GABOR FRAMES BY SAMPLING AND PERIODIZATION.

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Abstract. By sampling the window of a Gabor frame for $L^2(\mathbb{R})$ satisfying certain conditions, one obtains a Gabor frame for $l^2(\mathbb{Z})$. In this article this idea is extended to cover interrelations among Gabor frames for the four spaces $l^2(\mathbb{Z})$, $L^2([0;L])$ and $\mathbb{C}^L$. It is shown how the various types of Gabor frames are related by sampling and periodization and how general dual windows and canonical dual windows are also related by sampling and periodizations. Corresponding results about the expansion coefficients of functions with respect to the Gabor frames are shown. All this can be used for fast numerical computation of approximations to windows and expansion coefficients for the infinite-dimensional Gabor frames. An example is presented.

1. Introduction.

In the article "From continuous to Discrete Weyl-Heisenberg Frames Through Sampling" [11], Janssen shows that under certain conditions one can obtain a Gabor frame for $l^2(\mathbb{Z})$ by sampling the window function of a Gabor frame for $L^2(\mathbb{R})$ at the integers. Furthermore, it is shown that the canonical dual windows of the two Gabor frames are also related by sampling.

The purpose of this article is to show how and under which conditions Gabor frames for the four spaces $L^2(\mathbb{R})$, $l^2(\mathbb{Z})$, $L^2([0;L])$ and $\mathbb{C}^L$ are interrelated by samplings and periodizations. The situation is shown on figure 1.1:

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\textit{sampling}} & l^2(\mathbb{Z}) \\ \big\downarrow \textit{periodization} & & \big\downarrow \\ L^2([0;L]) & \longrightarrow & \mathbb{C}^L \end{array}$$

Figure 1.1. Relationship among the four different Gabor frames. Arrows to the right indicate that a Gabor frame can be obtained through sampling, and arrows down indicate periodization.

The top arrow on figure 1.1 represent the relation shown in [11]. The rightmost relation, from $l^2(\mathbb{Z})$ to $\mathbb{C}^L$ by periodization is also shown in [11] and [15].

For each relation on figure 1.1, three types of results are presented:

(1) If the window function of a Gabor frame satisfies a certain condition, then a sampling/periodization of the window generates a Gabor frame with the same parameters and frame bounds for one of the

other spaces. The canonical dual windows of the two Gabor frames are similarly related by sampling/periodization.

(2) For functions f satisfying a regularity condition, simple relations holds between the expansion coefficients of f in a Gabor frame, and the expansion coefficients of the sampled/periodized function in the sampled/periodized Gabor frame.
(3) If f, γ are dual windows of a Gabor frame and satisfies a certain condition, the sampled/periodized windows are also dual windows of the sampled/periodized Gabor frame.

The original result presented in [11] only holds when the time/frequency-shift parameters are compatible for both the involved Gabor frames. This can be overcome by dilating the frame elements, and these generalizations are presented as well. The only requirement turns out to be that the product of the parameters should be rational.

These results are the main results, and are all presented in section 3.

The results allows one to build a mathematical model of a continuous phenomena, using Gabor frames for $L^2(\mathbb{R})$ or $L^2([0;L])$, and then perform the numerical work by computing an approximation to the expansion coefficients using samples of the functions.

The error in using the expansion coefficients obtained from the sampled system as an approximation to the expansion coefficients from the continuous system is given by Proposition 3.10 and 3.16, and consist of high-frequency coefficients being aliased to the low-frequency coefficients, just as in the Fourier-case.

Moreover, since the sampled Gabor system has the same frame bounds as the continuous one, it is just as stable. Fast numerical algorithms for Gabor frames for $\mathbb{C}^L$ exists, both for computing dual windows and for computing expansion coefficients.

In the last section, section 4, another simple application of the results in section 3 is presented: It is shown how to approximate the canonical dual window of a Gabor frame for $L^2(\mathbb{R})$ by a finite number of Gabor atoms from that Gabor frame.

In section 2 the required background and notation is presented. It includes the definitions of the various operators used, commutation relations between the operators, window classes for the Gabor frames, frames and Gabor frames for each of the four spaces.

2. Basic theory

The four spaces used in this article, $L^2(\mathbb{R})$,$L^2(\mathbb{Z})$,$L^2([0;L])$ and $\mathbb{C}^L$ shares two common properties:

(1) They are all Hilbert spaces. This is used to apply common results from frame theory, presented in Section 2.5, to all four of them.
(2) The domains $\mathbb{R}, \mathbb{Z}, [0;L]$ and $\mathbb{C}$ are all locally compact Abelian (LCA) groups (the interval $[0;L]$ will always be thought of as a parameterization of the torus, $\mathbb{T}$). Most of the theory used in this article can be defined solely in terms of LCA-groups. This includes the Fourier-transform, the translation- and modulation operators, the sampling- and periodization operators, Gabor systems and the space S_0. We

shall not give the LCA-definition, but instead define everything for each of the four spaces.

2.1. **The Fourier transforms.** The proofs in this article will make frequent use of various types of Fourier transformations. They are defined by

$$\begin{aligned}
\mathcal{F}_{\mathbb{R}}\, L^1(\mathbb{R}) \to C_0(\mathbb{R}) \quad &: \quad (\mathcal{F}f)(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} dx \\
\mathcal{F}_{\mathbb{Z}} : l^2(\mathbb{Z}) \to L^2([0;1]) \quad &: \quad (\mathcal{F}f)(\omega) = \sum_{k\in\mathbb{Z}} f(k) e^{-2\pi i k x} \\
\mathcal{F}_{[0;L]} : L^2([0;L]) \to l^2(\mathbb{Z}) \quad &: \quad (\mathcal{F}f)(k) = \frac{1}{\sqrt{L}} \int_0^L f(x) e^{-2\pi i k x/L} dx \\
\mathcal{F}_L : \mathbb{C}^L \to \mathbb{C}^L \quad &: \quad (\mathcal{F}f)(k) = \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} f(j) e^{-2\pi i k j/L},
\end{aligned}$$

where C_0 is the space of continuous functions vanishing at infinity.

$\mathcal{F}_{\mathbb{R}}$ is the Fourier transform, $\mathcal{F}_{\mathbb{Z}}$ is a Fourier series, $\mathcal{F}_{[0;L]}$ is Fourier coefficients and $\mathcal{F}_L$ is the Discrete Fourier Transform. The notation might seem a bit heavy, but it helps to avoid confusion when several different transforms are involved.

The Fourier transform $\mathcal{F}_{\mathbb{R}}$ can be extended to a bounded, linear, unitary operator, $\mathcal{F}_{\mathbb{R}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. With this extension, all the transforms are unitary operators.

For brevity, we shall use the notation $\mathcal{F}$ and $\hat{f}$ for $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{R}} f$ *only*! The other types of Fourier transformation will always be written with a subscript.

2.2. **The translation, modulation and dilation operators.** Gabor frames for the four spaces are defined in terms of the translation and modulation operators on the spaces.

Definition 2.1. The translation operators:

$$\begin{aligned}
T_t : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \quad &: \quad (T_t f)(x) = f(x-t),\ x,t \in \mathbb{R} \\
T_j : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \quad &: \quad (T_j f)(k) = f(k-j),\ k,j \in \mathbb{Z} \\
T_t : L^2([0;L]) \to L^2([0;L]) \quad &: \quad (T_t f)(x) = f(x-t),\ x,t \in [0;L] \\
T_j : \mathbb{C}^L \to \mathbb{C}^L \quad &: \quad (T_j f)(k) = f(k-j),\ k,j \in \{0,\ldots,L-1\}.
\end{aligned}$$

The space $L^2([0;L])$ will always be though of as a space of periodic functions, such that if $f \in L^2([0;L])$ then $f(x) = f(x+nL)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Definition 2.2. The modulation operators:

$$\begin{aligned}
M_\omega : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \quad &: \quad (M_\omega f)(x) = e^{2\pi i \omega x} f(x),\ x,\omega \in \mathbb{R} \\
M_\omega : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \quad &: \quad (M_\omega f)(j) = e^{2\pi i \omega j} f(j),\ j \in \mathbb{Z}, \omega \in [0;1] \\
M_k : L^2([0;L]) \to L^2([0;L]) \quad &: \quad (M_k f)(x) = e^{2\pi i k x/L} f(x),\ x \in [0;L],\ k \in \mathbb{Z} \\
M_k : \mathbb{C}^L \to \mathbb{C}^L \quad &: \quad (M_k f)(j) = e^{2\pi i k j/L} f(j),\ j,k \in \{0,\ldots,L-1\}.
\end{aligned}$$

The space $L^2([0;L])$ only permit modulations with integer parameters, because the exponential factor must have period L. The modulation operator for $l^2(\mathbb{Z})$ is periodic in its parameter because $e^{2\pi i\omega j} = e^{2\pi i(\omega+n)j}$ for all integers j and n. This is known as the aliasing phenomena.

Definition 2.3. Let $d > 0$. The dilation operators

$$D_d : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \quad : \quad (D_d f)(x) = \sqrt{d} f(dx), \quad \forall x \in \mathbb{R}.$$
$$D_d : L^2([0;L]) \to L^2([0;\frac{L}{d}]) \quad : \quad (D_d g)(x) = \sqrt{d} g(dx), \quad x \in [0;\frac{L}{d}].$$

The dilation operators are unitary operator for all $d > 0$, and the following commutation relation with the translation and modulation operators holds:

$$D_d M_\omega T_t = M_{d\omega} T_{\frac{t}{d}} D_d \text{ on } L^2(\mathbb{R}) \tag{2.1}$$
$$D_d M_k T_t = M_k T_{\frac{t}{d}} D_d \text{ on } L^2([0;L]) \tag{2.2}$$

Notice that the parameter of the modulation operator is unchanged in (2.2), as opposed to (2.1). This is just an effect of the definition of the modulation operator.

For the various Fourier transforms and the modulation and translation operators the following commutator relations hold:

$$\mathcal{F}_\mathbb{R} M_\omega T_t = e^{2\pi i t\omega} M_{-t} T_\omega \mathcal{F}_\mathbb{R} \text{ on } L^2(\mathbb{R}) \tag{2.3}$$
$$\mathcal{F}_\mathbb{Z} M_\omega T_j = e^{2\pi i t\omega} M_{-j} T_\omega \mathcal{F}_\mathbb{Z} \text{ on } l^2(\mathbb{Z}) \tag{2.4}$$
$$\mathcal{F}_{[0;L]} M_k T_t = e^{2\pi i tk/L} M_{-t/L} T_k \mathcal{F}_{[0;L]} \text{ on } L^2([0;L]) \tag{2.5}$$
$$\mathcal{F}_L M_k T_j = e^{2\pi i jk/L} M_{-j} T_k \mathcal{F}_L \text{ on } C^L \tag{2.6}$$

2.3. **Window classes.** As a convenient window class for the four different types of Gabor frames we shall use the spaces $S_0(\mathcal{G})$, known as *Feichtinger's algebra,* which can be defined for LCA-groups, [4]. For the LCA-groups considered in this article this is the spaces $S_0(\mathbb{R})$, $l^1(\mathbb{Z})$, $\mathcal{A}([0;L])$ and $\mathbb{C}^L$. $S_0(\mathbb{R})$ and $\mathcal{A}([0;L])$ will be defined in the following.

Definition 2.4. A function $g \in L^2(\mathbb{R})$ belongs to *Feichtinger's algebra* $S_0(\mathbb{R})$ if

$$\|g\|_{S_0} = \int_{\mathbb{R}\times\mathbb{R}} \left| \int_\mathbb{R} g(t)\overline{\varphi(t-x)} e^{-2\pi i t\omega} dt \right| dx d\omega < \infty,$$

where φ is some window in the Schwartz-class of smooth, exponentially decaying functions.

Different windows φ in the definition of S_0 yields equivalent norms. $S_0(\mathbb{R})$ is invariant under translation, modulation, dilation and the Fourier transform. $S_0(\mathbb{R}) \subseteq L^p(\mathbb{R})$ for $1 \le p \le \infty$. Functions in $S_0(\mathbb{R})$ are continuous. This, along with other useful properties of $S_0(\mathbb{R})$ can be found in [7] and [9].

A simple, useful condition for membership of $S_0(\mathbb{R})$ is the following: If $f, f', f'' \in L^1(\mathbb{R})$, then $f \in S_0(\mathbb{R})$. This is proved in [13].

For Gabor frames on the interval $[0;L]$ we shall use the window class $S_0([0;L]) = \mathcal{A}([0;L])$ of functions on the interval having an absolutely convergent Fourier series, $\mathcal{A}([0;L]) = \mathcal{F}^{-1}_{[0;L]} l^1(\mathbb{Z})$. $\mathcal{A}([0;L])$ is a Banach space with respect to the norm

$$\|f\|_{\mathcal{A}([0;L])} = \left\|\mathcal{F}_{[0;L]}^{-1} f\right\|_{l^1(\mathbb{Z})},$$

and is a Banach algebra with respect to point-wise multiplication. If $g \in \mathcal{A}([0;L])$ then g is a continuous function.

2.4. **Sampling, periodization and the Poisson summation formula.** The process of sampling is well-defined on $S_0(\mathbb{R})$:

Definition 2.5. Let $\alpha > 0$ and $a \in \mathbb{N}$. The sampling operators are given by

$$\begin{aligned}
\mathcal{S}_\alpha : S_0(\mathbb{R}) \to l^1(\mathbb{Z}) &: (\mathcal{S}_\alpha f)(j) = \sqrt{\alpha} f(j\alpha), \quad \forall j \in \mathbb{Z} \\
\mathcal{S}_a : l^1(\mathbb{Z}) \to l^1(\mathbb{Z}) &: (\mathcal{S}_a f)(j) = \sqrt{a} f(ja), \quad \forall j \in \mathbb{Z} \\
\mathcal{S}_\alpha : C([0;\alpha L]) \to \mathbb{C}^L &: (\mathcal{S}_\alpha f)(j) = \sqrt{\alpha} f(j\alpha), \quad j = 0, ..., L-1 \\
\mathcal{S}_a : \mathbb{C}^{aL} \to \mathbb{C}^L &: (\mathcal{S}_a f)(j) = \sqrt{a} f(ja), \quad j = 0, ..., L-1
\end{aligned}$$

The fact that the sampling operator is a bounded operator from $S_0(\mathbb{R})$ into $l^1(\mathbb{Z})$ is proved in [7, lemma 3.2.11]. The factor $\sqrt{\alpha}$ appearing in the definition of the sampling operator gives the sampling operators the following important properties:

- Composition with dilations:

$$\mathcal{S}_a D_b = \mathcal{S}_{ab} \text{ on } S_0(\mathbb{R}) \tag{2.7}$$

$$\mathcal{S}_a D_b = \mathcal{S}_{ab} \text{ on } C([0;L]) \tag{2.8}$$

- If $f \in S_0(\mathbb{R})$, $\|S_\alpha f\|_{l^2} < C \|f\|_{S_0}$ for some $C > 0$ independent of α. This is proved in [9, prop. 11.1.4].

Definition 2.6. The periodization operators are given by

$$\begin{aligned}
\mathcal{P}_L : S_0(\mathbb{R}) \to \mathcal{A}([0;L]) &: \mathcal{P}_L g(x) = \sum_{k\in\mathbb{Z}} g(x+kL), \quad x \in [0;L] \\
\mathcal{P}_L : l^1(\mathbb{Z}) \to \mathbb{C}^L &: \mathcal{P}_L g(j) = \sum_{k\in\mathbb{Z}} g(j+kL), \quad j = 0, ..., L-1 \\
\mathcal{P}_M : \mathcal{A}([0;ML]) \to \mathcal{A}([0;L]) &: \mathcal{P}_M g(x) = \sum_{k\in\mathbb{Z}} g(x+kM), \quad x \in [0;L] \\
\mathcal{P}_M : \mathbb{C}^{ML} \to \mathbb{C}^L &: \mathcal{P}_M g(j) = \sum_{k\in\mathbb{Z}} g(j+kM), \quad j = 0, ..., L-1
\end{aligned}$$

The fact that $\mathcal{P}_L : S_0(\mathbb{R}) \to \mathcal{A}([0;L])$ is proved in [5] in the more general context of LCA-groups.

The famous Poisson summation formula can now be stated as relations between the various Fourier transforms and the sampling- and periodization operators.

Theorem 2.7. *The Poisson summation formula for the four spaces:*

$$\mathcal{P}_M = \mathcal{F}_{[0;M]}^{-1} \mathcal{S}_{1/M} \mathcal{F}_{\mathbb{R}} \text{ on } S_0(\mathbb{R}) \tag{2.9}$$

$$\mathcal{P}_M = \mathcal{F}_M^{-1} \mathcal{S}_{1/M} \mathcal{F}_{\mathbb{Z}} \text{ on } l^1(\mathbb{Z}) \tag{2.10}$$

$$\mathcal{P}_M = \mathcal{F}_M^{-1} \mathcal{S}_b \mathcal{F}_{[0;L]} \text{ on } \mathcal{A}([0;L]) \tag{2.11}$$

$$\mathcal{P}_M = \mathcal{F}_M^{-1} \mathcal{S}_b \mathcal{F}_L \text{ on } \mathbb{C}^L \tag{2.12}$$

Proof. The $S_0(\mathbb{R})$ case is proved in [9, p. 105]. The other results can be proved by careful modification of this proof. Alternatively, the result is proved for general LCA-groups in [8] with convergence in L^2 . □

2.5. **Frames for Hilbert spaces.** Since we will deal with Gabor frames for four different spaces, it will be beneficial to first note what can be said about frames for general Hilbert spaces.

Definition 2.8. A family of elements $\{e_j\}_{j\in J}$ in a separable Hilbert space $\mathcal{H}$ is called a frame if constants $0 < A \leq B < \infty$ exist such that

$$A \|f\|_{\mathcal{H}}^2 \leq \sum_{j\in J} |\langle f, e_j\rangle_{\mathcal{H}}|^2 \leq B \|f\|_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H}. \tag{2.13}$$

The constants A and B are called lower and upper frame bounds.

The *frame operator* of a frame $\{e_j\}_{j\in J}$ for a Hilbert space $\mathcal{H}$ is defined by

$$S : \mathcal{H} \to \mathcal{H} \quad : \quad Sf = \sum_j \langle f, e_j\rangle_{\mathcal{H}} e_j, \tag{2.14}$$

where the series defining Sf converges unconditionally for all $f \in \mathcal{H}$.

The condition (2.13) ensures that the frame operator is both bounded and invertible on $\mathcal{H}$.

The inverse frame operator can be used to give a decomposition of any function $f \in \mathcal{H}$:

$$f = \sum_j \langle f, S^{-1}e_j\rangle e_j, \quad \forall f \in \mathcal{H}. \tag{2.15}$$

The frame $\{S^{-1}e_j\}$ is known as the canonical dual frame.

The following simple lemma [2, lemma 5.3.3] shall be used frequently.

Lemma 2.9. *Let T be a unitary operator on $\mathcal{H}$, and assume that $\{e_j\}$ is a frame for $\mathcal{H}$ with frame operator S. Then $\{Te_j\}$ is also a frame for $\mathcal{H}$ with the same frame bounds.*

2.6. **Gabor systems.** Gabor systems can be defined for the four spaces using the corresponding translation and modulation operators.

Definition 2.10. Gabor systems for each of the four spaces:

$L^2(\mathbb{R})$ A Gabor system (g, α, β) for $L^2(\mathbb{R})$ is given by

$$(g, \alpha, \beta) = \{M_{m\beta}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}, \tag{2.16}$$

where $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$.

$l^2(\mathbb{Z})$ A Gabor system $\left(g, a, \frac{1}{M}\right)$ for $l^2(\mathbb{Z})$ is given by

$$\left(g, a, \frac{1}{M}\right) = \{M_{m/M}T_{na}g\}_{m=0,\dots,M-1,n\in\mathbb{Z}}, \tag{2.17}$$

where $g \in l^2(\mathbb{Z})$ and $a, M \in \mathbb{N}$.

$L^2([0;L])$ A Gabor system (g, a, b) for $L^2([0;L])$ given by

$$(g, a, b) = \{M_{mb}T_{na}g\}_{m\in\mathbb{Z},n=0,\dots,N-1}, \tag{2.18}$$

where $g \in L^2([0;L])$, $a, b, N \in \mathbb{N}$ and $L = Na$.

$\mathbb{C}^L$ A Gabor system for (g,a,b) for $\mathbb{C}^L$ is given by

$$(g,a,b) = \{M_{mb}T_{na}g\}_{m=0,...,M-1,n=0,...,N-1}\,, \tag{2.19}$$

where $g \in \mathbb{C}^L$, $a,b,M,N \in \mathbb{N}$ and $Mb = Na = L$.

The decomposition formula (2.15) simplifies when the frame is a Gabor frame, because the inverse frame operator commutes with the translation and modulation operator. This means that the canonical dual frame also has Gabor structure, i.e. that $\left(S^{-1}g,\alpha,\beta\right)$ is the canonical dual frame of (g,α,β) where $\gamma^0 = S^{-1}g$ is known as the *canonical dual window.*

The Gabor coefficients of an S_0-function with respect to an S_0-window belongs to l^1. The precise relation for the four spaces are as follows.

Proposition 2.11. *Summability of Gabor coefficients.*

$S_0(\mathbb{R})$ *Let* $g,\gamma \in S_0(\mathbb{R})$ *and* $\alpha,\beta > 0$. *Then*

$$\sum_{m,n\in\mathbb{Z}} |\langle \gamma, M_{m\beta}T_{n\alpha}g\rangle| < \infty$$

$l^1(\mathbb{Z})$ *Let* $g,\gamma \in l^1(\mathbb{Z})$ *and* $a,M \in \mathbb{N}$. *Then*

$$\sum_{m=0}^{M-1}\sum_{n\in\mathbb{Z}} \left|\left\langle \gamma, M_{m/M}T_{na}g\right\rangle\right| < \infty$$

$\mathcal{A}([0;L])$ *Let* $g,\gamma \in \mathcal{A}([0;L])$ *and* $N,b \in \mathbb{N}$ *with* $L = Na$. *Then*

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{N-1} |\langle \gamma, M_{mb}T_{na}g\rangle| < \infty$$

Proof. The $S_0(\mathbb{R})$ part of this proposition is proved in [9, Cor. 12.1.12] or in [7].

The $\mathcal{A}([0;L])$-part is trivial to prove, because

$$\langle \gamma, M_{mb}T_{na}g\rangle = \mathcal{F}_{[0;L]}(\gamma T_{na}g)(mb).$$

Since both $\gamma, T_{na}g \in \mathcal{A}([0;L])$, $\mathcal{F}_{[0;L]}(\gamma T_{na}g) \in l^1(\mathbb{Z})$, so the desired coefficients can be extracted from a finite number of l^1-sequences. The $l^1(\mathbb{Z})$-case has a similar short proof, using a convolution. □

If a Gabor frame is not a Riesz basis, it has more than one dual frame. The dual frames that have Gabor structure are characterized bye the Wexler-Raz relations:

Theorem 2.12. ***Wexler-Raz.*** *If* g *and* γ *both generates Gabor systems as in Definition 2.10 with finite upper frame bounds then they are dual windows if and only if*

$L^2(\mathbb{R})$ $\frac{1}{\alpha\beta}\left\langle \gamma, M_{m/\alpha}T_{n/\beta}g\right\rangle = \delta_m\delta_n, \quad m,n \in \mathbb{Z}$

$l^2(\mathbb{Z})$ $\frac{M}{a}\left\langle \gamma, M_{m/a}T_{nM}g\right\rangle = \delta_m\delta_n, \quad m = 0,...,a-1, n \in \mathbb{Z}$

$L^2([0;L])$ $\frac{N}{b}\left\langle \gamma, M_{mN}T_{nM/L}g\right\rangle = \delta_m\delta_n, \quad m \in \mathbb{Z}, n = 0,...,b-1$

$\mathbb{C}^L$ $\frac{MN}{L}\langle \gamma, M_{mN}T_{nM}g\rangle = \delta_m\delta_n, \quad m = 0,...,a-1, n = 0,...,b-1$

Proof. The original result for $L^2(\mathbb{R})$ and $\mathbb{C}^L$can be found in [16]. More rigorous proofs with a minimal sufficient condition appears in [10, 3]. □

Lemma Lemma 2.9 shows that if one on the unitary operators $\mathcal{F}_{\mathbb{R}}$, $\mathcal{F}_{\mathbb{Z}}$, $\mathcal{F}_{[0;L}$, $\mathcal{F}_L$, D_d on $L^2(\mathbb{R})$ and D_d on $L^2([0;L])$ is applied to each element of Gabor frame, the result is again a frame with the same frame bounds. Moreover, because of the commutation relations (2.1), (2.2) and (2.3)-(2.6), the result is even a Gabor frame, albeit with different parameters. Furthermore, the canonical dual frame of such a Gabor frame has a simple expression:

Lemma 2.13. *Dual Gabor frames under Fourier transforms and dilations.*

$\mathcal{F}_{\mathbb{R}}$: Let γ^0 be the canonical dual window of (g, α, β), $g \in L^2(\mathbb{R})$. The canonical dual window of $(\mathcal{F}_{\mathbb{R}} g, \beta, \alpha)$ is $\mathcal{F}_{\mathbb{R}}\gamma^0$.

$\mathcal{F}_{\mathbb{Z}}$: Let γ^0 be the canonical dual window of $\left(g, a, \frac{1}{M}\right)$, $g \in l^2(\mathbb{Z})$. The canonical dual window of $\left(\mathcal{F}_{\mathbb{Z}} g, \frac{1}{M}, a\right)$ is $\mathcal{F}_{\mathbb{Z}}\gamma^0$.

$\mathcal{F}_{[0;L]}$: Let γ^0 be the canonical dual window of (g, a, b), $g \in L^2([0;L])$. The canonical dual window of $\left(\mathcal{F}_{[0;L]} g, b, \frac{a}{L}\right)$ is $\mathcal{F}_{[0;L]}\gamma^0$.

$\mathcal{F}_L$: Let γ^0 be the canonical dual window of (g, a, b), $g \in \mathbb{C}^L$. The canonical dual window of $(\mathcal{F}_L g, b, a)$ is $\mathcal{F}_L\gamma^0$.

D_d on $L^2(\mathbb{R})$:Let γ^0 be the canonical dual window of (g, α, β), $g \in L^2(\mathbb{R})$. The canonical dual window of $\left(D_d g, \frac{\alpha}{d}, \beta d\right)$ is $D_d\gamma^0$.

D_d on $L^2([0;L])$:Let γ^0 be the canonical dual window of (g, a, b), $g \in L^2([0;L])$. The canonical window of $\left(D_d g, \frac{a}{d}, b\right)$ is $D_d\gamma^0$.

Proof. The proofs can be found by direct calculation. They are very simple, and almost identical, the only difference is which of the commutation relations (2.1), (2.2) and (2.3)-(2.6) to use. The first relation appears as [9, 6.36]. □

3. Between the spaces.

3.1. From $L^2(\mathbb{R})$ to $l^2(\mathbb{Z})$.

By sampling the window function of a Gabor frame for $L^2(\mathbb{R})$, then under certain conditions one obtains a Gabor frame for $l^2(\mathbb{Z})$. These conditions will be presented and discussed in this section.

Theorem 3.1. *Let $g \in S_0(\mathbb{R})$, $M, a \in \mathbb{N}$ and assume that $\left(g, a, \frac{1}{M}\right)$ is a Gabor frame for $L^2(\mathbb{R})$ with canonical dual window γ^0.*

Then $\left(\mathcal{S}_1 g, a, \frac{1}{M}\right)$ is a Gabor frame for $l^2(\mathbb{Z})$ with the same frame bounds and canonical dual window $\mathcal{S}_1\gamma^0$.

This result is originally proved in [11]. In the article, Janssen works with less restrictive conditions than listed here, the so-called condition R and condition A.

Condition R is a regularity condition, that require the function to have some smoothness and decay around sampling points. It is much weaker that requiring the function to be in S_0. Most remarkably, it does not require anything of the function in between the sampling points. It is the only requirement to prove that $\left(\mathcal{S}_1 g, a, \frac{1}{M}\right)$ has the same frame bounds as $\left(\mathcal{S} g, a, \frac{1}{M}\right)$. In [12] it is proved that condition R is satisfied for all functions in $S_0(\mathbb{R})$ for all sampling distances.

To prove that $S_1\gamma^0$ is the canonical dual, the additional Condition A is needed. It depends on the parameters α, β, and is in general hard to verify.

The easiest way of satisfying it, is to require the function to be in $S_0(\mathbb{R})$. This is proved in [9].

This covers the sampling of a window function $g \in S_0(\mathbb{R})$ at the integers. The following theorem generalizes the situation:

Theorem 3.2. *Let $g \in S_0(\mathbb{R})$, $\alpha, \beta > 0$ with $\alpha\beta = \frac{a}{M}$ for some $M, a \in \mathbb{N}$, and assume that (g, α, β) is a Gabor frame for $L^2(\mathbb{R})$ with canonical dual window γ^0. Then $\left(\mathcal{S}_{\alpha/a} g, a, \frac{1}{M}\right)$ is a Gabor frame for $l^2(\mathbb{Z})$ with the same frame bounds and canonical dual window $\mathcal{S}_{\alpha/a}\gamma^0$.*

Proof. By Lemma 2.9 and Lemma 2.13 $\left(D_{\alpha/a} g, a, \frac{1}{M}\right)$ is a Gabor frame for $L^2(\mathbb{R})$ with frame bounds A and B and canonical dual window $D_{\alpha/a}\gamma^0$. Using Theorem 3.1 the result follows, because by (2.7): $\mathcal{S}_1 D_{\alpha/a} g = \mathcal{S}_{\alpha/a} g$. □

The following proposition can be used to relate the expansion coefficients $c(m,n)$ of a function f in a Gabor frame for $L^2(\mathbb{R})$ with dual window γ to the expansion coefficients $d(m,n)$ of the sampled function $\mathcal{S}_{\alpha/a} f$ in a Gabor frame for $l^2(\mathbb{Z})$ with sampled dual window $\mathcal{S}_{\alpha/a}\gamma$.

Proposition 3.3. *Let $f, \gamma \in S_0(\mathbb{R})$, $\alpha, \beta > 0$ with $\alpha\beta = \frac{a}{M}$ for some $M, a \in \mathbb{N}$. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{m\beta} T_{n\alpha}\gamma \rangle_{L^2(\mathbb{R})}, \quad m, n \in \mathbb{Z}, \\ d(m,n) &= \left\langle \mathcal{S}_{\alpha/a} f, M_{m/M} T_{na} \mathcal{S}_{\alpha/a}\gamma \right\rangle_{l^2(\mathbb{Z})}, \quad m = 0, \ldots, M-1, n \in \mathbb{Z} \end{aligned}$$

then

$$d(m,n) = \sum_{j \in \mathbb{Z}} c(m - jM, n), \quad \forall m = 0, \ldots, M-1, n \in \mathbb{Z}$$

Proof. By Proposition 2.11 then $\sum_{j\in\mathbb{Z}} c(m - jM, n)$ is convergent.

Consider

$$\sum_{j\in\mathbb{Z}} c(m - jM, n) = \sum_{j\in\mathbb{Z}} \left\langle f, M_{(m-jM)\beta} T_{n\alpha}\gamma \right\rangle_{L^2(\mathbb{R})}.$$

Using the unitarity of $D_{\alpha/a}$ and (2.1):

$$\begin{aligned} \sum_{j\in\mathbb{Z}} c(m - jM, n) &= \sum_{j\in\mathbb{Z}} \left\langle D_{\alpha/a} f, M_{m/M-j} T_{na} D_{\alpha/a}\gamma \right\rangle_{L^2(\mathbb{R})} \\ &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} D_{\alpha/a} f \overline{\left(M_{m/M} T_{na} D_{\alpha/a}\gamma\right)(x)} e^{2\pi i j x} dx \\ &= \sum_{j\in\mathbb{Z}} \mathcal{F}_{\mathbb{R}} \left(D_{\alpha/a} f \overline{M_{m/M} T_{na} D_{\alpha/a}\gamma}\right)(j). \end{aligned}$$

In operator terms, the last line can be written as

$$\sum_{j\in\mathbb{Z}} c(m - jM, n) = \left(\mathcal{F}^{-1}_{[0;1]} \mathcal{S}_1 \mathcal{F}_{\mathbb{R}} \left(D_{\alpha/a} f \overline{M_{m/M} T_{na} D_{\alpha/a}\gamma}\right)\right)(0)$$

Since $D_{\alpha/a} f \overline{M_{m/M} T_{na} D_{\alpha/a}\gamma} \in S_0(\mathbb{R})$, the last line can be rewritten by the Poisson summation formula (2.9) and (2.7):

$$\begin{aligned}\sum_{j\in\mathbb{Z}} c(m-jM,n) &= \left(\mathcal{P}_1\left(D_{\alpha/a}f\overline{M_{m/M}T_{na}D_{\alpha/a}\gamma}\right)\right)(0)\\ &= \sum_{k\in\mathbb{Z}}\left(D_{\alpha/a}f\overline{M_{m/M}T_{na}D_{\alpha/a}\gamma}\right)(k)\\ &= \left\langle \mathcal{S}_{\alpha/a}f, M_{m/M}T_{na}\mathcal{S}_{\alpha/a}\gamma\right\rangle_{l^2(\mathbb{Z})}\\ &= d(m,n).\end{aligned}$$

□

Theorem 3.2 showed that if γ is the canonical dual window of g, $\mathcal{S}_{\alpha/a}\gamma$ will also be the canonical dual window of $\mathcal{S}_{\alpha/a}g$. This relation holds not only for the canonical dual window, but in fact for ALL dual windows $\gamma\in S_0(\mathbb{R})$.

Proposition 3.4. *Let (g,α,β), $g\in S_0(\mathbb{R})$, $\alpha,\beta>0$ with $\alpha\beta=\frac{a}{M}$ for some $M,a\in\mathbb{N}$, be a Gabor frame for $L^2(\mathbb{R})$ and let $\gamma\in S_0(\mathbb{R})$ be a dual window. Then $S_{\alpha/a}\gamma$ is a dual window of $\left(\mathcal{S}_{\alpha/a}g, a, \frac{1}{M}\right)$.*

Proof. Define $c\in l^2(\mathbb{Z}\times\mathbb{Z})$ and $d\in l^2(\{0,...,a-1\}\times\mathbb{Z})$ by

$$\begin{aligned}c(m,n) &= \left\langle \gamma, M_{m/\alpha}T_{n/\beta}g\right\rangle_{L^2(\mathbb{R})}, \quad m,n\in\mathbb{Z},\\ d(m,n) &= \left\langle \mathcal{S}_{\alpha/a}\gamma, M_{m/a}T_{nM}\mathcal{S}_{\alpha/a}g\right\rangle_{l^2(\mathbb{Z})}\\ &= \left\langle \mathcal{S}_{1/M\beta}\gamma, M_{m/a}T_{nM}\mathcal{S}_{1/M\beta}g\right\rangle_{l^2(\mathbb{Z})}, \quad m=0,...,M-1, n\in\mathbb{Z}\end{aligned}$$

It is well known, that if $g,\gamma\in S_0(\mathbb{R})$, (g,α,β) and (γ,α,β) has finite upper frame bounds, see e.g. [9, Chapter 6]. Since g,γ are dual windows, they satisfy the Wexler-Raz condition for $L^2(\mathbb{R})$, Theorem 2.12:

$$\frac{1}{\alpha\beta}c(m,n)=\delta_m\delta_n, \quad m,n\in\mathbb{Z}.$$

We wish to show that $\mathcal{S}_{\alpha/a}g$ and $\mathcal{S}_{\alpha/a}\gamma$ satisfies the Wexler-Raz condition for $l^2(\mathbb{Z})$:

$$\begin{aligned}&\frac{M}{a}\left\langle \mathcal{S}_{1/M\beta}\gamma, M_{m/a}T_{nM}\mathcal{S}_{1/M\beta}g\right\rangle_{l^2(\mathbb{Z})}\\ =\ &\frac{M}{a}d(m,n)\\ =\ &\delta_m\delta_n, \quad m=0,...,a-1, n\in\mathbb{Z}\end{aligned}$$

By Theorem 3.2, $\left(\mathcal{S}_{\alpha/a}g, a, \frac{1}{M}\right)$ and $\left(\mathcal{S}_{\alpha/a}\gamma, a, \frac{1}{M}\right)$ also has finite upper frame bounds. By Proposition 3.3:

$$\begin{aligned}\frac{M}{a}d(m,n) &= \sum_{j\in\mathbb{Z}}\frac{1}{\alpha\beta}c(m-ja,n)\\ &= \sum_{j\in\mathbb{Z}}\delta_{m-ja}\delta_n\\ &= \delta_m\delta_n, \quad \forall m=0,...,a-1, n\in\mathbb{Z}.\end{aligned}$$

This shows that $\mathcal{S}_{\alpha/a}g$ and $\mathcal{S}_{\alpha/a}\gamma$ satisfies the Wexler-Raz condition for $l^2(\mathbb{Z})$, and therefore they are dual windows. □

There are two remarks on Proposition 3.4 as compared to Theorem 3.2:

(1) In Theorem 3.2 the only assumption needed was $g \in S_0(\mathbb{R})$, because this implies $\gamma^0 \in S_0(\mathbb{R})$. In Proposition 3.4 we need to impose this as a separate condition.
(2) At first Proposition 3.4 seems a generalization of Theorem 3.2, but it is not. The main strength of Theorem 3.2 (besides the frame bounds relations) is the fact that $\mathcal{S}_{\alpha/a}\gamma^0$ is identified as a particular dual window of $\mathcal{S}_{\alpha/a}g$, namely the canonical dual. This allows for efficient approximation of canonical dual windows, like the method presented in Section 4. Proposition 3.4 does not offer this kind of application.

3.2. **From $L^2(\mathbb{R})$ to $L^2([0;L])$.** The results from the previous section can be easily extended to prove how a Gabor frame for $L^2([0;L])$ can be obtained from a Gabor frame for $L^2(\mathbb{R})$ by periodizing the window function.

Gabor frames for $L^2([0;L])$ are not as widely studied as Gabor frames for $L^2(\mathbb{R})$. From a practical point of view, they are interesting because they can be used to model continuous phenomena of finite duration.

Interrelations between Wilson bases for $L^2(\mathbb{R})$ and $L^2([0;L])$ are described in [1, Corollary 9.3.6].

Theorem 3.5. *Let $g \in S_0(\mathbb{R})$, $\alpha, \beta > 0$ with $\alpha\beta = \frac{b}{N}$ for some $N, b \in \mathbb{N}$ and assume that (g, α, β) is Gabor frame for $L^2(\mathbb{R})$ with canonical dual window γ^0.*

Then $\left(\mathcal{P}_{b/\beta}g, \alpha, b\right)$ is a Gabor frame for $L^2([0;\frac{b}{\beta}])$ with the same frame bounds and canonical dual window $\mathcal{P}_{b/\beta}\gamma^0$.

Proof. By Lemma 2.9 and Lemma 2.13, $(\mathcal{F}g, \beta, \alpha)$ is also a Gabor frame for $L^2(\mathbb{R})$ with frame bounds A and B and canonical dual window $\mathcal{F}\gamma^0$.

Since $\hat{g} \in S_0(\mathbb{R})$, Theorem 3.2 can be used: $\left(\mathcal{S}_{\beta/b}\mathcal{F}g, b, \frac{1}{N}\right)$ is a Gabor frame for $l^2(\mathbb{Z})$ with frame bounds A and B and canonical dual window $\mathcal{S}_{\beta/b}\mathcal{F}\gamma^0$.

By Lemma 2.9 and Lemma 2.13, $\left(\mathcal{F}^{-1}_{[0;\frac{b}{\beta}]}\mathcal{S}_{\beta/b}\mathcal{F}g, a, b\right)$ is a Gabor frame for $L^2([0;\frac{b}{\beta}])$ with frame bounds A and B and canonical dual window $\mathcal{F}^{-1}_{[0;\frac{b}{\beta}]}\mathcal{S}_{\beta/b}\mathcal{F}\gamma^0$.

By the Poisson summation formula (2.9), $\mathcal{F}^{-1}_{[0;\frac{b}{\beta}]}\mathcal{S}_{\beta/b}\mathcal{F} = \mathcal{P}_{b/\beta}$ and from this the result follows. □

Proposition 3.6. *Let $f, \gamma \in S_0(\mathbb{R})$, $\alpha, \beta > 0$ with $\alpha\beta = \frac{b}{N}$ for some $N, b \in \mathbb{N}$. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{m\beta}T_{n\alpha}\gamma\rangle_{L^2(\mathbb{R})}, \quad m,n \in \mathbb{Z} \\ d(m,n) &= \left\langle \mathcal{P}_{b/\beta}f, M_{mb}T_{na}\mathcal{P}_{b/\beta}\gamma\right\rangle_{L^2([0;\frac{b}{\beta}])}, \quad m \in \mathbb{Z}, n = 0, ..., N-1 \end{aligned}$$

then

$$d(m,n) = \sum_{j\in\mathbb{Z}} c(m, n-jN), \quad \forall m \in \mathbb{Z}, n = 0, ..., N-1.$$

Proof. By Proposition 2.11 then $\sum_{j\in\mathbb{Z}} c(m, n-jN)$ is convergent.

Consider

$$\sum_{j\in\mathbb{Z}} c(m, n-jN) = \sum_{j\in\mathbb{Z}} \langle f, M_{m\beta}T_{(n-jN)\alpha}\gamma\rangle_{L^2(\mathbb{R})}.$$

Using the unitarity of $D_{\alpha/a}$ and (2.1):

$$\begin{aligned}\sum_{j\in\mathbb{Z}} c(m,n-jN) &= \sum_{j\in\mathbb{Z}} \langle D_{b/\beta}f, M_{mb}T_{n/N-j}D_{b/\beta}\gamma\rangle_{L^2(\mathbb{R})} \\ &= \left\langle D_{b/\beta}f, M_{mb}T_{n/N}\sum_{j\in\mathbb{Z}} T_{-j}D_{b/\beta}\gamma\right\rangle_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} D_{b/\beta}f \overline{M_{mb}T_{n/N}\left(\sum_{j\in\mathbb{Z}} T_{-j}D_{b/\beta}\gamma\right)}(x)dx.\end{aligned}$$

The function $M_{mb}T_{n/N}\left(\sum_{j\in\mathbb{Z}} T_{-j}D_{b/\beta}\gamma\right)$ is periodic with period 1 and bounded because $D_{b/\beta}\gamma \in S_0(\mathbb{R})$. Since $D_{b/\beta}f \in L^1(\mathbb{R})$ also

$$D_{b/\beta}f\overline{M_{mb}T_{n/N}\left(\sum_{j\in\mathbb{Z}} T_{-j}D_{b/\beta}\gamma\right)} \in L^1(\mathbb{R}),$$

and it is therefore legal to split the outermost integration in parts and interchange summation and integration:

$$\begin{aligned}&\sum_{j\in\mathbb{Z}} c(m,n-jN) \\ =\;& \sum_{k\in\mathbb{Z}}\int_0^1 \left(D_{b/\beta}f\overline{M_{mb}T_{n/N}\left(\sum_{j\in\mathbb{Z}} T_{-j}D_{b/\beta}\gamma\right)}\right)(x-k)\,dx \\ =\;& \frac{b}{\beta}\int_0^1 \left(\sum_{k\in\mathbb{Z}} f(\frac{b}{\beta}x-\frac{b}{\beta}k)\right)\overline{e^{2\pi imbx}\sum_{j\in\mathbb{Z}}\gamma\left(\frac{b}{\beta}x-\frac{b}{\beta}\left(\frac{n}{N}-j\right)\right)}dx \\ =\;& \int_0^{\frac{b}{\beta}} \left(\sum_{k\in\mathbb{Z}} f(x-\frac{b}{\beta}k)\right)\overline{e^{2\pi im\beta x}\sum_{j\in\mathbb{Z}}\gamma\left(x-\frac{b}{\beta}\left(\frac{n}{N}-j\right)\right)}dx \\ =\;& \langle \mathcal{P}_{b/\beta}f, M_{mb}T_{n\alpha}\mathcal{P}_{b/\beta}\gamma\rangle_{L^2([0;\frac{b}{\beta}])}.\end{aligned}$$

□

Proposition 3.7. *Let (g,α,β) $g \in S_0(\mathbb{R})$, $\alpha,\beta > 0$ with $\alpha\beta = \frac{b}{N}$ for some $N, b \in \mathbb{N}$, be a Gabor frame for $L^2(\mathbb{R})$ and let $\gamma \in S_0(\mathbb{R})$ be a dual window. Then $\mathcal{P}_{b/\beta}\gamma$ is a dual window of $(\mathcal{P}_{b/\beta}g,\alpha,b)$.*

Proof. The proof has the same structure as that of Proposition 3.4, using Proposition 3.6 as the main ingredient. □

3.3. **From $L^2([0;L])$ to $\mathbb{C}^L$.** A Gabor frame for $\mathbb{C}^L$ can be obtained by sampling the window function of a Gabor frame for $L^2([0;L])$. The proofs are very similar to the proofs presented in section 3.1 for the $L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ case.

Theorem 3.8. *Let $g \in \mathcal{A}([0;L])$, $L, M, N, a, b \in \mathbb{N}$ with $L = Mb = Na$ and assume that (g, a, b) is a Gabor frame for $L^2([0;L])$ with canonical dual window γ^0.*

Then $(\mathcal{S}_1 g, a, b)$ is a Gabor frame for $\mathbb{C}^L$ with the same frame bounds and canonical dual window $\mathcal{S}_1\gamma^0$.

Proof. The proof can be found by carefully modifying the proof in [11] for the $L^2(\mathbb{R})$-case. The modifications consists in replacing integration over $\mathbb{R}$ with integrations over $[0;L]$ and similar changes. □

This covers the sampling of a window function $g \in \mathcal{A}([0;L])$ at the integers. The following theorem generalizes the situation:

Theorem 3.9. *Let $g \in \mathcal{A}([0;L_1])$, $L_2, M, N, a_2, b \in \mathbb{N}$ with $L_1 = Na_1$ and $L_2 = Mb = Na_2$. Assume that (g, a_1, b) is a Gabor frame for $L^2([0;L_1])$ with canonical dual window γ^0.*

Then $\left(\mathcal{S}_{L_1/L_2} g, a_2, b\right)$ is a Gabor frame for $\mathbb{C}^{L_2}$ with the same frame bounds and canonical dual window $\mathcal{S}_{L_1/L_2}\gamma^0$.

Proof. By Lemma 2.9 and Lemma 2.13, $\left(D_{L_1/L_2} g, a_2, b\right)$ is a Gabor frame for $L^2([0;L_2])$. From (2.8) and Theorem 3.8 the result follows. □

Proposition 3.10. *Let $f, \gamma \in A([0;L_1])$, $L_2, M, N, a_2, b \in \mathbb{N}$ with $L_1 = Na_1$ and $L_2 = Mb = Na_2$. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{mb} T_{na_1}\gamma\rangle_{L^2([0;L_1])}, \quad m \in \mathbb{Z}, n = 0, ..., N-1 \\ d(m,n) &= \left\langle \mathcal{S}_{L_1/L_2} f, M_{mb} T_{na_2} \mathcal{S}_{L_1/L_2}\gamma\right\rangle_{\mathbb{C}^L}, \ m = 0, ..., M-1, n = 0, ..., N-1 \end{aligned}$$

then

$$d(m,n) = \sum_{j\in\mathbb{Z}} c(m - jM, n), \quad \forall m = 0, ..., M-1, n = 0, ..., N-1$$

Proof. The proof is very similar to that of Proposition 3.10, using a different version of the Poisson summation formula, (2.11). □

Proposition 3.11. *Let (g, a_1, b) $g \in \mathcal{A}([0;L_1])$, $L_2, M, N, a_2, b \in \mathbb{N}$ with $L_1 = Na_1$ and $L_2 = Mb = Na_2$, be a Gabor frame for $L^2([0;L_1])$ and let $\gamma \in \mathcal{A}([0;L_1])$ be a dual window. Then $\mathcal{S}_{L_1/L_2}\gamma$ is a dual window of $\left(\mathcal{S}_{L_1/L_2} g, a_2, b\right)$.*

Proof. The proof has the same structure as that of Proposition 3.4, using Proposition 3.10 as the main ingredient. □

3.4. **From $l^2(\mathbb{Z})$ to $\mathbb{C}^L$.** Similarly to the results presented in the previous sections, one can obtain a Gabor frame for $\mathbb{C}^L$ by periodizing the window function of a Gabor frame for $l^2(\mathbb{Z})$.

Theorem 3.12. *Let $g \in l^1(\mathbb{Z})$, $M, N, a, b \in \mathbb{N}$ with $Mb = Na = L$ and assume that $\left(g, a, \frac{1}{M}\right)$ is a Gabor frame for $l^2(\mathbb{Z})$ with canonical dual window γ^0.*

Then $(\mathcal{P}_L g, a, b)$ is a Gabor frame for $\mathbb{C}^L$ with the same frame bounds and canonical dual window $\mathcal{P}_L\gamma^0$.

Proof. By Lemma 2.9 and Lemma 2.13, $(\mathcal{F}_{\mathbb{Z}}g, b, a)$ is a Gabor frame for $L^2([0;1])$ with frame bounds A and B and canonical dual window $\mathcal{F}_{\mathbb{Z}}\gamma^0$, and since $g \in l^1(\mathbb{Z})$ then $\mathcal{F}_{\mathbb{Z}}g \in \mathcal{A}([0;1])$.

Using Theorem 3.9, $\left(\mathcal{S}_{1/L}\mathcal{F}_{\mathbb{Z}}g, b, a\right)$ is a Gabor frame for $\mathbb{C}^L$ with frame bounds A and B and canonical dual window $\mathcal{S}_{1/L}\mathcal{F}_{\mathbb{Z}}\gamma^0$.

By Lemma 2.9 and Lemma 2.13, $\left(\mathcal{F}_L^{-1}\mathcal{S}_{1/L}\mathcal{F}_{\mathbb{Z}}g, a, b\right)$ is a Gabor frame for $\mathbb{C}^L$ with frame bounds A and B and canonical dual window $\mathcal{F}_L^{-1}\mathcal{S}_{1/L}\mathcal{F}_{\mathbb{Z}}\gamma^0$.

Using the Poisson summation formula (2.10), $\mathcal{F}_L^{-1}\mathcal{S}_{1/L}\mathcal{F}_{\mathbb{Z}}g = \mathcal{P}_L g$ and similarly $\mathcal{F}_L^{-1}\mathcal{S}_{1/L}\mathcal{F}_{\mathbb{Z}}\gamma^0 = \mathcal{P}_L\gamma^0$. □

Proposition 3.13. *Let $f, \gamma \in l^1(\mathbb{Z})$, $M, N, a, b \in \mathbb{N}$ with $Mb = Na = L$. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{mb}T_{na}\gamma\rangle_{l^2(\mathbb{Z})}, \quad m = 0, ..., M-1, n \in \mathbb{Z} \\ d(m,n) &= \langle \mathcal{P}_L f, M_{mb}T_{na}\mathcal{P}_L\gamma\rangle_{\mathbb{C}^L}, \ m = 0, ..., M-1, n = 0, ..., N-1, \end{aligned}$$

then

$$d(m,n) = \sum_{j\in\mathbb{Z}} c(m, n - jN), \quad \forall m = 0, ..., M-1, n = 0, ..., N-1$$

Proof. The proof closely follows that of Proposition 3.6. □

Proposition 3.14. *Let $\left(g, a, \frac{1}{M}\right)$ $g \in l^1(\mathbb{Z})$, $M, N, a, b \in \mathbb{N}$ with $Mb = Na = L$, be a Gabor frame for $l^2(\mathbb{Z})$ and let $\gamma \in l^1(\mathbb{Z})$ be a dual window. Then $\mathcal{P}_L\gamma$ is a dual window of $(\mathcal{P}_L g, a, b)$.*

Proof. The proof has the same structure as that of Proposition 3.4, using Proposition 3.13 as the main ingredient. □

3.5. **From $L^2(\mathbb{R})$ to $\mathbb{C}^L$.** With the results from the previous sections, any Gabor frame for $L^2(\mathbb{R})$, (g, α, β) with $g \in S_0(\mathbb{R})$ and rational sampling, $\alpha\beta \in \mathbb{Q}$, can be sampled and periodized to obtain a Gabor frame for $\mathbb{C}^L$ with corresponding sampled and periodized canonical dual window:

Theorem 3.15. *Let $g \in S_0(\mathbb{R})$, $\alpha\beta = \frac{a}{M} = \frac{b}{N}$ and $Mb = Na = L$ with $a, b, M, N, L \in \mathbb{N}$ and assume that (g, α, β) is a Gabor frame for $L^2(\mathbb{R})$ with canonical dual window γ^0.*

Then $\left(\mathcal{P}_L\mathcal{S}_{\alpha/a}g, a, b\right)$ is a Gabor frame for $\mathbb{C}^L$ with the same frame bounds and canonical dual window $\mathcal{P}_L\mathcal{S}_{\alpha/a}\gamma^0$.

Proof. Combine Theorem 3.2 and Theorem 3.12. Fully written out

$$\left(\mathcal{P}_L\mathcal{S}_{\alpha/a}\right)g(j) = \sqrt{\frac{\alpha}{a}}\sum_{k\in\mathbb{Z}} g\left(\frac{\alpha}{a}(j - kL)\right), \quad j = 0, ..., L-1.$$

Under the assumptions on the parameters, then $\mathcal{P}_L\mathcal{S}_{\alpha/a} = \mathcal{S}_{b/\beta/L}\mathcal{P}_{b/\beta}$ on $S_0(\mathbb{R})$. This shows that both ways on figure 1.1 produce the same result, so an alternative proof is to use Theorem 3.5 and Theorem 3.9. □

Similarly, the coefficients of a function f in a Gabor frame for $L^2(\mathbb{R})$ can be approximated by sampling and periodizing f and calculating the coefficients in a Gabor frame for $\mathbb{C}^L$:

Proposition 3.16. *Let $f,\gamma \in S_0(\mathbb{R})$, $\alpha,\beta > 0$ with $\alpha\beta = \frac{a}{M} = \frac{b}{N}$, $L = Mb = Na$ with $a,b,M,N,L \in \mathbb{N}$. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{m\beta}T_{n\alpha}\gamma\rangle_{L^2(\mathbb{R})}, \quad m,n \in \mathbb{Z} \\ d(m,n) &= \left\langle \mathcal{P}_L\mathcal{S}_{\alpha/a}f, M_{mb}T_{na}\mathcal{P}_L\mathcal{S}_{\alpha/a}\gamma\right\rangle_{\mathbb{C}^L}, \ m = 0,...,M-1, n = 0,...,N-1, \end{aligned}$$

then

$$d(m,n) = \sum_{j,k\in\mathbb{Z}} c(m-kM, n-jN), \quad \forall = 0,...,M-1, n = 0,...,N-1.$$

Proof. Combine Prop. 2.11 and Prop. 3.3, or alternatively Prop. 3.6 and Prop. 3.10. □

As with Theorem 3.15, the following proposition can be proved by combining the previous results.

Proposition 3.17. *Let $(g,\alpha\beta)$, $\alpha\beta = \frac{a}{M} = \frac{b}{N}$ and $Mb = Na = L$ with $a,b,M,N,L \in \mathbb{N}$, be a Gabor frame for $L^2(\mathbb{R})$ and let $\gamma \in S_0(\mathbb{R})$ be a dual window. Then $\mathcal{P}_L\mathcal{S}_{\alpha/a}\gamma$ is a dual window of $\left(\mathcal{P}_L\mathcal{S}_{\alpha/a}g, a, b\right)$.*

4. An application.

As an application of Proposition 3.16 one can calculate approximations to the canonical dual window γ^0 of a Gabor frame (g,α,β) for $L^2(\mathbb{R})$ using finite-dimensional methods. The idea is to use Theorem 3.15 describing the relationship between γ^0 and the canonical dual window $\mathcal{P}_L\mathcal{S}_{\alpha/a}\gamma^0 \in \mathbb{C}^L$ of a Gabor frame $\left(\mathcal{P}_L\mathcal{S}_{\alpha/a}g, a, b\right)$ for $\mathbb{C}^L$. This will give sampled values of a periodization of the canonical dual window γ^0.

To obtain a continuous function $\gamma^0_{M,N}$ approximating γ^0 from the finite sequence $\mathcal{P}_L\mathcal{S}_{\alpha/a}\gamma^0$, some kind of interpolation scheme is needed. In [12], Norbert Kaiblinger uses B-splines and shows that this is sufficient to get convergence in S_0 (and therefore also $L^2(\mathbb{R})$.

The following results uses another simple interpolation scheme: Using the Gabor frame (g,α,β) itself.

The result is presented only for M,N being even. The extension to odd M,N is trivial.

Theorem 4.1. *Let $g \in S_0(\mathbb{R})$, $\alpha,\beta > 0$ with $\alpha\beta = \frac{a}{M}$ and assume that (g,α,β) is a Gabor frame for $L^2(\mathbb{R})$ with canonical dual window γ^0.*

For each even $M,N \in \mathbb{N}$ such that $L = Mb = Na$ with $a,b,L \in \mathbb{N}$ denote the canonical dual window of $\left(\mathcal{P}_L\mathcal{S}_{\alpha/a}g, a, b\right)$ by $\varphi_{M,N}$. Define $d_{M,N} \in \mathbb{C}^{M\times N}$ by $d_{M,N}(m,n) = \langle\varphi_{M,N}, M_{mb}T_{na}\varphi_{M,N}\rangle_{\mathbb{C}^L}$ and $\gamma^0_{M,N} \in S_0(\mathbb{R})$ by

$$\gamma^0_{M,N} = \sum_{n=-N/2}^{N/2-1}\sum_{m=-M/2}^{M/2-1} d_{M,N}(m,n)M_{m\beta}T_{n\alpha}g. \tag{4.1}$$

Then $\gamma^0_{M,N} \to \gamma^0$ as $M,N \to \infty$.

Proof. Define $c \in l^1(\mathbb{Z}\times\mathbb{Z})$ by $c(m,n) = \left\langle\gamma^0, M_{m\beta}T_{n\alpha}\gamma^0\right\rangle_{L^2(\mathbb{R})}$. Then

$$\gamma^0 = \sum_{m,n\in\mathbb{Z}} c(m,n)M_{m\beta}T_{n\alpha}g. \tag{4.2}$$

This is the standard frame expansion, since γ^0 and g are dual windows.

By (4.1) and (4.3) both γ^0 and $\gamma^0_{M,N}$ can be written using the frame (g, α, β). Subtracting them gives

$$\gamma^0 - \gamma^0_{M,N} = \sum_{m,n\in\mathbb{Z}} r_{M,N}(m,n) M_{m\beta} T_{n\alpha} g,$$

where

$$r_{M,N}(m,n) = \begin{cases} c(m,n) - d_{M,N}(m,n) & \text{if } -\frac{M}{2} \le m \le \frac{M}{2} - 1 \text{ and } -\frac{N}{2} \le n \le \frac{N}{2} - 1 \\ c(m,n) & \text{otherwise} \end{cases}$$

By Proposition 3.16,

$$d_{M,N}(m,n) = \sum_{j,k\in\mathbb{Z}} c(m - kM, n - jN), \tag{4.3}$$

for all $-\frac{M}{2} \le m \le \frac{M}{2} - 1$ and $-\frac{N}{2} \le n \le \frac{N}{2} - 1$. This gives the following expression of the residual r:

$$r_{M,N}(m,n) = \begin{cases} -\sum_{j,k\in\mathbb{Z}\setminus\{0\}} c_{M,N}(m - kM, n - jN) & \text{if } -\frac{M}{2} \le m \le \frac{M}{2} - 1 \text{ and } -\frac{N}{2} \le n \le \frac{N}{2} - 1 \\ c(m,n) & \text{otherwise} \end{cases}$$

Since the translation and modulation operators are unitary, we get the following estimate.

$$\begin{aligned} \left\|\gamma^0 - \gamma^0_{M,N}\right\| &\le \|g\| \sum_{m,n\in\mathbb{Z}} |r_{M,N}(m,n)| \\ &\le \|g\| \sum_{n=-N/2}^{N/2-1} \sum_{m=-M/2}^{M/2-1} \sum_{j,k\in\mathbb{Z}\setminus\{0\}} |c_{M,N}(m - kM, n - jN)| + \\ &\quad + \|g\| \sum_{m\notin\left\{-\frac{M}{2},\dots,\frac{M}{2}-1\right\}} \sum_{n\notin\left\{-\frac{N}{2},\dots,\frac{N}{2}-1\right\}} |c(m,n)| \end{aligned}$$

In the last term, only the coefficients outside the rectangle indexed by $m \in \left\{-\frac{M}{2}, \dots, \frac{M}{2} - 1\right\}$ and $n \in \left\{-\frac{N}{2}, \dots, \frac{N}{2} - 1\right\}$ appears, and they all appear twice. From this

$$\left\|\gamma^0 - \gamma^0_{M,N}\right\| \le 2\|g\| \sum_{m\notin\left\{-\frac{M}{2},\dots,\frac{M}{2}-1\right\}} \sum_{n\notin\left\{-\frac{N}{2},\dots,\frac{N}{2}-1\right\}} |c(m,n)|$$

When $M, N \to \infty$, the last term goes to zero. □

The trivial change needed for odd M or N consist in replacing $-\frac{M}{2}$ by $-\frac{M-1}{2}$ and $\frac{M}{2} - 1$ by $\frac{M-1}{2}$ and similarly for N.

Since each $\gamma^0_{M,N}$ is a finite linear combination of Gabor atoms from (g, α, β), they inherit properties from g: Since $g \in S_0(\mathbb{R})$ then each $\gamma^0_{M,N} \in S_0(\mathbb{R})$. Similarly, if g or $\hat{g}$ has exponential decay, then so does $\gamma^0_{M,N}$ or $\hat{\gamma}^0_{M,N}$.

To use this method, the two main numerical calculations to be carried out are the inversion of the frame operator for $(g_{M,N}, a, b)$ and the calculation of the coefficients $d_{M,N} \in \mathbb{C}^{M\times N}$. Algorithms based on FFTs and matrix-factorisations can be found in [14]. These calculations can be performed in $\mathcal{O}(Lq) + \mathcal{O}(NM \log M)$, where $\frac{q}{p}$ is the oversampling factor written as an irreducible fraction.

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