. *A Gabor system $\left(g^D, a, b\right)$ for $\mathbb{C}^L$ with $g \in \mathbb{C}^L$, $a, b \in \mathbb{N}$ is given by*

$$g_{m,n}^D(l) = e^{2\pi i m b l/L} g(l - na), \quad , m = 0,\dots,\frac{L}{b}-1, n = 0,\dots,\frac{L}{a}-1. \tag{2.12}$$

For more information about Gabor systems and frames, see the books [10,6,8,9].

We consider the following symmetries of discrete signals:

Definition 2.3. *Let $g \in \mathbb{C}^L$. We say that g is* whole point even *(WPE) if*

$$g(l) = \overline{g(-l)} = \overline{g(L-l)}$$

for $l = 0, \ldots, L-1$. This implies that $g(0)$ must always be real, and so must $g\left(\frac{L}{2}+1\right)$ if L is even.

Definition 2.4. *Let $g \in \mathbb{C}^L$. We say that g is* half point even *(HPE) if*

$$g(l) = \overline{g(L-1-l)}$$

for $l = 0, \ldots, L-1$. This implies that $g\left(\frac{L-1}{2}\right)$ must be real if L is odd.

If g is HPE then

$$\hat{g}(k) = h(k)\, e^{\pi i k/L}, \tag{2.13}$$

where h is some real-valued signal: $h \in \mathbb{R}^L$.

It is relevant to consider these two symmetries, because WPE and HPE windows are a requirement for discrete Wilson bases [5] and the MDCT [17,18].

Definition 2.5. *The basis functions $w_{m,n} \in \mathbb{C}^L$ of a Wilson basis for $\mathbb{C}^L$ with $M \in \mathbb{N}$ channels and where $g \in \mathbb{C}^L$ is WPE if $c_t = 0$ or HPE if $c_t = \frac{1}{2}$ are given by:*

If $m = 0$:

$$w_{0,n}(l) = g(l - 2nM)$$

If m is odd and less than M:

$$\begin{aligned} w_{m,n}(l) &= \sqrt{2}\sin(2\pi\frac{m}{2M}(l+c_t))g(l-2nM) \\ w_{m+M,n}(l) &= \sqrt{2}\cos(2\pi\frac{m}{2M}(l+c_t))g(l-(2n+1)M) \end{aligned}$$

If m is even and less than M:

$$\begin{aligned} w_{m,n}(l) &= \sqrt{2}\cos\left(2\pi\frac{m}{2M}(l+c_t)\right)g(l-2nM) \\ w_{m+M,n}(l) &= \sqrt{2}\sin\left(2\pi\frac{m}{2M}(l+c_t)\right)g(l-(2n+1)M) \end{aligned}$$

If $m = M$ and M is even:

$$w_{M,n}(l) = (-1)^l g(l - 2nM)$$

else if $m = M$ and M is odd:

$$w_{M/2,n}(l) = (-1)^l g(l-(2n+1)M).$$

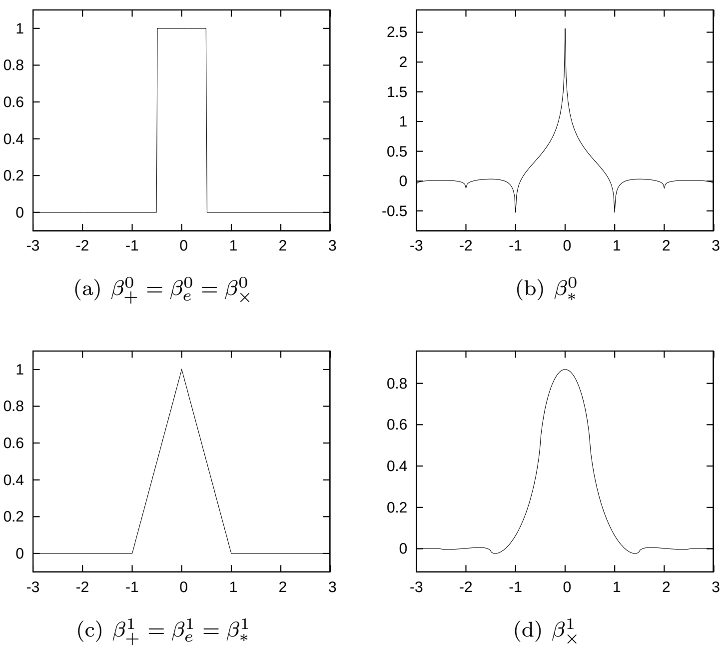


Fig. 1. The figure shows the B-splines of order 0 and 1.

3. Symmetric splines for the real line

The β_+^α spline defined by (1.2) is not symmetric for α not being an integer, which means in cannot be used as a window function with Wilson bases. In the following, we shall present three ways to overcome this problem.

The first way is to add β_+^α to its own reverse:

$$\beta_e^\alpha(x) = \frac{\beta_+^\alpha(x) + \beta_+^\alpha(-x)}{2}. \tag{3.1}$$

In the Fourier domain, this has the expression

$$\widehat{\beta_e^\alpha} = \frac{\widehat{\beta_+^\alpha} + \overline{\widehat{\beta_+^\alpha}}}{2} = \Re\left(\widehat{\beta_+^\alpha}\right). \tag{3.2}$$

Another way of looking at this is, that β_e^α is the even part of β_+^α. This can be seen from (3.1).

In [3] Blu and Unser's presents a family of symmetric, fractional splines, β_*^α. These splines coincides with the regular B-splines only when α is an odd integer. In particular, β_*^0 is not the box function. For our purpose, namely Gabor analysis, this is a downside, because the box function often (depending on the parameters α and β) generates a tight Gabor frame. We will therefore define another family of B-splines, $\beta_\times^\alpha$, which coincides with the regular B-splines for α being an even integer.

The definition of β_*^α and $\beta_\times^\alpha$ comes from substituting the normal power function x^α with its signed and unsigned $|x|^\alpha$ counterparts. The *signed power* of a real number is given by:

$$s^{\langle\alpha\rangle} = |s|^{\alpha-1} s = |s|^\alpha \operatorname{sign}(s), \quad \forall s \in \mathbb{R}. \tag{3.3}$$

We can now define the three types of B-splines:

Definition 3.1. *Let $\alpha > -\frac{1}{2}$. Then $\beta_e^\alpha, \beta_*^\alpha, \beta_\times^\alpha \in L^2(\mathbb{R})$ are given in the Fourier domain by*

$$\widehat{\beta_e^\alpha}(\omega) = \Re\left(sinc(\omega)^{\alpha+1}\right), \quad \omega \in \mathbb{R}, \tag{3.4}$$

$$\widehat{\beta_*^\alpha}(\omega) = |sinc(\omega)|^{\alpha+1}, \quad \omega \in \mathbb{R}, \tag{3.5}$$

$$\widehat{\beta_\times^\alpha}(\omega) = sinc(\omega)^{\langle\alpha+1\rangle}, \quad \omega \in \mathbb{R}. \tag{3.6}$$

We shall use the notation β^α in results that hold for all three type of B-splines. When $\alpha > -\frac{1}{2}$ then sinc raised to $\alpha+1$ is an $L^2(\mathbb{R})$ function, and because the Fourier transform maps $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, then $\beta^\alpha \in L^2(\mathbb{R})$.

It is clear from the definition that $\beta_e^\alpha = \beta_\times^\alpha$ for even values of α, and $\beta_e^\alpha = \beta_*^\alpha$ for odd values of α. Contrary to the regular B-splines then $\beta_\times^\alpha$ is not compactly supported for odd values of α, and β_*^α is not compactly supported for even values of α. For integer values of α, β_e^α is equal to the normal, compactly supported B-splines. Figure 1 shows the B-splines of order 0 and 1, where these differences are most visible. All the splines quickly start to resemble the Gaussian function as $\alpha \to \infty$. Figure 2 shows the three symmetric B-splines for order $\alpha = 0.3$.

From the definition and (2.7) it can be seen that the splines have the following simple convolution properties for $\alpha > -\frac{1}{2}$:

$$\beta_e^{\alpha+1} = \beta_e^\alpha * \beta_+^0 \tag{3.7}$$

$$\beta_*^{\alpha+1} = \beta_*^\alpha * \beta_*^0 \tag{3.8}$$

$$\beta_\times^{\alpha+1} = \beta_*^\alpha * \beta_+^0 \tag{3.9}$$

In [22], pure time domain expressions are given for β_+^α and β_*^α. By adapting the proof, the same decay result can be shown for β_e^α and $\beta_\times^\alpha$:

$$|\beta^\alpha(x)| \leq C_t |x|^{-(\alpha+2)}, \tag{3.10}$$

where C_t is a constant independent of x.

The decay in frequency is straightforward to show from the definition:

$$\left|\widehat{\beta^\alpha}(\omega)\right| \leq C_f |\omega|^{-(\alpha+1)}, \tag{3.11}$$

where C_f is a constant independent of ω. Using the decay results, it is easy to show that $\beta^\alpha \in W(\mathbb{R})$ when $\alpha \geq 0$ and $\widehat{\beta^\alpha} \in W(\mathbb{R})$ when $\alpha > 0$.

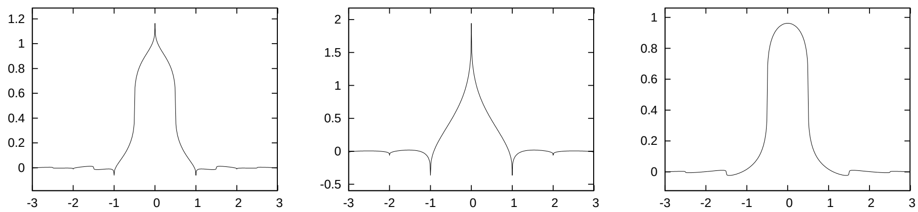


Fig. 2. The three splines of order $\alpha = 0.3$. From left to right: β_e^α, β_*^α and $\beta_\times^\alpha$.

The three types of splines all form a partition of unity (PU):

$$\sum_{k\in\mathbb{Z}} \beta^\alpha (x+k) = 1, \quad a.e. x \in \mathbb{R}. \tag{3.12}$$

Since $\beta^\alpha \in L^1(\mathbb{R})$, this is a consequence of the fact that

$$\widehat{\beta^\alpha}(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{0\} \end{cases} \tag{3.13}$$

For a simple proof, see [23] .

4. Discrete splines by sampling and periodization

In the papers [16,13,15,19] a theory for finite, discrete Gabor frames obtained from sampled and periodized Gabor frames for $\mathbb{C}^L$ has been developed. The main results are as follows: Suppose that $g \in L^2(\mathbb{R})$ (satisfying certain conditions) generates a Gabor frame for $L^2(\mathbb{R})$. Then

- $g^D \in \mathbb{C}^L$ obtained from g by sampling and periodization will generate a Gabor frame for $\mathbb{C}^L$ with the same (or better) frame bounds.
- The canonical dual/tight windows of these two Gabor systems are related by sampling and periodization as well.
- If $f \in L^2(\mathbb{R})$ and $f^D \in \mathbb{C}^L$ are also related by sampling and periodization then

$$\left\langle f^D, g^D_{m,n} \right\rangle = \sum_{r\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} \left\langle f, g_{m+\frac{rL}{b}, n+\frac{sL}{a}} \right\rangle . \tag{4.1}$$

  This corresponds to an aliasing in time and frequency of the expansion coefficients from the continuous Gabor system, very similar to the well known phenomena for Fourier series and the DFT.

This theory is the reason for trying to define finite, discrete fractional splines by sampling and periodization.

We define the finite, discrete splines by sampling and periodization of their continuous counterparts. We shall denote splines that are WPE by a capital 'W' and similarly denote splines that are HPE by a capital 'H', see Definition 2.3 and 2.4:

Definition 4.1. *We define the discrete splines* $C_{L,a}^{\alpha,+W}$, $C_{L,a}^{\alpha,eW}$, $C_{L,a}^{\alpha,*W}$, $C_{L,a}^{\alpha,\times W}$, $C_{L,a}^{\alpha,+H}$, $C_{L,a}^{\alpha,eH}$, $C_{L,a}^{\alpha,*H}$, $C_{L,a}^{\alpha,\times H}$ *by sampling and periodization of their continuous counterparts:*

$$C_{L,a}^{\alpha,W}(l) = \frac{1}{a}\sum_{n\in\mathbb{Z}}\beta^{\alpha}\left(\frac{l}{a}+nN\right), \quad l=0,\ldots,L-1 \tag{4.2}$$

$$C_{L,a}^{\alpha,H}(l) = \frac{1}{a}\sum_{n\in\mathbb{Z}}\beta^{\alpha}\left(\frac{2l+1}{2a}+nN\right), \quad l=0,\ldots,L-1. \tag{4.3}$$

The splines $C_{L,a}^{\alpha,+W}$, $C_{L,a}^{\alpha,\times W}$ $C_{L,a}^{\alpha,+H}$ $C_{L,a}^{\alpha,\times H}$ *are defined for* $\alpha \geq 0$, *while* $C_{L,a}^{\alpha,*W}$ *and* $C_{L,a}^{\alpha,*H}$ *are only defined for* $\alpha > 0$. *This is because the periodization of* β_*^0 *is divergent.*

To find Fourier domain expressions for $\alpha > 0$ for these splines, the main tool will be the Poisson summation formula:

Lemma 4.1 (The Poisson summation formula). *. Let* $f \in L^2(\mathbb{R})$ *then*

$$\sum_n f(x+nN) = \frac{1}{L}\sum_k \hat{f}\left(\frac{k}{N}\right)e^{2\pi ikx/N}, \quad a.e.\, x\in\mathbb{R}. \tag{4.4}$$

When both $f, \hat{f} \in W(\mathbb{R})$, the Poisson summation formula hold with pointwise convergence everywhere, [10], and this is exactly the case for β^α when $\alpha > 0$.

We shall make use of *Hurwitz' zeta function* [12], also known as the generalized zeta function:

$$\zeta(z,v) = \sum_{k=0}^{\infty}\frac{1}{(k+v)^z}, \quad z>1,\, v>0 \tag{4.5}$$

In the following, we set $\beta = \alpha + 1$ to shorten the formulas. Using the Poisson summation formula (4.4) and Hurwitz' zeta function, we obtain Fourier domain expressions for for the splines defined in Definition 4.1:

Proposition 4.1. *For* $\alpha > 0$ *we have the following. For even* a *and* $m = 1,\ldots,L-1$.

$$\widehat{C_{L,a}^{\alpha,eW}}(m) = \frac{1}{\sqrt{L}}\Re\left(\left(\frac{\sin\left(\frac{\pi m}{N}\right)}{\pi a}\right)^{\beta}\left(\zeta\left(\beta,\frac{m}{L}\right)+e^{-\pi i\beta}\zeta\left(\beta,1-\frac{m}{L}\right)\right)\right) \tag{4.6}$$

$$\widehat{C_{L,a}^{\alpha,*W}}(m) = \frac{1}{\sqrt{L}}\left|\frac{\sin\left(\frac{\pi m}{N}\right)}{\pi a}\right|^{\beta}\left(\zeta\left(\beta,\frac{m}{L}\right)+\zeta\left(\beta,1-\frac{m}{L}\right)\right) \tag{4.7}$$

$$\widehat{C_{L,a}^{\alpha,\times W}}(m) = \frac{1}{\sqrt{L}}\left(\frac{\sin\left(\frac{\pi m}{N}\right)}{\pi a}\right)^{\langle\beta\rangle}\left(\zeta\left(\beta,\frac{m}{L}\right)-\zeta\left(\beta,1-\frac{m}{L}\right)\right) \tag{4.8}$$

For odd a and $m = 1, \ldots, L-1$ we get the expressions

$$\widehat{C_{L,a}^{\alpha,eW}}(m) = \frac{1}{\sqrt{L}} \Re \left(\left(\frac{\sin\left(\frac{\pi m}{N}\right)}{2\pi a} \right)^{\beta} \left((-1)^{-\beta} \zeta\left(\beta, 1 - \frac{m}{2L}\right) + \zeta\left(\beta, \frac{m}{2L}\right) + \right.\right.$$
$$\left.\left. + \zeta\left(\beta, \frac{1}{2} - \frac{m}{2L}\right) + (-1)^{\beta} \zeta\left(\beta, \frac{1}{2} + \frac{m}{2L}\right) \right) \right) \quad (4.9)$$

$$\widehat{C_{L,a}^{\alpha,\times W}}(m) = \frac{1}{\sqrt{L}} \left(\frac{\sin\left(\frac{\pi m}{N}\right)}{2\pi a} \right)^{\langle\beta\rangle} \left(\zeta\left(\beta, \frac{m}{2L}\right) - \zeta\left(\beta, 1 - \frac{m}{2L}\right) + \right.$$
$$\left. + \zeta\left(\beta, \frac{1}{2} - \frac{m}{2L}\right) - \zeta\left(\beta, \frac{1}{2} + \frac{m}{2L}\right) \right). \quad (4.10)$$

The expression (4.7) holds for both even and odd a. For $m = 0$ and all values of a we get

$$\widehat{C_{L,a}^{\alpha,eW}}(0) = \widehat{C_{L,a}^{\alpha,*W}}(0) = \widehat{C_{L,a}^{\alpha,\times W}}(0) = \frac{1}{\sqrt{L}}. \quad (4.11)$$

For a derivation, see the appendix.

The discrete splines defined in this manner does not satisfy the discrete equivalent of the convolution properties (3.7)-(3.9). Instead, because these splines are formed from sampling and periodization, they have the subsampling property

$$C_{L,a}^{\alpha,W}(k) = \sqrt{c} C_{Lc,ac}^{\alpha,W}(ck), \quad k = 0, \ldots, L-1, \quad (4.12)$$

where $c \in \mathbb{N}$. This means, that if one picks out a regularly spaced subsequence of the splines containing the first element, then this sequence is again a spline of the same order and type. Another consequence is that interleaving a WPE spline with the corresponding HPE spline will form the WPE spline (differently scaled) of twice L and a.

5. Discrete splines by the DFT

In the previous section we defined finite, discrete splines by means of sampling and periodization. In this section we will define finite, discrete splines by transferring the continuous definition to the discrete case. We wish to create both WPE and HPE spline functions. For this the following simple observations can be used:

- The convolution of two WPE functions is again a WPE function.
- The convolution of an HPE and a WPE function is a again an HPE function.

However, it does not hold that the convolution of two HPE functions is again a HPE function. Contrary, it is a WPE function shifted by one sample.

We can construct splines by repeated convolution of a rectangular function, just as in the continuous case. The zeroth order splines is defined by a sampling and periodization of the box spline. Because the box spline is compactly supported, the periodization consists of only two terms:

Definition 5.1. *Let* $a = \{0, \ldots, L-1\}$. *We define* $D_{L,a}^{0,+W}, D_{L,a}^{0,+H} \in \mathbb{C}^L$ *by:*

$$D_{L,a}^{0,+W}(l) = \frac{1}{\sqrt{a}}\left(\beta_+^0\left(\frac{l-L}{a}\right) + \beta_+^0\left(\frac{l}{a}\right)\right), \tag{5.1}$$

$$D_{L,a}^{0,+H}(l) = \frac{1}{\sqrt{a}}\left(\beta_+^0\left(\frac{l+\frac{1}{2}-L}{a}\right) + \beta_+^0\left(\frac{l+\frac{1}{2}}{a}\right)\right), \tag{5.2}$$

for $l = 0, \ldots, L-1$.

By standard trigonometric formulas, we find that the DFT of $D_{L,a}^{0,W}$ is

$$\widehat{D_{L,a}^{0,+W}}(k) = \frac{2}{\sqrt{L}}\left(\sum_{l=1}^{\lceil (a-1)/2\rceil} b(l)\cos(2\pi kl/L)\right) \tag{5.3}$$

$$= \begin{cases} \frac{\sin\left(a\frac{\pi l}{L}\right)}{\sin\left(\frac{\pi l}{L}\right)} & \text{if } a \text{ is odd} \\ \frac{\sin\left((a-1)\frac{\pi l}{L}\right)+\sin\left((a+1)\frac{\pi l}{L}\right)}{2\sin\left(\frac{\pi l}{L}\right)} & \text{if } a \text{ is even} \end{cases} \tag{5.4}$$

where

$$b(l) = \begin{cases} \frac{1}{2} & \text{if } l = 0 \text{ or } l = \frac{a}{2} \\ 1 & \text{otherwise.} \end{cases} \tag{5.5}$$

We define the discrete, fractional WPE B-splines in the same way as their continuous counterparts.

Definition 5.2. *Let* $\alpha > -1$ *and* $a = \{0, \ldots, L-1\}$. *We define* $D_{L,a}^{\alpha,eW}, D_{L,a}^{\alpha,*W}, D_{L,a}^{\alpha,\times W} \in \mathbb{C}^L$

$$\widehat{D_{L,a}^{\alpha,eW}} = L^{\frac{\alpha}{2}}\Re\left(\left(\widehat{D_{L,a}^{\alpha,W}}\right)^{\alpha+1}\right) \tag{5.6}$$

$$\widehat{D_{L,a}^{\alpha,*W}} = L^{\frac{\alpha}{2}}\left|\widehat{D_{L,a}^{\alpha,W}}\right|^{\alpha+1} \tag{5.7}$$

$$\widehat{D_{L,a}^{\alpha,\times W}} = L^{\frac{\alpha}{2}}\left(\widehat{D_{L,a}^{\alpha,W}}\right)^{\langle\alpha+1\rangle} \tag{5.8}$$

The constant term $L^{\frac{\alpha}{2}}$ is necessary to ensure the proper normalization. It comes from the $\sqrt{L}$ appearing in (2.8).

The Fourier domain methods used so far need to be modified for defining the HPE splines, because the Fourier transform of an HPE function is not real valued. Instead we will use the convolution properties (3.7)-(3.9) as the definition. We define the zero-order splines by:

The unsigned power spline of order zero can be defined in the Fourier domain using (2.13):

Definition 5.3. *Let* $a = \{0, \ldots, L-1\}$. *We define* $D_{L,a}^{0,*H} \in \mathbb{C}^L$ *in the Fourier domain by*

$$\widehat{D_{L,a}^{0,*H}}(k) = \begin{cases} \left|\widehat{B_{L,a}^{0,+HD}}(k)\right| e^{\pi ik/L} & \text{if } 0 \leq k < \left\lfloor \frac{L}{2} \right\rfloor \\ -\left|\widehat{B_{L,a}^{0,+HD}}(k)\right| e^{\pi ik/L} & \text{otherwise} \end{cases}.$$

Using the two zero order HPE splines just defined, we can define all the HPE splines:

Definition 5.4. *Let* $a = \{0, \ldots, L-1\}$ *and* $\alpha > 0$. *We define* $D_{L,a}^{\alpha,eH}$, $D_{L,a}^{\alpha,*H}$, $D_{L,a}^{\alpha,\times H} \in \mathbb{C}^L$ *in the Fourier domain by:*

$$\widehat{D_{L,a}^{\alpha,eH}} = L^{\frac{\alpha}{2}} \Re\left(\left(\widehat{D_{L,a}^{0,W}}\right)^{\alpha}\right) \widehat{D_{L,a}^{0,+H}}, \tag{5.9}$$

$$\widehat{D_{L,a}^{\alpha,*H}} = L^{\frac{\alpha}{2}} \left|\widehat{D_{L,a}^{0,W}}\right|^{\alpha} \widehat{D_{L,a}^{0,*H}}, \tag{5.10}$$

$$\widehat{D_{L,a}^{\alpha,\times H}} = L^{\frac{\alpha}{2}} \left|\widehat{D_{L,a}^{0,W}}\right|^{\alpha} \widehat{D_{L,a}^{0,H}}. \tag{5.11}$$

6. Numerical results

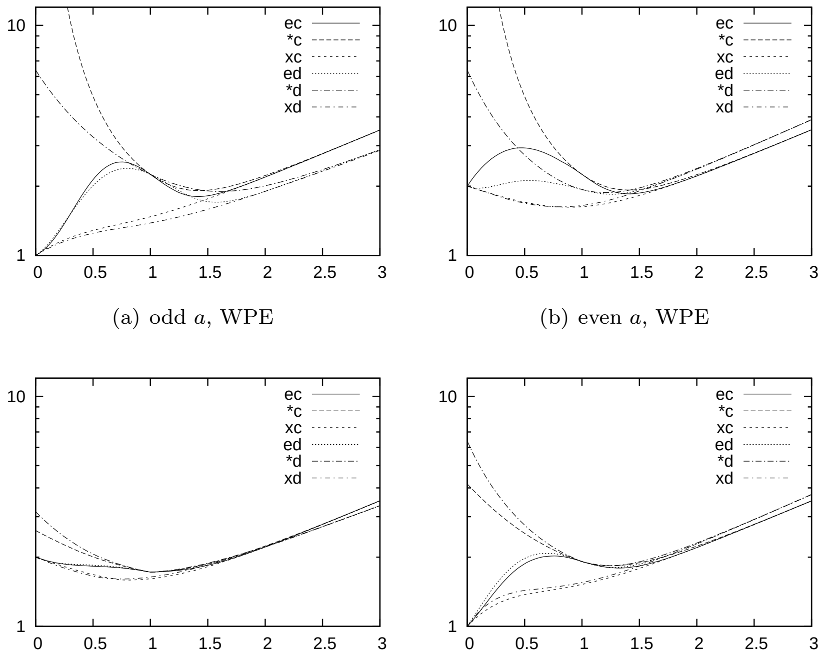
In this section we consider Gabor frames and Wilson bases as defined in Definition 2.2 and 2.5.

The six different types of splines (both in WPE and HPE variation) have been implemented in the Linear Time Frequency Toolbox (LTFAT) available from `http://www.univie.ac.at/nuhag-php/ltfat` as the function `pbspline` (Periodic B-spline). Implementations of Gabor systems, Wilson bases and frame bound calculations are available as well.

Implementations of Hurwitz' zeta function needed for the 'C' splines can (among other places) be found in GSL (the GNU Scientific Library), Maple and Mathematica. Octave uses the GSL implementation and Matlab uses the Maple implementation.

Figure 3 show a comparison of the frame bound ratio $\frac{B}{A}$ of Gabor frames using splines windows with different values of α. The windows that consistently generates the lowest frame bound ratios are the signed power splines, $C^{\alpha,\times}$ and $D^{\alpha,\times}$, and the unsigned power splines $C^{\alpha,*}$ and $D^{\alpha,*}$ generate the highest. The even splines, $C^{\alpha,e}$ and $D^{\alpha,e}$, oscillates between these two, as would be expected.

For values of α close to zero, using a WPE spline when a is odd and an HPE spline when a is even gives much lower frame bounds that doing the opposite. The cause of this is that $D_{L,a}^{0,+W}$ for a odd and $D_{L,a}^{0,+H}$ for a even consists of only the values 0 and 1. Because $D_{L,a}^{0,+W}$ forms a PU, then also $g(k) = \left|D_{L,a}^{0,+W}(k)\right|^2$ forms a PU. This is exactly the requirement for generating a tight Gabor frame for this case.



(a) odd a, WPE

(b) even a, WPE

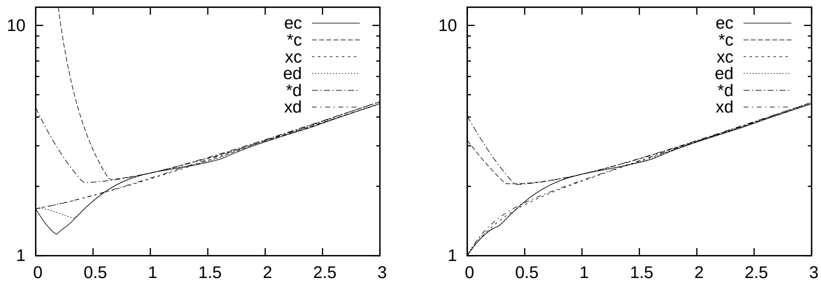
(c) odd a, HPE

(d) even a, HPE

Fig. 3. The figure show the frame bound ratio $\frac{B}{A}$ of a Gabor frame using one of the six types of splines as window function. The parameters for the left plot are $L = 48$, $a = 3$ and $M = 4$. For the right plot they are $L = 96$, $a = 6$ and $M = 8$. The top row shows WPE windows and the bottom row shows HPE windows.

Figure 4 show the same investigation for Wilson bases. We have constructed a Wilson basis for which the unmodulated terms form a PU. This means that we must choose the parameter a for the splines as $a = 2M$, and therefore the value of a cannot be odd. We see similar result as for Gabor frames, except that the 'e' splines may sometimes generate frames with a lower frame bound ratio than those generated by the 'x' splines.

Figure 5 shows the effect of using a window that forms a PU in image compression. The image used is a standard test image, the 'cameraman', which has a large background area of almost constant color. For each of the three windows $C^{0,eW}$, $C^{0.4,eW}$ and the Gaussian, the test image has been heavily compressed so as to produce visible artifacts of the compression. The two spline windows that form a PU provides a smooth resolution of the background, while the Gaussian produces a visibly disturbing pattern in the background. The box function on the other hand produces a sharp image but with visible horizontal and vertical lines, because of



(a) WPE (b) HPE

Fig. 4. The figure show the frame bound ratio $\frac{B}{A}$ of a Wilson basis as defined in Definition 2.5 using one of the six types of splines as window function. The parameters are $L = 36$ using $M = 6$ channels. The left figure shows a Wilson basis with $c_t = 0$ using WPE splines as window. The right plot show a Wilson basis with $c_t = \frac{1}{2}$ using HPE splines as windows.

the sharp cutoff.[a] The image produced by the fractional spline $C^{0.4,eW}$ shows a smoother background without disturbing lines or patterns.

7. Conclusion

We have introduced two families of finite, discrete, symmetric B-splines suitable for discrete Gabor analysis:

- A family of discrete splines constructed by sampling and periodization of the continuous splines. These splines are suitable for Gabor analysis if we are working with signals coming from a sampling of continuous signals, and desire a precise estimate of sampling errors. To compute these splines, evaluation of Hurwitz' zeta function is needed, which can be slow if not properly implemented.
- A family of discrete splines defined in the same manner as their continuous counterparts. These splines are very similar to the splines produced by sampling and periodization, but lack the close relationship to the continuous splines. They are very fast to compute, requiring only a single DFT.

For Gabor systems, the splines formed by raising the sinc function to a signed power consistently generated Gabor frames with a lower frame bound ratio than the other types of splines.

An implementation of the different type of splines discussed in this paper is available in the LTFAT toolbox as the function `pbspline`. By default this function will return a spline of type $B^{e,D}_{L,a}$, as this spline coincides with the normal B-splines

[a]When using this window a 2D Wilson transform is almost the same transform as the 2D block DCT that is used in JPEG encoding of images.

(a) Original

(b) Box WPE window

(c) 'ec' WPE spline window with $\alpha = .4$

(d) Gauss window

Fig. 5. The figure shows a test done on a standard test image, the Cameraman. The image in the upper left corner is the original 256x256 greyscale image. The other 3 images have been compressed using a 2D Wilson basis with $M = 16$ channels and different window functions as specified below each image. The specified window was used for synthesis and the corresponding canonical dual window was used for analysis. The 1% largest coefficients have been kept, and all other coefficients set to zero.

for all integer orders, and because it is fast and reliable to compute (computation does not depend on an external library to be available).

Appendix A. Derivation of the periodic, discrete fractional splines by sampling and periodization

We wish to find an explicit expression for the DFT of

$$C_{L,a}^{\alpha,+W}(l) = \frac{1}{a}\sum_{n}\beta_{+}^{\alpha}\left(\frac{l}{a} + nN\right). \tag{A.1}$$

To derive this, we will need to consider the following two sums: For the first one, we expand the sum in (4.5) to run through all whole numbers:

$$\sum_{k\in\mathbb{Z}} (z+k)^{-\beta} = \sum_{k=1}^{\infty} (z-k)^{-\beta} + \sum_{k=0}^{\infty} (z+k)^{-\beta} \tag{A.2}$$

$$= \sum_{k=0}^{\infty} (z-(k+1))^{-\beta} + \sum_{k=0}^{\infty} (z+k)^{-\beta} \tag{A.3}$$

$$= (-1)^{-\beta} \zeta(\beta, 1-z) + \zeta(\beta, z) \tag{A.4}$$

For the second one, we consider a sum with alternating signs. We split it for k even and k odd and use (A.4):

$$\sum_{k\in\mathbb{Z}} \frac{\left((-1)^k\right)^{\beta}}{(z+k)^{\beta}} = \sum_{k\in\mathbb{Z}} \frac{1}{(z+2k)^{\beta}} + \sum_{k\in\mathbb{Z}} \frac{(-1)^{\beta}}{(z+2k+1)^{\beta}} \tag{A.5}$$

$$= 2^{-\beta} \left(\sum_{k\in\mathbb{Z}} \left(\frac{z}{2}+k\right)^{-\beta} + (-1)^{\beta} \sum_{k\in\mathbb{Z}} \left(\frac{z+1}{2}+k\right)^{-\beta} \right) \tag{A.6}$$

$$= 2^{-\beta} \Bigg((-1)^{-\beta} \zeta\left(\beta, 1-\frac{z}{2}\right) + \zeta\left(\beta, \frac{z}{2}\right) + $$

$$+ \zeta\left(\beta, \frac{1}{2}-\frac{z}{2}\right) + (-1)^{\beta} \zeta\left(\beta, \frac{1}{2}+\frac{z}{2}\right) \Bigg) \tag{A.7}$$

Using the Poisson summation formula (4.4) and the well-known aliasing of high-mode waves to low mode waves (also a form of Poisson-summation) we obtain

$$C_{L,a}^{\alpha,+W}(l) = \frac{1}{a} \sum_{n} \beta_{+}^{\alpha} \left(\frac{l}{a} + nN\right) \tag{A.8}$$

$$= \frac{1}{L} \sum_{k\in\mathbb{Z}} \operatorname{sinc}\left(\frac{k}{N}\right)^{\alpha+1} e^{2\pi i k l/L} \tag{A.9}$$

$$= \frac{1}{L} \sum_{m=0}^{L-1} \left(\sum_{k\in\mathbb{Z}} \operatorname{sinc}\left(\frac{m+kL}{N}\right)^{\alpha+1} \right) e^{2\pi i l m/L} \tag{A.10}$$

$$= \frac{1}{L} \sum_{m=0}^{L-1} \left(\sum_{k\in\mathbb{Z}} \operatorname{sinc}\left(\frac{m}{N}+ka\right)^{\alpha+1} \right) e^{2\pi i l m/L} \tag{A.11}$$

for $j = 0, \ldots, L-1$ and $\alpha > 0$. The infinite sum is absolute convergent because we have assumed $\alpha > 0$. From the last expression we recognize the discrete Fourier transform of $C_{L,a}^{\alpha,+W}$:

$$\widehat{C_{L,a}^{\alpha,+W}}(m) = \frac{1}{\sqrt{L}} \sum_{k\in\mathbb{Z}} \operatorname{sinc}\left(\frac{m}{N}+ka\right)^{\alpha+1}, \quad m = 0, \ldots, L-1. \tag{A.12}$$

To get rid of the infinite sum, we rewrite for $m = 1, \ldots, L-1$:

$$\widehat{C_{L,a}^{\alpha,+W}}(m) = \frac{1}{\sqrt{L}}\sum_{k\in\mathbb{Z}}\operatorname{sinc}\left(\frac{m}{N}+ka\right)^{\alpha+1} \tag{A.13}$$

$$= \frac{1}{\sqrt{L}}\sum_{k\in\mathbb{Z}}\left(\frac{\sin\left(\frac{\pi m}{N}+\pi ka\right)}{\frac{\pi m}{N}+\pi ka}\right)^{\alpha+1} \tag{A.14}$$

$$= \frac{1}{\sqrt{L}}\sin\left(\frac{\pi m}{N}\right)^{\alpha+1}\sum_{k\in\mathbb{Z}}\frac{\left((-1)^{ka}\right)^{\alpha+1}}{\left(\frac{\pi m}{N}+\pi ka\right)^{\alpha+1}} \tag{A.15}$$

$$= \frac{1}{\sqrt{L}}\left(\frac{\sin\left(\frac{\pi m}{N}\right)}{\pi a}\right)^{\alpha+1}\sum_{k\in\mathbb{Z}}\frac{\left((-1)^{ka}\right)^{\alpha+1}}{\left(\frac{m}{L}+k\right)^{\alpha+1}} \tag{A.16}$$

The last expression (A.16) simplifies when a is even. We therefore first treat the case when a is even and use (A.4) with $\beta = \alpha + 1$:

$$\widehat{B_{L,a}^{\alpha,+WC}}(m) \tag{A.17}$$

$$= \frac{1}{\sqrt{L}}\left(\frac{\sin\left(\frac{\pi m}{N}\right)}{\pi a}\right)^{\alpha+1}\sum_{k\in\mathbb{Z}}\frac{1}{\left(\frac{m}{L}+k\right)^{\alpha+1}} \tag{A.18}$$

$$= \frac{1}{\sqrt{L}}\left(\frac{\sin\left(\frac{\pi m}{N}\right)}{\pi a}\right)^{\alpha+1}\left((-1)^{-(\alpha+1)}\zeta\left(\alpha+1,1-\frac{m}{L}\right)+\zeta\left(\alpha+1,\frac{m}{L}\right)\right) \tag{A.19}$$

To treat the case when a is odd, we use use (A.7) on (A.16):

$$\widehat{B_{L,a}^{\alpha,+WC}}(m) = \frac{1}{\sqrt{L}}\left(\frac{\sin\left(\frac{\pi m}{N}\right)}{\pi a}\right)^{\alpha+1}\sum_{k\in\mathbb{Z}}\frac{\left((-1)^{k}\right)^{\alpha+1}}{\left(\frac{m}{L}+k\right)^{\alpha+1}} \tag{A.20}$$

$$= \frac{1}{\sqrt{L}}\left(\frac{\sin\left(\frac{\pi m}{N}\right)}{2\pi a}\right)^{\alpha+1}\left((-1)^{-(\alpha+1)}\zeta\left(\alpha+1,1-\frac{m}{2L}\right)+\zeta\left(\alpha+1,\frac{m}{2L}\right)+\right.$$

$$\left.+\zeta\left(\alpha+1,\frac{1}{2}-\frac{m}{2L}\right)+(-1)^{\alpha+1}\zeta\left(\alpha+1,\frac{1}{2}+\frac{m}{2L}\right)\right) \tag{A.21}$$

For $m = 0$ we get trivially from the properties of the sinc function that

$$\widehat{C_{L,a}^{\alpha,+W}}(0) = \frac{1}{\sqrt{L}}. \tag{A.22}$$

The derivations of $\widehat{C_{L,a}^{\alpha,*W}}$ and $\widehat{C_{L,a}^{\alpha,\times W}}$ are very similar.

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