Advanced Topics in Numerical and Computational Bifurcation Analysis: Assignment Projects

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Instructions and Submission Deadline

The objective of these projects is to demonstrate your ability to

- develop and implement algorithms for bifurcation problems of dynamical systems,
- apply basic techniques from numerical analysis to check that your algorithm is correct, that is, that it produces the correct result in a numerically stable way if the problem is well-posed,
- verify the correctness of your implementation with suitable test examples,
- apply your implementation in a mathematically sound way, and
- interpret the results of your computations in way that provides insight to a given problem.

The projects are research oriented and open-ended, that is, they provide only some guidance on what one could investigate instead of closed exam-style questions. Pick a problem from a project that you find interesting and follow its direction until you have enough material to submit your report. You may also chose to combine parts from different projects, for example, two theoretical (computational) parts from different projects into a large purely theoretical (computational) project.

Project report. Present your results in the form of a short academic paper of 6-8 pages (excluding title page). Please note that we will not consider more than 8 pages for marking! Focus on describing your work in a way that demonstrate your abilities as outlined in the list above, which may serve as a guideline for contents. Please include your **full name, DTU student number (if applicable), postal and e-mail address** on the cover page of your paper.

Submission. Together with your report, collect all files that are necessary to reproduce the computational results presented in your report in a single archive file (format ZIP or TAR.GZ). Upload this archive file to the DTU CampusNet group “ANBA 2011” as explained on the summer school home page. Please make sure that all computations can be executed (no files are missing) before uploading your archive file.

Deadline. Deadline for submission is midnight of Monday August 1 (MESZ).

Four-Cell Brusselator *(by Claudia Wulff)*

The aim of this project is to study symmetric periodic orbits in the 4 cell Brusselator, which emerge in equivariant Hopf-bifurcations from symmetric equilibria. This problem is studied in Section 5.2 of [1].

The equations of the 4-cell Brusselator are ($j = 1, 3, 5, 7$, $x_i := x_{i-8}$ for $i > 8$)

$$\begin{aligned} x_j' &= A - (B+1)x_j + x_j^2 x_{j+1} + \lambda(-3x_j + x_{j+2} + x_{j+4} + x_{j+6})/1000, \\ x_{j+1}' &= Bx_j - x_j^2 x_{j+1} + \lambda(-3x_{j+1} + x_{j+3} + x_{j+5} + x_{j+7})/1000, \end{aligned}$$

where we set $A = 2.0$, $B = 5.9$.

Theoretical part

Complete the missing details of the example in Section 5.2 of [1], that is, write down how the symmetry group S_4 acts, prove that the 4-cell Brusselator is S_4 equivariant, show that the given equilibrium is an S_4 symmetric equilibrium and check that its imaginary eigenspace is absolutely irreducible. Check that in the example the given starting plane for the bifurcating symmetric periodic orbits is indeed invariant under the axial symmetry group as claimed in the paper.

Suggestions for further studies

Compute the other axial symmetry groups at this equilibrium and the corresponding starting planes for the bifurcating periodic orbits with these spatio-temporal symmetry groups.

Computational part

Numerically continue the symmetric periodic orbits found above using SYMPERCON and explore what bifurcations occur. You may use SYMPERCON's example files for the 4 Cell Brusselator under sympercon/examples as a starting point.

Suggestions for further studies

Numerically continue the other symmetric periodic orbits bifurcating from the equilibrium in the 4 cell systems obtained by reduction with respect to the spatial symmetry of the other axial symmetry groups of the symmetric equilibrium.

Compute loci of (equivariant) saddle-node- and (equivariant) Hopf-bifurcation points. Compute these curves for the 4-cell Brusselator in the A-B-parameter plane. Try to add information about bifurcations of the symmetric periodic orbits to this diagram.

References

[1] C. Wulff, A. Schebesch. *Numerical continuation of symmetric periodic orbits.* SIAM J. Appl. Dyn. Syst. 5(3), pp. 435–475, 2006.
Preprint: `http://personal.maths.surrey.ac.uk/st/C.Wulff/preprints/symper.pdf`

Lagrangian Relative Equilibria *(by Claudia Wulff)*

The aim of this project is to study Lagrangian relative equilibria in the three body problem, and relative periodic orbits that bifurcate off these equilibria. We consider 3 identical bodies of mass 1 in $\mathbb{R}^3$ acted upon only by the forces they exert on each other. These forces are assumed to be given by 3 identical copies of a potential energy function V (one for each pair of bodies) which depends only on the distance between the bodies. Writing p_j for the momenta conjugate to the positions q_j, $q = (q_1, q_2, q_3)$, $p = (p_1, p_2, p_3)$, the Hamiltonian is

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{3} |p_j|^2 + V(r_{12}) + V(r_{23}) + V(r_{13}) \quad \text{where} \quad r_{ij} = |q_i - q_j|, \quad V(r) = -\frac{1}{r}. \quad (1)$$

Excluding collisions, the configuration space Q is

$$Q = \{q = (q_1, \ldots, q_3) \in \mathbb{R}^9, \quad q_i \neq q_j \text{ for } i \neq j\}$$

and the phase space is $P = Q \times \mathbb{R}^9 \subset \mathbb{R}^{18}$. The equations of motion are

$$\dot{q}_j = p_j, \qquad \dot{p}_j = \sum_{i \neq j} \frac{q_i - q_j}{r_{ij}^3}, \qquad j = 1, \ldots, 3. \quad (2)$$

The angular momentum is $\mathbf{L}(q,p) = \sum_{j=1}^{3} q_j \wedge p_j$. Without loss of generality, the centre of mass of the systems can be assumed to be fixed at 0 restricting the configuration space to

$$Q^0 = \{q \in Q \; : \; \sum_{j=1}^{3} q_j = 0\}$$

with corresponding phase space $P^0 = Q^0 \times \mathbb{R}^6 \subseteq \mathbb{R}^{12}$. The 3-identical-body Hamiltonian (2) has the symmetry group

$$\Gamma = \mathrm{O}(3) \times S_N.$$

Theoretical part

Explain how Γ acts on the three bodies, using for example [1, Section 4] or your lecture notes. Then derive an explicit formula for Lagrange relative equilibria and of the rotation frequency as a function of the position vector q of the Lagrange relative equilibrium. To do this you may either review [1, Section 1, in particular 1.D.2], or compute the isotropy subgroup K of the Lagrange relative equilibria, reduce (2) to $\mathrm{Fix}(K)$, write down the symmetry group of the reduced system and then solve for relative equilibria on $\mathrm{Fix}(K)$.

Suggestions for further studies

Investigate the dynamics near the Lagrange relative equilibrium in the planar three body problem, for example, check whether planar (R)POs bifurcate off.

Numerically continue Lagrangian relative equilibria for the (non-planar) three-body problem with centre of mass fixed at 0 (that is, restricted to P^0) and compute the linearization in the corotating frame of the Lagrange RE in the (non-planar) three-body problem on P^0. Explain why there is a purely imaginary eigenvalue of multiplicity 3 and relate each imaginary eigenvalue to bifurcating (R)POs or to symmetry.

Study Lagrange tetrahedral solutions in the 4 body problem, theoretically or numerically.

References

[1] K.R. Meyer and G.R. Hall. *Introduction to Hamiltonian dynamical systems and the N-body problem.* Springer-Verlag, New York, 1992.

[2] C. Wulff and A. Schebesch. *Numerical continuation of Hamiltonian relative periodic orbits.* J. Nonlinear Science, 18(4), pp. 343–390, 2008.

Bifurcation Analysis of a Micro-Actuator *(by Harry Dankowicz)*

The aim of this project is to recompute the results presented in [1] using multiple shooting rather than TC-HAT. Use the multiple-shooting toolbox provided by Harry Dankowicz as a starting point.

In a first step, continue hybrid periodic orbits starting at initial points given in [1] until grazing is detected. Use grazing analysis in order to construct an initial guess along the post-grazing branch of limit cycles and use this initial guess to "branch-switch" to post-grazing solutions; see also [2].

In a second step, implement algorithms for saddle-node- and period-doubling continuation problems, and reproduce the bifurcation curves presented in [1].

References

[1] Kang, W., Thota, P., Wilcox, B. and Dankowicz, H. *Bifurcation analysis of a microactuator using a new toolbox for continuation of hybrid system trajectories.* Journal of Computational and Nonlinear Dynamics 4 (1), pp. 1-8, 2009.

[2] Harry Dankowicz, Michael Katzenbach. *Discontinuity-induced bifurcations in models of mechanical contact, capillary adhesion, and cell division: A common framework.* Physica D: Nonlinear Phenomena, In Press, Accepted Manuscript, Available online 7 May 2011.

Bifurcation Curves of RPOs *(by Frank Schilder)*

The aim of this project is to implement methods for the computation of loci of (relative) saddle-node- and (relative) period-doubling bifurcation points of relative periodic orbits (RPOs). For simplicity, we assume that the RPO has trivial isotropy sub-group and that its symmetry group has dimension $d \in \{0, 1\}$.

Show that under the above assumptions an RPO can be written as a hybrid periodic orbit of a suitably unfolded ODE as defined in [1, 2]. Implement a multiple shooting method for RPOs using the toolbox provided by Harry Dankowicz as a starting point. Test your toolbox with symmetric periodic solutions of the 4-cell Brusselator defined in the project *Four-Cell Brusselator* and with figure-eight choreographis of the three-body problem (2); compare your results with the analysis carried out in [2].

Implement algorithms for the continuation of (relative) saddle-node and (relative) period-doubling bifurcation points. Use these algorithms to compute loci of symmetry-increasing and symmetry-decreasing bifurcation points for symmetric periodic orbits of the 4-cell Brusselator in the A-B-parameter plane.

Suggestions for further studies

Modify the potential V in Equation (1) to $V(r) = -1/r^{\alpha}$, $\alpha > 0$. Investigate what bifurcations occur on families of rotating figure-eight choreographies under variation of angular momentum and α. See [2] for a bifurcation analysis under variation of angular momentum. Extend these results to the case $\alpha \neq 1$.

References

[1] C. Wulff and A. Schebesch. *Numerical continuation of Hamiltonian relative periodic orbits.* J. Nonlinear Science, 18(4), pp. 343–390, 2008.

[2] C. Wulff and F. Schilder. *Numerical bifurcation of Hamiltonian relative periodic orbits.* SIAM J. Appl. Dyn. Syst. 8, no. 3, pp. 931966, 2009.