Hybrid Dynamical Systems, Multiple Shooting Notes

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Outline

Matlab event handling

- Introduce event functions so that integration *interrupts* when the event surface is reached transversally in
  - a direction of decreasing values of the event function;
  - a direction of increasing values of the event function; or
  - independently of nature of variations in value of the event function
- Apply state jump function, change mode, and repeat.
- If necessary, compute useful partial derivatives at terminal point for later reference.

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Mathematical model

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} q \\ \dot{q} \\ \omega t \ mod\ 2\pi \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{S}^1 \tag{1}$$

Vector fields

$$\mathbf{f}_{\text{smooth}}(\mathbf{x}) = \begin{pmatrix} x_2 \\ \frac{1}{m}\left(A\cos x_3 - cx_2 - kx_1\right) \\ \omega \end{pmatrix} \tag{2}$$

Mathematical model

Event functions

$$h_{\text{impact}}(\mathbf{x}) = q_c - x_1 \tag{3}$$

$$h_{\text{phase}}(\mathbf{x}) = 2\pi - x_3 \tag{4}$$

$$h_{\text{turning}}(\mathbf{x}) = x_2 \tag{5}$$

Mathematical model

State jump functions

$$\mathbf{g}_{\text{impact}}(\mathbf{x}) = \begin{pmatrix} x_1 \\ -ex_2 \\ x_3 \end{pmatrix} \tag{6}$$

$$\mathbf{g}_{\text{phase}}(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 - 2\pi \end{pmatrix} \tag{7}$$

$$\mathbf{g}_{\text{identity}}(\mathbf{x}) = \mathbf{x} \tag{8}$$

Mathematical model

The *index vector* takes one of three distinct values:

$$\mathbf{I}_1 = (\mathfrak{smooth},\mathfrak{impact}) \tag{9}$$

$$\mathbf{I}_2 = (\mathfrak{smooth},\mathfrak{phase}) \tag{10}$$

$$\mathbf{I}_3 = (\mathfrak{smooth},\mathfrak{turning}) \tag{11}$$

Outline

Zero problem

Denote by x_i, $i = 1, \ldots, n$, the sequence of starting points for each of n trajectory segments.

The multiple shooting method

Consider the zero problem $F = 0$, where

$$F(x_1, \ldots, x_n, p) = \begin{pmatrix} x_2 - g_{\mathfrak{e}_1}\left(\Phi_{\mathfrak{m}_1}\left(t_1(x_1, p), x_1, p\right), p\right) \\ \vdots \\ x_1 - g_{\mathfrak{e}_n}\left(\Phi_{\mathfrak{m}_n}\left(t_n(x_n, p), x_n, p\right), p\right) \end{pmatrix} \tag{12}$$

where p denotes a vector of system parameters and the $t_i(x_i, p)$'s are implicitly defined by

$$h_{\mathfrak{e}_i}\left(\Phi_{\mathfrak{m}_i}\left(t_i(x_i, p), x_i, p\right), p\right) = 0 \tag{13}$$

Zero problem

Recall that

$$\partial_t \Phi_{\mathfrak{m}}(t,x,p) = f_{\mathfrak{m}}(\Phi_{\mathfrak{m}}(t,x,p),p), \Phi_{\mathfrak{m}}(0,x,p) = x \tag{14}$$

and thus

$$\partial_t \Phi_{\mathfrak{m},x}(t,x,p) = f_{\mathfrak{m},x}(\Phi_{\mathfrak{m}}(t,x,p),p) \cdot \Phi_{\mathfrak{m},x}(t,x,p) \tag{15}$$

$$\Phi_{\mathfrak{m},x}(0,x,p) = Id \tag{16}$$

and

$$\partial_t \Phi_{\mathfrak{m},p}(t,x,p) = f_{\mathfrak{m},x}(\Phi_{\mathfrak{m}}(t,x,p),p) \cdot \Phi_{\mathfrak{m},p}(t,x,p)$$

$$+ f_{\mathfrak{m},p}(\Phi_{\mathfrak{m}}(t,x,p),p) \tag{17}$$

$$\Phi_{\mathfrak{m},p}(0,x,p) = 0 \tag{18}$$

Implicit differentiation

Since the $t_i\left(x_i, p\right)$'s are implicitly defined by

$$h_{\mathfrak{e}_i}\left(\Phi_{\mathfrak{m}_i}\left(t_i\left(x_i, p\right), x_i, p\right), p\right) = 0 \tag{19}$$

it follows by implicit differentiation that

$$t_{i,x}\left(x_i, p\right) = -\frac{h_{\mathfrak{e}_i,x}}{h_{\mathfrak{e}_i,x} \cdot f_{\mathfrak{m}_i}} \cdot \Phi_{\mathfrak{m}_i,x}\left(t_i\left(x_i, p\right), x_i, p\right) \tag{20}$$

and

$$t_{i,p}\left(x_i, p\right) = -\frac{h_{\mathfrak{e}_i,x} \cdot \Phi_{\mathfrak{m}_i,p}\left(t_i\left(x_i, p\right), x_i, p\right) - h_{\mathfrak{e}_i,p}}{h_{\mathfrak{e}_i,x} \cdot f_{\mathfrak{m}_i}} \tag{21}$$

where the omitted arguments are $\left(\Phi_{\mathfrak{m}_i}\left(t_i\left(x_i, p\right), x_i, p\right), p\right)$.

The Jacobian of the zero problem

It follows that

$$F_x = \begin{pmatrix} \Lambda_1 & Id & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & \cdots & Id \\ Id & 0 & \cdots & \Lambda_n \end{pmatrix} \tag{22}$$

where

$$\Lambda_i = -g_{\mathfrak{e}_i,x} \cdot \left(Id - \frac{f_{\mathfrak{m}_i} \cdot h_{\mathfrak{e}_i,x}}{h_{\mathfrak{e}_i,x} \cdot f_{\mathfrak{m}_i}} \right) \cdot \Phi_{\mathfrak{m}_i,x} \tag{23}$$

The Jacobian of the zero problem

Moreover,

$$F_p = \begin{pmatrix} -g_{\mathfrak{e}_1,x} \cdot \left(Id - \frac{f_{\mathfrak{m}_1} \cdot h_{\mathfrak{e}_1,x}}{h_{\mathfrak{e}_1,x} \cdot f_{\mathfrak{m}_1}} \right) \cdot \Phi_{\mathfrak{m}_1,p} - g_{\mathfrak{e}_1,x} \cdot \frac{f_{\mathfrak{m}_1} \cdot h_{\mathfrak{e}_1,p}}{h_{\mathfrak{e}_1,x} \cdot f_{\mathfrak{m}_1}} - g_{\mathfrak{e}_1,p} \\ \vdots \\ -g_{\mathfrak{e}_n,x} \cdot \left(Id - \frac{f_{\mathfrak{m}_n} \cdot h_{\mathfrak{e}_n,x}}{h_{\mathfrak{e}_n,x} \cdot f_{\mathfrak{m}_n}} \right) \cdot \Phi_{\mathfrak{m}_n,p} - g_{\mathfrak{e}_n,x} \cdot \frac{f_{\mathfrak{m}_n} \cdot h_{\mathfrak{e}_n,p}}{h_{\mathfrak{e}_n,x} \cdot f_{\mathfrak{m}_n}} - g_{\mathfrak{e}_n,p} \end{pmatrix} \tag{24}$$

The Jacobian of the zero problem

In the single-shot, continuous case:

$$F_x = \left(\; Id - \Phi_x \; \right) \tag{25}$$

$$F_p = \left(\; -\Phi_p \; \right) \tag{26}$$