Lecture Notes on

NUMERICAL ANALYSIS
of
BIFURCATION PROBLEMS

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Foreword

These notes correspond to an expanded version of the lectures given in the "Summer" School on Nonlinear Equations at the Technical University of Hamburg in March 1997. The author thanks the organizers, Professor Andreas Griewank and Professor Florian Jarre, for having created this opportunity.

The material in these notes constitutes an introduction to elementary bifurcation theory and an introduction to basic numerical methods for analyzing bifurcations on the computer.

The theoretical point of view may be viewed as "classical", but it is constructive and applicable to numerical computation. Most current presentations of bifurcation theory rely on explicit center manifold reduction, followed by transformation to normal form. This is the right approach for studying new bifurcations theoretically, but the techniques are hardly useful for numerical computation.

Chapters with "Theory" in their titles (Chapters 1,4,6, 11) may be skipped by the mathematically ill-prepared reader (and perhaps also by the mathematically well-prepared reader!), as the immediately following chapter derives or illustrates the key results in an informal way. Several chapters may appear incomplete (they are!), in particular Chapters 11 and 13.

These lecture notes and related documents and software are available by FTP from ftp.cs.concordia.ca. Log in as "anonymous" and give your full email address as password. Type "bin" (binary mode) before transferring any files. The following files are relevant :

pub/doedel/doc/hamburg.ps.Z	the complete version of these notes
pub/doedel/doc/ndes.ps.Z	an overview paper with "future directions" and references
pub/doedel/auto/auto.ps.Z	the AUTO97 software reference manual
pub/doedel/auto/auto.tar.Z	the AUTO97 software package

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Chapter 1

The Implicit Function Theorem

We first prove

1.1 The Contraction Theorem

Let $G : \mathcal{B} \to \mathcal{B}$, $\mathcal{B}$ a Banach space, e.g., $\mathcal{B} = \mathrm{R}^n$.

Suppose for some $\rho > 0$, $0 \leq K_0 < 1$, and some $x_0 \in \mathcal{B}$ we have

i) $\|G(u) - G(v)\| \leq K_0 \|u - v\| \quad \forall u, v \in S_\rho(x_0)$,

ii) $\|G(x_0) - x_0\| \leq (1 - K_0)\rho$.

Above $S_\rho(x_0)$ is the closed ball of radius ρ centered at x_0. Then the equation $x = G(x), x \in \mathcal{B}$ has one and only one solution $x_* \in S_\rho(x_0)$ and x_* is the limit of the sequence

$$x_{k+1} = G(x_k), k = 0, 1, 2, \cdots .$$

PROOF. $x_1 = G(x_0)$. $\quad \|x_1 - x_0\| = \|G(x_0) - x_0\| \leq (1 - K_0)\rho \leq \rho$.

Thus $x_1 \in S_\rho(x_0)$. Inductively assume $x_0, x_1, \cdots, x_n \in S_\rho(x_0)$. Then for $k \leq n$ we have

$$\begin{aligned} \|x_{k+1} - x_k\| &= \|G(x_k) - G(x_{k-1})\| \leq K_0 \|x_k - x_{k-1}\| \\ &= \cdots \leq K_0^k \|x_1 - x_0\| \\ &\leq K_0^k (1 - K_0)\rho. \end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - x_0\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\| \\
&\leq (K_0^n + K_0^{n-1} + \cdots + 1)(1 - K_0)\rho \\
&= (1 - K_0^{n+1})\rho \\
&\leq \rho.
\end{aligned}$$

Hence $x_k \in S_\rho(x_0)$ for all k.

Next show that $\{x_k\}$ is a Cauchy sequence :

$$\begin{aligned}
\|x_{k+n} - x_k\| &\leq \|x_{k+n} - x_{k+n-1}\| + \cdots + \|x_{k+1} - x_k\| \\
&\leq (K_0^{n-1} + K_0^{n-2} + \cdots + 1)\ K_0^k(1 - K_0)\rho \\
&= (1 - K_0^n)K_0^k\rho \\
&\leq K_0^k\rho.
\end{aligned}$$

Given $\epsilon > 0$ choose k such that $K_0^k\rho < \frac{1}{2}\ \epsilon$.

Then $\|x_{k+\ell} - x_{k+m}\| \leq \|x_{k+\ell} - x_k\| + \|x_{k+m} - x_k\| \leq 2K_0^k\rho < \epsilon$, independent of ℓ, m.

Thus $\{x_k\}$ is a Cauchy sequence and therefore converges, with $\lim x_k = x_*$, $\quad x_* \in S_\rho(x_0)$.

$$\begin{array}{ccc}
x_{k+1} & = & G(x_k) \\
\downarrow & & \downarrow \\
x_* & & x_*
\end{array}$$

Since G is continuous (by (i)) we have $x_* = G(x_*)$.

Uniqueness : Let $x = G(x)$ and $y = G(y)$, $x, y \in S_\rho(x_0)$.

Then

$$\|x - y\| = \|G(x) - G(y)\| \leq K_0\|x - y\|, \quad K_0 < 1.$$

Contradiction. $\bullet$

1.2 The Implicit Function Theorem

Let $F\ :\ \mathcal{B} \times \mathrm{R}^n \to \mathcal{B}$ satisfy

(i) $F(u_0, \lambda_0) = 0,\ u_0 \in \mathcal{B},\ \lambda_0 \in \mathrm{R}^n$.

(ii) $F_u(u_0, \lambda_0)$ is nonsingular with bounded inverse,

$$\|F_u(u_0, \lambda_0)^{-1}\| \leq M.$$

(iii) Lipschitz continuity of F and F_u : $\forall u, v \in S_\rho(u_0)$, and $\forall \lambda, \mu \in S_\rho(\lambda_0)$,

$$\begin{aligned}
\|F(u, \lambda) - F(v, \mu)\| &\leq K_L(\|u - v\| + \|\lambda - \mu\|), \\
\|F_u(u, \lambda) - F_u(v, \mu)\| &\leq K_L(\|u - v\| + \|\lambda - \mu\|).
\end{aligned}$$

Then there exists $\delta \leq \rho$ and a unique function $u(\lambda)$ such that

(i) $F(u(\lambda), \lambda) = 0, \quad \forall \lambda \in S_\delta(\lambda_0)$,

(ii) $u(\lambda_0) = u_0$,

(iii) $u(\lambda) \in C(S_\delta(\lambda_0))$.

REMARK. If $F(u_0, \lambda_0) = 0$ and if $F_u(u_0, \lambda_0)$ is invertible with $\|F_u(u_0, \lambda_0)^{-1}\| \leq M$ then u_0 is said to be an *isolated* solution of $F(u, \lambda_0) = 0$.

Thus, isolation (plus Lipschitz continuity assumptions) implies the local existence of a unique branch *solution branch* $u = u(\lambda)$.

PROOF.

$$\begin{aligned} F(u,\lambda) = 0 \quad &\Longleftrightarrow \quad F_u^0 u = F_u^0 u - F(u,\lambda) \qquad (F_u^0 \equiv F_u(u_0, \lambda_0)) \\ &\Longleftrightarrow \quad u = \underbrace{F_u^{0^{-1}}[F_u^0 u - F(u,\lambda)].}_{\equiv G(u,\lambda)} \end{aligned}$$

Thus $F(u, \lambda) = 0 \Longleftrightarrow$ u is a fixed point of $G(\cdot, \lambda)$. (Note that the corresponding fixed point iteration is in fact the Chord Method for solving $F(u, \lambda) = 0$.)

Verify the conditions of the Contraction Theorem:

Pick $u, v \in S_{\rho_1}(u_0)$ and any *fixed* $\lambda \in S_{\rho_1}(\lambda_0)$, ρ_1 to be chosen later. Then

$G(u, \lambda) - G(v, \lambda) = F_u^{0^{-1}}\{F_u^0[u - v] - [F(u, \lambda) - F(v, \lambda)]\}$.

By the Fundamental Theorem of Calculus (FTC) :

$$\begin{aligned} F(u,\lambda) - F(v,\lambda) \quad &= \quad \int_0^1 \frac{d}{dt} F(tu + (1-t)v, \lambda) dt \\ &= \quad \int_0^1 F_u(tu + (1-t)v, \lambda) dt \ [u - v] \\ &\equiv \quad \hat{F}_u(u, v, \lambda)[u - v]. \ \ (\sim \text{Mean Value Theorem}) \end{aligned}$$

Now

$$\|G(u,\lambda) - G(v,\lambda)\| \leq M \|F_u^0 - \hat{F}_u(u,v,\lambda)\| \cdot \|u - v\|$$

$$= M \| \int_0^1 F_u(u_0, \lambda_0) - F_u(tu + (1-t)v, \lambda) \ dt \| \cdot \ \|u - v\|$$

$$\leq M \int_0^1 \|F_u(u_0, \lambda_0) - F_u(tu + (1-t)v, \lambda)\| \ dt \cdot \ \|u - v\|$$

$$\leq M \int_0^1 K_L\{\|u_0 - \underbrace{(tu + (1-t)v)}_{\in S_{\rho_1}(u_0)}\| + \|\lambda_0 - \lambda\|\}\ dt \cdot\ \|u - v\|$$

$$\leq \underbrace{M\ K_L 2\rho_1}_{\equiv K_0}\|u - v\|.$$

Thus if we take $\rho_1 < \frac{1}{2M\ K_L}$ then $K_0 < 1$.

Verify condition (ii) of the Contraction Theorem :

$$\begin{aligned}
\|G(u_0, \lambda) - & \underbrace{u_0}_{=G(u_0,\lambda_0)}\| = \|G(u_0, \lambda) - G(u_0, \lambda_0)\| \\
= & \ \|F_u^{0^{-1}}\{F_u^0 u_0 - F(u_0, \lambda)\} - F_u^{0^{-1}}\{F_u^0 u_0 - F(u_0, \lambda_0)\}\| \\
= & \ \|F_u^{0^{-1}}\{F(u_0, \lambda_0) - F(u_0, \lambda)\}\| \\
\leq & \ M\ K_L\|\lambda - \lambda_0\| \leq M\ K_L\rho. \\
& (\rho \leq \rho_1 \text{ to be chosen.})
\end{aligned}$$

We want this to be $\leq (1 - K_0)\rho_1$. So choose $\rho \leq \frac{(1-K_0)\rho_1}{M\ K_L}$.

Finally we prove continuity. Let $\lambda_1, \lambda_2 \in S_\rho(\lambda_0)$ with corresponding solutions $u(\lambda_1), u(\lambda_2)$.

Then

$$\|u(\lambda_1) - u(\lambda_2)\| = \|G(u(\lambda_1), \lambda_1) - G(u(\lambda_2), \lambda_2)\|$$

$$\leq \|G(u(\lambda_1), \lambda_1) - G(u(\lambda_2), \lambda_1)\| + \|G(u(\lambda_2), \lambda_1) - G(u(\lambda_2), \lambda_2)\|$$

$$\leq K_0\|u(\lambda_1) - u(\lambda_2)\|\ +\ \|F_u^{0^{-1}}[F_u^0 u(\lambda_2) - F(u(\lambda_2), \lambda_1)] - F_u^{0^{-1}}[F_u^0 u(\lambda_2) - F(u(\lambda_2), \lambda_2)]\|$$

$$\leq \underbrace{K_0}_{<1}\|u(\lambda_1) - u(\lambda_2)\| + M\ K_L\|\lambda_1 - \lambda_2\|.$$

Hence

$$\|u(\lambda_1) - u(\lambda_2)\| \leq \frac{MK_L}{1 - K_0}\|\lambda_1 - \lambda_2\|. \qquad \bullet$$

1.3 The Banach Lemma

Let $L : \mathcal{B} \to \mathcal{B}$, with $\|L\| < 1$. Then $(I+L)^{-1}$ exists and

$$\|(I+L)^{-1}\| \le \frac{1}{1-\|L\|}.$$

PROOF. Suppose $I+L$ is not invertible. Then there exists $y \in \mathcal{B}$ such that $(I+L)y = 0$, $y \neq 0$. So $y = -Ly$, and

$$\|y\| = \|Ly\| \le \|L\| \cdot \|y\| < \|y\|.$$

Contradiction. Thus $(I+L)^{-1}$ exists.

Bound the inverse :

$$(I+L)(I+L)^{-1} = I,$$

$$(I+L)^{-1} = I - L(I+L)^{-1},$$

$$\|(I+L)^{-1}\| \le 1 + \underbrace{\|L\|}_{<1} \|(I+L)^{-1}\|,$$

$$\|(I+L)^{-1}\| \le \frac{1}{1-\|L\|}\ . \qquad \bullet$$

1.4 Differentiability

LEMMA.

Under the conditions of the IFT, there exists $M_1 > 0$, $\delta > 0$, such that $F_u(u,\lambda)^{-1}$ exists and

$$\|F_u(u,\lambda)^{-1}\| \le M_1 \quad \text{in} \quad S_\delta(u_0) \times S_\delta(\lambda_0).$$

PROOF.

$$\begin{aligned} F_u(u,\lambda) &= F_u^0 + F_u(u,\lambda) - F_u^0 \qquad (F_u^0 \equiv F_u(u_0,\lambda_0)) \\ &= F_u^0\{I + \underbrace{F_u^{0^{-1}}(F_u(u,\lambda) - F_u^0)}_{\equiv L}\}. \end{aligned}$$

$$\|L\| \le MK_L\{\|u-u_0\| + \|\lambda-\lambda_0\|\} \le MK_L 2\delta.$$

Choose $\delta < \frac{1}{2MK_L}$ (as in the IFT). Then $(I+L)^{-1}$ exists and

$$\|(I+L)^{-1}\| \le \frac{1}{1-MK_L 2\delta},$$

and $F_u(u,\lambda)^{-1}$ exists and

$$\|F_u(u,\lambda)^{-1}\| = \|F_u^{0^{-1}}(I+L)^{-1}\| \leq \frac{M}{1-MK_L 2\delta} \equiv M_1. \qquad \bullet$$

THEOREM.

If, in addition to the assumptions of the IFT, $F_\lambda(u,\lambda)$ is continuous in $S_\rho(u_0)\times S_\rho(\lambda_0)$ then $u(\lambda)$ has a continuous derivative $u_\lambda(\lambda)$ on $S_\delta(u_0)\times S_\delta(\lambda_0)$.

PROOF.

Using the definition of (Fréchet) derivative, we are given that

$$\exists F_u(u,\lambda) \ni F(u,\lambda) - F(v,\lambda) = F_u(u,\lambda)(u-v) + R_1(u,v,\lambda),$$

$$\frac{\|R_1(u,v,\lambda)\|}{\|u-v\|} \to 0 \text{ as } \|u-v\| \to 0,$$

$$\exists F_\lambda(u,\lambda) \ni F(u,\lambda) - F(u,\mu) = F_\lambda(u,\lambda)(\lambda-\mu) + R_2(u,\lambda,\mu),$$

$$\frac{\|R_2(u,\lambda,\mu)\|}{\|\lambda-\mu\|} \to 0 \text{ as } \|\lambda-\mu\| \to 0,$$

We must show

$$\exists u_\lambda(\lambda) \ni u(\lambda) - u(\mu) = u_\lambda(\lambda)(\lambda-\mu) + r(\lambda,\mu),$$

$$\frac{\|r(\lambda,\mu)\|}{\|\lambda-\mu\|} \to 0 \text{ as } \|\lambda-\mu\| \to 0.$$

Now

$$\begin{aligned} 0 &= F(u(\lambda),\lambda) - F(u(\mu),\mu) \\ &= F(u(\lambda),\lambda) - F(u(\mu),\lambda) + F(u(\mu),\lambda) - F(u(\mu),\mu) \\ &= F_u(u(\lambda),\lambda)\ (u(\lambda)-u(\mu)) + R_1(u(\lambda),u(\mu),\lambda)\ +\ F_\lambda(u(\mu),\lambda)\ (\lambda-\mu) + R_2(u(\mu),\lambda,\mu). \end{aligned}$$

$$u(\lambda) - u(\mu) = -\underbrace{F_u(u(\lambda),\lambda)^{-1}}_{\text{exists}}\{F_\lambda(u(\mu),\lambda)\ (\lambda-\mu) - (R_1+R_2)\}$$

$$= -\underbrace{F_u(u(\lambda),\lambda)^{-1}}_{\text{bounded}}\{F_\lambda(u(\lambda),\lambda)\ (\lambda-\mu)\ -\ \underbrace{F_\lambda(u(\lambda),\lambda)-F_\lambda(u(\mu),\lambda)\ (\lambda-\mu)-(R_1+R_2)}_{\text{r}}\}.$$

Consider each term of r, "forgetting" the harmless factor $F_u(u(\lambda),\lambda)^{-1})$:

Since u and F_λ are continuous, we have

$$\frac{\|(F_\lambda(u(\lambda),\lambda)-F_\lambda(u(\mu),\lambda))(\lambda-\mu)\|}{\|\lambda-\mu\|}\to 0 \text{ as } \|\lambda-\mu\|\to 0.$$

Also,

$$\frac{\|R_2(u(\lambda),\lambda,\mu)\|}{\|\lambda-\mu\|}\to 0,\ \text{since } F_\lambda \text{ exists },$$

and

$$\frac{\|R_1(u(\lambda),u(\mu),\lambda)\|}{\|\lambda-\mu\|} = \underbrace{\frac{\|R_1(u(\lambda),u(\mu),\lambda)\|}{\|u(\lambda)-u(\mu)\|}}_{\to 0 \text{ since } F_u \text{ exists}} \quad \underbrace{\frac{\|u(\lambda)-u(\mu)\|}{\|\lambda-\mu\|}}_{\text{bounded (see proof of IFT)}}$$

Thus $u_\lambda(\lambda) = -F_u(u(\lambda),\lambda)^{-1}F_\lambda(u(\lambda),\lambda)$.

To prove that u_λ is continuous it suffices to show that $F_u(u(\lambda),\lambda)^{-1}$ is continuous

$$\begin{aligned} &\|F_u(u(\lambda),\lambda)^{-1}-F_u(u(\mu),\mu)^{-1}\| \\ &= \|F_u(u(\lambda),\lambda)^{-1}\{F_u(u(\mu),\mu)-F_u(u(\lambda),\lambda)\}F_u(u(\mu)\mu)^{-1}\| \\ &\le M_1^2K_L\{\|u(\mu)-u(\lambda)\|+\|\mu-\lambda\|\}. \qquad \bullet \end{aligned}$$

REMARK. In fact, if F_λ is Lipschitz continuous (F_u already assumed Lipschitz continuous) then u_λ is Lipschitz continuous. More generally, it can be shown that if F is C^K then $u(\lambda)$ is C^K, i.e., u inherits the degree of continuity of G.

Chapter 2

Examples of IFT Application

2.1 A Predator-Prey Model

$$\begin{cases} u_1' = & 3u_1(1-u_1) - u_1u_2 - \lambda(1-e^{-5u_1}), \\ u_2' = & -u_2 + 3u_1u_2. \end{cases}$$

Here u_1 may be thought of as "fish" and u_2 as "sharks", while the term $\lambda(1-e^{-5u_1})$ represents "fishing", with "fishing-quota" λ. When $\lambda = 0$ the *stationary solutions* are

$$\left.\begin{array}{ll} 3u_1(1-u_1) - u_1u_2 & = 0 \\ -u_2 + 3u_1u_2 & = 0 \end{array}\right\} \Rightarrow (u_1, u_2) = (0,0),\ (1,0),\ (\frac{1}{3}, 2).$$

Jacobian matrix

$$J = \begin{pmatrix} 3 - 6u_1 - u_2 - 5\lambda e^{5u_1} & -u_1 \\ 3u_2 & -1 + 3u_1 \end{pmatrix} = J(u_1, u_2; \lambda)$$

$$J(0,0;0) = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \text{ eigenvalues 3,-1} \quad \text{(unstable)}$$

$$J(1,0;0) = \begin{pmatrix} -3 & -1 \\ 0 & 2 \end{pmatrix}, \text{ eigenvalues -3,2} \quad \text{(unstable)}$$

$$J(\frac{1}{3}, 2; 0) = \begin{pmatrix} -1 & -\frac{1}{3} \\ 6 & 0 \end{pmatrix}, \text{ eigenvalues } \begin{cases} (-1-\mu)(-\mu) + 2 = 0 \\ \mu^2 + \mu + 2 = 0 \\ \mu_\pm = \frac{-1 \pm \sqrt{-7}}{2} \\ \mathrm{Re}(\mu_\pm) < 0 \quad \text{(stable)} \end{cases}$$

All three Jacobians at $\lambda = 0$ are nonsingular. Thus, by the IFT, all three stationary points have continuations for (small) $\lambda \neq 0$.

In this problem we can *explicitly* find all solutions (see Figure 2.1) :

I :

$$(u_1, u_2) = (0, 0).$$

II :

$$u_2 = 0, \qquad \lambda = \frac{3u_1(1-u_1)}{1-e^{-5u_1}}.$$

$$\text{Note that} \lim_{u_1 \to 0} \lambda = \lim_{u_1 \to 0} \frac{3(1-2u_1)}{5e^{-5u_1}} = \frac{3}{5}.$$

III :

$$u_1 = \frac{1}{3}, \qquad \frac{2}{3} - \frac{1}{3}\, u_2 - \lambda(1 - e^{-5/3}) = 0 \;\Rightarrow\; u_2 = 2 - 3\lambda(1 - e^{-5/3}).$$

These *solution branches* intersect at two *branch points*, one of which is $(u_1, u_2, \lambda) = (0, 0, 3/5)$.

Stability of branch I :

$$J((0,0);\lambda) = \begin{pmatrix} 3-5\lambda & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{eigenvalues } \; 3-5\lambda, -1.$$

Hence the trivial solution is unstable if $\lambda < 3/5$, and stable if $\lambda > 3/5$, as indicated in Figure 2.2.

Stability of branch II : This branch has no stable positive solutions.

Stability of branch III : At $\lambda_H \approx 0.67$ (Solution 8 in Figure 2.2) the complex eigenvalues cross the imaginary axis. This crossing is a *Hopf bifurcation*. Beyond λ_H there are *periodic solutions* whose period T increases as λ increases; see Figure 2.3 for some representative orbits. The period becomes infinite at $\lambda = \lambda_\infty \approx 0.7$. This final orbit is in fact a *heteroclinic cycle*.

From Figure 2.2 we can deduce the solution behavior for (slowly) increasing λ :

- Branch III is followed until λ_H.
- Periodic solutions of increasing period until $\lambda = \lambda_\infty$.
- Collapse to trivial solution (Branch I).

EXERCISE. Use AUTO to repeat the numerical calculations (demo pp2). Sketch phase plane diagrams for $\lambda = 0, 0.5, 0.68, 0.70, 0.71$.

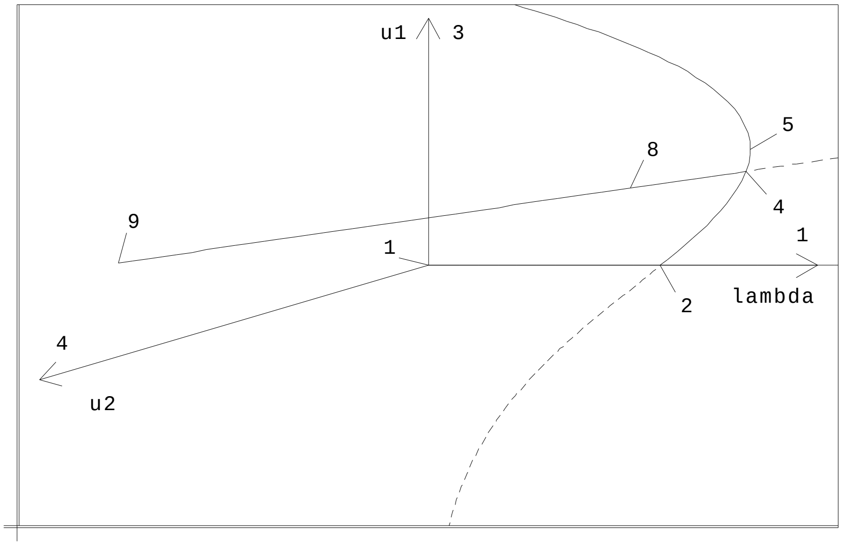

Figure 2.1: Stationary solution branches of the predator-prey model. Solution 2 is the branch point that will be discussed later. Solution 4 is another branch point, while Solution 8 is a Hopf bifurcation point.

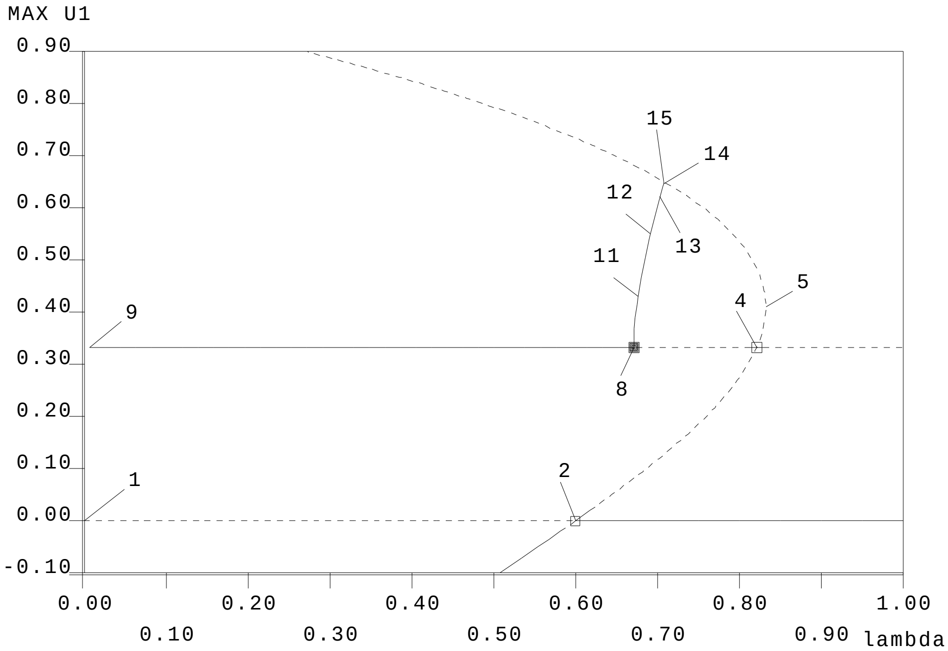

Figure 2.2: Bifurcation diagram of the predator-prey model. The periodic solution branch is also shown. For stationary solutions the vertical axis is simply u_1, while for periodic solutions $max\ u_1$ is plotted. Solid/dashed lines denote stable/unstable solution. Open squares are branch points; the solid square is a Hopf bifurcation.

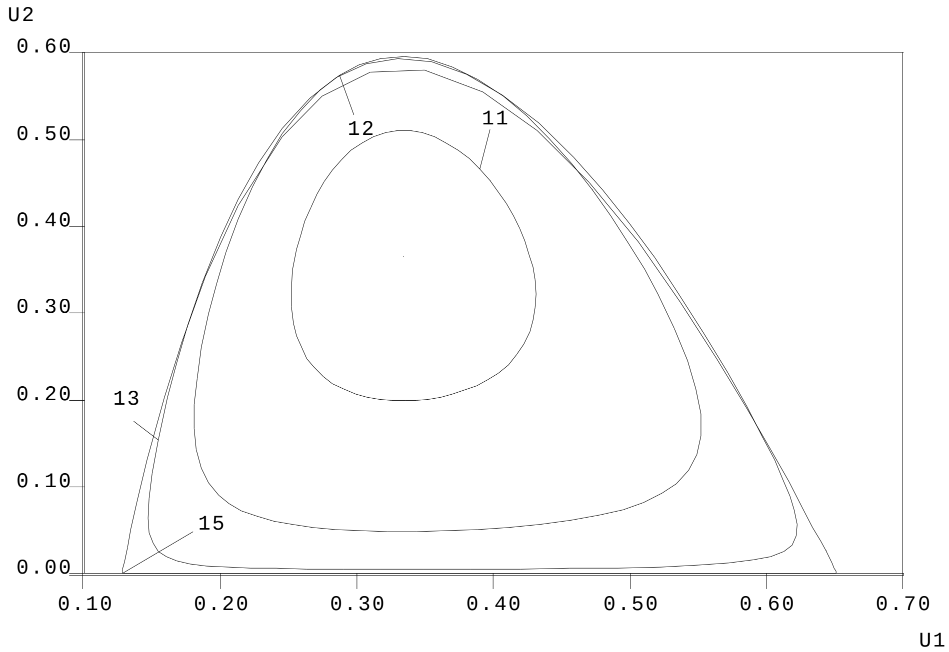

Figure 2.3: Some periodic solutions of the predator-prey model. The final orbits are very close to a heteroclinic cycle.

2.2 The Bratu Problem

$$\begin{cases} u''(x) - \lambda e^{u(x)} &= 0, \quad \forall x \in [0,1], \\ u(0) = u(1) &= 0. \end{cases}$$

If $\lambda = 0$ then $u(x) \equiv 0$ is a solution. We'll prove it is isolated, so that there is a continuation $u = \tilde{u}(\lambda)$, for $|\lambda|$ small. Consider

$$\left.\begin{array}{rcl} u'' - \lambda e^u &=& 0, \\ u(0) = 0, u'(0) &=& q, \end{array}\right\} \Rightarrow u = u(x; q, \lambda).$$

We want to solve $\underbrace{u(1; q, \lambda)}_{\equiv F(q,\lambda)} = 0$ for $|\lambda|$ small. $F(0,0) = 0$.

We must show (IFT) that $F_q(0,0) \equiv u_q(1; 0, 0) \neq 0$:

$$\left.\begin{array}{rcl} u_q^{''} - \lambda_0 e^{u_0} u_q &=& 0, \\ u_q(0) = 0, u_q^{'}(0) &=& 1, \end{array}\right\} \text{ where } u_0 \equiv 0.$$

Now $u_q(x; 0, 0)$ satisfies

$$\begin{cases} u_q^{''} = 0, \\ u_q(0) = 0, u_q^{'}(0) = 1. \end{cases}$$

Hence $u_q(x; 0, 0) = x, \quad u_q(1; 0, 0) = 1 \neq 0.$

EXERCISE. Recompute the solution branch of the Bratu problem as represented in Figures 2.4 and 2.5. (AUTO demo exp.)

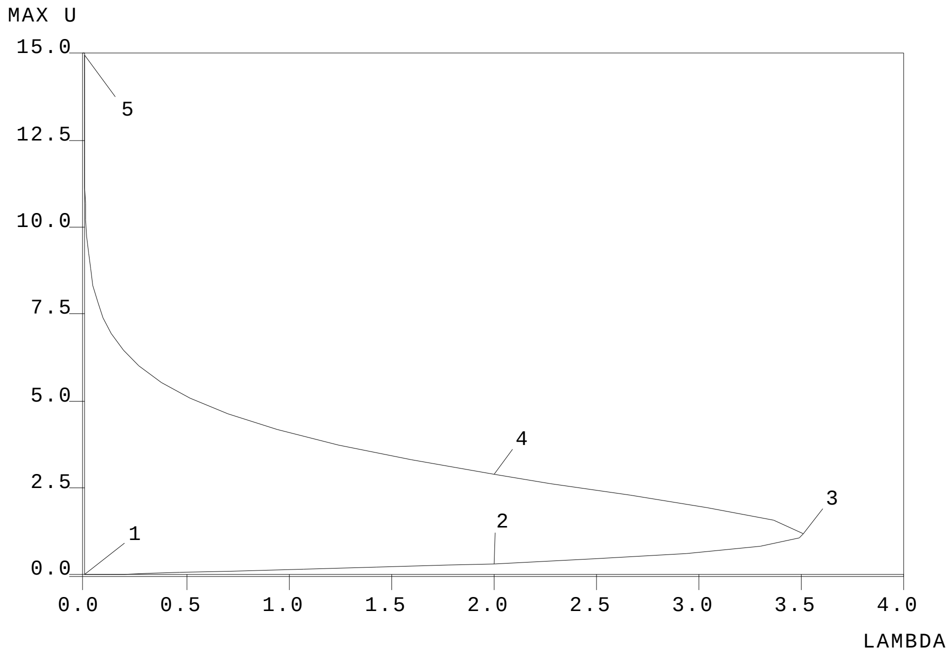

Figure 2.4: Bifurcation diagram of the Bratu equation. Note that there are two solutions for $0 < \lambda < \lambda_C$, where $\lambda_C \approx 3.51$. There is one solution for $\lambda = \lambda_C$ and for $\lambda \leq 0$, and are no solutions for $\lambda > \lambda_C$.

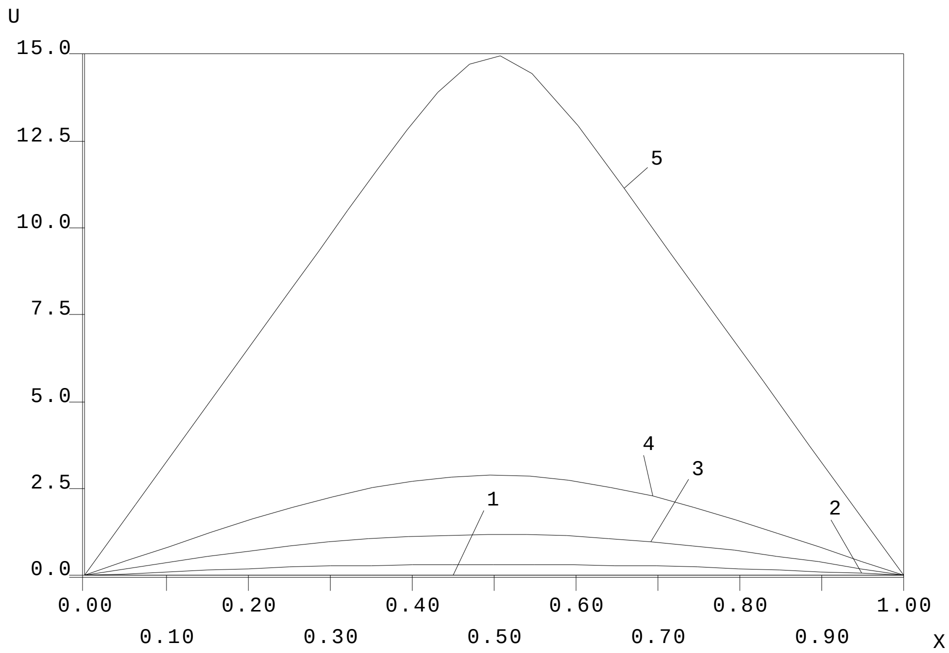

Figure 2.5: Some solutions to the Bratu equation.

2.3 A Nonlinear Eigenvalue Problem

The equations for column buckling (from nonlinear electricity theory) are :

$$\begin{aligned} &u_1''(x) + \lambda u_1(x) = 0, \\ &u_3'(x) + \tfrac{1}{2}u_1'(x)^2 + \lambda\beta = 0, \end{aligned}$$

for $x \in [0,1]$, with

$$\begin{aligned} &u_1(0) = u_1(1) = 0, \\ &u_3(0) = -u_3(1) = c \quad (c > 0), \end{aligned}$$

where λ is a stress (to be determined) and β is another physical constant. Take $\beta = 1$. Note that the boundary conditions are "overspecified" (to determine λ).

Rewrite the equations as a first order system :

$$\left\{ \begin{array}{lll} u_1' & = u_2, & u_1(0) = u_1(1) = 0, \\ u_2' & = -\lambda u_1, & \\ u_3' & = -\frac{1}{2}u_2^2 - \lambda, & u_3(0) = -u_3(1) = c. \end{array} \right. \tag{2.1}$$

Note that if $u_1 \equiv u_2 \equiv 0$ then $u_3' = -\lambda$, so that $u_3(x) = c - \lambda x$, with $u_3(0) = c$ and $u_3(1) = c - \lambda = -c$. Thus we must have $\lambda = 2c$, so that $u_3(x) = c - 2cx = c(1-2x)$. Hence

$$u_1 \equiv u_2 \equiv 0, \quad u_3(x) = c(1-2x), \quad \lambda = 2c,$$

is a solution for all c.

Are these solutions isolated? Formal setup : Consider

$$\left. \begin{array}{ll} u_1' = u_2, & u_1(0) = 0, \\ u_2' = -\lambda u_1, & u_2(0) = p, \\ u_3' = -\frac{1}{2}\, u_2^2 - \lambda, & u_3(0) = c. \end{array} \right\} \Rightarrow \underline{u} = \underline{u}(x; p, \lambda, c).$$

We must have

$$\left. \begin{array}{lll} u_1(1; p, \lambda, c) & = & 0, \\ u_3(1; p, \lambda, c) + c & = & 0, \end{array} \right\} \quad \sim \quad F(p, \lambda; c) = 0,$$

$$F \;:\; \mathrm{R}^2 \times \mathrm{R} \to \mathrm{R}^2.$$

So the question is : Is $(F_p \mid F_\lambda)(c)$ nonsingular about the basic solution branch above ?

To answer the above question quickly, we omit explicit construction of F : The linearized homogeneous equations obtained by linearization of Equation (2.1) about $u_1, u_2, u_3, \lambda, c$ with respect to u_1, u_2, u_3, λ, acting on v_1, v_2, v_3, μ, are

$$\left\{ \begin{array}{ll} v_1' = v_2, & v_1(0) = v_1(1) = 0, \\ & \\ v_2' = -\lambda v_1 - \mu u_1, & \\ & \\ v_3' = -u_2\, v_2 - \mu, & v_3(0) = -v_3(1) = 0. \end{array} \right.$$

In particular, the linearized homogeneous equations about $u_1 \equiv u_2 \equiv 0$, $u_3(x) = c(1-2x)$, $\lambda = 2c$, are

$$\left.\begin{array}{ll} v_1' = v_2, & v_1(0) = v_1(1) = 0, \\ & \\ v_2' = -2cv_1, & \\ & \\ v_3' = -\mu, & v_3(0) = -v_3(1) = 0. \end{array}\right\} \quad \Rightarrow \quad \mu = 0, \ v_3 \equiv 0.$$

Now, if $2c \neq k^2\pi^2$, $k = 1, 2, 3 \cdots$, then

$$v_1'' + 2cv_1 = 0,$$

$$v_1(0) = v_1(1) = 0,$$

has the unique solution $v_1 \equiv 0$, and hence also $v_2 \equiv 0$. Thus if $c \neq \frac{1}{2}k^2\pi^2$ then the basic solution branch is locally unique. However, if $c = \frac{1}{2}k^2\pi$ then the linearization is singular, and there may be bifurcations. (In fact there are : buckled states.)

EXERCISE. Use AUTO to compute the first few bifurcating branches of the buckling problem.

2.4 The Pendulum Equation

2.4.1 Derivation

The equation of a damped pendulum subject to a constant torque is given by

$$m\ R\phi''(t) + \underbrace{\epsilon m\phi'(t)}_{\text{damping}} + mg\sin\phi(t) = \underbrace{I}_{\text{torque}},$$

$$\phi''(t) + \frac{\epsilon}{R}\phi'(t) + \frac{g}{R}\sin\phi(t) = \frac{I}{mR}.$$

Scale time :

$$s = ct, \quad \phi' = \frac{d\phi}{dt} = \frac{d\phi}{ds}\frac{ds}{dt} = c\dot{\phi}, \quad \phi'' = c^2\ddot{\phi}.$$

$$c^2\ddot{\phi}(s) + \frac{\epsilon c}{R}\dot{\phi}(s) + \frac{g}{R}\sin\phi(s) = \frac{I}{mR},$$

$$\ddot{\phi} + \frac{\epsilon}{Rc}\dot{\phi} + \frac{g}{Rc^2}\sin\phi = \frac{I}{mRc^2}.$$

Choose c such that $\frac{g}{Rc^2} = 1$, i.e., $c = \sqrt{g/R}$,

$$\tilde{\epsilon} \equiv \frac{\epsilon}{Rc}, \qquad \tilde{I} = \frac{I}{mRc^2}.$$

Then the equation becomes $\ddot{\phi} + \tilde{\epsilon}\ \dot{\phi} + \sin\phi = \tilde{I}$, or, dropping the $\tilde{}$ and using $'$,

$$\boxed{\phi'' + \epsilon\ \phi' + \sin\phi = I}\ .$$

Rotations are solutions that satisy $\phi(t+T) = \phi(t) + 2\pi$ for all t, or, equivalently,

$$\begin{aligned} \phi(T) &= \overbrace{\phi(0)}^{(\equiv 0)} + 2\pi \qquad (= 2\pi), \\ \phi'(T) &= \phi'(0), \end{aligned}$$

where T is the *period.*

2.4.2 The Undamped Pendulum

First consider the undamped, unforced pendulum $(\epsilon = I = 0)$:

$$\phi'' + \sin\phi = 0.$$

Suppose the initial data for a rotation are $\phi(0) = 0$, and $\phi'(0) = p > 0$. We have $\phi(T) = 2\pi$ and $\phi'(T) = \phi'(0) = p$. Integration gives

$$\begin{aligned} \int_0^t \phi'\phi'' dt + \int_0^t \phi' \sin\phi\ dt &= 0, \\ \frac{1}{2}\phi'^2|_0^t - \cos\phi|_0^t &= 0, \\ \\ \frac{1}{2}\phi'(t)^2 - \cos\phi(t) &= \frac{1}{2}p^2 - 1, \\ \underbrace{\frac{1}{2}\phi'(t)^2}_{\text{kinetic energy}} \quad + \quad \underbrace{1 - \cos\phi(t)}_{\text{potential energy}} \quad &= \frac{1}{2}p^2. \end{aligned}$$

Thus

$$\begin{aligned} \phi'(t) = \frac{d\phi}{dt} &= \sqrt{p^2 - 2 + 2\cos\phi(t)}\ , \\ \frac{dt}{d\phi} &= \frac{1}{\sqrt{p^2 - 2 + 2\cos\phi}}\ , \\ \int_0^{2\pi} \frac{dt}{d\phi}\ d\phi &= \int_0^{2\pi} \frac{1}{\sqrt{p^2 - 2 + 2\cos\phi}}\ d\phi, \\ T &= \int_0^{2\pi} \frac{1}{\sqrt{p^2 - 2 + 2\cos\phi}}\ d\phi. \end{aligned}$$

We see that

$$T \to 0 \text{ as } p \to \infty,$$

and

$$T \to \int_0^{2\pi} \frac{1}{\sqrt{2 + 2\cos\phi}}\ d\phi = \infty \text{ as } \ p \to 2.$$

In fact, rotations exists for all $p > 2$.

2.4.3 The Forced, Damped Pendulum

Now consider the forced, damped pendulum :

$$\phi''(t) + \epsilon\ \phi' + \sin\phi = I,$$

with $\phi(0) = 0$ (which sets the phase), and $\phi'(0) = p$. We write the solution as $\phi = \phi(t; p, I, \epsilon)$. Do there exist rotations, i.e., does there exist T such that

$$\phi(T; p, I, \epsilon) = 2\pi, \qquad \text{and} \qquad \phi'(T; p, I, \epsilon) = p \quad ?$$

THEOREM.

Let ϕ_0 be a rotation of period T_0 of the undamped, unforced pendulum :

$$\phi_0'' + \sin\phi_0 = 0,$$

$$\phi_0(0) = 0, \qquad \phi_0'(0) = p_0,$$
$$\phi_0(T_0) = 2\pi, \qquad \phi_0'(T_0) = p_0.$$

Then there exist (smooth) functions

$$T = T(p, \epsilon), \qquad I = I(p, \epsilon),$$

with

$$T(p_0, 0) = T_0, \qquad I(p_0, 0) = 0,$$

such that $\phi(t; p, I(p, \epsilon), \epsilon)$ is a rotation of period $T(p, \epsilon)$ of the damped, forced pendulum

$$\phi'' + \epsilon\ \phi' + \sin\phi = I,$$

that is,

$$\begin{array}{lcllcl} \phi(T(p,\epsilon); p, I(p,\epsilon), \epsilon) & = & 2\pi, & \quad \phi'(T(p,\epsilon), p, I(p,\epsilon), \epsilon) & = & p, \\ \phi(0; p, I(p,\epsilon), \epsilon) & = & 0, & \quad \phi'(0; p, I(p,\epsilon), \epsilon) & = & p, \end{array}$$

for all (p, ϵ) sufficiently close to $(p_0, 0)$.

PROOF.

The Jacobian matrix with respect to T, I of the algebraic system

$$\begin{aligned} \phi(T; p, I, \epsilon) - 2\pi &= 0, \\ \phi'(T; p, I, \epsilon) - p &= 0, \end{aligned}$$

evaluated at $p = p_0$, $T = T_0$, $I = \epsilon = 0$, is

$$J_0 = \begin{pmatrix} \phi_0' & \phi_I^0 \\ \phi_0'' & \phi_I^{0\prime} \end{pmatrix} (T_0).$$

We must show that $\det J_0 \neq 0$:

$$\phi_0'' + \sin\phi_0 = 0 \;\Rightarrow\; \phi_0''(T_0) = -\sin(\phi_0(T_0)) = -\sin(2\pi) = 0,$$

$$\phi_0'(T_0) = p_0 \neq 0.$$

Thus $\det J_0 \neq 0$ if $\phi_I^{0'}(T_0) \neq 0$. Here ϕ_I satisfies

$$\phi_I'' + \epsilon\ \phi_I' + \phi_I \cos\phi = 1, \qquad \phi_I(0) = \phi_I'(0) = 0.$$

In particular,

$$\phi_I^{0''} + \phi_I^0 \cos\phi_0 = 1, \qquad \phi_I^0(0) = \phi_I^{0'}(0) = 0.$$

From

$$\phi_I^{0''}\phi_0' + \phi_I^0 \cos\phi_0\ \phi_0' \;=\; \phi_0',$$

and

$$\phi_0'' + \sin\phi_0 = 0 \;\Rightarrow\; \phi_0''' + \cos\phi_0\ \phi_0' = 0,$$

we have

$$\phi_I^{0''}\ \phi_0' - \phi_I^0\ \phi_0''' \;=\; \phi_0'.$$

Integrate :

$$\int_0^{T_0} \phi_I^{0''}\phi_0' - \int_0^{T_0} \phi_I^0\phi_0''' \;=\; \int_0^{T_0} \phi_0' \;=\; 2\pi,$$

$$\phi_I^{0'}\phi_0'|_0^{T_0} - \int_0^{T_0} \phi_I^{0'}\phi_0'' - \phi_I^0 \underbrace{\phi_0''}_{-\sin\phi_0} |_0^{T_0} + \int_0^{T_0} \phi_I^{0'}\phi_0'' \;=\; 2\pi,$$

$$\phi_I^{0'}(T_0)\underbrace{\phi_0'(T_0)}_{p_0} - \underbrace{\phi_I^{0'}(0)}_{0}\phi_0'(0) = 2\pi.$$

Hence $\phi_I^{0'}(T_0) = 2\pi/p_0 \neq 0$. $\bullet$

EXERCISE. Run AUTO demo pen to compute solutions to the equations of two torsion-coupled pendula.

Chapter 3

Continuation

Consider the equation

$$G(u,\lambda)=0, \qquad u, G(\cdot,\cdot)\in \mathrm{R}^n, \quad \lambda \in \mathrm{R}.$$

Let $x \equiv (u,\lambda)$. Then the equation can be written

$$G(x)=0, \qquad G:\mathrm{R}^{n+1}\to \mathrm{R}^n.$$

3.1 Regular Solutions

A solution x_0 of $G(x)=0$ is *regular* if $G_x^0 \equiv G_x(x_0)$ has maximal rank, i.e., if $\mathrm{Rank}(G_x^0)=n$.

In the parameter formulation, $G(u,\lambda)=0$, we have

$$\mathrm{Rank}(G_x^0) \;=\; \mathrm{Rank}(G_u^0 \mid G_\lambda^0) \;=\; n \quad \Longleftrightarrow \quad \begin{cases} \text{(i) } G_u^0 \text{ is nonsingular,} \\[1ex] \quad \text{or} \\[1ex] \text{(ii) } \begin{cases} \mathit{dim}\ \mathcal{N}(G_u^0)=1, \\ \text{and} \\ G_\lambda^0 \notin \mathcal{R}(G_u^0). \end{cases} \end{cases}$$

THEOREM. Let $x_0 \equiv (u_0,\lambda_0)$ be a regular solution of $G(x)=0$. Then, near x_0, there exists a unique one-dimensional continuum of solutions $x(s)$ with $x(0)=x_0$.

PROOF. If $\mathrm{Rank}(G_u^0 \mid G_\lambda^0)=n$ then either G_u^0 is nonsingular and by the IFT we have $u=u(\lambda)$ near x_0, or else we can interchange colums in G_x^0 to see that the solution can locally be parametrized by one of the components of u. Thus a unique *solution branch* passes through a regular solution. •

REMARKS. Such a continuum of solutions is often referred to as a *solution branch*. See Figure 3.1 for a graphical interpretation.

Case (ii) above is that of a *simple fold*, to be discussed in detail in Chapters 4 and 5.

3.2 Parameter Continuation

Here the continuation parameter is taken to be λ. Suppose we have a solution (u_0, λ_0) of

$$G(u, \lambda) = 0,$$

as well as the direction vector $\dot{u}_0$. Here $\dot{u} \equiv du/d\lambda$. We want to compute the solution u_1 at $\lambda_1 \equiv \lambda_0 + \Delta\lambda$. See Figure 3.2 for a graphical interpretation.

To find the solution u_1 we use Newton's method

$$\left.\begin{aligned} G_u(u_1^{(\nu)}, \lambda_1)\ \Delta u_1^{(\nu)} &= -G(u_1^{(\nu)}, \lambda_1) \\ u_1^{(\nu+1)} &= u_1^{(\nu)} + \Delta u_1^{(\nu)}. \end{aligned}\right. \qquad \nu = 0, 1, 2, \cdots .$$

Take

$$u_1^{(0)} = u_0 + \Delta\lambda\ \dot{u}_0.$$

If $G_u(u_1, \lambda_1)$ is nonsingular and $\Delta\lambda$ sufficiently small then the convergence theory for Newton's method assures us that this iteration will converge.

After convergence, the new direction $\dot{u}_1$ can be obtained by solving

$$G_u(u_1, \lambda_1)\dot{u}_1 = -G_\lambda(u_1, \lambda_1).$$

This equation follows from differentiating $G(u(\lambda), \lambda) = 0$ with respect to λ at $\lambda = \lambda_1$. Note that in practice the calculation of $\dot{u}_1$ can be done without another LU factorization of $G_u(u_1, \lambda_1)$. Thus the extra work to find $\dot{u}_1$ is negligible.

EXAMPLE. (The Bratu problem)

$$u''(x) + \lambda e^{u(x)} = 0 \quad \text{for} \quad x \in [0, 1], \qquad u(0) = 0, \qquad u(1) = 0.$$

If $\lambda = 0$ then $u(x) \equiv 0$ is an isolated solution (See Section 2.2.)

Discretize by introducing a mesh,

$$\{0 = x_0 < x_1 < \cdots < x_N = 1\},$$

$$x_j - x_{j-1} = h, \quad (1 \leq j \leq N), \qquad h = 1/N.$$

The discrete equations are :

$$\frac{u^{j+1} - 2u^j + u^{j-1}}{h^2} + \lambda e^{u^j} = 0, \qquad j = 1, \cdots, N-1,$$

with $u^0 = u^N = 0$. Let

$$U \equiv (u^1, u^2, \cdots, u^{N-1}).$$

Then we can write the above as

$$G(U, \lambda) = 0,$$

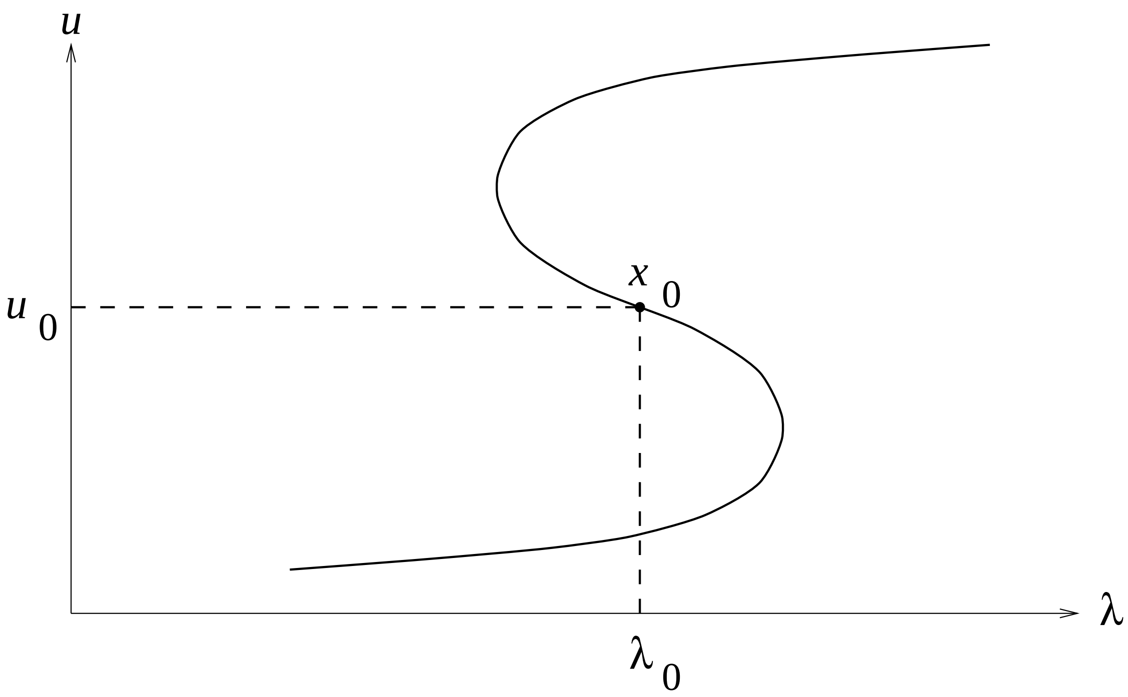

Figure 3.1: A solution branch of $G(u, \lambda) = 0$. Note the two folds.

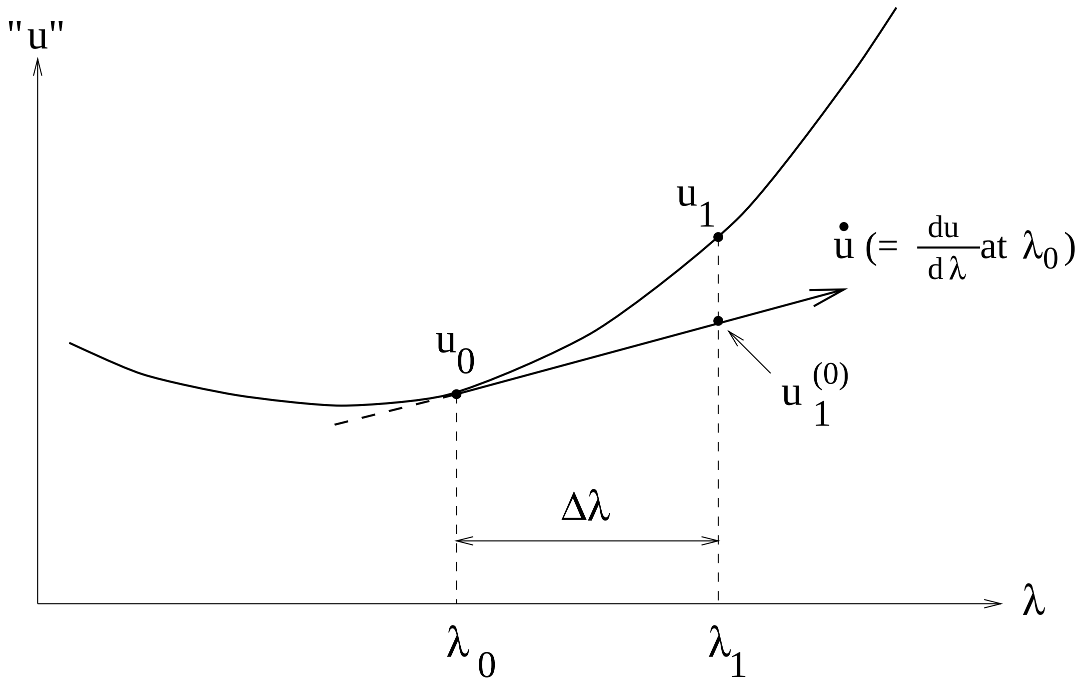

Figure 3.2: Graphical interpretation of parameter continuation.

where

$$G \; : \; \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}^n, \qquad n \equiv N-1.$$

Parameter continuation : Suppose we have λ_0, U_0 and $\dot{U}_0$. Set

$$\lambda_1 = \lambda_0 + \Delta\lambda.$$

Newton's method :

$$\left.\begin{array}{c} G_U(U_1^{(\nu)}, \lambda_1)\Delta U_1^{(\nu)} = -G(U_1^{(\nu)}, \lambda_1) \\ \\ U_1^{(\nu+1)} = U_1^{(\nu)} + \Delta U_1^{(\nu)} \end{array}\right. \qquad \nu = 0, 1, 2, \cdots,$$

with $U_1^{(0)} = U_0 + \Delta\lambda \dot{U}_0$. After convergence find $\dot{U}_1$ from

$$G_U(U_1, \lambda_1)\dot{U}_1 = -G_\lambda(U_1, \lambda_1).$$

Repeat the above procedure to find U_2, U_3, $\cdots$. Above

$$G_U(U, \lambda) = \begin{pmatrix} -\frac{2}{h^2} + \lambda e^{u^1} & \frac{1}{h^2} & & \\ \frac{1}{h^2} & -\frac{2}{h^2} + \lambda e^{u^2} & \frac{1}{h^2} & \\ & \cdot & \cdot & \cdot \\ & & \frac{1}{h^2} & -\frac{2}{h^2} + \lambda e^{u^{N-1}} \end{pmatrix}.$$

Thus we must solve a tridiagonal system for each Newton iteration.

The solution branch has a fold (see Figure 2.4) where the parameter continuation method will fail.

3.3 Keller's Pseudo-Arclength Continuation

This method allows continuation of a branch past folds. (See, for example, AUTO demo exp, as illustrated in Figure 2.4.) Suppose we have a solution (u_0, λ_0) of $G(u, \lambda) = 0$, as well as the direction vector of the solution branch $(\dot{u}_0, \dot{\lambda}_0)$. Pseudo-arclength continuation consists of solving the following equations for (u_1, λ_1) :

$$G(u_1, \lambda_1) = 0, \qquad (u_1 - u_0)^* \dot{u}_0 + (\lambda_1 - \lambda_0)\dot{\lambda}_0 - \Delta s = 0.$$

See Figure 3.3 for a graphical interpretation.

Newton's method :

$$\begin{pmatrix} (G_u^1)^{(\nu)} & (G_\lambda^1)^{(\nu)} \\ \\ \dot{u}_0^* & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \Delta u_1^{(\nu)} \\ \Delta \lambda_1^{(\nu)} \end{pmatrix} = - \begin{pmatrix} G(u_1^{(\nu)}, \lambda_1^{(\nu)}) \\ \\ (u_1^{(\nu)} - u_0)^* \dot{u}_0 - (\lambda_1^{(\nu)} - \lambda_0)\dot{\lambda}_0 \; - \; \Delta s \end{pmatrix}.$$

Next direction vector :

$$\begin{pmatrix} G_u^1 & G_\lambda^1 \\ \\ \dot{u}_0^* & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{\lambda}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix}.$$

Note that

- In practice $(\dot{u}_1, \dot{\lambda}_1)$ can be computed with just one extra backsubstitution.

- The orientation of the branch is preserved if Δs is sufficiently small.

- The direction vector must be rescaled, so that indeed $\| \dot{u}_1 \|^2 + \dot{\lambda}_1^2 = 1$.

THEOREM.

The Jacobian of the pseudo-arclength system is nonsingular at a regular solution point.

PROOF.

Let $x \equiv (u, \lambda) \in \mathrm{R}^{n+1}$. Then pseudo-arclength continuation can be written as

$$G(x_1) = 0,$$

$$(x_1 - x_0)^* \dot{x}_0 - \Delta s = 0, \qquad (\| \dot{x}_0 \| = 1).$$

(See Figure 3.4 for a graphical interpretation.) The matrix in Newton's method at $\Delta s = 0$ is $\begin{pmatrix} G_x^0 \\ \dot{x}_0^* \end{pmatrix}$. At a regular solution $\mathcal{N}(G_x^0) = \mathrm{Span}\{\dot{x}_0\}$. We must show that $\begin{pmatrix} G_x^0 \\ \dot{x}_0^* \end{pmatrix}$ is nonsingular at a regular solution.

If on the contrary $\begin{pmatrix} G_x^0 \\ \dot{x}_0^* \end{pmatrix}$ is singular then

$$G_x^0 z = 0 \quad \text{and} \quad \dot{x}_0^* z = 0,$$

for some vector $z \neq 0$. Thus $z = c\dot{x}_0$, for some constant c. But then

$$0 \ = \ \dot{x}_0^* z \ = \ c\dot{x}_0^* \dot{x}_0 \ = \ c \| \dot{x}_0 \|^2 \ = \ c,$$

so that $z = 0$, which is a contradiction. $\bullet$

EXAMPLE. Use pseudo-arclength continuation for the discretized Bratu problem. Then the matrix in Newton's method is bordered tridiagonal :

$$\begin{pmatrix}
\bullet & \bullet & & & & & & & \bullet \\
\bullet & \bullet & \bullet & & & & & & \bullet \\
 & \bullet & \bullet & \bullet & & & & & \bullet \\
 & & \bullet & \bullet & \bullet & & & & \bullet \\
 & & & \bullet & \bullet & \bullet & & & \bullet \\
 & & & & \bullet & \bullet & \bullet & & \bullet \\
 & & & & & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{pmatrix}.$$

We shall now show to solve such linear systems efficiently.

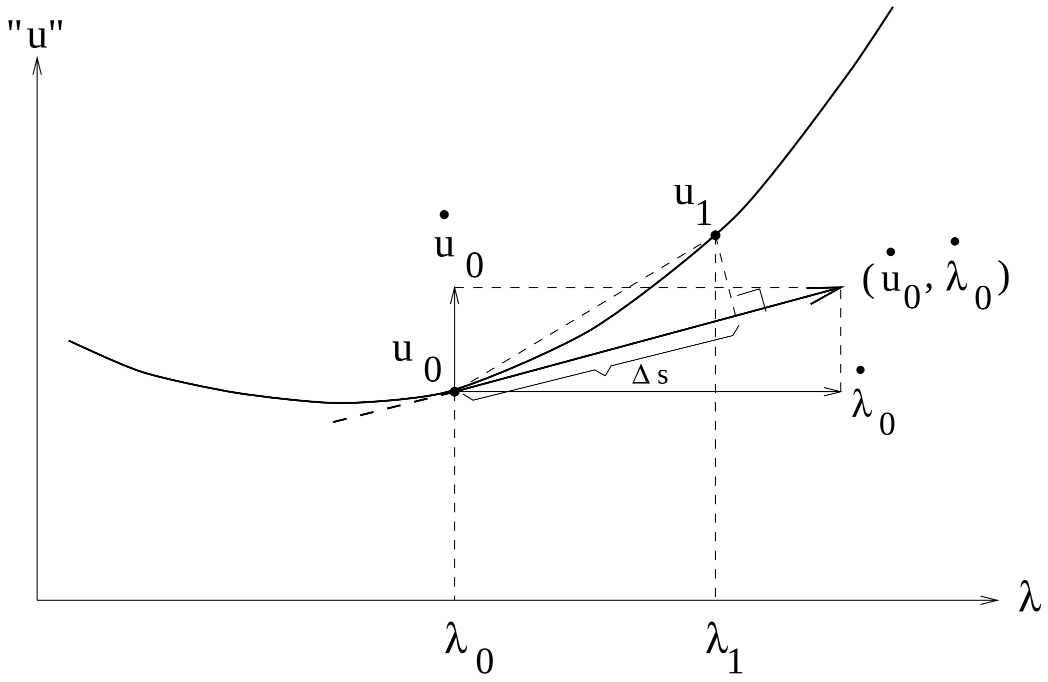

Figure 3.3: Graphical interpretation of pseudo-arclength continuation.

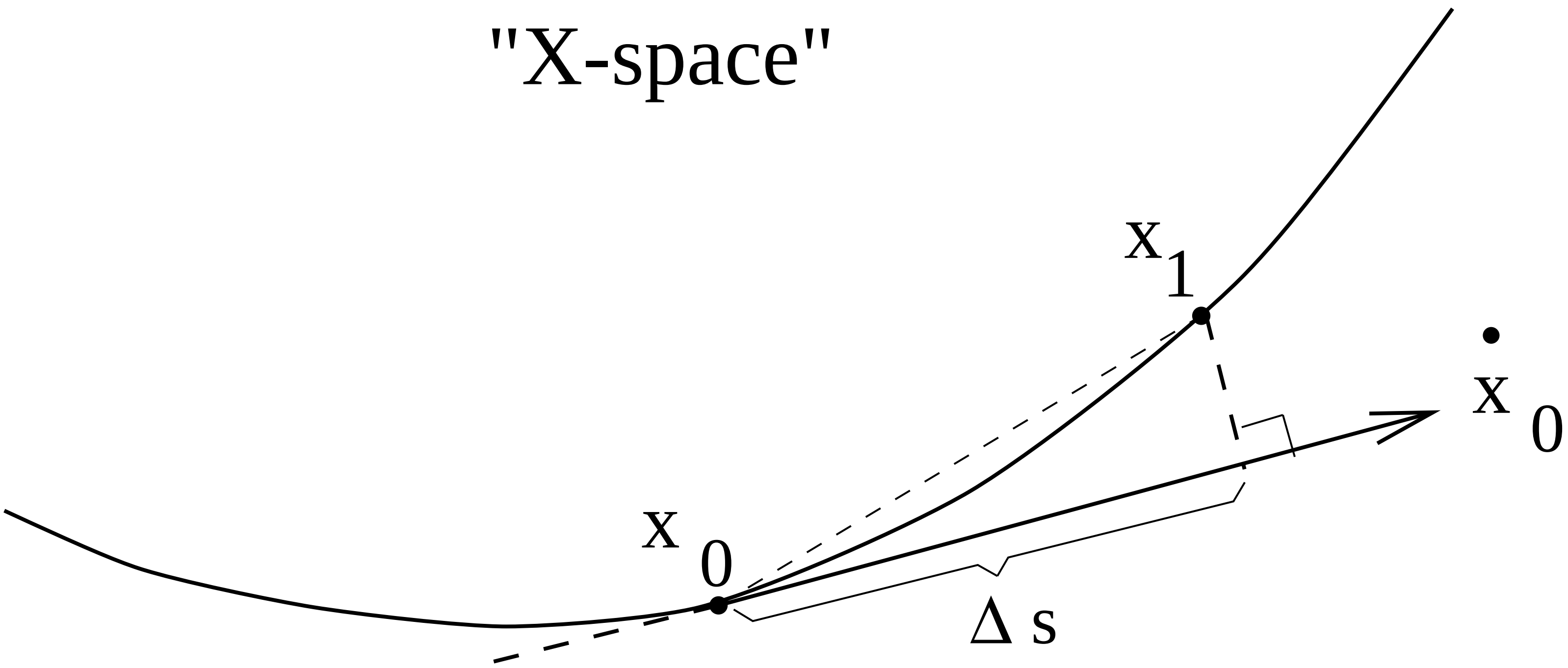

Figure 3.4: Parameter-independent graphical interpretation of pseudo-arclength continuation.

3.4 The Bordering Lemma

LEMMA.

Let $\mathcal{A} = \begin{pmatrix} A & c \\ b^* & d \end{pmatrix}$. Then

(a) A nonsingular $\Rightarrow$ $\mathcal{A}$ nonsingular iff $d \neq b^* A^{-1} c$,

(b) $dim\ \mathcal{N}(A) = dim\ \mathcal{N}(A^*) = 1 \Rightarrow$ $\mathcal{A}$ nonsingular if $\begin{cases} c \notin \mathcal{R}(A), \\ b \notin \mathcal{R}(A^*). \end{cases}$

(c) If $dim\ \mathcal{N}(A) \geq 2$ then $\mathcal{A}$ is singular.

PROOF.

(a) In this case $\mathcal{A} = \begin{pmatrix} A & 0 \\ b^* & 1 \end{pmatrix} \begin{pmatrix} I & A^{-1}c \\ 0^* & e \end{pmatrix}$,

where $e = d - b^* A^{-1} c$. Clearly $\mathcal{A}$ is nonsingular iff $e \neq 0$.

(b) Suppose $\mathcal{A}$ is singular :

$$\begin{aligned} Az + \xi c &= 0, \\ b^* z + \xi d &= 0. \end{aligned}$$

If $\xi \neq 0$ then $c \in \mathcal{R}(A)$ which contradicts the assumptions.

If $\xi = 0$ and $z \neq 0$ then

$$\mathcal{N}(A) = \text{Span}\{z\} \text{ and } b \in \mathcal{N}(A)^{\perp} = \mathcal{R}(A^*),$$

which also contradicts the assumptions.

(c) follows easily from a rank argument. •

3.5 The Bordering Algorithm

The linear systems in Newton's method for pseudo-arclength continuation are of the form

$$\begin{pmatrix} A & c \\ b^* & d \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}.$$

If A is a sparse matrix whose LU decomposition can be found relatively cheaply (e.g., if A is tridiagonal), then the following *bordered LU-decomposition* will be efficient:

$$\begin{pmatrix} A & c \\ b^* & d \end{pmatrix} = \begin{pmatrix} L & 0 \\ \beta^* & 1 \end{pmatrix} \begin{pmatrix} U & \gamma \\ 0^* & \delta \end{pmatrix}.$$

After decomposing $\boxed{A = LU}$ (which may require pivoting) we compute γ, β and δ from

$$\boxed{L\gamma = c},$$

$$\boxed{U^*\beta = b},$$

$$\boxed{\delta = d - \beta^*\gamma}.$$

The linear system can then be written as

$$\begin{pmatrix} L & 0 \\ \beta^* & 1 \end{pmatrix} \underbrace{\begin{pmatrix} U & \gamma \\ 0^* & \delta \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}}_{\equiv \begin{pmatrix} \hat{f} \\ \hat{h} \end{pmatrix}} = \begin{pmatrix} f \\ h \end{pmatrix} .$$

and we can compute the solution (x, z) through the following steps :

$$\boxed{L\hat{f} = f},$$

$$\boxed{\hat{h} = h - \beta^*\hat{f}},$$

$$\boxed{z = \hat{h}/\delta},$$

$$\boxed{Ux = \hat{f} - z\gamma}.$$

THEOREM.

This algorithm works if A and the full matrix $\mathcal{A} \equiv \begin{pmatrix} A & c \\ b^* & d \end{pmatrix}$ are nonsingular.

PROOF.

We must have $\delta \neq 0$. But from the above

$$\begin{aligned} \delta &= d - \beta^*\gamma = d - (U^{*\ -1}b)^*(L^{-1}c) \\ &= d - b^*U^{-1}L^{-1}c = d - b^*(LU)^{-1}c \\ &= d - b^*A^{-1}c \end{aligned}$$

which is nonzero by Conclusion (a) of the Bordering Lemma.

Chapter 4

Fold Theory

4.1 Formal Definition

Suppose $G(u_0, \lambda_0) = 0$, where

$$G \ : \ \mathcal{B} \times \mathrm{R} \to \mathcal{B}, \qquad (\text{ or } \ G \ : \ \mathcal{B}_1 \times \mathrm{R} \to \mathcal{B}_2 \ ),$$

where $\mathcal{B}, \mathcal{B}_1$ and $\mathcal{B}_2$) are Banach spaces. An important practical case is

$$G \ : \ \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}^n.$$

DEFINITION.

Assume

$$\mathcal{N}(G_u^0) = \mathrm{Span}\{\phi\}, \quad \mathcal{N}(G_u^{0^*}) = \mathrm{Span}\{\psi\}, \text{ and } G_\lambda^0 \notin \mathcal{R}(G_u^0),$$

with $\|\phi\| = \|\psi\| = 1$, $G_u^0 \equiv G_u(u_0, \lambda_0)$. Then (u_0, λ_0) is called *a simple fold.*

What is the solution set of $G(u, \lambda) = 0$ near (u_0, λ_0) ? Write

$$u = u_0 + \epsilon\phi + v, \qquad \text{where } v^* \ \phi = 0.$$

Then

$$\phi^* \ (u - u_0) \ = \ \epsilon \underbrace{\phi^*\phi}_{=1} \ + \ \underbrace{\phi^* v}_{=0} \ = \ \epsilon,$$

i.e., ϵ is the length of projection of $u - u_0$ onto ϕ.

Consider

$$\left.\begin{array}{rcl} G(u_0 + \epsilon\phi + v, \lambda) & = & 0, \\ \\ \phi^* \ v & = & 0, \end{array}\right\} \sim \ F(v, \lambda; \epsilon) = 0.$$

We have $F(0, \lambda_0; 0) = 0$ and

$$(F_v^0 \mid F_\lambda^0) \equiv \Big(F_v(0, \lambda_0; 0) \mid F_\lambda(0, \lambda_0; 0)\Big) = \begin{pmatrix} G_u^0 & G_\lambda^0 \\ \phi^* & 0 \end{pmatrix} \text{ is nonsingular.}$$

To see this, consider

$$\begin{pmatrix} G_u^0 & G_\lambda^0 \\ \phi^* & 0 \end{pmatrix} \begin{pmatrix} w \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus $G_u^0 w = -\mu \underbrace{G_\lambda^0}_{\notin \mathcal{R}(G_u^0)}$, from which $\mu = 0$ and $w = \alpha\phi$. Also $\phi^* w = 0$, from which $\alpha\phi^*\phi = 0$, so that $\alpha = 0$ and hence $w = 0$. Thus $(F_v^0 \mid F_\lambda^0)$ is nonsingular. Hence, using the IFT, we have shown

THEOREM.

Suppose $G(u_0, \lambda_0) = 0$, with

$$\mathcal{N}(G_u^0) = \text{Span}\{\phi\}, \quad \mathcal{N}(G_u^{0^*}) = \text{Span}\{\psi\}, \quad G_\lambda^0 \notin \mathcal{R}(G_u^0).$$

Then the equation $G(u, \lambda) = 0$, has, for small ϵ, a unique solution branch

$$(\ u(\epsilon)\ ,\ \lambda(\epsilon)\ )\ =\ (\ u_0 + \epsilon\phi + v(\epsilon)\ ,\ \lambda(\epsilon)\ ),$$

with

$$v(0) = 0, \quad \lambda(0) = \lambda_0, \quad \phi^*\ v(\epsilon) = 0,$$

Moreover if G is k times differentiable near (u_0, λ_0) then so is $(u(\epsilon), \lambda(\epsilon))$. •

4.2 Construction of the Coefficients

Taylor expand in ϵ to construct local approximations to $v(\epsilon)$, and hence to $u(\epsilon)$, and $\lambda(\epsilon)$.

$$v(\epsilon) = \underbrace{v(0)}_{=0}\ +\ \epsilon v'(0)\ +\ \frac{1}{2}\epsilon^2 v''(0)\ +\ \cdots\ ,$$

$$\lambda(\epsilon) = \underbrace{\lambda(0)}_{=\lambda_0}\ +\ \epsilon \lambda'(0)\ +\ \frac{1}{2}\epsilon^2 \lambda''(0)\ +\ \cdots\ .$$

Write $v_0' = v'(0)$, etc. Now

$$\phi^*\ v(\epsilon) \equiv 0 \quad \Rightarrow \quad \phi^*\ v_0' = 0, \quad \phi^*\ v_0'' = 0, \cdots\ ,$$

and

$$G(\ u_0 + \epsilon\phi + v(\epsilon)\ ,\ \lambda(\epsilon)\ )\ \equiv\ \tilde{G}(\epsilon)\ \equiv\ 0, \qquad \text{for all small } \epsilon.$$

$$\Rightarrow\ \left.\frac{d^k \tilde{G}}{d\epsilon^k}\right|_{\epsilon=0} = 0, \quad k = 0, 1, 2, \cdots.$$

k=1 :

$$\begin{aligned} \tilde{G}'(\epsilon) = \quad & G_u(\ u_0 + \epsilon\phi + v(\epsilon)\ ,\ \lambda(\epsilon)\ )\ \ (\phi + v'(\epsilon)) \\ + \quad & G_\lambda(\ u_0 + \epsilon\phi + v(\epsilon)\ ,\ \lambda(\epsilon)\ )\ \ \lambda'(\epsilon)\ . \end{aligned}$$

$$\begin{aligned} \Rightarrow\ 0 \ = \ \tilde{G}'(0) \ = \ G_u^0(\phi + v_0') + G_\lambda^0 \lambda_0' \ = \ G_u^0 v_0' + G_\lambda^0 \lambda_0' \\ \Rightarrow\ \underbrace{\psi^* G_u^0}_{=0} v_0' + \underbrace{\psi^* G_\lambda^0}_{\neq 0} \lambda_0' = 0 \\ \Rightarrow\ \boxed{\lambda_0' = 0}\ , \quad G_u^0 v_0' = 0 \end{aligned}$$

$$\left.\begin{aligned} v_0' \ &= \ \alpha\phi \\ \phi^* v_0' \ &= \ 0 \end{aligned}\right\} \ \Rightarrow\ \alpha = 0, \quad \boxed{v_0' = 0}\ .$$

k=2 :

$$\begin{aligned} \tilde{G}''(\epsilon) \ = \ & G_{uu}(\cdot)\ (\phi + v'(\epsilon))^2 \ + \ 2G_{u\lambda}(\cdot)\ (\phi + v'(\epsilon))\ \lambda'(\epsilon) \\ + \ & G_{\lambda\lambda}(\cdot)\ \lambda'(\epsilon)^2 \ + \ G_u(\cdot)\ v''(\epsilon) \ + \ G_\lambda(\cdot)\ \lambda''(\epsilon), \end{aligned}$$

$$\begin{aligned} \Rightarrow\ 0 = \tilde{G}''(0) = G_{uu}^0 \phi\phi + G_u^0 v_0'' + \lambda_0'' G_\lambda^0, \\ \Rightarrow\ \psi^* G_{uu}^0 \phi\phi + \underbrace{\psi^* G_u^0}_{=0} v_0'' + \underbrace{\psi^* G_\lambda^0}_{\neq 0} \lambda_0'' = 0, \\ \Rightarrow\ \boxed{\lambda_0'' = -\tfrac{\psi^* G_{uu}^0 \phi\phi}{\psi^* G_\lambda}}\ . \end{aligned}$$

Then we can solve

$$G_u^0 v_0'' = -G_{uu}^0 \phi\phi - \lambda_0'' G_\lambda^0\ .$$

DEFINITION. A simple fold with $\lambda_0'' \neq 0$ is called a *simple quadratic fold*. See Figure 3.1 for a graphical interpretation.

COROLLARY.

Under the conditions of the Theorem, and with G sufficiently differentiable, we have, for small ϵ,

$$u(\epsilon) \ = \ u_0 \ + \ \epsilon\phi \ + \ \tfrac{1}{2}\,\epsilon^2 v_0'' \ + \ O(\epsilon^3),$$

$$\lambda(\epsilon) \ = \ \lambda_0 \ - \ \tfrac{1}{2}\,\epsilon^2\ \tfrac{\psi^* G_{uu}^0 \phi\phi}{\psi^* G_\lambda^0} \ + \ O(\epsilon^3)\ . \qquad \bullet$$

EXAMPLE. Let $G(u, \lambda) = u^2 - \lambda$. Then

$$G(0,0) = 0, \qquad G_u^0 = 0,$$

$$\mathcal{N}(G_u^0) = \mathrm{Span}\{(1)\}, \qquad \mathcal{N}(G_u^{0^*}) = \mathrm{Span}\{(1)\},$$

$$G_\lambda^0 = -1 \ \notin\ \mathcal{R}(G_u^0), \qquad G_{uu}^0 = 2,$$

$$\lambda_0'' \ = \ \frac{-\psi^* G_{uu}^0 \phi\phi}{\psi^* G_\lambda^0} = \frac{-2}{-1} \ = \ 2 \ > \ 0.$$

4.3 A Simplified Derivation

Given the conditions of the Theorem above, let $x = (u, \lambda)$. Then $G_x = (G_u | G_x)$. What is $\mathcal{N}(G_x^0)$?

$$(G_u^0 | G_\lambda^0)\binom{v}{\mu} = 0, \quad v \in \mathcal{B}, \ \mu \in \mathrm{R}.$$

$$\Rightarrow \ G_u^0 v = -\mu \underbrace{G_\lambda^0}_{\notin \mathcal{R}(G_u^0)},$$

Thus $\mu = 0$, $G_u^0 v = 0$, and $v = \alpha\phi$. Hence $\mathcal{N}(G_x^0) = \mathrm{Span}\{\binom{\phi}{0}\}$ (1-D null space). Similarly we have $\mathcal{N}(G_x^0)^* = \mathrm{Span}\{0\}$. So consider (with a new ϕ !)

$$G(x_0) = 0, \quad \mathcal{N}(G_x^0) = \mathrm{Span}\{\phi\}, \quad \mathcal{N}(G_x^{0^*}) = \{0\},$$

Let $x = x_0 + \epsilon\phi + y$, with $\phi^* y = 0$, and consider

$$\left.\begin{array}{rcl} G(x_0 + \epsilon\phi + y) & = & 0, \\ \\ \phi^* y & = & 0, \end{array}\right\} \sim \ F(y; \epsilon) = 0.$$

Claim : $F_y^0 = \binom{G_x^0}{\phi^*}$ is nonsingular. To see this, suppose

$$\begin{pmatrix} G_x^0, \\ \phi^* \end{pmatrix} z = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{l} G_x^0 \ z = 0, \\ \phi^* \ z = 0. \end{array} \right.$$

Thus $z = \alpha\phi$, $\alpha\phi^*\phi = 0$, and hence $\alpha = 0$, $z = 0$. Thus $y = y(\epsilon)$ near $\epsilon = 0$, and

$$x \ = \ x(\epsilon) \ = \ x_0 + \epsilon\phi + y(\epsilon).$$

This derivation not only covers the case of the fold, where $\mathcal{N}(G_u^0)$ is one-dimensional and $G_\lambda^0 \notin \mathcal{R}(G_u^0)$, but also the case of a regular solution that is not a fold, namely, when G_u^0 is nonsingular. In the latter case the null space is also one-dimensional, namely,

$$\mathcal{N}(G_x^0) = \mathcal{N}((G_u^0 | G_\lambda)) = \mathrm{Span}\{\binom{-G_u^{0^{-1}} G_\lambda}{1}\}.$$

Thus the derivation applies, in fact, to any regular solution point.

4.4 Change of Stability

THEOREM.

Let (u_0, λ_0) be a simple quadratic fold. Also assume that zero is an algebraically simple eigenvalue of G_u^0. Then an eigenvalue of G_u crosses zero "as we pass around the fold".

PROOF.

Near a fold (u_0, λ_0) we have

$$u(\epsilon) \ = \ u_0 \ + \ \epsilon\phi_0 \ + \ \overbrace{v(\epsilon)}^{\text{h.o.t.}} ,$$

where $v(0) = 0$, $v(\epsilon)^*\phi_0 = 0$. We have $u(0) = u_0$, $\dot{u}(0) = \phi_0$. (Here ϕ_0 is the null vector of G_u^0.) Also

$$\lambda(\epsilon) \ = \ \lambda_0 \ - \ \frac{1}{2}\epsilon^2 \frac{\psi^* G_{uu}^0 \phi_0\phi_0}{\psi^* G_\lambda^0} \ + \ \underbrace{r(\epsilon)}_{\text{h.o.t.}} .$$

We have $\lambda(0) = \lambda_0$, $\dot{\lambda}(0) = 0$.

Consider the parametrized eigenvalue-vector problem

$$\left.\begin{aligned} & G_u(u(\epsilon), \lambda(\epsilon)) \ \phi \ - \ \sigma \ \phi = 0, \\ & \phi^* \ \phi_0 \ - \ 1 \ = \ 0, \end{aligned}\right\} \ \sim \ F(\phi, \sigma; \epsilon) = 0.$$

We have $F(\phi_0, 0; 0) = 0$, and

$$(F_\phi^0 \mid F_\sigma^0) = \begin{pmatrix} G_u^0 & -\phi_0 \\ \phi_0^* & 0 \end{pmatrix} ,$$

which, in the finite dimensional case, is an $n+1$ by $n+1$ nonsingular matrix :

$$\begin{pmatrix} G_u^0 & -\phi_0 \\ \phi_0^* & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow \ \begin{cases} G_u^0 \ p \ = \ q\phi_0, \\ \phi_0^* \ p \ = \ 0. \end{cases}$$

If $q \neq 0$ then $\phi_0 \in \mathcal{R}(G_u^0)$, and hence $\psi^*\phi_0 = 0$. But this is impossible, since $\sigma = 0$ is algebraically simple; see the Remark below. Hence $q = 0$ and $p = \alpha\phi_0$. But then, since $\phi_0^* p = 0$, it follows that $\alpha = 0$ and hence $p = 0$. Therefore it follows from the IFT that, for small $|\epsilon|$, $\phi = \phi(\epsilon)$, $\sigma = \sigma(\epsilon)$, with $\phi(0) = \phi_0$, $\sigma(0) = 0$.

Thus we can write

$$\begin{cases} G_u(u(\epsilon), \lambda(\epsilon)) \ \phi(\epsilon) \ - \ \sigma(\epsilon) \ \phi(\epsilon) = 0, \\ \phi(\epsilon)^* \ \phi_0 - 1 = 0. \end{cases}$$

Differentiate with respect to ϵ, and evaluate at $\epsilon = 0$:

$$G_{uu}^0 \underbrace{\dot{u}(0)}_{=\phi_0} \phi_0 + \underbrace{\dot{\lambda}_0}_{=0} G_{u\lambda}^0 \phi_0 + G_u^0 \dot{\phi}(0) - \dot{\sigma}(0)\phi_0 - \underbrace{\sigma(0)}_{=0} \dot{\phi}(0) = 0,$$

$$\Rightarrow \ \psi^* G_{uu}^0 \phi_0\phi_0 \ - \ \dot{\sigma}(0) \ \psi^*\phi_0 = 0,$$

$$\Rightarrow \ \dot{\sigma}(0) \ = \ \frac{\psi^* G_{uu}^0 \phi_0\phi_0}{\underbrace{\psi^*\phi_0}_{\neq 0}} \ \neq \ 0.$$

Thus $\sigma(0) = 0$, with $\dot{\sigma}(0) \neq 0$, i.e., $\sigma(\epsilon)$ passes through zero. $\bullet$

REMARK. If the zero eigenvalue is not algebraically simple then the left and right null vector can be orthogonal; consider for example the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ : \qquad A^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{N}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{N}(A^*) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Note that the zero eigenvalue of A is not algebraically simple here, and that $\psi^*\phi = 0$.

Chapter 5

Computation of Folds

5.1 Simple Folds

Recall from Section 3.1 (or Section 4.1) that a regular solution $x_0 \equiv (u_0, \lambda_0)$ of $G(u, \lambda) = 0$ is called a *simple fold* if

$$dim\ \mathcal{N}(G_u^0) = 1 \qquad and \qquad G_\lambda^0 \notin \mathcal{R}(G_u^0).$$

From differentiating $G(u(s), \lambda(s)) = 0$ we have

$$G_u(u(s), \lambda(s))\dot{u}(s) + G_\lambda(u(s), \lambda(s))\dot{\lambda}(s) = 0, \qquad \text{in particular} \quad G_u^0 \dot{u}_0 = -\dot{\lambda}_0 G_\lambda^0\ .$$

At a fold we have

$$G_\lambda^0 \notin \mathcal{R}(G_u^0) \ \Rightarrow \ \dot{\lambda}_0 = 0.$$

Thus $G_u^0 \dot{u}_0 = 0$ and since $dim\ \mathcal{N}(G_u^0) = 1$ we have

$$\mathcal{N}(G_u^0) = \text{Span}\{\dot{u}_0\}.$$

Differentiating again, we have

$$G_u^0 \ddot{u}_0 + G_\lambda^0 \ddot{\lambda}_0 + G_{uu}^0 \dot{u}_0 \dot{u}_0 + 2G_{u\lambda}^0 \dot{u}_0 \dot{\lambda}_0 + G_{\lambda\lambda}^0 \dot{\lambda}_0 \dot{\lambda}_0 = 0.$$

At a simple fold (u_0, λ_0) let

$$\mathcal{N}(G_u^0) = \text{Span}\{\phi\}, \qquad (\phi = \dot{u}_0),$$

and

$$\mathcal{N}((G_u^0)^*) = \text{Span}\{\psi\}.$$

Multiply by ψ^* and use $\dot{\lambda}_0 = 0$ and $\psi \perp \mathcal{R}(G_u^0)$ to find

$$\psi^* G_\lambda^0 \ddot{\lambda}_0 + \psi^* G_{uu}^0 \phi\phi = 0.$$

Here $\psi^* G_\lambda^0 \neq 0$ since $G_\lambda^0 \notin \mathcal{R}(G_u^0)$. Thus $\boxed{\ddot{\lambda}_0 = \frac{-\psi^* G_{uu}^0 \phi\phi}{\psi^* G_\lambda^0}.}$ If the *curvature* $\ddot{\lambda}_0 \neq 0$ then (u_0, λ_0) is called a *simple quadratic fold*. (See also Section 4.2.)

5.2 The Extended System

To continue a fold in two parameters we use the *extended system*

$$\boxed{\begin{aligned} &G(u, \lambda, \mu) = 0, \\ &G_u(u, \lambda, \mu)\phi = 0, \\ &\phi^*\phi_0 - 1 = 0. \end{aligned}}$$

Here $\mu \in \mathrm{R}$ is a second parameter in the equations.

The vector ϕ_0 belongs to a 'reference solution' $(u_0, \phi_0, \lambda_0, \mu_0)$. In practice this is the latest computed solution point on the branch.

The above system has the form

$$F(U, \mu) = 0, \qquad U \equiv (u, \phi, \lambda), \qquad F \; : \; \mathrm{R}^{2n+1} \times \mathrm{R} \to \mathrm{R}^{2n+1},$$

or, using the "uniform" formulation,

$$F(X) = 0, \qquad X \equiv (U, \mu), \qquad F \; : \; \mathrm{R}^{2n+2} \to \mathrm{R}^{2n+1}.$$

5.3 Parameter Continuation

First consider continuing a solution

$$(u_0, \phi_0, \lambda_0) \qquad \text{at } \mu = \mu_0,$$

in μ (although, in practice, pseudo-arclength is used).

By the IFT there is a smooth solution branch

$$U(\mu) = (u(\mu), \phi(\mu), \lambda(\mu)),$$

if the Jacobian

$$F_U^0 \equiv \frac{dF}{dU}(U_0) = \begin{pmatrix} G_u^0 & O & G_\lambda^0 \\ G_{uu}^0\phi_0 & G_u^0 & G_{u\lambda}^0\phi_0 \\ 0^* & \phi_0^* & 0 \end{pmatrix},$$

is nonsingular.

THEOREM.

A simple quadratic fold with respect to λ can be continued locally using the second parameter μ as continuation parameter.

PROOF.

Suppose F_U^0 is singular. Then

$$(i) \qquad G_u^0 x + z G_\lambda^0 = 0,$$

$$(ii) \qquad G_{uu}^0 \phi_0 x + G_u^0 y + z G_{u\lambda}^0 \phi_0 = 0,$$

$$(iii) \qquad \phi_0^* y = 0,$$

for some

$$x, y \in \mathrm{R}^n, \qquad z \in \mathrm{R}.$$

Since $G_\lambda^0 \notin \mathcal{R}(G_u^0)$ we have from (i) that $z = 0$, and hence

$$x = c_1 \phi_0, \text{ for some } c_1 \in \mathrm{R}.$$

Multiply (ii) on the left by ψ_0^* to get

$$c_1 \ \psi_0^* G_{uu}^0 \phi_0 \phi_0 = 0.$$

Thus $c_1 = 0$ because by assumption $\psi_0^* G_{uu}^0 \phi_0 \phi_0 \neq 0$. So $x = 0$, and from (ii) we now have

$$G_u^0 y = 0, \text{ i.e., } y = c_2 \phi_0.$$

But then by (iii) $c_2 = 0$. Thus

$$x = y = z = 0,$$

and hence F_U^0 must be nonsingular. $\bullet$

REMARK. The theorem does not require zero to be an algebraically simple eigenvalue of G_u^*.

Thus, for example, G_u^0 may have the form

$$G_u^0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

provided the fold remains simple and quadratic.

Natural parameter continuation fails at folds with respect to μ on the solution branch to $F(U,\mu) = 0$. Such singular points would correspond to cusps, branch points, or isola formation points for $G(u, \lambda, \mu)$.

5.4 Pseudo-Arclength Continuation

Treat μ as one of the unknowns, and compute a solution branch

$$X(s) \equiv (u(s), \phi(s), \lambda(s), \mu(s)),$$

to

$$F(X) \equiv \begin{cases} G(u, \lambda, \mu) = 0, \\ G_u(u, \lambda, \mu)\phi = 0, \\ \phi^* \phi_0 - 1 = 0, \end{cases} \tag{5.1}$$

and the added pseudo-arclength equation

$$(u - u_0)^* \dot{u}_0 + (\phi - \phi_0)^* \dot{\phi}_0 + (\lambda - \lambda_0)\dot{\lambda}_0 + (\mu - \mu_0)\dot{\mu}_0 - \Delta s = 0. \tag{5.2}$$

As before,

$$(\dot{u}_0, \dot{\phi}_0, \dot{\lambda}_0, \dot{\mu}_0),$$

is the direction of the branch at the current solution point

$$(u_0, \phi_0, \lambda_0, \mu_0).$$

The Jacobian of F with respect to u, ϕ, λ *and* μ at

$$X_0 = (u_0, \phi_0, \lambda_0, \mu_0),$$

is now

$$F_X^0 \equiv \frac{dF}{dX}(X_0) \equiv \begin{pmatrix} G_u^0 & O & G_\lambda^0 & G_\mu^0 \\ G_{uu}^0 \phi_0 & G_u^0 & G_{u\lambda}^0 \phi_0 & G_{u\mu}^0 \phi_0 \\ 0^* & \phi_0^* & 0 & 0 \end{pmatrix}.$$

For pseudo-arclength continuation to work we need only check that F_X^0 has full rank.

For a simple quadratic fold with respect to λ this follows from the above Theorem.

Otherwise, if

$$\psi_0^* G_{uu}^0 \phi_0 \phi_0 \neq 0,$$

and

$$G_\lambda^0 \in \mathcal{R}(G_u^0), \qquad G_\mu^0 \notin \mathcal{R}(G_u^0),$$

i.e., if we have a simple quadratic fold with respect to μ, then we can apply the theorem to F_X^0 with the second last column struck out to see that F_X^0 still has full rank.

5.5 Efficient Implementation

The system (5.1), (5.2) can be written as

$$\begin{aligned} F(X) &= 0, \\ (X - X_0)^* \dot{X}_0 - \Delta s &= 0 = 0. \end{aligned} \tag{5.3}$$

As we have seen above, if

$$\psi_0^* G_{uu}^0 \phi_0 \phi_0 \neq 0, \text{ and if either } G_\lambda^0 \notin \mathcal{R}(G_u^0), \text{ or } G_\mu^0 \notin \mathcal{R}(G_u^0),$$

then after interchanging the last two columns, if necessary, we can put the Jacobian

$$\begin{pmatrix} F_X^0 \\ \\ \dot{X}_0 \end{pmatrix} \quad \text{in the form} \quad \begin{pmatrix} A & c \\ b^* & d \end{pmatrix},$$

where A is nonsingular.

REMARK. Here we assume that the Chord method is used to solve (5.3). Then the Jacobian of (5.3) is always evaluated at a solution, which is necessary to satisy the assumptions in the algorithm below. However, it is possible to make some adjustments that allow full Newton.

The linear systems then have the form

$$\begin{pmatrix} A & c \\ b^* & d \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix},$$

or

$$\begin{pmatrix} A & 0 \\ b^* & e \end{pmatrix} \begin{pmatrix} I & A^{-1}c \\ O & 1 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix},$$

where $e = d - b^* A^{-1} c$.

These systems can be solved efficiently as follows :

$$\boxed{\begin{aligned} & A\hat{c} = c, \\ & e = d - b^*\hat{c} \qquad (\neq 0), \\ & A\hat{f} = f, \\ & z = (h - b^*\hat{f})/e, \\ & x = \hat{f} - z\hat{c}. \end{aligned}}$$

Note that the above is simply a variation on the Bordering Algorithm of Section 3.5.

Next we show how to solve the linear systems above that have A as coefficient matrix. We see that such systems have the form

$$\begin{pmatrix} \tilde{A} & O & c_1 \\ B & \tilde{A} & c_2 \\ 0^* & \phi_0^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

where

$$\mathcal{N}(\tilde{A}) = \mathrm{Span}\{\phi_0\}, \quad \text{and} \quad \mathcal{N}(\tilde{A}^*) = \mathrm{Span}\{\psi\},$$

$$\tilde{A} = G_u^0, \qquad B = G_{uu}^0\phi_0, \qquad c_1 = G_\lambda^0, \qquad c_2 = G_{u\lambda}^0\phi_0,$$

or $c_1 = G_\mu^0$, $c_2 = G_{u\mu}^0\phi_0$, in case $G_\lambda^0 \in \mathcal{R}(G_u^0)$, $G_\mu^0 \notin \mathcal{R}(G_u^0)$.

e have from the first of the equations that

$$\tilde{A}x = f - zc_1.$$

Thus we need $(f - zc_1) \in \mathcal{R}(\tilde{A})$, i.e., $\psi^*(f - zc_1) = 0$, or

$$\boxed{z = z_0 \equiv \tfrac{\psi^* f}{\psi^* c_1}, \qquad (\ \psi^* c_1 \neq 0 \text{ by assumption}).}$$

Then

$$x = x_p + \alpha\phi_0.$$

A particular solution x_p is easily found by solving

$$\boxed{\tilde{A}x_p = f - z_0 c_1}$$

using row and column pivoting as has been shown earlier.

From the second of the equations we have

$$\tilde{A}y = g - Bx - z_0 c_2,$$

and thus we need $\psi^*(g - Bx - z_0 c_2) = 0$, i.e.,

$$\psi^* g - \psi^* B(x_p + \alpha\phi_0) - z_0\psi^* c_2 = 0,$$

from which

$$\boxed{\alpha = \alpha_0 \equiv \tfrac{\psi^* g - z_0\psi^* c_2 - \psi^* B x_p}{\psi^* B\phi_0},}$$

where $\psi^* B\phi_0 = \psi^* G_{uu}^0\phi_0\phi_0 \neq 0$ for a quadratic fold.

Now y also has the form

$$y = y_p + \beta\phi_0,$$

and as before a particular solution can be be computed from

$$\boxed{\tilde{A}y_p = g - B(x_p + \alpha_0\phi_0) - z_0 c_2.}$$

But by the third equation we have $\phi_0^* y = h$, i.e.,

$$\phi_0^*(y_p + \beta\phi_0) = h,$$

so that

$$\boxed{\beta = \tfrac{h - \phi_0^* y_p}{\phi_0^*\phi_0}.}$$

Chapter 6

Bifurcation Theory

6.1 The Algebraic Bifurcation Equation

DEFINITION.

Suppose (u_0, λ_0) is a solution of

$$G(u, \lambda) = 0,$$

where $G : \mathcal{B}_1 \times \mathrm{R} \to \mathcal{B}_2$. Write $x \equiv (u, \lambda)$, so that $G(x_0) = 0$, and assume

$$\mathcal{N}(G_x^0) = \mathrm{Span}\{\phi_1, \phi_2\}, \quad \mathcal{N}(G_x^{0^*}) = \mathrm{Span}\{\psi\}.$$

Then x_0 is called *simple singular point.*

This is a singular case that one would not ordinarily ("generically") expect to occur anywhere along a solution branch. However, many equation are not generic, so that, in fact, this situation often arises in practice. We shall show that it can give rise to *branching* (or *bifurcation*), i.e., the intersection of solution branches. In the practically important case where $\mathcal{B}_1 = \mathcal{B}_2 = \mathrm{R}^n$ the linear operator G_x^0 is a matrix with n rows and $n+1$ columns. The rank of such a matrix is at most n. The above assumption then means that the rank is in fact $n-1$.

6.1.1 A special case

In the "parameter formulation" $G(u, \lambda) = 0$, consider the case

$$\mathcal{N}(G_u^0) = \mathrm{Span}\{\phi\}, \quad \mathcal{N}(G_u^{0^*}) = \mathrm{Span}\{\psi\}, \quad G_\lambda^0 \in \mathcal{R}(G_u^0).$$

(Compare this to the fold case in Section 5.1 or Section 4.1.) Then the null space of $(G_u^0 \mid G_\lambda^0)\binom{v}{\mu} = 0$ is as follows : If $\mu = 0$ then we must have $v = \phi$. If $\mu \neq 0$ then $G_u^0 v = -\mu G_\lambda^0$, where $G_\lambda^0 \in \mathcal{R}(G_u^0)$. Thus

$$\mathcal{N}(G_x^0) \;=\; \mathrm{Span}\{\begin{pmatrix}\phi\\0\end{pmatrix}, \begin{pmatrix}v\\1\end{pmatrix}\}, \qquad \text{(a 2-D null space).}$$

EXAMPLE. (To motivate the analytical technique.)

$$G(u,\lambda) = u\lambda, \qquad G(0,0) = 0,$$

$$G_x = (G_u \mid G_\lambda) = (\lambda \mid u), \qquad G_x^0 = (0,0),$$
$$\mathcal{N}(G_x^0) = \text{Span}\{\tbinom{1}{0}, \tbinom{0}{1}\}, \quad \mathcal{N}(G_x^{0^*}) = \text{Span}\{(1)\}.$$

Consider the "perturbed problem"

$$F(x,\mu) = G(u,\lambda) + \mu\xi,$$

where $\xi \notin \mathcal{R}(G_x^0)$, for example, $\xi = -1$, i.e.,

$$F(x,\mu) = u\lambda - \mu.$$

$$F(x,\mu) = 0 \iff \mu = u\lambda, \quad \text{i.e., } \mu = \mu(u,\lambda).$$

The solutions of $G(u,\lambda) = 0$ are on the "level curve" $\mu(u,\lambda) = 0$.

6.1.2 The general case

Now consider the general case :

$$G \;:\; \mathcal{B}_1 \times \; R \to \mathcal{B}_2,$$

with $G(x_0) = 0$, $x_0 = (u_0, \lambda_0)$,

$$\mathcal{N}(G_x^0) = \text{Span}\{\phi_1, \phi_2\}, \quad \mathcal{N}(G_x^{0^*}) = \text{Span}\{\psi\},$$

where $\|\phi_1\| = \|\phi_2\| = \|\psi\| = 1$, and $\phi_1^*\phi_2 = 0$.

Define

$$F(x,\mu) = G(x) + \mu\psi, \qquad \mu \in \; R.$$

(In fact, any $\xi \notin \mathcal{R}(G_x^0)$ will do instead of ψ.) Then $F(x_0, 0) = G(x_0) = 0$ and

$$F_x(x,\mu) = G_x(x), \qquad F_\mu^0 \equiv F_\mu(x_0 0) = \psi,$$
$$F_x^0 \equiv F_x(x_0, 0) = G_x^0, \quad F_\mu(x,\mu) = \psi.$$

Let $\mathcal{B} = \mathcal{B}_1 \times$ R and for $x \in \mathcal{B}$ write

$$x = x_0 + \epsilon_1\phi_1 + \epsilon_2\phi_2 + y,$$

where we require

$$y^*\phi_i = 0, \qquad i = 1, 2.$$

Then $(x - x_0)^*\phi_i = \epsilon_i, i = 1, 2$, i.e., ϵ_i measures the length of the projection of $(x - x_0)$ onto the null vector ϕ_i.

Consider the equations

$$\left.\begin{array}{rcl} G(x_0 + \epsilon_1\phi_1 + \epsilon_2\phi_2 + y) + \mu\psi & = & 0, \\ y^*\phi_1 & = & 0, \\ y^*\phi_2 & = & 0. \end{array}\right\} \sim \ \tilde{F}(y, \mu; \epsilon_1, \epsilon_2) = 0. \tag{6.1}$$

Here

$$\tilde{F} \ : \ \mathcal{B} \times \mathrm{R} \times \mathrm{R}^2 \to \mathcal{B}_2 \times \mathrm{R}^2.$$

For example, if $\mathcal{B}_1 \ = \ \mathcal{B}_2 \ = \ \mathrm{R}^n$, then $\mathcal{B} = \mathrm{R}^{n+1}$ and

$$\tilde{F} \ : \ \underbrace{\mathrm{R}^{n+1}}_{y} \times \underbrace{\mathrm{R}}_{\mu} \times \underbrace{\mathrm{R}^2}_{\epsilon_1,\epsilon_2} \to \mathrm{R}^{n+2} \ ,$$

that is, $\tilde{F} \ : \ \mathrm{R}^{n+4} \to \mathrm{R}^{n+2}$. Therefore we expect solution "surfaces", i.e., 2-dimensional solution manifolds. We have

$$\tilde{F}(0,0;0,0) \ = \ 0.$$

The Fréchet derivative of the system (6.1) is

$$\Big(\tilde{F}_y(y,\mu;\epsilon_1,\epsilon_2)\Big|\tilde{F}_\mu(y,\mu;\epsilon_1,\epsilon_2)\Big) = \begin{pmatrix} G_x(x_0 + \epsilon_1\phi_1 + \epsilon_2\phi_2 + y) & \psi \\ \phi_1^* & 0 \\ \phi_2^* & 0 \end{pmatrix}.$$

We now show that

$$(\tilde{F}_y^0 \mid \tilde{F}_\mu^0) \equiv \Big(\tilde{F}_y(0,0;0,0) \ \Big| \ \tilde{F}_\mu(0,0;0,0)\Big) = \begin{pmatrix} G_x^0 & \psi \\ \phi_1^* & 0 \\ \phi_2^* & 0 \end{pmatrix},$$

is nonsingular. If

$$\begin{pmatrix} G_x^0 & \psi \\ \phi_1^* & 0 \\ \phi_2^* & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then

$$\begin{array}{lll} G_x^0 p = -q\psi & \Rightarrow & q = 0, \ p = \alpha_1\phi_1 + \alpha_2\phi_2, \quad (\text{ since } \psi \notin \mathcal{R}(G_x^0) \text{ }), \\ \phi_1^* p = 0 & \Rightarrow & \phi_1^*(\alpha_1\phi_1 + \alpha_2\phi_2) = 0, \ \Rightarrow \ \alpha_1 = 0, \\ \phi_2^* p = 0 & \Rightarrow & \phi_2^*(\alpha_1\phi_1 + \alpha_2\phi_2) = 0, \ \Rightarrow \ \alpha_2 = 0. \end{array}$$

Thus $p = 0$ and, using the IFT, we have proved

THEOREM.

Near $\mu = 0$, $y = 0$, $\epsilon_1 = \epsilon_2 = 0$, the equation

$$\tilde{F}(y, \mu; \epsilon_1, \epsilon_2) = 0,$$

defines a unique solution manifold

$$y = y(\epsilon_1, \epsilon_2), \quad \mu = \mu(\epsilon_1, \epsilon_2),$$

with $y(0,0) = 0$, $\mu(0,0) = 0$. •

This manifold is as smooth as $\tilde{F}$ (i.e., G) is. Thus, assuming G sufficiently smooth, we have the Taylor expansion

$$\mu(\epsilon_1,\epsilon_2) = \epsilon_1\ \mu_1^0 + \epsilon_2\ \mu_2^0 + \frac{1}{2}\ \epsilon_1^2\ \mu_{11}^0 + \epsilon_1\epsilon_2\ \mu_{12}^0 + \frac{1}{2}\ \epsilon_2^2\ \mu_{22}^0 + \cdots,$$

$$y(\epsilon_1,\epsilon_2) = \epsilon_1\ y_1^0 + \epsilon_2\ y_2^0 + \frac{1}{2}\ \epsilon_1^2\ y_{11}^0 + \epsilon_1\epsilon_2\ y_{12}^0 + \frac{1}{2}\ \epsilon_2^2\ y_{22}^0 + \cdots.$$

where, for example,

$$\mu_i^0 = \frac{d\mu}{d\epsilon_i}\Big|_{\epsilon_1=\epsilon_2=0}\ .$$

REMARK. The solutions of $G(x) = 0$ near x_0 correspond to the level curves

$$\mu(\epsilon_1,\epsilon_2) = 0.$$

This gives a relation between ϵ_1 and ϵ_2 to be used in

$$x = x_0 + \epsilon_1\phi_1 + \epsilon_2\phi_2 + y(\epsilon_1,\epsilon_2).$$

6.1.3 Construction of the coefficients

We now show how to construct the coefficients in the expansion of $\mu(\epsilon_1,\epsilon_2)$ and $y(\epsilon_1,\epsilon_2)$. We have, for all small ϵ_1, ϵ_2,

$$G\Big(x_0 + \epsilon_1\phi_1 + \epsilon_2\phi_2 + y(\epsilon_1,\epsilon_2)\Big)\ +\ \mu(\epsilon_1,\epsilon_2)\ \psi\ =\ 0,$$

$$G_1 \equiv \frac{dG}{d\epsilon_1} = G_x(\cdot)(\phi_1 + y_1) + \mu_1\psi, \quad G_2 = G_x(\cdot)(\phi_2 + y_2) + \mu_2\ \psi,$$

where

$$G_x(\cdot)\ \equiv\ G_x\Big(x_0 + \epsilon_1\phi_1 + \epsilon_2\phi_2 + y(\epsilon_1,\epsilon_2)\Big).$$

Let

$$G_1^0 \equiv \frac{dG}{d\epsilon_1}\Big|_{\epsilon_1=\epsilon_2=0}\ , \qquad \text{etc.}$$

Then

$$0\ =\ G_1^0\ =\ G_x^0(\phi_1 + y_1^0) + \mu_1^0\psi\ =\ G_x^0\ y_1^0 + \mu_1^0\psi$$

$$\Rightarrow\ \underbrace{\psi^* G_x^0}_{=0} y_1^0 + \mu_1^0\underbrace{\psi^*\psi}_{=1} = 0,\ \Rightarrow\ \boxed{\mu_1^0 = 0}\ ,\ y_1^0 = \alpha_{11}\phi_1 + \alpha_{12}\phi_2.$$

Similarly,

$$0\ =\ G_2^0\ =\ G_x^0(\phi_2 + y_2^0) + \mu_2^0\psi\ =\ G_x^0\ y_2^0 + \mu_2^0\ \psi,$$

$$\Rightarrow\ \psi^* G_x^0 y_2^0 + \mu_2^0\psi^*\psi = 0\ \Rightarrow\ \boxed{\mu_2^0 = 0}\ ,\ y_2^0 = \alpha_{21}\phi_1 + \alpha_{22}\phi_2\ .$$

By (6.1) we also have, for small ϵ_1, ϵ_2, that $\phi_i^* y(\epsilon_1, \epsilon_2) = 0$, from which

$$\phi_i^* \ \{\epsilon_1 \ y_1^0 \ + \ \epsilon_2 \ y_2^0 \ + \ \frac{1}{2} \ \epsilon_1^2 \ y_{11}^0 \ + \ \epsilon_1 \epsilon_2 \ y_{12}^0 \ + \ \frac{1}{2} \ \epsilon_1^2 \ y_{11}^0 \ + \ \cdots\} \ = \ 0.$$

$$\Rightarrow \ \phi_i^* \ y_j^0 = 0, \quad \phi_i^* \ y_{jk}^0 = 0, \ \cdots$$

Thus

$$\left.\begin{array}{rcl} \phi_i^* \ (\alpha_{11}\phi_1 + \alpha_{12}\phi_2) & = & 0, \\ \phi_i^* \ (\alpha_{21}\phi_1 + \alpha_{22}\phi_2) & = & 0, \end{array}\right\} \ i = 1, 2 \quad \Rightarrow \quad \alpha_{ij} = 0, \ \boxed{y_1^0 = y_2^0 = 0} \ .$$

The second order terms are obtained as follows :

$$\begin{array}{rcllll} G_{11} & = & G_{xx}(\cdot)(\phi_1 + y_1)^2 & + & G_x(\cdot)y_{11} & + \ \mu_{11}\psi, \\ G_{12} & = & G_{xx}(\cdot)(\phi_1 + y_1)(\phi_2 + y_2) & + & G_x(\cdot)y_{12} & + \ \mu_{12}\psi, \\ G_{21} & = & G_{xx}(\cdot)(\phi_2 + y_2)^2 & + & G_x(\cdot)y_{22} & + \ \mu_{22}\psi. \end{array}$$

$$\begin{array}{rcl} G_{11}^0 & = & G_{xx}^0\phi_1\phi_1 \ + \ G_x^0 y_{11}^0 \ + \ \mu_{11}^0\psi = 0, \\ G_{12}^0 & = & G_{xx}^0\phi_1\phi_2 \ + \ G_x^0 y_{12}^0 \ + \ \mu_{12}^0\psi = 0, \\ G_{22}^0 & = & G_{xx}^0\phi_2\phi_2 \ + \ G_x^0 y_{22}^0 \ + \ \mu_{22}^0\psi = 0. \end{array}$$

$$\psi^* G_{11}^0 = 0 \ \Rightarrow \ \psi^* G_{xx}^0 \phi_1\phi_1 + \mu_{11}^0 \psi^*\psi = 0, \ \boxed{\mu_{11}^0 = -\psi^* G_{xx}^0 \phi_1\phi_1} \ ,$$

$$\psi^* G_{12}^0 = 0 \ \Rightarrow \ \psi^* G_{xx}^0 \phi_1\phi_2 + \mu_{12}^0 \psi^*\psi = 0, \ \boxed{\mu_{12}^0 = -\psi^* G_{xx}^0 \phi_1\phi_2} \ ,$$

$$\psi^* G_{22}^0 = 0 \ \Rightarrow \ \psi^* G_{xx}^0 \phi_2\phi_2 + \mu_{22}^0 \underbrace{\psi^*\psi}_{=1} = 0, \ \boxed{\mu_{22}^0 = -\psi^* G_{xx}^0 \phi_2\phi_2} \ .$$

Thus

$$\mu(\epsilon_1, \epsilon_2) = -\frac{1}{2}\left(\psi^* G_{xx}^0 \phi_1\phi_1\right)\epsilon_1^2 - \left(\psi^* G_{xx}^0 \phi_1\phi_2\right)\epsilon_1\epsilon_2 - \frac{1}{2}\left(\psi^* G_{xx}^0 \phi_2\phi_2\right)\epsilon_2^2 \ + \ O(\epsilon^3).$$

To get y_{11}^0, solve

$$G_x^0 \ y_{11}^0 = -\underbrace{G_{xx}^0 \phi_1\phi_1 - \mu_{11}^0 \psi}_{\in \mathcal{R}(G_x^0)}.$$

$$\phi_i^* y_{11}^0 = 0 \ \Rightarrow \ y_{11}^0 \text{ contains no component in } \mathcal{N}(G_x^0),$$

i.e., y_{11}^0 is unique. Similarly for y_{12}^0, y_{22}^0.

To lowest order approximation, the solutions of $G(x) = 0$ correspond to the level curves

$$\underbrace{(\psi^* G_{xx}^0 \phi_1\phi_1)}_{c_{11}}\epsilon_1^2 + 2\underbrace{(\psi^* G_{xx}^0 \phi_1\phi_2)}_{c_{12}}\epsilon_1\epsilon_2 + \underbrace{(\psi^* G_{xx}^0 \phi_2\phi_2)}_{c_{22}}\epsilon_2^2 = 0.$$

This is the *Algebraic Bifurcation Equation* (ABE), which is of the form

$$c_{11}\epsilon_1^2 + 2c_{12}\epsilon_1\epsilon_2 + c_{22}\epsilon_2^2 = 0,$$

with discriminant

$$\Delta \equiv c_{12}^2 - c_{11}c_{22}.$$

We can also write

$$\mu(\epsilon_1,\epsilon_2) \;=\; -\frac{1}{2}(c_{11}\epsilon_1^2 + 2c_{12}\epsilon_1\epsilon_2 + c_{22}\epsilon_2^2) \;+\; O(\epsilon^3).$$

6.2 Exchange of Stability

Suppose that, using the parameter formulation, we have $G(u_0,\lambda_0)=0$, with

$$\mathcal{N}(G_u^0) = \mathrm{Span}\{\phi_0\}, \qquad \mathcal{N}(G_u^{0^*}) = \mathrm{Span}\{\psi\}, \qquad G_\lambda^0 \in \mathcal{R}(G_u^0).$$

Assume that, in addition, zero is an algebraically simple eigenvalue of G_u^0.

In the ABE

$$c_{11}\;\epsilon_1^2 \;+\; 2c_{12}\;\epsilon_1\epsilon_2 \;+\; c_{22}\;\epsilon_2^2 \;=\; 0,$$

assume that

$$\Delta \equiv c_{12}^2 - c_{11}c_{22} > 0,$$

and also that

$$c_{11} \neq 0, \quad \text{ so that } (\epsilon_1,\epsilon_2) = (1,0) \text{ is not a root },$$

We have (see Section 6.1.1)

$$\begin{pmatrix} u(\epsilon) \\ \lambda(\epsilon) \end{pmatrix} = \begin{pmatrix} u_0 \\ \lambda_0 \end{pmatrix} + \epsilon_1 \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix} + \epsilon_2 \begin{pmatrix} \phi_1 \\ 1 \end{pmatrix} + \cdots,$$

where

$$G_u^0\;\phi_1 \;=\; -G_\lambda^0, \qquad \epsilon \;=\; (\epsilon_1,\epsilon_2).$$

Note that since $(\epsilon_1,\epsilon_2)=(1,0)$ is not a root, neither of the branches is "vertical" at (u_0,λ_0) (see Figure 6.7).

Consider the equations

$$\left.\begin{aligned} &G_u\Big(u(\epsilon),\lambda(\epsilon)\Big)\;\phi - \sigma\;\phi \;=\; 0, \\ &\phi^*\phi_0 - 1 = 0, \end{aligned}\right\} \sim\; F(\phi,\sigma;\epsilon) = 0,$$

for which $F(\phi_0,0;0)=0$.

Here $(F_\phi^0 \mid F_\sigma^0)$ is nonsingular, by the same argument used for the case of a fold. (We need algebraic simplicity here.) Thus, for small ϵ,

$$\phi = \phi(\epsilon), \qquad \sigma = \sigma(\epsilon),$$

and

$$G_u\Big(u(\epsilon),\lambda(\epsilon)\Big)\;\phi(\epsilon) - \sigma(\epsilon)\;\phi(\epsilon) \;=\; 0.$$

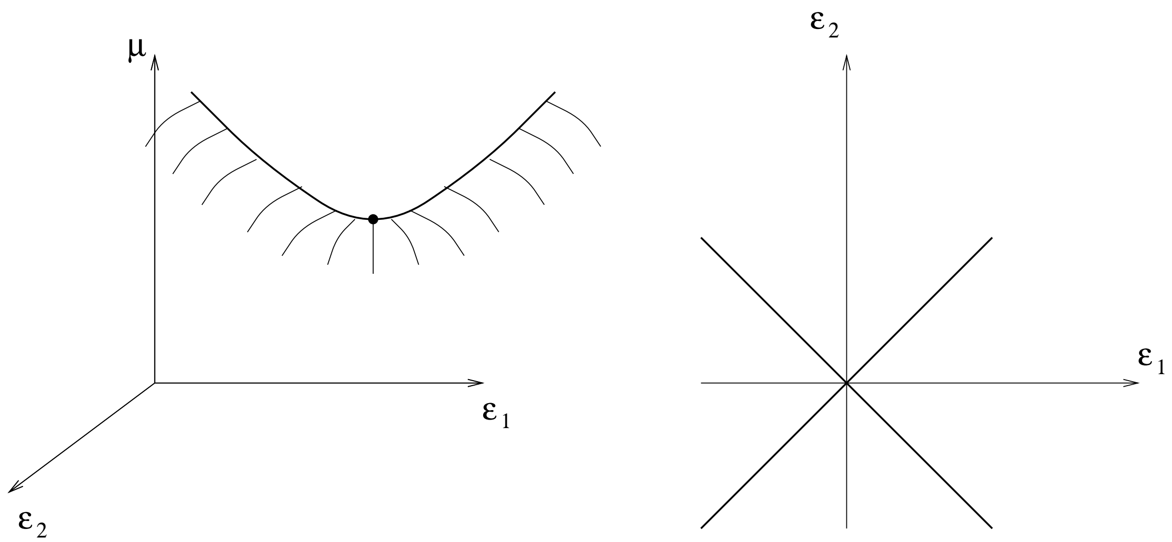

Figure 6.1: Graph of $\mu(\epsilon_1, \epsilon_2)$ with level curves (to lowest order) for the case $\Delta > 0$. This is a hyperbolic point ("bifurcation"), which is structurally stable under $\mathcal{O}(\epsilon^3)$ perturbations.

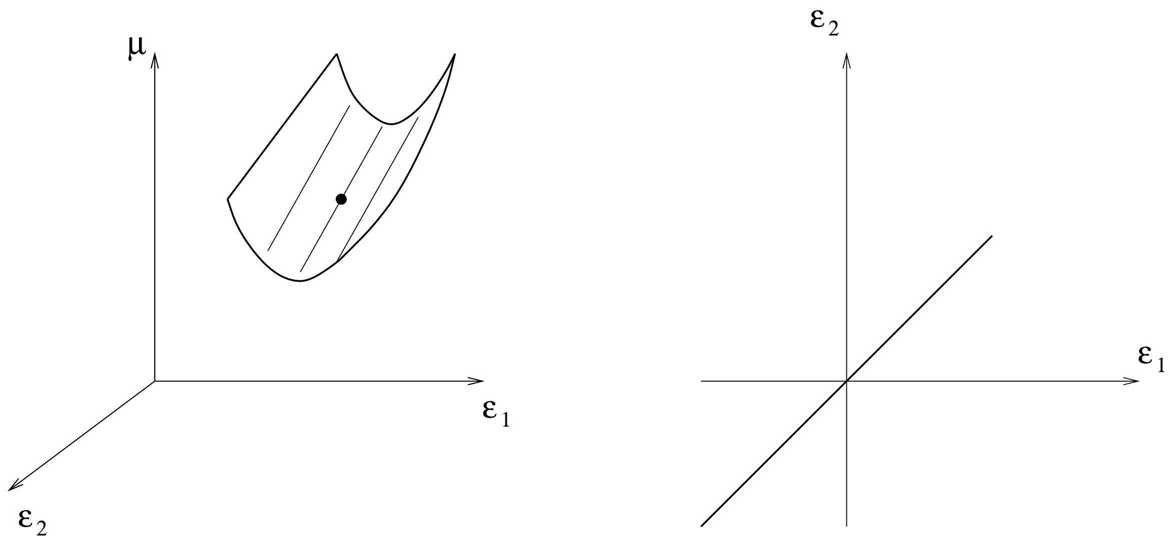

Figure 6.2: Graph of $\mu(\epsilon_1, \epsilon_2)$ with level curves (to lowest order) for the case $\Delta = 0$. This is a parabolic point, which is not structurally stable.

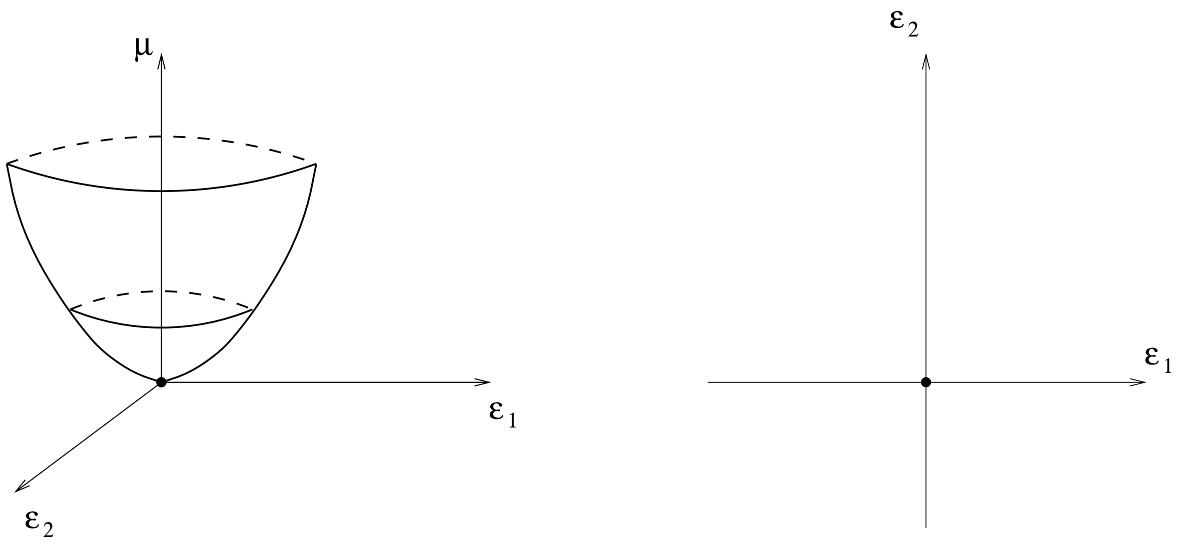

Figure 6.3: Graph of $\mu(\epsilon_1, \epsilon_2)$ with level curves (to lowest order) for the case $\Delta < 0$. This is a elliptic point ("isola formation point"), which is structurally stable.

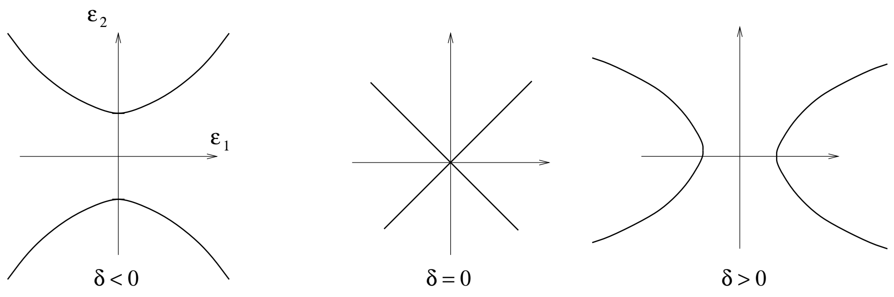

Figure 6.4: Graphs of the level curves $\mu(\epsilon_1, \epsilon_2) = \delta$ (to lowest order) for the case $\Delta > 0$ (bifurcation).

Differentiate with respect to ϵ_1 or ϵ_2, evaluated at $\epsilon_1 = \epsilon_2 = 0$:

$$G^0_{uu}\dot{u}(0)\phi_0 + G^0_{u\lambda}\phi_0\dot{\lambda}(0) + G^0_u\dot{\phi}(0) - \dot{\sigma}(0)\ \phi_0 - \underbrace{\sigma(0)}_{=0}\dot{\phi}(0) = 0.$$

$$\Rightarrow\ \psi^* G^0_{uu}\dot{u}(0)\phi_0 + \psi^* G^0_{u\lambda}\phi_0\dot{\lambda}(0) - \dot{\sigma}(0)\psi^*\phi_0 = 0.$$

$$\Rightarrow \begin{cases} \epsilon = \epsilon_1 & : \quad \psi^* G^0_{uu}\phi_0\phi_0 - \dot{\sigma}_1(0)\ \psi^*\phi_0 = 0, \\ \epsilon = \epsilon_2 & : \quad \psi^* G^0_{uu}\phi_0\phi_1 + \psi^* G_{u\lambda}\phi_0 - \dot{\sigma}_2(0)\ \psi^*\phi_0 = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\sigma}_1(0) &= \psi^* G^0_{uu}\phi_0\phi_0/\psi^*\phi_0, \\ \dot{\sigma}_2(0) &= (\psi^* G^0_{uu}\phi_0\phi_1 + \psi^* G^0_{u\lambda}\phi_0)/\psi^*\phi_0. \end{cases}$$

where $\psi^*\phi_0 \neq 0$, since the zero eigenvalue is algebraic simple.

The coefficients in the ABE are :

$$\begin{cases} c_{11} = \psi^* G_{xx}\Phi_1\Phi_1 = \psi^*\Big((G^0_{uu}\ G^0_{u\lambda})\ \Big|\ (G^0_{u\lambda}\ G^0_{\lambda\lambda})\Big)\binom{\phi_0}{0}\binom{\phi_0}{0} = \psi^* G^0_{uu}\phi_0\phi_0, \\ c_{12} = \psi^* G_{xx}\Phi_1\Phi_2 = \psi^*\Big(\cdots\Big)\binom{\phi_0}{0}\binom{\phi_1}{1} = \psi^* G^0_{uu}\phi_0\phi_1 + \psi^* G^0_{u\lambda}\phi_0, \\ c_{22} = \psi^* G_{xx}\Phi_2\Phi_2 = \ \cdots\ . \end{cases}$$

Thus

$$\nabla\sigma(0)\ \equiv\ \begin{pmatrix} \dot{\sigma}_1(0) \\ \dot{\sigma}_2(0) \end{pmatrix}\ =\ c\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix},\ \text{where } c = \frac{1}{\psi^*\phi_0}.$$

The direction of the "level curve" $\sigma(\epsilon_1, \epsilon_2) = 0$ at $\epsilon_1 = \epsilon_2 = 0$ is orthogonal to $\nabla\sigma(0)$, i.e. this direction is $c\begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix}$. Now $\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix}$ is not a root, for if so,

$$c_{11}(-c_{12})^2\ +\ 2c_{12}(-c_{12})(c_{11})\ +\ c_{22}(c_{11})^2\ =\ 0,$$

and, since $c_{11} \neq 0$ by assumption, it follows that

$$c_{12}^2 - 2c_{12}^2 + c_{11}c_{22} = 0,$$

i.e.,

$$c_{11}c_{22} - c_{12}^2 = 0,$$

which contradicts $\Delta \neq 0$.

Thus we have, to first order, the picture shown in Figure 6.6. It is not difficult to check that, to lowest order, the level set $\sigma(\epsilon_1, \epsilon_2) = 0$ lies between branch 1 and branch 2 in the right half plane, and hence also in the left hand plane. A possible stability assignment is shown in Figure 6.7.

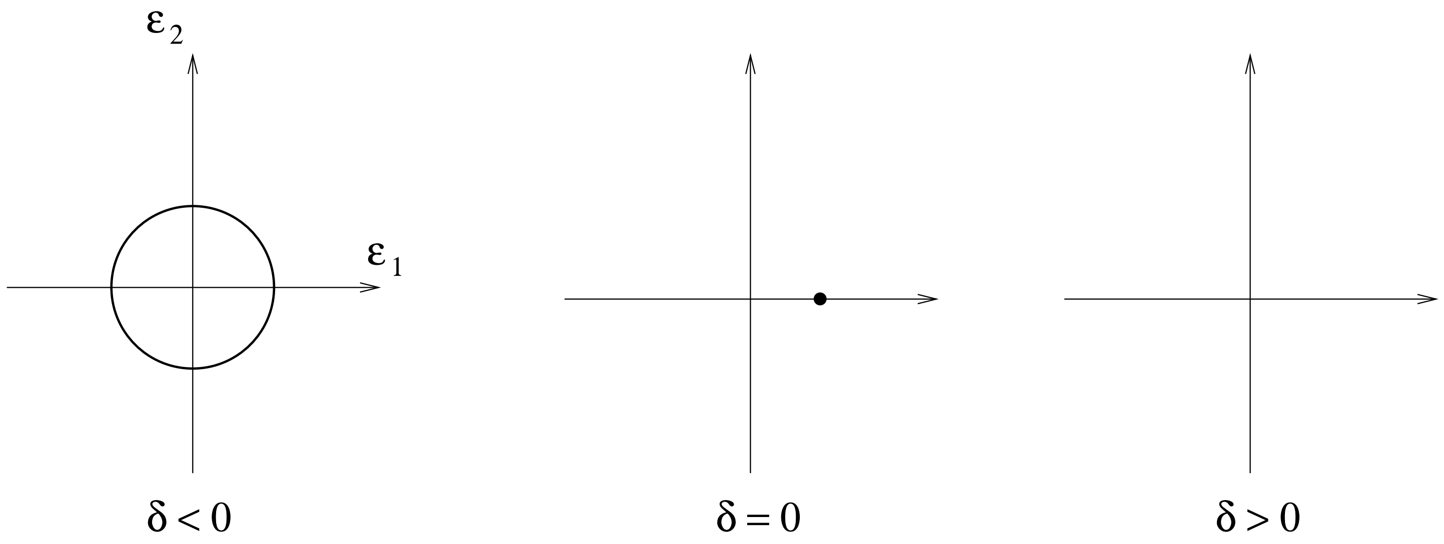

Figure 6.5: Graphs of the level curves $\mu(\epsilon_1, \epsilon_2) = \delta$ (to lowest order) for the case $\Delta < 0$ (isola formation point).

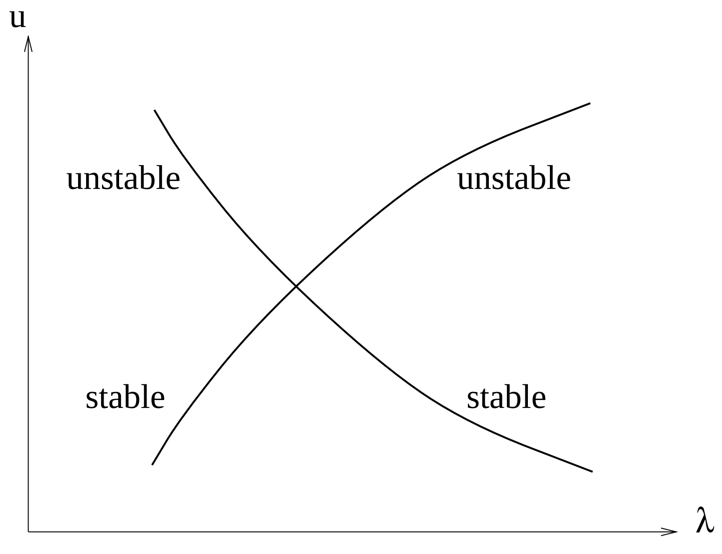

Figure 6.6: Local representation of the solution curves of $G(u, \lambda) = 0$, and the level set $\sigma(\epsilon_1, \epsilon_2) = 0$ (to lowest order).

6.3 Higher Dimensional Nullspace

6.3.1 Problem formulation

Suppose $G(u_0, \lambda_0) = 0$, with

$$\mathcal{N}(G_u^0) = \mathrm{Span}\{\hat{\phi}_1, \hat{\phi}_2\}, \quad \mathcal{N}(G_u^{0^*}) = \mathrm{Span}\{\psi_1, \psi_2\}, \quad G_\lambda^0 \in \mathcal{R}(G_u^0).$$

Let $x \equiv (u, \lambda)$. Then $G_x^0 = (G_u^0 | G_\lambda^0)$, and

$$(G_u^0 \mid G_\lambda^0) \binom{v}{\mu} = 0, \quad \Rightarrow \quad G_u^0 v = -\mu G_\lambda^0,$$

from which

$$\begin{cases} \mu = 0, & \Rightarrow \quad v = \alpha_1 \hat{\phi}_1 + \alpha_2 \hat{\phi}_2, \\ \\ \mu \neq 0, (\mu = 1), & \Rightarrow \quad \begin{cases} G_\lambda^0 = 0, & v = \beta_1 \hat{\phi}_1 + \beta_2 \hat{\phi}_2, \\ G_\lambda^0 \neq 0, & v = v_p + \beta_1 \hat{\phi}_1 + \beta_2 \hat{\phi}_2. \end{cases} \end{cases}$$

So the null space is 3-dimensional,

$$\mathcal{N}(G_x^0) = \mathrm{Span}\left\{ \begin{pmatrix} \hat{\phi}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \hat{\phi}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} v_p \\ 1 \end{pmatrix} \right\} \equiv \mathrm{Span}\left\{ \phi_1, \phi_2, \phi_3 \right\}.$$

Next, consider $G_x^{0^*}$:

$$\begin{pmatrix} G_u^{0^*} \\ \\ G_\lambda^{0^*} \end{pmatrix} v = 0 \;\; \Rightarrow \;\; \begin{cases} G_u^{0^*} v &= 0 \quad \Rightarrow \;\; v = \alpha_1 \psi_1 + \alpha_2 \psi_2, \\ \\ v^* G_\lambda^0 &= 0 \quad \Rightarrow \;\; \alpha_1 \underbrace{\psi_1^* G_\lambda^0}_{=0} + \alpha_2 \underbrace{\psi_2^* G_\lambda^0}_{=0} = 0. \end{cases}$$

Thus α_1 and α_2 are arbitrary. Hence we have

$$\mathcal{N}(G_x^0) = \mathrm{Span}\{\phi_1, \phi_2\, \phi_3\} \qquad (3 - D),$$

$$\mathcal{N}(G_x^{0^*}) = \mathrm{Span}\{\psi_1, \psi_2\} \qquad (2 - D),$$

where we may take

$$\psi_i^* \; \psi_j = \delta_{ij}, \qquad \phi_i^* \; \phi_j = \delta_{ij}.$$

Consider

$$\tilde{F}(y, \mu_1, \mu_2; \epsilon_1, \epsilon_2, \epsilon_3) \equiv \begin{cases} G(\underbrace{x_0 + \epsilon_1 \phi_1 + \epsilon_2 \phi_2 + \epsilon_3 \phi_3 + y}_{x}) + \mu_1 \xi_1 + \mu_2 \xi_2, \\ \\ y^* \; \phi_i = 0, \qquad i = 1, 2, 3. \end{cases}$$

where

$$\xi_1, \xi_2 \ \notin \ \underbrace{\mathcal{R}(G_x^0)}_{=\mathcal{N}(G_x^{0^*})^\perp} \ ,$$

i.e.,

$$\xi_i \ \not\perp \ \mathcal{N}(G_x^{0^*}).$$

For example, take

$$\xi_i = \psi_i, \quad i = 1, 2, \qquad \text{where } \psi_1^* \ \psi_2 = 0.$$

Note that $(y, \mu_1, \mu_2, \epsilon_1, \epsilon_2, \epsilon_3) = (0, 0, 0, 0, 0, 0)$ is a solution. and that $\epsilon_i = (x - x_0)^* \phi_i$.

THEOREM.

Near $(y, \mu_1, \mu_2, \epsilon_1, \epsilon_2, \epsilon_3) = (0, 0, 0, 0, 0, 0)$, the equation

$$\tilde{F}(y, \mu_1, \mu_2; \epsilon_1, \epsilon_2, \epsilon_3) = 0,$$

defines a unique (smooth) solution manifold

$$y = y(\epsilon_1, \epsilon_2, \epsilon_3), \quad \mu_1 = \mu_1(\epsilon_1, \epsilon_2, \epsilon_3), \quad \mu_2 = \mu_2(\epsilon_1, \epsilon_2, \epsilon_3),$$

$$y(0, 0, 0) = 0, \quad \mu_1(0, 0, 0) = 0, \quad \mu_2(0, 0, 0) = 0.$$

PROOF.

$$(\tilde{F}_{y_1}^0 \mid \tilde{F}_{\mu_1}^0 \mid \tilde{F}_{\mu_2}^0) = \begin{pmatrix} G_x^0 & \psi_1 & \psi_2 \\ \phi_1^* & 0 & 0 \\ \phi_2^* & 0 & 0 \\ \phi_3^* & 0 & 0 \end{pmatrix}.$$

Suppose $(\tilde{F}_{y_1}^0 \mid \tilde{F}_{\mu_1}^0 \mid \tilde{F}_{\mu_2}^0)$ is singular. Then

$$G_x^0 p = -q_1 \psi_1 - q_2 \psi_2.$$

for some p, q_1 and q_2. But the RHS is in $\mathcal{N}(G_x^{0*})$ or, equivalently, in $\mathcal{R}(G_x^0)^\perp$. Thus $q_1 = q_2 = 0$, $G_x^0 p = 0$, and hence $p = \alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3$.

Finally

$$\phi_i^* p = 0 \quad \Rightarrow \quad \alpha_i = 0 \quad \Rightarrow \quad p = 0. \qquad \bullet$$

REMARK. The solution set of $G(x) = 0$ near x_0 has the form

$$x \ = \ x_0 \ + \ \epsilon_1 \phi_1 \ + \ \epsilon_2 \phi_2 \ + \ \epsilon_3 \phi_3 \ + \ y(\epsilon_1, \epsilon_2, \epsilon_3),$$

with the ϵ_i constrained by the "level set" relations

$$\mu_1(\epsilon_1, \epsilon_2, \epsilon_3) \ = \ 0, \qquad \mu_2(\epsilon_1, \epsilon_2, \epsilon_3) \ = \ 0.$$

6.3.2 Construction of the coefficients

From

$$G(\ x_0 + \epsilon_1\phi_1 \ + \ \epsilon_2\phi_2 \ + \ \epsilon_3\phi_3 \ + \ y(\epsilon_1, \epsilon_2, \epsilon_3)\ ) \ + \ \mu_1(\epsilon_1, \epsilon_2, \epsilon_3)\ \psi_1 \ + \ \mu_2(\epsilon_1, \epsilon_2, \epsilon_3)\ \psi_2 \ = \ 0,$$

for all small ϵ_j, $j = 1, 2, 3$, it follows that

$$0 = \frac{dG}{d\epsilon_j}\Big|_{\underline{\epsilon}=0} \equiv G_j^0 = G_x^0 y_j^0 + \mu_{1j}^0\psi_1 + \mu_{2j}^0\psi_2 \quad \Rightarrow \quad \boxed{\mu_{1j}^0 = \mu_{2j}^0 = 0}\ ,$$

and

$$y_j^0 \ = \ \alpha_{1j}\phi_1 \ + \ \alpha_{2j}\phi_2 \ + \ \alpha_{3j}\phi_3\ .$$

Also

$$\phi_i^*\ y(\epsilon_1, \epsilon_2, \epsilon_3) \equiv 0 \quad \Rightarrow \quad \phi_i^*\ y_j^0 = 0.$$

and hence $\alpha_{ij} = 0$, and $\boxed{y_j^0 = 0}$, $\quad (i, j = 1, 2, 3)$.

The second order terms are obtained as follows :

$$\psi_i^* G_{jk}^0 = 0 \ \Rightarrow \ \psi_i^* G_{xx}^0 \phi_j\phi_k + \mu_{1jk}^0\psi_i^*\psi_1 + \mu_{2jk}^0\psi_i^*\psi_2 = 0.$$

Hence

$$\boxed{\mu_{1jk}^0 \ = \ -\psi_1^*\ G_{xx}^0\ \phi_j\ \phi_k}\ , \qquad \boxed{\mu_{2jk}^0 \ = \ -\psi_2^*\ G_{xx}^0\ \phi_j\ \psi_k}\ .$$

Thus the level sets are implicitly defined, to lowest order, by the equations

$$\sum_{ij}\ (\psi_1^* G_{xx}^0 \phi_i\phi_j)\ \epsilon_i\ \epsilon_j = 0,$$

$$\sum_{ij}\ (\psi_2^* G_{xx}^0 \phi_i\phi_j)\ \epsilon_i\ \epsilon_j = 0.$$

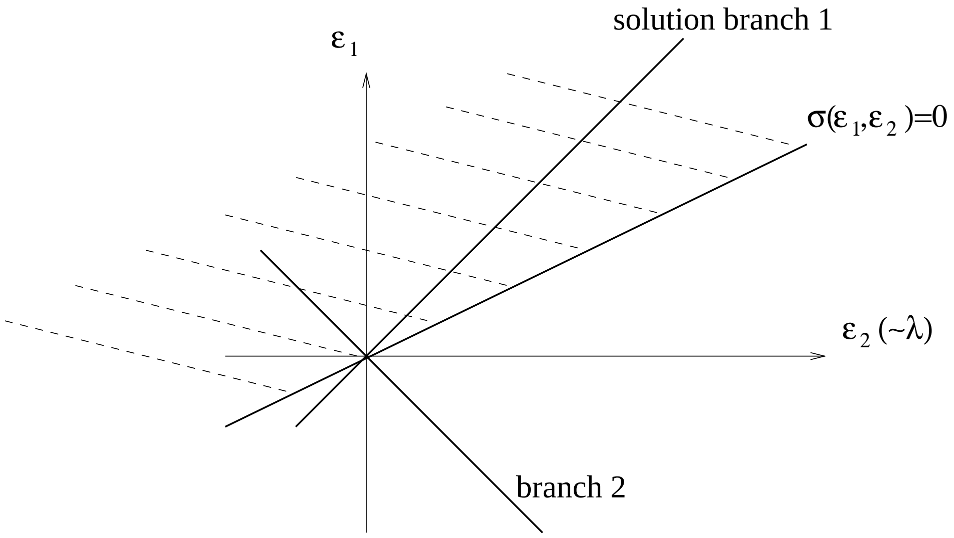

Figure 6.7: Solution curve of $G(u, \lambda) = 0$ near the with bifurcation point, with (possible) stability assignment.

Chapter 7

Numerical Treatment of Bifurcations

7.1 Simple Singular Points

Let $G : \mathrm{R}^{n+1} \rightarrow \mathrm{R}^n$. A solution $x_0 \equiv x(s_0)$ of $G(x) = 0$ is called a *simple singular point* if $G_x^0 \equiv G_x(x_0)$ has rank $n-1$. (See also Section 6.1.) In the parameter formulation, where $G_x^0 = (G_u^0 \mid G_\lambda^0)$, we have that $x_0 = (u_0, \lambda_0)$ is a simple singular point iff

$$(i) \qquad dim\ \mathcal{N}(G_u^0) = 1, \qquad G_\lambda^0 \in \mathcal{R}(G_u^0),$$

or

$$(ii) \qquad dim\ \mathcal{N}(G_u^0) = 2, \qquad G_\lambda^0 \notin \mathcal{R}(G_u^0).$$

An example of case (i) is

$$G(u, \lambda) = u(u - \lambda) \quad \text{at} \quad (u_0, \lambda_0) = (0, 0),$$

where $G_u^0 = 0$ and $G_\lambda^0 = 0$.

An example of case (ii) is

$$G(u, \lambda) = \begin{pmatrix} \lambda - u_1^2 - u_2^2 \\ u_1 u_2 \end{pmatrix},$$

at $\lambda = 0$, $u_1 = u_2 = 0$. Here

$$G_u^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad G_\lambda^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Suppose we have a solution branch $x(s)$ of $G(x) = 0$, where s is some parametrization. Let $x_0 \equiv (u_0, \lambda_0)$ be a simple singular point. Then we must have

$$\mathcal{N}(G_x^0) = \mathrm{Span}\{\phi_1, \phi_2\}, \qquad \mathcal{N}(G_x^{0^*}) = \mathrm{Span}\{\psi\}.$$

We also have

$$G(x(s)) = 0, \qquad G^0 = G(x_0) = 0,$$

$$G_x(x(s))\dot{x}(s) = 0, \qquad G^0_x \dot{x}_0 = 0,$$

$$G_{xx}(x(s))\dot{x}(s)\dot{x}(s) + G_x(x(s))\ddot{x}(s) = 0, \qquad G^0_{xx}\dot{x}_0\dot{x}_0 + G^0_x\ddot{x}_0 = 0.$$

Thus $\dot{x}_0 = \alpha\phi_1 + \beta\phi_2$, for some $\alpha, \beta \in \mathrm{R}$, and

$$\psi^* G^0_{xx}(\alpha\phi_1 + \beta\phi_2)(\alpha\phi_1 + \beta\phi_2) + \underbrace{\psi^* G^0_x}_{=0}\ddot{x}_0 = 0,$$

$$\boxed{\underbrace{(\psi^* G^0_{xx}\phi_1\phi_1)}_{c_{11}}\alpha^2 + 2\underbrace{(\psi^* G^0_{xx}\phi_1\phi_2)}_{c_{12}}\alpha\beta + \underbrace{(\psi^* G^0_{xx}\phi_2\phi_2)}_{c_{22}}\beta^2 = 0.}$$

This is the Algebraic bifurcation equation (ABE; see Section 6.1 for a more precise derivation). We want solution pairs (α, β) of the ABE with not both α and β equal to zero. If the *discriminant*

$$\Delta \equiv c_{12}^2 - c_{11}c_{22} \ > \ 0,$$

then the ABE has two distinct real nontrivial solution pairs (α_1, β_1) and (α_2, β_2), which are unique up to scaling. In such case we have a *bifurcation*, (or *branch point*) i.e., two distinct branches pass through x_0. See Figure 7.1 for a graphical interpretation.

7.2 Branch Switching

7.2.1 Computing the bifurcation direction

The direction vector of a bifurcating branch satisfies

$$\dot{x}_0 = \alpha\phi_1 + \beta\phi_2,$$

where (α, β) is a root of the ABE

$$c_{11}\alpha^2 + 2c_{12}\alpha\beta + c_{22}\beta^2 = 0.$$

We can take ϕ_1 to be the direction of the "given" branch at the branch point, i.e.,

$$\phi_1 = \dot{x}_0.$$

Then

$$(\alpha_1, \beta_1) = (1, 0)$$

must be a root of the ABE and we must have

$$c_{11} = 0.$$

If $\Delta > 0$ then also
$$c_{12} \neq 0.$$
The second root then satisfies
$$\alpha_2/\beta_2 = -c_{22}/2c_{12}.$$
To evaluate c_{12} and c_{22} we need the nullvectors :

ϕ_1 : We already have chosen $\phi_1 = \dot{x}_0$.

ϕ_2 : Choose $\phi_2 \perp \phi_1$. Then ϕ_2 is a nullvector of
$$F_x^0 = \begin{pmatrix} G_x^0 \\ \dot{x}_0^* \end{pmatrix}.$$
Note that F_x^0 is the Jacobian of the extended pseudo-arclength system at x_0 (see Section 3.3). The null space of F_x^0 is indeed one-dimensional as will be shown in Section 7.3.

ψ : $\begin{pmatrix} \psi \\ 0 \end{pmatrix}$ is a simple nullvector of
$$(F_x^0)^* = ((G_x^0)^* \mid \dot{x}_0).$$

REMARK. Left and right null vectors of a matrix can be computed at little extra cost once the matrix has been LU decomposed.

After determining the coefficients α_2 and β_2, scale the direction vector
$$x_0' \equiv \alpha_2\phi_1 + \beta_2\phi_2,$$
of the bifurcating branch so that
$$\| x_0' \| = 1.$$

7.2.2 Switching branches

The first solution x_1 on the bifurcating branch can be computed from :
$$G(x_1) = 0,$$
$$(x_1 - x_0)^* x_0' - \Delta s = 0, \tag{7.1}$$
where
$$x_0' \text{ is the direction of the bifurcating branch,}$$
with initial approximation
$$x_1^{(0)} = x_0 + \Delta s\ x_0'.$$
For a graphical interpretation see Figure 7.2. From the preceding discussion we see that computation of x_0' requires evaluation of G_{xx}^0.

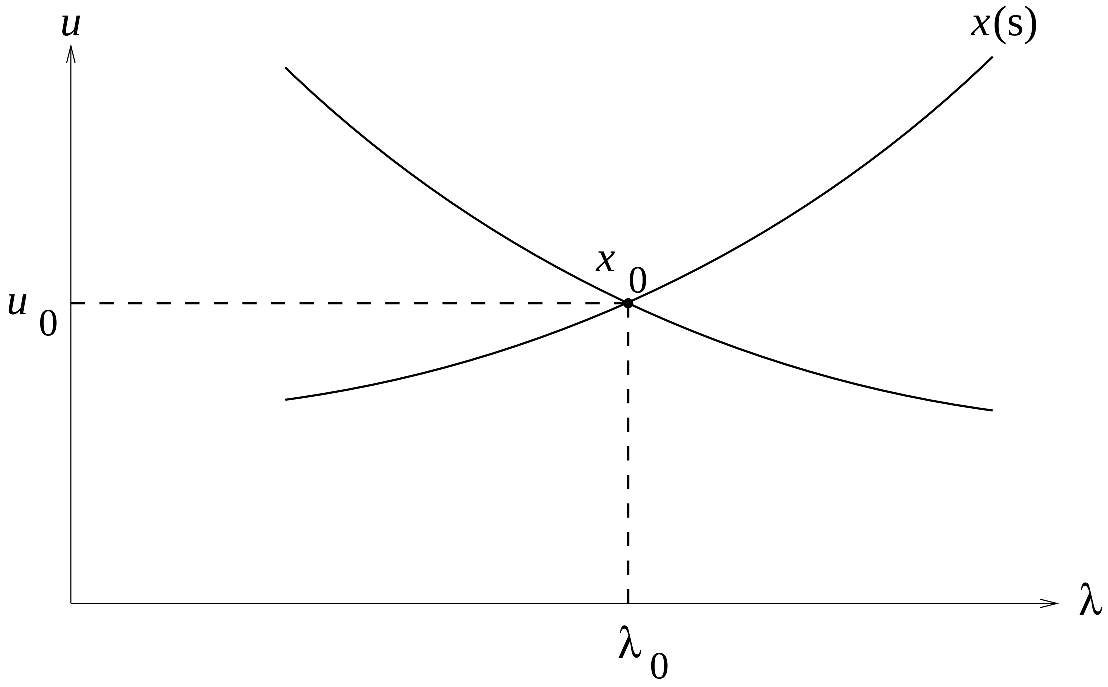

Figure 7.1: A solution branch $x(s) = (u(s), \lambda(s))$ of $G(u, \lambda) = 0$ with a branch point x_0.

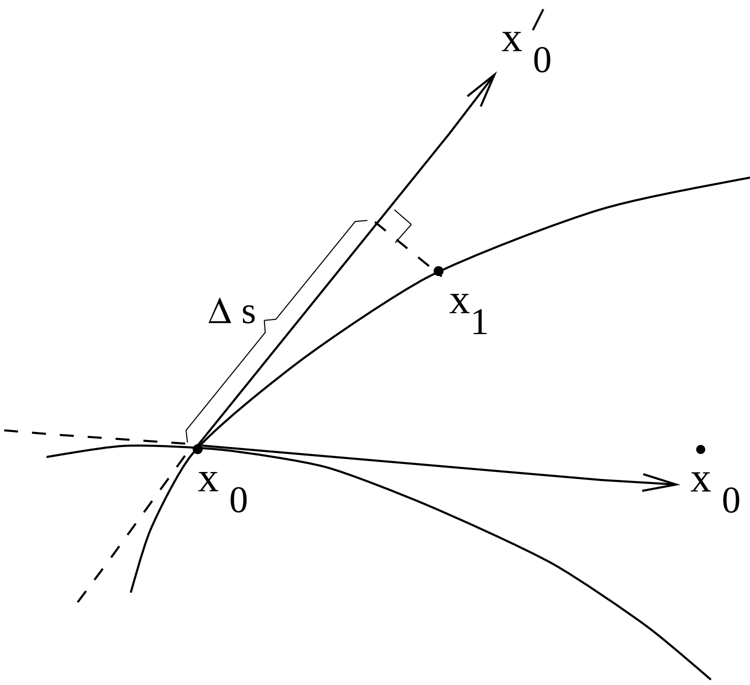

Figure 7.2: Switching branches using the correct bifurcation direction.

7.2.3 Simplified branch switching

Instead of solving Equations (7.1) for taking the first step on the bifurcating branch, solve :

$$G(x_1) = 0,$$

$$(x_1 - x_0)^* \phi_2 - \Delta s = 0.$$

where ϕ_2 is the second null vector of G_x^0 with, as above in Section 7.2.1,

$$\phi_2 \perp \phi_1, \qquad (\phi_1 = \dot{x}_0).$$

For a graphical interpretation see Figure 7.3. Although there may be situations where this method must fail, it appears to work very well in almost all applications. The advantage of this technique is that it does not require second derivatives.

7.3 Detection of Bifurcation Points

Consider the pseudo-arclength system of Section 3.3,

$$F(x; s) \equiv \begin{pmatrix} G(x) \\ \\ (x - x_0)^* \dot{x}_0 - s \end{pmatrix}, \tag{7.2}$$

where we take x_0 to be a simple singular point. Then

$$F_x^0 \equiv F_x(x_0; s_0) = F_x(x_0) = \begin{pmatrix} G_x^0 \\ \dot{x}_0^* \end{pmatrix}.$$

As above in Section 7.2.1, take $\phi_1 = \dot{x}_0$, for the first nullvector of G_x^0, and choose the second nullvector ϕ_2 such that $\phi_2^* \phi_1 = 0$. Thus

$$F_x^0 = \begin{pmatrix} G_x^0 \\ \phi_1^* \end{pmatrix}.$$

We see that ϕ_2 is also a nullvector of F_x^0 but that ϕ_1 is not. Therefore F_x^0 has a one-dimensional nullspace.

Also, the null vector of F_x^{0*} is given by

$$\Psi \equiv \begin{pmatrix} \psi \\ 0 \end{pmatrix},$$

where ψ is the null vector of $(G_x^0)^*$.

THEOREM.

Let $x_0 = x(s_0)$ be a simple singular point on a smooth solution branch $x(s)$ of $G(x) = 0$. Let $F(x;s)$ be as above. Assume that

$$\text{the discriminant } \Delta \text{ is positive,}$$

and

$$0 \text{ is an algebraically simple eigenvalue of } F_x^0 \equiv \begin{pmatrix} G_x^0 \\ \dot{x}_0^* \end{pmatrix}.$$

Then det F_x changes sign at x_0.

PROOF.

Consider the parametrized eigenvalue problem

$$F_x(x(s))\ \phi(s) \ = \ \kappa(s)\ \phi(s),$$

where $\kappa(s)$ and $\phi(s)$ are smooth near s_0, with

$$\kappa(s_0) = 0 \quad \text{and} \quad \phi(0) = \phi_2,$$

i.e., the eigen pair $(\kappa(s), \phi(s))$ is the continuation of $(0, \phi_2)$. This can be done because we have assumed that 0 is an algebraically simple eigenvalue. Differentiation and evaluation at s_0, using $\kappa(0) = 0$, $\dot{x}_0 = \phi_1$ and $\phi(0) = \phi_2$, gives

$$F_{xx}^0 \phi_1 \phi_2 + F_x^0 \dot{\phi}(s_0) = \dot{\kappa}_0 \phi_2,$$

and upon multiplying this on the left by Ψ^* we find

$$\dot{\kappa}_0 = \frac{\Psi^* F_{xx}^0 \phi_1 \phi_2}{\Psi^* \phi_2} = \frac{(\psi^*, 0) \begin{pmatrix} G_{xx}^0 \\ 0 \end{pmatrix} \phi_1 \phi_2}{\Psi^* \phi_2} = \frac{\psi^* G_{xx}^0 \phi_1 \phi_2}{\Psi^* \phi_2} = \frac{c_{12}}{\Psi^* \phi_2},$$

where the left and right nullvectors cannot be orthogonal, i.e.,

$$\Psi^* \phi_2 \neq 0,$$

and where c_{12} is a coefficient of the ABE. By assumption $\Delta \neq 0$. As before this implies that $c_{12} \neq 0$ and hence $\dot{\kappa}_0 \neq 0$. •

The detection of simple singular points in practical computations is based upon the above result. During continuation we monitor the determinant of the the pseudo-arclength Jacobian matrix $\begin{pmatrix} G_x \\ \dot{x}^* \end{pmatrix}$. If a sign change is detected then an iterative method such as Müller's method (with bracketing) can be used to accurately locate the singular point. For large systems one should compute a scaled determinant in order to avoid overflow.

REMARK. We have proved that the determinant of the system 7.2 changes sign. Note that this does not prove that the determinant of the pseudo-arclength Jacobian changes sign. However, the latter follows from a similar, if somewhat more elaborate argument.

The following theorem states that there must be a bifurcation at x_0. This result can be proven by degree theory.

THEOREM.

Let $x(s)$ be a smooth solution branch of

$$F(x; s) = 0,$$

where

$$F : \mathrm{R}^{n+1} \times \mathrm{R} \to \mathrm{R}^{n+1} \quad \text{is} \quad C^1,$$

and assume that

$$\det\, F_x(x(s); s) \text{ changes sign at } s = s_0.$$

Then $x(s_0)$ is a bifurcation point, i.e., every open neigborhood of x_0 contains a solution of $F(x; s) = 0$ that does not lie on $x(s)$.

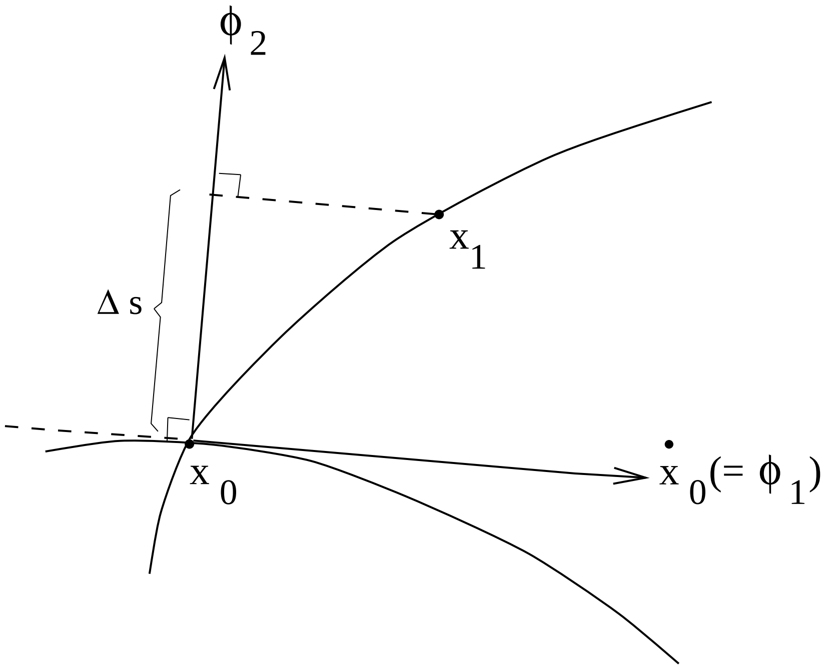

Figure 7.3: Switching branches using the orthogonal direction.

Chapter 8

Examples of Bifurcations

8.1 A Predator-Prey Model

In the 2-species predator-prey model of Section 2.1 we have

$$G_x = (G_{u_1}\ G_{u_2}\ G_\lambda) = \begin{pmatrix} 3-6u_1-u_2-5\lambda e^{-5u_1} & -u_1 & -(1-e^{-5u_1}) \\ 3u_2 & -1+3u_1 & 0 \end{pmatrix}.$$

At the branch point $u_1 = u_2 = 0$, $\lambda = 3/5$ (See Figure 2.1 and Figure 2.2), we have $x_0 = (0, 0, 3/5)$ and

$$G_x^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\mathcal{N}(G_x^0) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad \mathcal{N}(G_x^{0^*}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

$$G_{xx} = \begin{pmatrix} (-6+25\lambda e^{-5u_1}, -1, -5e^{-5u_1}) & (-1,0,0) & (-5e^{-5u_1},0,0) \\ (0,3,0) & (3,0,0) & (0,0,0) \end{pmatrix}.$$

$$G_{xx}^0 = \begin{pmatrix} (9,-1,-5) & (-1,0,0) & (-5,0,0) \\ (0,3,0) & (3,0,0) & (0,0,0) \end{pmatrix}.$$

Thus

$$\psi^* G_{xx}^0 \phi_1\phi_1 = \psi^* \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi_1 = \psi^* \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0,$$

$$\psi^* G_{xx}^0 \phi_1\phi_2 = \psi^* \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi_2 = \psi^* \begin{pmatrix} -5 \\ 0 \end{pmatrix} = -5,$$

$$\psi^* G_{xx}^0 \phi_2\phi_2 = \psi^* \begin{pmatrix} 9 & -1 & -5 \\ 0 & 3 & 0 \end{pmatrix} \phi_2 = \psi^* \begin{pmatrix} 9 \\ 0 \end{pmatrix} = 9.$$

Therefore the ABE is $-10\alpha\beta + 9\beta^2 = 0$. The ABE has two linearly independent solutions, namely, $\binom{\alpha}{\beta} = \binom{1}{0}, \binom{9}{10}$. Thus the (non-normalized) directions of the two bifurcation branches at x_0 are

$$\dot{x}_0 = (1)\phi_1 + (0)\phi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{\lambda} \end{pmatrix},$$

and

$$\dot{x}_0 = (9)\phi_1 + (10)\phi_2 = \begin{pmatrix} 10 \\ 0 \\ 9 \end{pmatrix} = \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{\lambda} \end{pmatrix}.$$

8.2 A Nonlinear Eigenvalue Problem

8.2.1 Variation of parameters

$$v'(t) = A(t)\ v(t) + f(t) \quad \Rightarrow \quad v(t) = V(t)\ [v(0) + \int_0^t V(s)^{-1} f(s)\ ds],$$

where

$$\begin{aligned} V'(t) &= A(t)\ V(t), \\ V(0) &= I. \end{aligned}$$

Here $V(t)$ is the fundamental solution matrix. Apply this to $v'' + \lambda v = f$, rewritten as

$$\begin{cases} v_1' &= v_2, \\ v_2' &= -\lambda v_1 + f, \end{cases}$$

or

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Then

$$V(t) = \begin{pmatrix} \cos\sqrt{\lambda}t & \sin\sqrt{\lambda}t/\sqrt{\lambda} \\ -\sqrt{\lambda}\sin\sqrt{\lambda}t & \cos\sqrt{\lambda}t \end{pmatrix}, \qquad V^{-1}(t) = \begin{pmatrix} \cos\sqrt{\lambda}t & -\sin\sqrt{\lambda}t/\sqrt{\lambda} \\ \sqrt{\lambda}\sin\sqrt{\lambda}t & \cos\sqrt{\lambda}t \end{pmatrix},$$

$$V^{-1}(s)\ f(s) = \begin{pmatrix} -\sin\sqrt{\lambda}s\ f(s)/\sqrt{\lambda} \\ \cos\sqrt{\lambda}s\ f(s) \end{pmatrix},$$

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \cos\sqrt{\lambda}t & \sin\sqrt{\lambda}t/\sqrt{\lambda} \\ -\sqrt{\lambda}\sin\sqrt{\lambda}t & \cos\sqrt{\lambda}t \end{pmatrix} \left[\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} + \int_0^t \begin{pmatrix} -\sin\sqrt{\lambda}s\ f(s)/\sqrt{\lambda} \\ \cos\sqrt{\lambda}s\ f(s) \end{pmatrix} ds \right].$$

Hence

$$v_1(t) \;=\; \cos(\sqrt{\lambda}t)\, v_1(0) + \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} v_2(0) - \cos\sqrt{\lambda}t \int_0^t \frac{\sin\sqrt{\lambda}s\ f(s)}{\sqrt{\lambda}}\, ds + \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} \int_0^t \cos\sqrt{\lambda}s\, f(s)\, ds.$$

For the specific problem

$$\begin{cases} v'' + \lambda v = f, \\ \\ v(0) = v'(0) = 0, \end{cases}$$

we have

$$v(t) \;=\; v_1(t) \;=\; \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} \int_0^t \cos\sqrt{\lambda}s\ f(s)\ ds \;-\; \frac{\cos\sqrt{\lambda}t}{\sqrt{\lambda}} \int_0^t \sin\sqrt{\lambda}s\ f(s)\ ds.$$

8.2.2 Singular points

Consider the nonlinear boundary value problem

$$\begin{cases} u'' + \lambda(u + u^2) = 0, \\ u(0) \;=\; u(1) \;=\; 0, \end{cases}$$

which has $u(t) \equiv 0$ as solution for all λ. Equivalently, we want the solution

$$u \;=\; u(t; p, \lambda),$$

of the initial value problem

$$\begin{cases} u'' + \lambda(u + u^2) = 0, \\ u(0) = 0, \quad u'(0) = p, \end{cases}$$

that satisfies

$$G(p, \lambda) \;\equiv\; u(1; p, \lambda) \;=\; 0.$$

Here

$$G \;:\; \mathrm{R} \;\times\; \mathrm{R} \;\to\; \mathrm{R},$$

with $G(0, \lambda) = 0$ for all λ. Let

$$u_p(t; p, \lambda) = (\frac{du}{dp})(t; p, \lambda), \qquad G_p(p, \lambda) = u_p(1; p, \lambda), \text{ etc.}$$

Then u_p $(= u_p^0)$ satisfies

$$\begin{cases} u_p'' + \lambda(1 + 2u)u_p = 0, \\ u_p(0) = 0, \quad u_p'(0) = 1, \end{cases}$$

which, about $u \equiv 0$, gives

$$\begin{cases} u_p'' + \lambda u_p = 0, \\ u_p(0) = 0, \quad u_p'(0) = 1. \end{cases}$$

By the variation of parameters formula

$$u_p(t;p,\lambda) \;=\; \sin\sqrt{\lambda}t/\sqrt{\lambda} \qquad \lambda \geq 0, \quad \text{(independent of } p).$$

$$G_p(0,\lambda) \;=\; u_p(1;p,\lambda) \;=\; \sin(\sqrt{\lambda})/\sqrt{\lambda} \;=\; 0, \qquad \text{if } \lambda = \lambda_k \equiv (k\pi)^2.$$

(The λ_k are branch points.)

Next, u_λ satisfies

$$\begin{cases} u_\lambda'' \;+\; u \;+\; u^2 \;+\; \lambda(1+2u)u_\lambda = 0, \\ u_\lambda(0) = 0, \quad u_\lambda'(1) = 0, \end{cases}$$

which, about $u \equiv 0$, gives

$$\begin{cases} u_\lambda'' \;+\; \lambda u_\lambda = 0, \\ u_\lambda(0) = 0, \quad u_\lambda'(0) = 0. \end{cases}$$

from which,

$$u_\lambda(t;p,\lambda) \equiv 0, \qquad G_\lambda(0,\lambda) \;=\; u_\lambda(1;0,\lambda) \;=\; 0,$$

which holds, in particular, at $\lambda = \lambda_k = (k\pi)^2$.

Thus, so far we know :

$$G(0,\lambda) = 0 \qquad \text{for all } \lambda,$$

and if $\lambda = (k\pi)^2$ then, with $x \equiv (p,\lambda)$,

$$G_x^0 \;\equiv\; (\; G_p(0,\lambda) \mid G_\lambda(0,\lambda)\;) \;=\; (0 \mid 0),$$

$$\mathcal{N}(G_x^0) = \text{Span}\{\tbinom{1}{0},\tbinom{0}{1}\}, \quad (2D), \qquad \mathcal{N}(G_x^{0^*}) = \text{Span}\{(1)\}, \quad (1D).$$

Thus $p = 0$, $\lambda = \lambda_k = (k\pi)^2$, are simple singular points.

8.2.3 Construction of the ABE

Construct the ABE :

$$(\psi^* G_{xx}^0 \phi_1\phi_1)\alpha^2 + 2(\psi^* G_{xx}^0\phi_1\phi_2)\alpha\beta + (\psi^* G_{xx}^0\phi_2\phi_2)\beta^2 = 0,$$

where $\psi = 1$,

$$G_{xx}^0 = \Big(\; (G_{pp}^0 \mid G_{p\lambda}^0) \;\Big|\; (G_{\lambda p}^0 | G_{\lambda\lambda}^0)\;\Big),$$

$$G_{pp}^0 = G_{pp}(0,\lambda_k) = u_{pp}(1;0,\lambda_k), \quad \text{etc.}$$

If the ABE has 2 independent roots then the bifurcation directions are

$$\begin{pmatrix} \dot{p} \\ \dot{\lambda} \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \dot{p} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}.$$

Since $p \equiv 0$ is a solution branch, one direction is $\binom{\dot{p}}{\dot{\lambda}} = \binom{0}{1}$.

Now u_{pp} satisfies

$$\begin{cases} u''_{pp} + 2\lambda u_p^2 + \lambda(1+2u)u_{pp} = 0, \\ u_{pp}(0) = u'_{pp}(0) = 0, \end{cases}$$

which, about $u \equiv 0$, $\lambda = \lambda_k$, gives

$$\begin{cases} u''_{pp} + \lambda_k u_{pp} = -2\lambda_k u_p^2 = -2\sin^2\sqrt{\lambda}t, \\ u_{pp}(0) = u'_{pp}(0) = 0. \end{cases}$$

$$\begin{cases} u''_{pp} + (k\pi)^2 u_{pp} = -2\sin^2(k\pi t), \\ u_{pp}(0) = u'_{pp}(0) = 0. \end{cases}$$

By the variation of parameters formula

$$\begin{aligned} u_{pp}(t;p,\lambda) &= \frac{\sin k\pi t}{k\pi}\int_0^t \cos k\pi s(-2\sin^2 k\pi s)\,ds \;-\; \frac{\cos k\pi t}{k\pi}\int_0^t \sin k\pi s(-2\sin^2 k\pi s)\,ds \\ &= -\frac{2\sin k\pi t}{k\pi}\int_0^t \sin^2 k\pi s\;\cos k\pi s\,ds \;+\; \frac{2\cos k\pi t}{k\pi}\int_0^t \sin^3 k\pi s\,ds, \end{aligned}$$

and hence

$$G_{pp}(0,\lambda_k) = u_{pp}(1;0,\lambda_k) = \frac{2}{3(k\pi)^2}(1-(-1)^k) = \begin{cases} 0\,, & k \text{ even}, \\ \frac{4}{3(k\pi)^2}\,, & k \text{ odd}. \end{cases}$$

Next, $u_{p\lambda}$ satisfies

$$\begin{cases} u''_{p\lambda} + (1+2u)u_p + 2\lambda u_p u_\lambda + \lambda(1+2u)u_{p\lambda} = 0, \\ u_{p\lambda}(0) = u'_{p\lambda}(0) = 0. \end{cases}$$

which, about $u \equiv 0$, $\lambda = \lambda_k = (k\pi)^2$, with $u_\lambda \equiv 0$, gives

$$\begin{cases} u''_{p\lambda} + k^2\pi^2 u_{p\lambda} = -u_p = -\frac{\sin k\pi t}{k\pi}, \\ u_{p\lambda}(0) = u'_{p\lambda}(0) = 0. \end{cases}$$

Using the variation of parameters formula

$$\begin{aligned} u_{p\lambda}(t;p,\lambda) &= \frac{\sin k\pi t}{k\pi}\int_0^t \cos k\pi s(\frac{-\sin k\pi s}{k\pi})\,ds \;-\; \frac{\cos k\pi t}{k\pi}\int_0^t \sin k\pi s(\frac{-\sin k\pi s}{k\pi})\,ds \\ &= \frac{\sin k\pi t}{(k\pi)^2}\int_0^t \sin k\pi s\cos k\pi s\,ds \;+\; \frac{\cos k\pi t}{(k\pi)^2}\int_0^t \sin^2 k\pi s\,ds, \end{aligned}$$

and hence

$$G_{p\lambda}(0,\lambda_k) = u_{p\lambda}(1;0,\lambda_k) = \frac{1}{2(k\pi)^2}(-1)^k = \begin{cases} \frac{1}{2(k\pi)^2}, & k \text{ even}, \\ \frac{-1}{2(k\pi)^2}, & k \text{ odd}. \end{cases}$$

Finally $u_{\lambda\lambda}$ satisfies

$$\begin{cases} u''_{\lambda\lambda} \;+\; (1+2u)u_\lambda \;+\; (1+2u)u_\lambda \;+\; 2\lambda u_\lambda^2 \;+\; \lambda(1+2u)u_{\lambda\lambda} = 0, \\ u_{\lambda\lambda}(0) = u'_{\lambda\lambda}(0) = 0, \end{cases}$$

which, about $u=0$, $\lambda=\lambda_k=k^2\pi^2$, with $u_\lambda \equiv 0$, gives

$$\begin{cases} u''_{\lambda\lambda}+k^2\pi^2 u_{\lambda\lambda}=0, \\ u_{\lambda\lambda}(0)=u'_{\lambda\lambda}(0)=0, \end{cases}$$

so that

$$u_{\lambda\lambda}(t;p,\lambda)\equiv 0, \quad G_{\lambda\lambda}(0,\lambda_k) \;=\; u_{\lambda\lambda}(1;0,\lambda_k) \;=\; 0.$$

Hence

$$G^0_{xx}=\Big((G^0_{pp}|G^0_{p\lambda})\Big|(G^0_{\lambda p}|G^0_{\lambda\lambda})\Big)=\begin{cases}\Big((0|\frac{1}{2(k\pi)^2})\Big|(\frac{1}{2(k\pi)^2}|0)\Big), & k \text{ even}, \\ \\ \Big((\frac{4}{3(k\pi)^2}|\frac{-1}{2(k\pi)^2})\Big|(\frac{-1}{2(k\pi)^2}|0)\Big), & k \text{ odd}.\end{cases}$$

The coefficients of the ABE are

$$\psi^* G^0_{xx}\phi_1\phi_1=\begin{cases} 0, & k \text{ even}, \\ \\ \frac{4}{3(k\pi)^2}, & k \text{ odd},\end{cases} \qquad \psi^* G^0_{xx}\phi_1\phi_2=\begin{cases} \frac{1}{2(k\pi)^2}, & k \text{ even}, \\ \\ \frac{-1}{2(k\pi)^2}, & k \text{ odd},\end{cases}$$

$$\psi^* G^0_{xx}\phi_2\phi_2=\begin{cases} 0, & k \text{ even}, \\ \\ 0, & k \text{ odd}.\end{cases}$$

Thus the ABE is

$$\begin{cases} \alpha\beta=0, & k \text{ even}, & \text{roots} \;: (\alpha,\beta)=(1,0),\;(0,1), \\ \\ \frac{4}{3}\alpha^2-\frac{1}{2}\alpha\beta=0, & k \text{ odd}, & \text{roots} \;: (\alpha,\beta)=(0,1),\;(3,8). \end{cases}$$

The directions of the bifurcation branches are:

$$\dot{x} \;=\; \begin{pmatrix}\dot{p}\\ \dot{\lambda}\end{pmatrix} \;=\; \alpha\phi_1+\beta\phi_2,$$

where

$$\begin{pmatrix}\dot{p}\\ \dot{\lambda}\end{pmatrix}=\begin{cases} \binom{0}{1},\;\; \binom{1}{0}, & k \text{ even}, \;\; (\text{“pitch fork bifurcation”}), \\ \\ \binom{0}{1},\;\; \binom{3}{8}, & k \text{ odd}, \;\; (\text{“transcritical bifurcation”}). \end{cases}$$

EXERCISE. Check the above calculation (!). Use the AUTO demo bvp to compute some bifurcating branches. Do the numerical results support the analytical results? Also carry out the above analysis and AUTO computations for

$$\begin{cases} u''+\lambda(u+u^3)=0, \\ u(0)=u(1)=0. \end{cases}$$

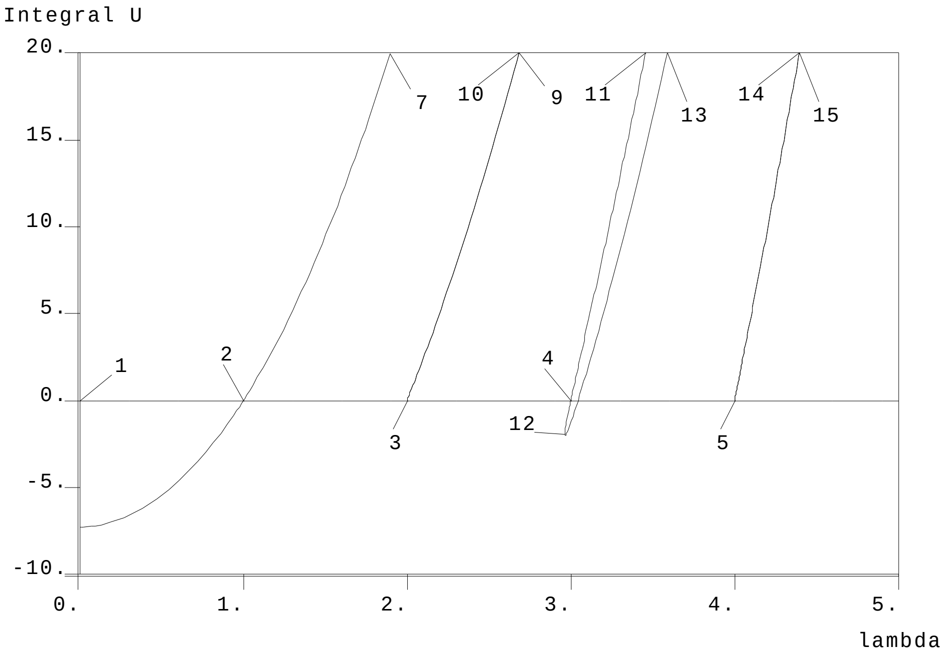

Figure 8.1: The first 4 bifurcating branches of the nonlinear eigenvalue problem.

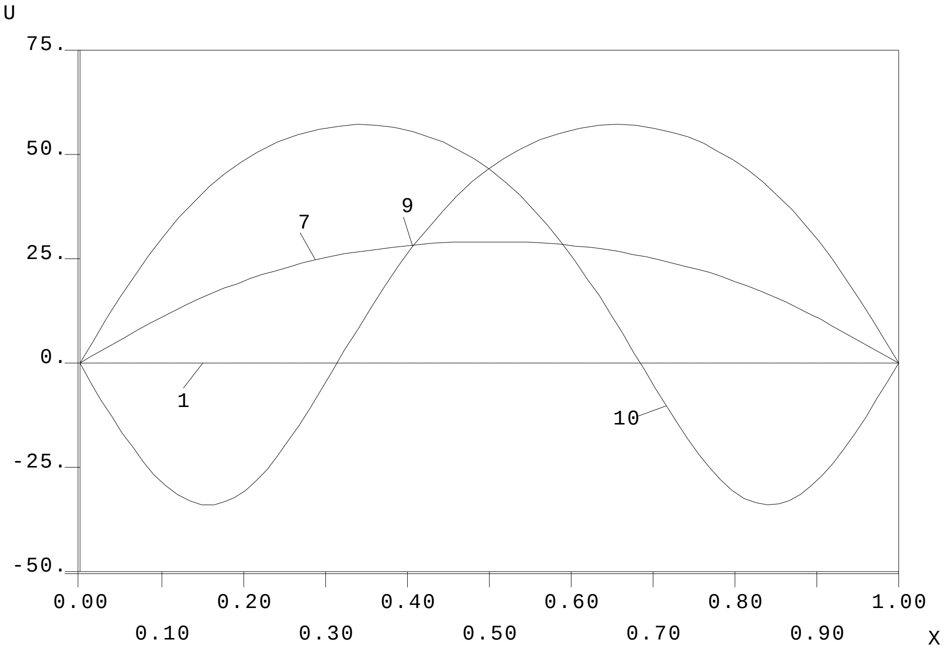

Figure 8.2: Some solutions of the nonlinear eigenvalue problem.

Chapter 9

Complex Bifurcations

9.1 Complexification of a Fold

9.1.1 An introductory example

Consider

$$G(u,\lambda)=0, \qquad \text{where } G(u,\lambda)\equiv u^2-\lambda,$$

Then $u=\pm\sqrt{\lambda}$ (see Figure 9.1). Let $x\equiv(u,\lambda)$ and $x_0=(0,0)$. Then $G(x_0)=0$ and

$$G'(x)\ \equiv\ G_x\ =\ (\ G_u\ ,\ G_\lambda\ )\ =\ (2u,-1), \qquad G'(x_0)\ \equiv\ G_x^0\ =\ (0,-1),$$
$$\mathcal{N}(G_x^0)\ =\ \mathrm{Span}\{\underbrace{\tbinom{1}{0}}_{\phi}\}, \qquad \mathcal{N}(G_x^{0^*})\ =\ \{0\}.$$

In the "uniform formulation", x_0 is not a singular point :

$$\left.\begin{aligned} G(x_0+\epsilon\phi+y)\ &=0,\\ y^*\phi\ &=0, \end{aligned}\right\}\ \sim\ F(y,\epsilon)=0, \quad F(0,0)=0,$$

$$F_y(y,\epsilon)=\begin{pmatrix} G_x(x_0+\epsilon\phi+y)\\ \phi^* \end{pmatrix},$$

$$F_y^0=F_y(0,0)=\begin{pmatrix} G_x^0\\ \phi^*\end{pmatrix}=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix} \text{ is nonsingular.}$$

Hence, by the IFT, there is a unique solution branch $y=y(\epsilon)$, with $y(0)=0$, near $\epsilon=0$.

In the "parameter formulation", x_0 is a fold (see also Section 4.1) :

$$G(u,\lambda)=u^2-\lambda,$$
$$G(0,0)=0, \qquad G_u^0\ =\ G_u(0,0)\ =\ 0, \qquad G_\lambda^0\ =\ -1,$$
$$\mathcal{N}(G_u^0)\ =\ \mathrm{Span}\{(1)\}, \qquad \mathcal{N}(G_u^{0^*})=\mathrm{Span}\{(1)\},$$
$$G_\lambda^0\ \notin\ \mathcal{R}(G_u^0).$$

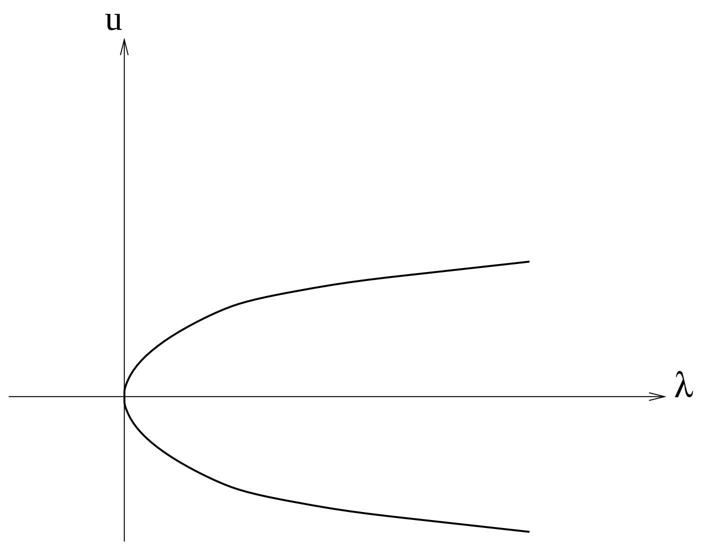

Figure 9.1: Solution curve of $u^2 - \lambda = 0$.

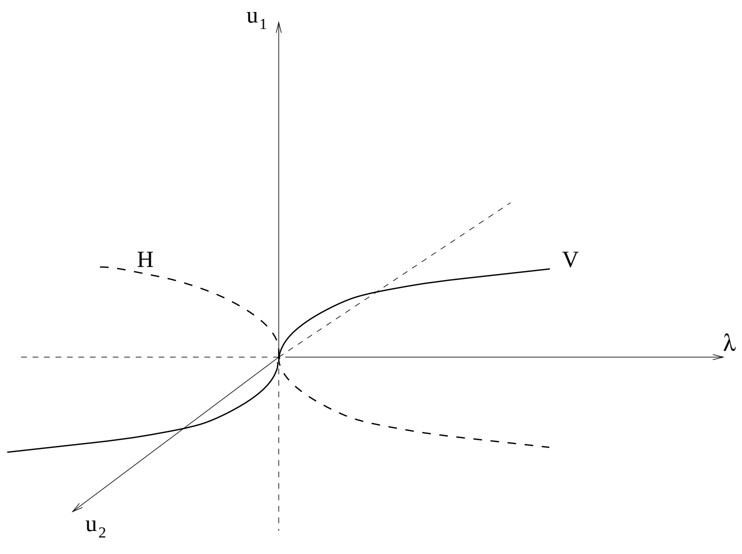

Figure 9.2: Solution curves of $u^2 - \lambda = 0$ if $u \in \mathcal{C}$.

Now consider the case where u is allowed to be complex (see Figure 9.2) :

$$u^2 - \lambda = 0, \qquad u \in \mathcal{C}, \lambda \in \mathrm{R}.$$

$$u = \begin{cases} \pm\sqrt{\lambda} & \text{if} \quad \lambda \geq 0, \\ \\ \pm\, i\, \sqrt{-\lambda} & \text{if} \quad \lambda \leq 0. \end{cases}$$

Equivalent real formulation :

$$u = u_1 + i\ u_2,$$

$$G(u,\lambda) \ = \ G(u_1 + i\ u_2, \lambda) \ = \ (u_1 + i\ u_2)^2 - \lambda \ = \ u_1^2 - u_2^2 - \lambda + 2i\ u_1 u_2.$$

$$G(u,\lambda) = 0 \ \Rightarrow \ \begin{cases} u_1^2 - u_2^2 - \lambda = 0, \\ \\ u_1\ u_2 = 0, \end{cases}$$

with solution

$$\begin{cases} u_2 = 0, & u_1^2 - \lambda = 0, \\ \\ u_1 = 0, & u_2^2 + \lambda = 0. \end{cases}$$

Bifurcation analysis:

$$u \equiv (u_1, u_2), \quad x \equiv (u, \lambda), \quad G(x) = \begin{cases} u_1^2 - u_2^2 - \lambda, \\ u_1\ u_2, \end{cases}$$

with $\qquad G(0) = 0, \qquad G_x(x) \ = \ (\ G_{u_1} \mid G_{u_2} \mid G_\lambda\ ),$

$$G_x(x) \ = \ \begin{pmatrix} 2u_1 & -2u_2 & -1 \\ u_2 & u_1 & 0 \end{pmatrix}, \quad G_x^0 \ \equiv \ G_x(0) \ = \ \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{N}(G_x^0) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{N}(G_x^{0*}) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Thus $x_0 = (0,0)$ is a simple singular point.

$$G_{xx}(x) \ = \ \Big(\ (G_{u_1u_1}|G_{u_1u_2}|G_{u_1\lambda}) \ \Big|\ (G_{u_2u_1}|G_{u_2u_2}|G_{u_2\lambda}) \ \Big|\ (G_{\lambda u_1}|G_{\lambda u_2}|G_{\lambda\lambda})\ \Big)$$

$$= \left(\begin{matrix} (2 & 0 & 0) \\ (0 & 1 & 0) \end{matrix} \ \middle|\ \begin{matrix} (0 & -2 & 0) \\ (1 & 0 & 0) \end{matrix} \ \middle|\ \begin{matrix} (0 & 0 & 0) \\ (0 & 0 & 0) \end{matrix} \right) \ = \ G_{xx}^0\ ,$$

The ABE is

$$\psi^*(G_{xx}^0\phi_1\phi_1)\ \epsilon_1^2 \ + \ 2(\psi^* G_{xx}^0 \phi_1\phi_2)\ \epsilon_1\epsilon_2 \ + \ (\psi^* G_{xx}^0\phi_2\phi_2)\ \epsilon_2^2 = 0,$$

which gives $\epsilon_1\epsilon_2 = 0$, with roots, $(\epsilon_1, \epsilon_2) = (1,0),\ (0,1)$. Thus, as shown in see Figure 9.3, the linear approximations to the bifurcation branches are

$$\begin{pmatrix} u_1 \\ u_2 \\ \lambda \end{pmatrix} = \epsilon \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} u_1 \\ u_2 \\ \lambda \end{pmatrix} = \epsilon \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

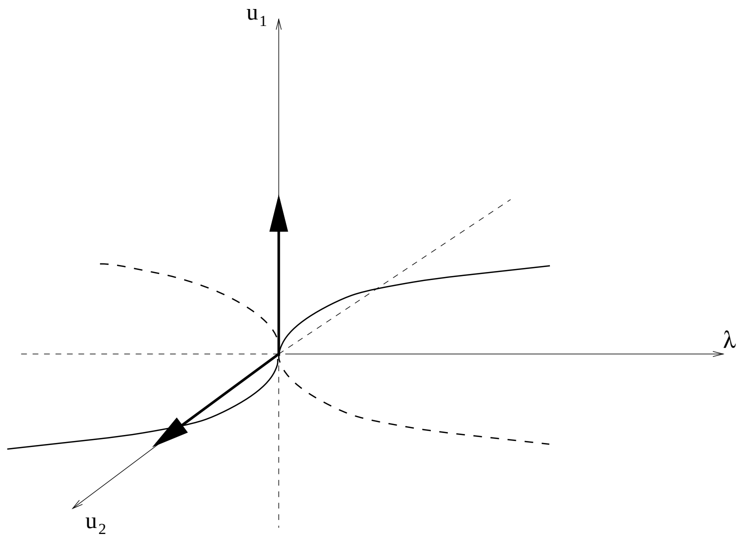

Figure 9.3: Solution curves of $u^2 - \lambda = 0$ if $u \in \mathcal{C}$, with bifurcation direction vectors.

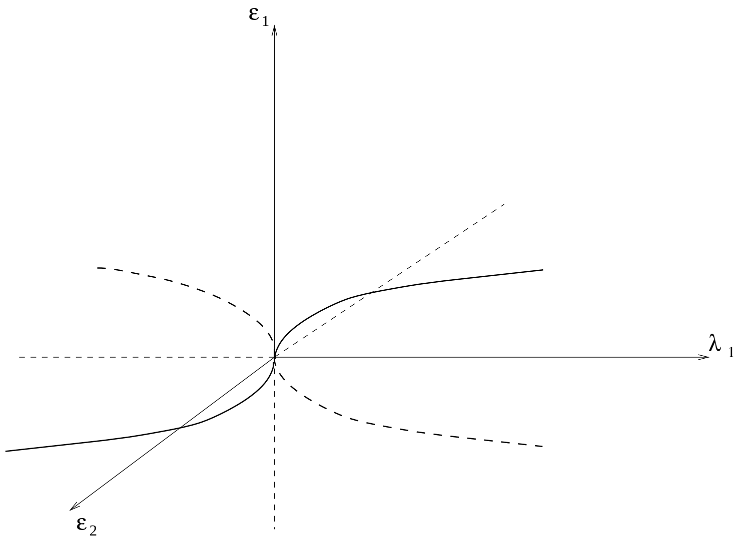

Figure 9.4: Representation of the solution curves of $G(u, \lambda) = 0$ if $u \in \mathcal{C}$. (Case $\lambda_0'' < 0$.)

9.1.2 The general case

Now consider the general case :

$$G \;:\; \mathcal{B}_1 \;\times\; \mathrm{R} \;\to\; \mathcal{B}_2, \qquad G(u_0,\lambda_0) = 0,$$

with

$$\mathcal{N}(G_u^0) = \mathrm{Span}\{\phi\}, \qquad \mathcal{N}(G_u^{0^*}) = \mathrm{Span}\{\psi\}, \quad G_\lambda^0 \notin \mathcal{R}(G_u^0).$$

Also assume the fold is quadratic,

$$\psi^* G_{uu}^0 \phi\phi \;\neq\; 0.$$

Complexification:

$$G \;:\; \underbrace{\mathcal{B}_1 \;\times\; i\mathcal{B}_1}_{\equiv \mathcal{C}_1} \times \mathcal{C} \;\to\; \underbrace{\mathcal{B}_2 \;\times\; i\mathcal{B}_2}_{\equiv \mathcal{C}_2},$$

where, to simplify the derivation, we also allow $\lambda \in \mathcal{C}$. Later we shall restrict to $\lambda \in \mathrm{R}$.

The derivation of the solution behavior near (u_0, λ_0) formally proceeds as in the real case (see Section 6.1.2), although we shall now find two distinct branches :

$$\begin{cases} u = u_0 \;+\; \epsilon\,\phi \;+\; v, \qquad \epsilon \in \mathcal{C}, \\ \qquad\qquad v^*\,\phi = 0. \end{cases}$$

Get

$$\begin{cases} u \;=\; u(\epsilon) \;=\; u_0 \;+\; \epsilon\phi \;+\; \cdots\;, \\ \lambda \;=\; \lambda(\epsilon) \;=\; \lambda_0 \;-\; \frac{1}{2}\,\epsilon^2\lambda_0'' \;+\; \cdots, \end{cases}$$

where $\lambda_0'' = \psi^* G_{uu}^0 \phi\phi / \psi^* G_\lambda^0$ is real and nonzero. To second order

$$\lambda = \lambda_0 - \frac{1}{2}\,\epsilon^2\lambda_0'', \quad \lambda = \lambda_1 + i\lambda_2, \quad \epsilon = \epsilon_1 + i\epsilon_2, \quad \lambda_0 \in \mathrm{R}.$$

$$\Rightarrow\; \lambda_1 + i\lambda_2 \;=\; \lambda_0 - \frac{1}{2}(\epsilon_1^2 - \epsilon_2^2 + 2i\epsilon_1\epsilon_2)\;\lambda_0'',$$

$$\Rightarrow\; \lambda_1 - \lambda_0 + \frac{1}{2}(\epsilon_1^2 - \epsilon_2^2)\lambda_0'' + i(\lambda_2 + \epsilon_1\epsilon_2\lambda_0'') = 0,$$

$$\Rightarrow\; \begin{cases} \lambda_1 - \lambda_0 + \frac{1}{2}(\epsilon_1^2 - \epsilon_2^2)\lambda_0'' = 0, \\ \\ \lambda_2 + \epsilon_1\epsilon_2\lambda_0'' = 0. \end{cases}$$

Now restrict again to $\lambda \in \mathrm{R}$, i.e. $\lambda_2 = 0$:

$$\begin{cases} \lambda_1 - \lambda_0 + \frac{1}{2}(\epsilon_1^2 - \epsilon_2^2)\lambda_0'' = 0, \\ \\ \qquad \epsilon_1\;\epsilon_2 = 0, \end{cases} \quad\Rightarrow\quad \begin{cases} \epsilon_1 = 0 \;:\; \lambda_1 = \lambda_0 + \frac{1}{2}\;\epsilon_2^2\lambda_0'', \\ \\ \epsilon_2 = 0 \;:\; \lambda_1 = \lambda_0 - \frac{1}{2}\;\epsilon_1^2\lambda_0'', \end{cases}$$

(see Figure 9.4) and, to lowest order,

$$u = u_0 + \epsilon\phi = u_0 + (\epsilon_1 + i\epsilon_2)\underbrace{\phi}_{\in \mathrm{R}} \;=\; \begin{cases} u_0 + i\epsilon_2\phi, & \text{if } \epsilon_1 = 0, \\ \\ u_0 + \epsilon_1\phi, & \text{if } \epsilon_2 = 0. \end{cases}$$

EXERCISE. Compute the branch of complex solutions that bifurcates from the fold in the complexified Bratu problem. (AUTO demo ezp.)

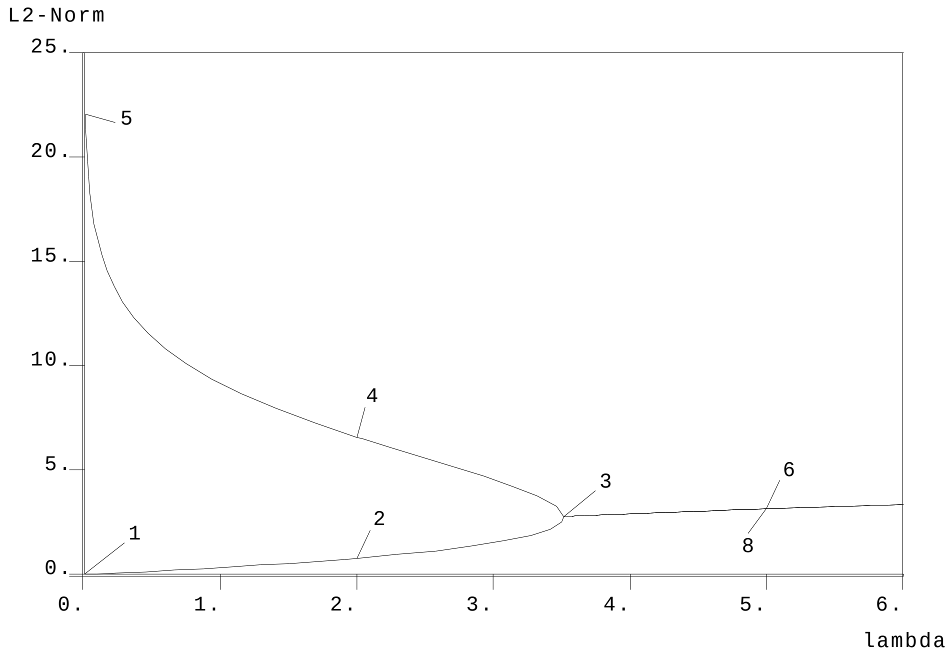

Figure 9.5: The bifurcation diagram of the complex Bratu equation.

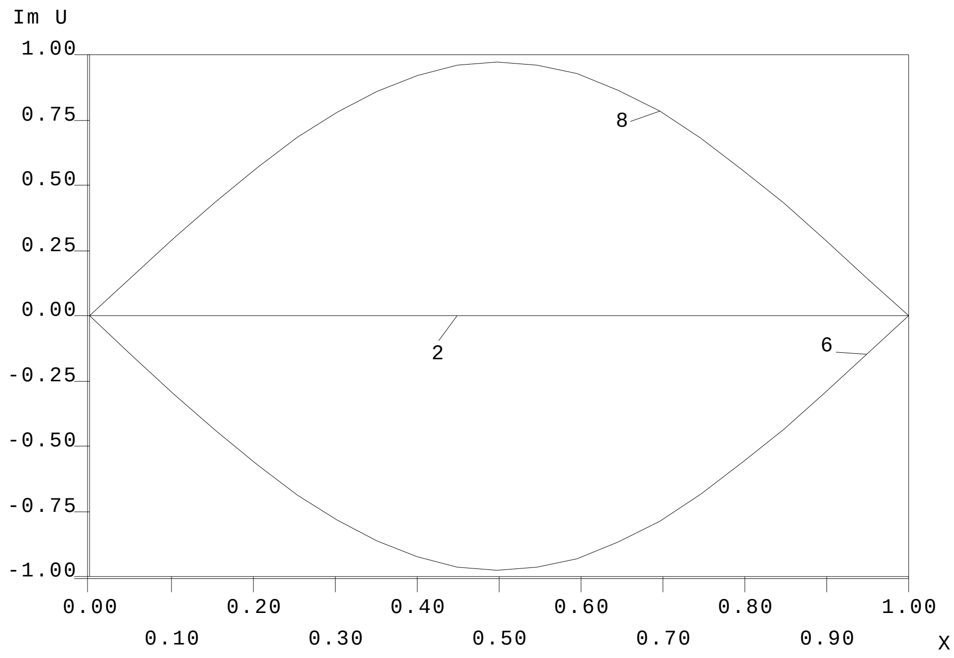

Figure 9.6: The imaginary part of some solutions to the complex Bratu equation.

9.2 Complexification of a Simple Bifurcation

Consider

$$G \ : \ \mathcal{B}_1 \ \times \ \mathrm{R} \ \rightarrow \ \mathcal{B}_2, \qquad \mathcal{B}_1, \mathcal{B}_2 \ \text{ real Banach spaces,}$$

with $x \equiv (u, \lambda)$, $G(\underbrace{u_0, \lambda_0}_{x_0}) = 0$, $\mathcal{N}(G_x^0) = \mathrm{Span}\{\phi_1, \phi_2\}$, $\mathcal{N}(G_x^{0^*}) = \mathrm{Span}\{\psi\}$.

Complexify :

$$G \ : \ \underbrace{\mathcal{B}_1 \ \times \ i\mathcal{B}_1}_{C_1} \times \mathcal{C} \ \rightarrow \ \underbrace{\mathcal{B}_2 \ \times \ i\mathcal{B}_2}_{C_2} \ .$$

As in the fold case, we let $\lambda \in \mathcal{C}$ to simplify the derivation. Later we restrict to $\lambda \in \mathrm{R}$. Write

$$x = x_0 + \epsilon_1 \phi_1 + \epsilon_2 \phi_2 + y,$$

$$\phi_i^* \ y \ = \ 0 \ \in \ \mathcal{C}, \qquad i = 1, 2,$$

with x, ϵ_1, ϵ_2, y complex, and x_0, ϕ_0, ϕ_1, real. Consider

$$\left.\begin{array}{c} G(x_0 + \epsilon_1\phi_1 + \epsilon_2\phi_2 + y) \ + \ \mu \ \psi = 0, \\ \phi_i^* \ y \ = \ 0, \qquad i = 1, 2, \end{array}\right\} \sim \ F(y, \mu; \epsilon_1, \epsilon_2) = 0,$$

with $F(0, 0; 0, 0) = 0$.

Precisely as in the real case, one finds that $(\ F_y^0 \mid F_\mu^0 \ )$ is nonsingular, so that by the IFT

$$y = y(\epsilon_1, \epsilon_2), \quad \mu = \mu(\epsilon_1, \epsilon_2), \quad \text{ for } \quad |\epsilon_i| \ \text{ small,}$$

with $y(0, 0) = 0, \quad \mu(0, 0) = 0$.

As in Section 6.1.3, we have to lowest order,

$$\mu(\epsilon_1, \epsilon_2) = -\frac{1}{2} \underbrace{(\psi^* G_{xx}^0 \phi_1 \phi_1)}_{c_{11} \in \mathrm{R}} \epsilon_1^2 - \underbrace{(\psi^* G_{xx}^0 \phi_1 \phi_2)}_{c_{12} \in \mathrm{R}} \epsilon_1 \epsilon_2 - \frac{1}{2} \underbrace{(\psi^* G_{xx}^0 \phi_2 \phi_2)}_{c_{22} \in \mathrm{R}} \epsilon_2^2,$$

Assume that $\Delta = c_{12}^2 - c_{11} c_{22} > 0$, and write

$$\epsilon_1 = r_1 + i \ s_1, \qquad \epsilon_2 = r_2 + i \ s_2, \qquad \mu = \mu_1 + i \mu_2.$$

Then

$$\mu_1 + i\mu_2 \ = \ -\frac{1}{2}[c_{11}(r_1 + i \ s_1)^2 \ + \ 2c_{12}(r_1 + i \ s_1)(r_2 + i \ s_2) \ + \ c_{22}(r_2 + i \ s_2)^2]$$

$$= \ -\frac{1}{2}[c_{11}(r_1^2 - s_1^2) + 2c_{12}(r_1 r_2 - s_1 s_2) + c_{22}(r_2^2 - s_2^2)] - \ \frac{i}{2} \ [2c_{11} r_1 s_1 + 2c_{12}(r_1 s_2 + r_2 s_1) + 2c_{22} r_2 s_2].$$

$$\Rightarrow \ \left\{ \begin{array}{l} \mu_1 = -\frac{1}{2}\Big[(c_{11} r_1^2 + 2c_{12} r_1 r_2 + c_{22} r_2^2) - (c_{11} s_1^2 + 2c_{12} s_1 s_2 + c_{22} s_2^2)\Big], \\ \mu_2 = -\Big[c_{11} r_1 s_1 + c_{12}(r_1 s_2 + r_2 s_1) + c_{22} r_2 s_2\Big]. \end{array} \right. \tag{9.1}$$

The ABE are obtained by setting

$$\mu_1 \ = \ 0, \qquad \mu_2 \ = \ 0,$$

which, in the real case, $s_1 = s_2 = 0$, reduce to the previous

$$c_{11} r_1^2 + 2c_{12} r_1 r_2 + c_{22} r_2^2 = 0.$$

9.2.1 An elementary example

Consider

$$G(u,\lambda) = 0, \qquad \text{where} \qquad G(u,\lambda) \equiv u^2 - \lambda^2,$$

Then $u = \pm\lambda$ (see Figure 9.7). Let $x = (u,\lambda)$, $x_0 = (u_0,\lambda_0) = (0,0)$. We have

$$G_x = (2u_1, -2\lambda), \quad G_x^0 \equiv G_x(0,0) = (0,0), \quad G_{xx} = \big((2,0),(0,-2)\big) = G^0_{xx},$$

and

$$\mathcal{N}(G_x^0) = \text{Span}\{\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\phi_1}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\phi_2}\}, \qquad \mathcal{N}(G_x^{0^*}) = \text{Span}\{\underbrace{(1)}_{\psi}\}.$$

$$\Rightarrow \begin{cases} c_{11} &= \psi^* G^0_{xx}\phi_1\phi_1 &= (2,0)\begin{pmatrix}1\\0\end{pmatrix} &= 2, \\ c_{12} &= \psi^* G^0_{xx}\phi_1\phi_2 &= (2,0)\begin{pmatrix}0\\1\end{pmatrix} &= 0, \\ c_{22} &= \psi^* G^0_{xx}\phi_2\phi_2 &= (0,-2)\begin{pmatrix}0\\1\end{pmatrix} &= -2. \end{cases}$$

From (9.1), the ABE are

$$\begin{cases} (r_1^2 - r_2^2) - (s_1^2 - s_2^2) &= 0, \\ \\ r_1 s_1 - r_2 s_2 &= 0, \end{cases}$$

and we have, to first order,

$$x = \underbrace{x_0}_{=0} + \epsilon_1\phi_1 + \epsilon_2\phi_2,$$

$$\Rightarrow \begin{pmatrix}u\\ \lambda\end{pmatrix} = (r_1 + i\, s_1)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + (r_2 + i\, s_2)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}r_1 + i\, s_1\\ r_2 + i\, s_2\end{smallmatrix}\right).$$

Now restrict again to real λ. Thus $s_2 = 0$, and

$$\begin{pmatrix}u\\ \lambda\end{pmatrix} = \begin{pmatrix}r_1 + is_1\\ r_2\end{pmatrix}.$$

The reduced ABE are

$$\begin{cases} r_1^2 - r_2^2 - s_1^2 = 0, \\ r_1 s_1 = 0, \end{cases} \quad \Rightarrow \quad \begin{cases} r_1 = 0 \; : \; r_2^2 + s_1^2 &= 0 \;\Rightarrow\; r_2 = s_1 = 0, \\ s_1 = 0 \; : \; r_1^2 - r_2^2 &= 0. \end{cases}$$

Now consider real perturbations,

$$G(\overbrace{u,\lambda}^{x}) + \mu_1\xi, \qquad \mu_1 \in \mathrm{R}, \quad \xi \notin \mathcal{R}(G_x^0), \quad \xi \in \mathrm{R}.$$

Again restrict to real λ, so that $s_2 = 0$. The level set for $\mu = \mu_1$, which is shown in Figures 9.9 and 9.10, satisfies

$$\begin{cases} \mu = -(r_1^2 - r_2^2) + s_1^2, \\ r_1 s_1 = 0, \end{cases} \quad \Rightarrow \quad \begin{cases} r_1 = 0, \quad & r_2^2 + s_1^2 \;=\; \mu_1, \\ s_1 = 0, \quad & r_1^2 - r_2^2 \;=\; -\mu_1. \end{cases}$$

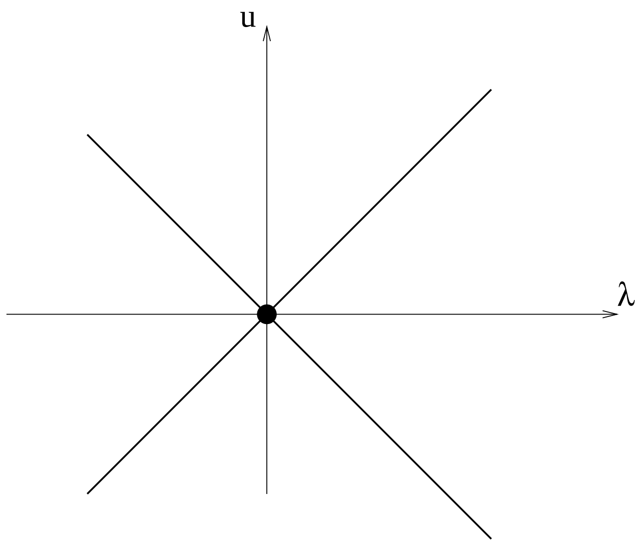

Figure 9.7: Solution curves of $u^2 - \lambda^2 = 0$.

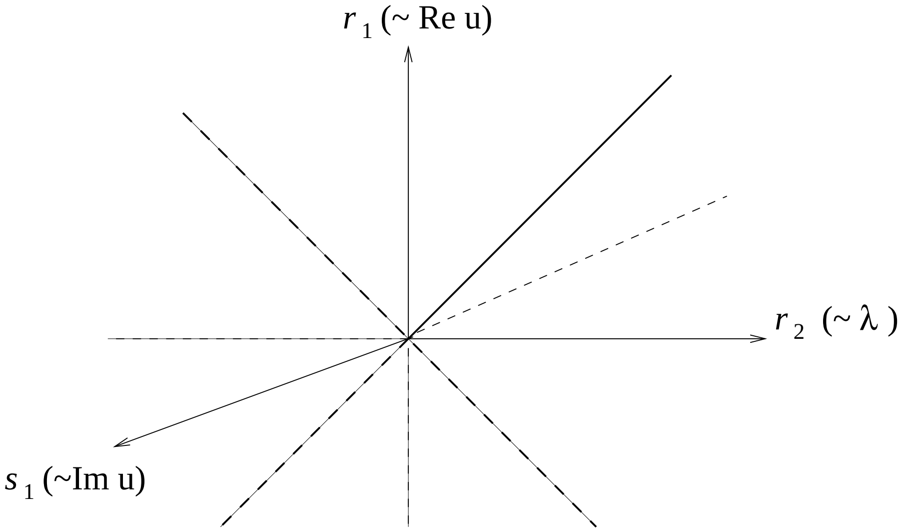

Figure 9.8: Solution curves of $u^2 - \lambda^2 = 0$ in r_1, r_2, s_1 space. Note that $s_1 = 0$ along the solution curves, i.e., there are no complex solutions.

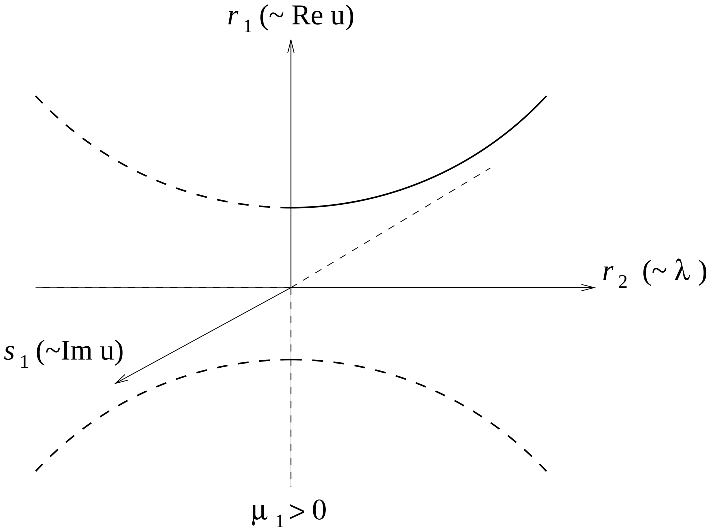

Figure 9.9: Solution curves of the perturbed problem if $\mu_1 < 0$. There are no complex solutions.

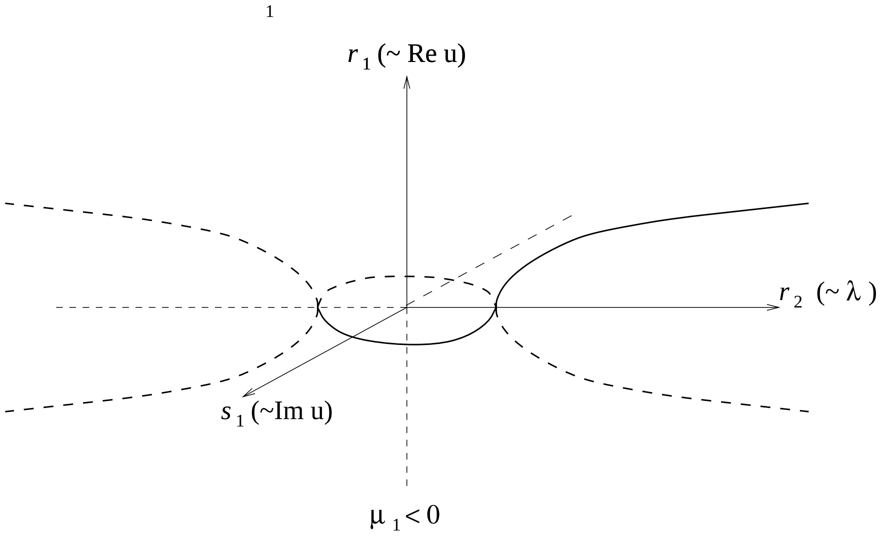

Figure 9.10: Solution curves of the perturbed problem if $\mu_1 > 0$. There is a complex solution branch (a loop) that connects the two real branches.

9.2.2 A more general example

Suppose that for $G(u_0, \lambda_0) = 0$,

$$\mathcal{N}(G_u^0) = \text{Span}\{\hat{\phi}_1\}, \qquad \mathcal{N}(G_u^{0^*}) = \text{Span}\{\psi\}, \qquad G_\lambda^0 \in \mathcal{R}(G_u^0),$$

so that, with $G_x^0 = (G_u^0)|G_\lambda^0)$,

$$\mathcal{N}(G_x^0) = \text{Span}\{\underbrace{\left(\begin{smallmatrix}\hat{\phi}_1\\0\end{smallmatrix}\right)}_{\equiv\phi_1}, \underbrace{\left(\begin{smallmatrix}\hat{\phi}_2\\1\end{smallmatrix}\right)}_{\equiv\phi_2}\}, \qquad \mathcal{N}(G_x^{0^*}) = \text{Span}\{\psi\},$$

where

$$\hat{\phi}_2^* \ \hat{\phi}_1 = 0, \qquad G_u^0 \ \hat{\phi}_2 = -G_\lambda^0.$$

The real ABE is

$$\underbrace{(\psi_1^* G_{xx}^0 \phi_1 \phi_2)}_{c_{11}} \ \epsilon_1^2 \ + \ 2\underbrace{(\psi^* F_{xx}^0 \phi_1 \phi_2)}_{c_{12}} \ \epsilon_1 \epsilon_2 \ + \ \underbrace{(\psi^* F_{xx}^0 \phi_2 \phi_2)}_{c_{22}} \ \epsilon_2^2 \ = \ 0.$$

Assume that

$$\Delta \equiv c_{12}^2 - c_{11} c_{22} > 0.$$

Also assume that $c_{11} \neq 0$, so that $(\epsilon_1, \epsilon_2) = (1, 0)$ is not a root.

Then

$$\begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} u_0 \\ \lambda_0 \end{pmatrix} + \epsilon_1 \begin{pmatrix} \hat{\phi}_1 \\ 0 \end{pmatrix} + \epsilon_2 \begin{pmatrix} \hat{\phi}_2 \\ 1 \end{pmatrix} + \ \cdots \ ,$$

with, for both branches, $\dot{\lambda}_0 \neq 0$ at (u_0, λ_0), i.e., the branches pass nonvertically through (u_0, λ_0) (see Figure 9.11).

Now allow u to be complex. Let λ be real, so that $s_2 = 0$ in $\epsilon_2 = r_2 + i s_2$. Consider real perturbation only, so that $\mu_2 = 0$ in $\mu = \mu_1 + i\mu_2$. From Equation 9.1, the level sets are

$$\begin{cases} c_{11} r_1^2 + 2c_{12} r_1 r_2 + c_{22} r_2^2 - c_{11} s_1^2 \ = \ \mu_1, \\ \\ c_{11} r_1 s_1 + c_{12} r_2 s_1 \ = \ 0, \end{cases}$$

$$\Rightarrow \begin{cases} s_1 = 0 & \Rightarrow \ c_{11} r_1^2 + 2c_{12} r_1 r_2 + c_{22} r_2^2 = \mu_1, \quad \text{(Level set of the real problem.)} \\ \\ r_1 = -\frac{c_{12}}{c_{11}} r_2 & \Rightarrow \ c_{11}(\frac{c_{12}}{c_{11}} r_2)^2 + 2c_{12}(-\frac{c_{12}}{c_{11}} r_2) r_2 + c_{22} r_2^2 - c_{11} s_1^2 = \mu_1, \end{cases}$$

$$\Rightarrow \begin{cases} \underbrace{c_{11}}_{\neq 0} r_1^2 + 2c_{12} r_1 r_2 + c_{22} r_2 r_2 \ = \ \mu_1, \\ \\ \underbrace{(c_{12}^2 - c_{11} c_{22})}_{>0} r_2^2 + c_{11}^2 s_1^2 \ = \ -c_{11} \mu_1. \end{cases}$$

See Figure 9.12 for a representation of these level curves. We also have

$$\begin{pmatrix} u \\ \lambda \end{pmatrix} \ = \ \begin{pmatrix} u_0 \\ \lambda_0 \end{pmatrix} + \epsilon_1 \begin{pmatrix} \hat{\phi}_1 \\ 0 \end{pmatrix} + \epsilon_2 \begin{pmatrix} \hat{\phi}_2 \\ 1 \end{pmatrix} \ = \ \begin{pmatrix} u_0 + (r_1 + i \ s_1)\hat{\phi}_1 + r_2 \hat{\phi}_2 \\ \lambda_0 + r_2 \end{pmatrix}.$$

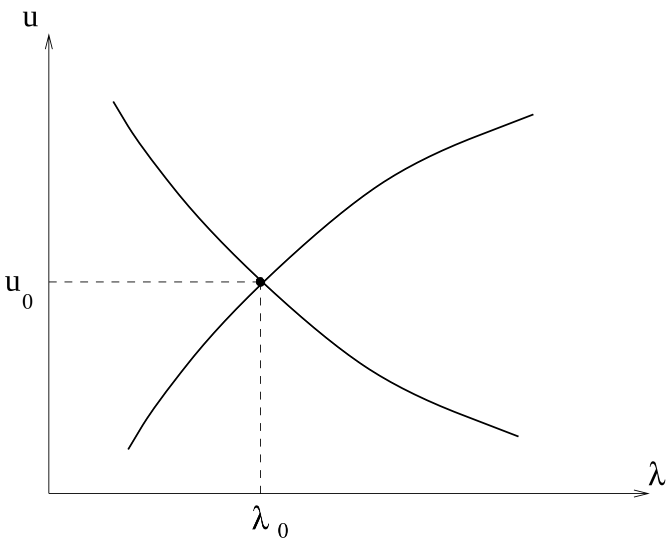

Figure 9.11: Solution curves of $G(u, \lambda) = 0$, with bifurcation point (u_0, λ_0).

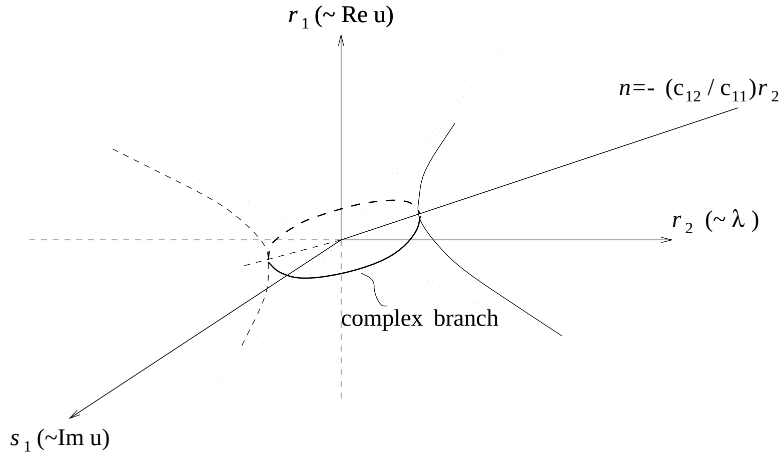

Figure 9.12: Solution curves of the perturbed problem.

Chapter 10

Computation of Hopf Bifurcations

10.1 A Linear Example

Consider the linear problem :

$$\begin{cases} u_1' = -\lambda u_1 - u_2, \\ u_2' = u_1, \end{cases} \tag{10.1}$$

which can also be written as

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} -\lambda & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

which is of the form

$$u'(t) \;=\; \underbrace{A(\lambda)\; u(t)}_{G(u,\lambda)}.$$

with stationary solutions, $u_1 = u_2 = 0$, for all λ. The Jacobian matrix

$$G_u(u,\lambda) \;=\; A(\lambda) \;=\; G_u(0,\lambda),$$

has eigenvalues

$$\mu_{1,2} \;=\; (\lambda \pm \sqrt{\lambda^2 - 4})/2.$$

Assume that $-2 < \lambda < 2$. Consider the initial value problem with $u(0) = p$, i.e.,

$$u_1(0) = p_1, \qquad u_2(0) = p_2.$$

Then

$$u(t) \;=\; e^{tA(\lambda)}\, p \;=\; V(\lambda)\; e^{t\Lambda(\lambda)}\; V^{-1}(\lambda)\; p \;=\; V(\lambda) \begin{pmatrix} e^{t\mu_1(\lambda)} & 0 \\ 0 & e^{t\mu_2(\lambda)} \end{pmatrix} V^{-1}(\lambda)\; p,$$

where

$$A(\lambda)\; V(\lambda) \;=\; V(\lambda)\; \Lambda(\lambda), \qquad A(\lambda) \;=\; V(\lambda)\; \Lambda(\lambda)\; V^{-1}(\lambda).$$

$$\Lambda(\lambda) \;=\; \begin{pmatrix} \mu_1(\lambda) & 0 \\ 0 & \mu_2(\lambda) \end{pmatrix}, \qquad V(\lambda) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

Thus $u(t) \to 0$ if $\lambda < 0$, and $u(t) \to \infty$ if $\lambda > 0$, i.e., the stationary solution is stable/unstable if λ is negative/positive.

However, if $\lambda = 0$, then

$$A_0 \equiv A(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mu_1 = i,\ \mu_2 = -i, \quad V_0 \equiv V(0) \ = \ \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad V_0^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

$$\begin{aligned} u(t) \ &= \ V_0 \, e^{t\Lambda} \, V_0^{-1} \, p \ = \ \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ &= \ \frac{1}{2}\begin{pmatrix} e^{it} + e^{-it} & i({}^{it} - e^{-it}) \\ -i(e^{it} - e^{-it}) & e^{it} + e^{-it} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

which is periodic, with period 2π, for any p_1, p_2. We can fix the phase by setting, for example, $p_2 = 0$. This leaves a one-parameter family of periodic solutions. (See Figures 10.1 and 10.2.). For nonlinear problems the branch of periodic solutions is generally not "vertical".

EXERCISE. Use AUTO to compute stationary solutions, Hopf bifurcations, and periodic solutions to the equation in this section, as well as for the following equations.

$$\begin{cases} u_1' = -\lambda u_1 - u_2, \\ u_2' = u_1(1 - u_1), \end{cases} , \tag{10.2}$$

$$\begin{cases} u_1' = -\lambda u_1 - u_2, \\ u_2' = u_1(1 - u_1^2). \end{cases} \tag{10.3}$$

Draw phase diagrams for $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, and determine the exact periodic solutions. Hint : see the pendulum example in Section 2.4.

10.2 Hopf Bifurcation

THEOREM.

Suppose that a complex conjugate pair of eigenvalues

$$\alpha(s) \pm i\beta(s),$$

of $f_u(u(\lambda), \lambda)$ crosses the imaginary axis transversally, i.e., for some λ_0 we have

$$\alpha(\lambda_0) = 0, \qquad \beta(\lambda_0) \neq 0, \qquad \text{and} \qquad \dot{\alpha}(\lambda_0) \neq 0.$$

Also assume that there are no other eigenvalues on the imaginary axis. Then there is a *Hopf bifurcation* from a stationary solution branch $(u(\lambda), \lambda)$ of

$$u' = f(u, \lambda),$$

i.e., a branch of periodic solutions bifucates from the stationary solution at (u_0, λ_0). •

Note that the assumption also imply that f_u^0 is nonsingular, so that stationary solution branch can be parametrized locally in λ.

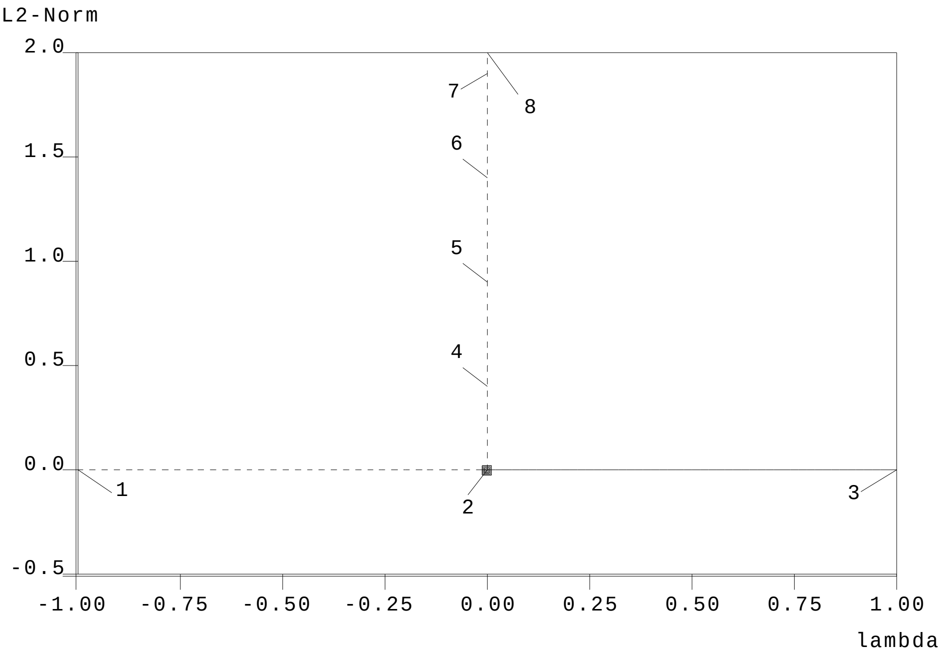

Figure 10.1: Bifurcation diagram of the linear Hopf bifurcation problem.

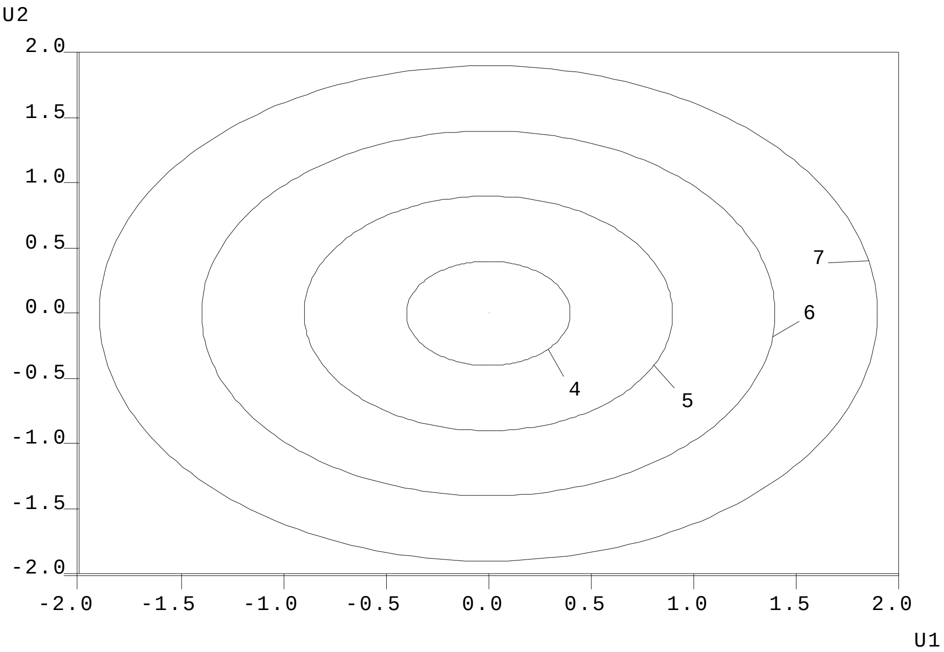

Figure 10.2: A phase plot of some periodic solutions.

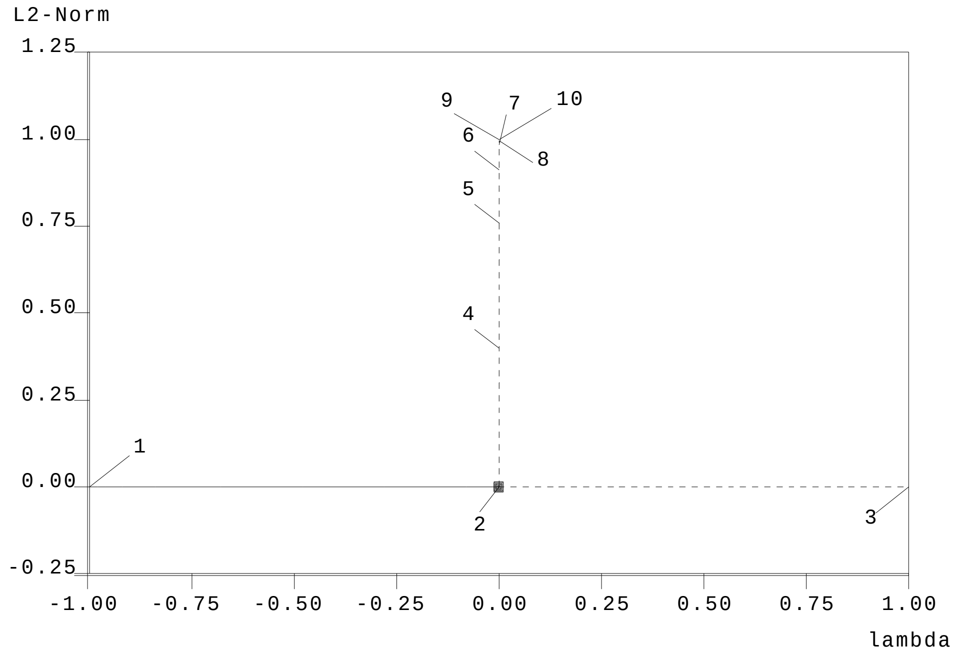

Figure 10.3: Bifurcation diagram for Equation (10.2).

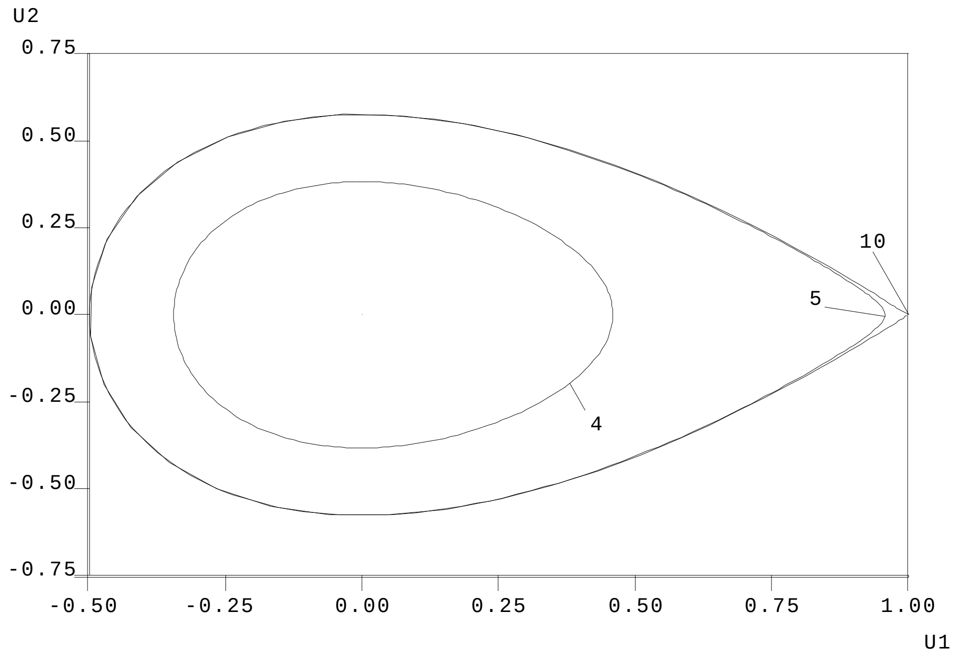

Figure 10.4: A phase plot of some periodic solutions to Equation (10.2).

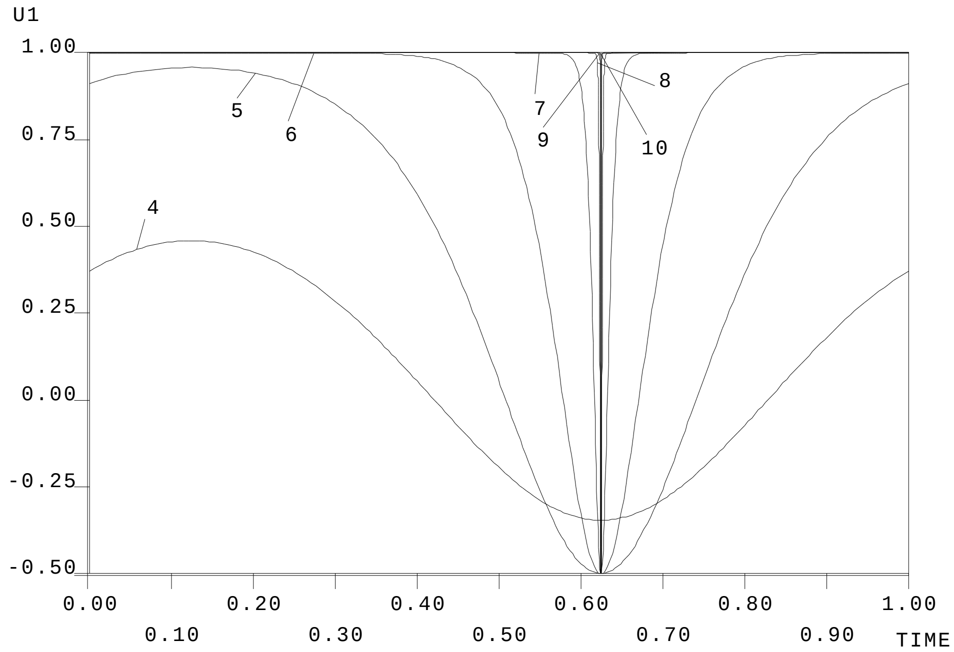

Figure 10.5: Solution component u_1 as a function of the scaled time variable t for Equation (10.2).

10.3 Transversality

THEOREM. The eigenvalue crossing in the Hopf Bifurcation Theorem is transversal if

$$\mathrm{Re}(\psi_0^*[f_{uu}^0(f_u^0)^{-1}f_\lambda^0 - f_{u\lambda}^0]\phi_0) \neq 0.$$

PROOF.

Since the eigenvalue $i\beta(\lambda_0)$ is algebraically simple, we have locally a smooth solution branch to the parametrized right and left eigenvalue-eigenvector problems :

$$(a) \qquad f_u(u(\lambda),\lambda)\ \phi(\lambda) = \kappa(\lambda)\ \phi(\lambda),$$

$$(b) \qquad \psi(\lambda)^* f_u(u(\lambda),\lambda) = \kappa(\lambda)\ \psi^*(\lambda),$$

$$(c) \qquad \psi(\lambda)^*\phi(\lambda) = 1, \quad \text{(and also } \ \phi(\lambda)^*\phi(\lambda) = 1),$$

with $\kappa(\lambda_0) = i\beta(\lambda_0)$. Above * denotes conjugate transpose. Differentiation gives

$$(a) \qquad f_{uu}\dot{u}\phi + f_{u\lambda}\phi + f_u\dot{\phi} = \dot{\kappa}\phi + \kappa\dot{\phi},$$

$$(b) \qquad \psi^* f_{uu}\dot{u} + \psi^* f_{u\lambda} + \dot{\psi}^* f_u = \dot{\kappa}\psi^* + \kappa\dot{\psi}^*,$$

$$(c) \qquad \dot{\psi}^*\phi + \psi^*\dot{\phi} = 0.$$

Multiply (a) on the left by ψ^* and (b) on the right by ϕ to get

$$(a) \qquad \psi^* f_{uu}\dot{u}\phi + \psi^* f_{u\lambda}\phi + \psi^* f_u \dot{\phi} = \dot{\kappa}\psi^*\phi + \kappa\psi^*\dot{\phi},$$

$$(b) \qquad \psi^* f_{uu}\dot{u}\phi + \psi^* f_{u\lambda}\phi + \dot{\psi}^* f_u \phi = \dot{\kappa}\psi^*\phi + \kappa\dot{\psi}^*\phi.$$

Adding the above, and using $\underbrace{\psi^* f_u}_{=\kappa\psi^*}\dot{\phi} + \dot{\psi}^* \underbrace{f_u\phi}_{=\kappa\phi} = \kappa(\psi^*\dot{\phi} + \dot{\psi}^*\phi) = \kappa\frac{d}{d\lambda}(\underbrace{\psi^*\phi}_{=1}) = 0$, we find $\dot{\kappa} = \psi^*[f_{uu}\dot{u} + f_{u\lambda}]\phi$. From differentiating $f(u(\lambda),\lambda) = 0$ with respect to λ we have

$$\dot{u} = -(f_u)^{-1} f_\lambda,$$

so that

$$\boxed{\dot{\kappa} = \psi^*[-f_{uu}(f_u)^{-1} f_\lambda + f_{u\lambda}]\phi.}$$

Thus the crossing is transversal if $\dot{\alpha}(s_0) = \mathrm{Re}(\dot{\kappa}_0) = \mathrm{Re}(\psi_0^*[f_{uu}^0(f_u^0)^{-1}f_\lambda^0 - f_{u\lambda}^0]\phi_0) \neq 0$. •

REMARK. This condition of the Theorem is also required for the nonsingularity of the extended system given below and for the numerical stability of the solution algorithm.

10.4 Persistence of Hopf Bifurcation

The *extended system* for computing Hopf bifurcations is

$$F(u,\phi,\beta,\lambda;\mu) \equiv \begin{cases} f(u,\lambda,\mu) = 0, \\ f_u(u,\lambda,\mu)\phi - i\beta\phi = 0, \\ \phi^*\phi_0 - 1 = 0, \end{cases}$$

$$F \quad : \quad \mathrm{R}^n \times \mathrm{C}^n \times \mathrm{R}^2 \times \mathrm{R} \quad \rightarrow \quad \mathrm{R}^n \times \mathrm{C}^n \times \mathrm{C},$$

to which we want to compute a solution branch

$$(u,\phi,\beta,\lambda,\mu),$$

with

$$u \in \mathrm{R}^n, \qquad \phi \in \mathrm{C}^n, \qquad \beta,\lambda,\mu \in \mathrm{R}.$$

Above ϕ_0 belongs to a 'reference solution'

$$(u_0,\phi_0,\beta_0,\lambda_0,\mu_0),$$

typically the last computed solution point on a branch.

First consider parametrizing in the second parameter μ, i.e., we seek a solution branch

$$(u(\mu),\phi(\mu),\beta(\mu),\lambda(\mu)).$$

Of course, pseudo-arclength continuation is used in practical computations.

The derivative with respect to $(u, \phi, \beta, \lambda)$ at the solution point $(u_0, \phi_0, \beta_0, \lambda_0, \mu_0)$, is

$$\begin{pmatrix} f_u^0 & O & 0 & f_\lambda^0 \\ \\ f_{uu}^0\phi_0 & f_u^0 - i\beta_0 I & -i\phi_0 & f_{u\lambda}^0\phi_0 \\ \\ 0^* & \phi_0^* & 0 & 0 \end{pmatrix}, \tag{10.4}$$

which is of the form

$$\begin{pmatrix} A & O & 0 & c_1 \\ C & D & -i\phi_0 & c_2 \\ 0^* & \phi_0^* & 0 & 0 \end{pmatrix}.$$

Here

$$A = f_u^0 \quad \text{(nonsingular)}, \qquad C = f_{uu}^0\phi_0, \qquad c_1 = f_\lambda^0, \qquad c_2 = f_{u\lambda}^0\phi_0, \qquad D = f_u^0 - i\beta_0 I,$$

with

$$\mathcal{N}(D) = \text{Span}\{\phi_0\}, \qquad \mathcal{N}(D^*) = \text{Span}\{\psi_0\},$$
$$\psi_0^*\phi_0 = 1, \qquad \phi_0^*\phi_0 = 1.$$

THEOREM.

If the eigenvalue crossing is transversal, i.e., if $\text{Re}(\dot\kappa_0) \neq 0$, then the Jacobian matrix 10.4 is nonsingular. Hence there locally exists a solution branch

$$(u(\mu), \phi(\mu), \beta(\mu), \lambda(\mu))$$

to the extended system $F(u, \phi, \beta, \lambda; \mu)$, i.e., the Hopf bifurcation persists under small perturbation of the second problem parameter μ.

PROOF.

We prove this by constructing a solution

$$x \in \text{R}^n, \quad y \in \text{C}^n, \quad z_1, z_2 \in \text{R},$$

to

$$\begin{pmatrix} A & O & 0 & c_1 \\ C & D & -i\phi_0 & c_2 \\ 0^* & \phi_0^* & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

where

$$f \in \text{R}^n, \quad g \in \text{C}^n, \quad h \in \text{C}.$$

From the first equation

$$Ax + z_2 c_1 = f,$$

we have

$$x = A^{-1}f - z_2 A^{-1} c_1.$$

The second equation can then be written

$$CA^{-1}f - z_2 CA^{-1}c_1 + Dy - z_1 i\phi_0 + z_2 c_2 = g.$$

Multiply on the left by ψ_0^* to get

$$\psi_0^* CA^{-1}f - z_2 \psi_0^* CA^{-1}c_1 - z_1 i\psi_0^*\phi_0 + z_2\psi_0^* c_2 = \psi_0^* g. \tag{10.5}$$

Recall that $\psi_0^*\phi_0 = 1$.

Define

$$\tilde{f} \equiv CA^{-1}f, \qquad \tilde{c}_1 \equiv CA^{-1}c_1.$$

Computationally $\tilde{f}$ and $\tilde{c}_1$ are obtained from

$$\boxed{A\hat{f} = f, \quad \tilde{f} = C\hat{f}}\,, \qquad \boxed{A\hat{c}_1 = c_1, \quad \tilde{c}_1 = C\hat{c}_1}\,.$$

Equation (10.5) can now be written as

$$iz_1 + \psi_0^*(\tilde{c}_1 - c_2)z_2 = \psi_0^*(\tilde{f} - g).$$

Separate real and imaginary part of this equation and use the fact that z_1 and z_2 are real to get

$$\mathrm{Re}(\psi_0^*[\tilde{c}_1 - c_2])z_2 = \mathrm{Re}(\psi_0^*[\tilde{f} - g]),$$

$$z_1 + \mathrm{Im}(\psi_0^*[\tilde{c}_1 - c_2])z_2 = \mathrm{Im}(\psi_0^*[\tilde{f} - g]),$$

from which

$$\boxed{z_2 = \frac{\mathrm{Re}\big(\psi_0^*[\tilde{f}-g]\big)}{\mathrm{Re}\big(\psi_0^*[\tilde{c}_1-c_2]\big)}}\,, \qquad \boxed{z_1 = -z_2\mathrm{Im}(\psi_0^*[\tilde{c}_1 - c_2]) + \mathrm{Im}(\psi_0^*[\tilde{f} - g])}\,.$$

Now solve for x in

$$\boxed{Ax = f - z_2 c_1}\,,$$

and compute a particular solution y_p to

$$\boxed{Dy = g - Cx + iz_1\phi_0 - z_2 c_2}\,.$$

Then

$$y = y_p + \alpha\phi_0, \qquad \alpha \in \mathrm{C}.$$

The third equation reads

$$\phi_0^* y = \phi_0^* y_p + \alpha\phi_0^*\phi_0 = h,$$

from which, using $\phi_0^*\phi_0 = 1$,

$$\boxed{\alpha = h - \phi_0^* y_p}\,.$$

The above construction can be carried out if

$$\mathrm{Re}(\psi_0^*[\tilde{c}_1 - c_2]) \neq 0.$$

But, using the definition of $\tilde{c}_1$ and c_2, we have

$$\begin{aligned}
\mathrm{Re}(\psi_0^*[\tilde{c}_1 - c_2]) &= \mathrm{Re}(\psi_0^*[CA^{-1}c_1 - c_2]) \\
&= \mathrm{Re}(\psi_0^*[f_{uu}^0\phi_0(f_u^0)^{-1}f_\lambda^0 - f_{u\lambda}^0\phi_0]) \\
&= \mathrm{Re}(\psi_0^*[f_{uu}^0(f_u^0)^{-1}f_\lambda^0 - f_{u\lambda}^0]\phi_0) \\
&= \mathrm{Re}(\dot{\kappa}_0) \neq 0. \qquad \bullet
\end{aligned}$$

10.5 The Continuation Algorithm

The construction in the proof of the Theorem above can be used in the algorithm for solving the equations the Chord method :

- First one computes the left and right null vectors ψ_0 and ϕ_0 of the complex matrix D. This can be done at very little extra computational cost once D has been LU decomposed.
- Then the "boxed" steps in the proof are carried out in order of appearance.

Additional computational savings are possible, using the fact that certain calculations are real or involve real variables.

For large n most of the computational effort is spent on LU decomposing A and D. Here $A = f_u^0$ is real and assumed nonsingular, and $D = f_u^0 - i\beta_0 I$ is complex with a one-dimensional nullspace. The decomposition of D requires row and column pivoting.

In practice we treat μ as an additional unknown and add the pseudo-arclength relation

$$(u - u_0)^*\dot{u}_0 + (\phi - \phi_0)^*\dot{\phi}_0 + (\beta - \beta_0)\dot{\beta}_0 + (\lambda - \lambda_0)\dot{\lambda}_0 + (\mu - \mu_0)\dot{\mu}_0 - \Delta s = 0.$$

Newton's method applied to the enlarged system leads to linear systems with matrices of the form

$$\begin{pmatrix} A & c \\ b^* & d \end{pmatrix}.$$

The efficient solution of such bordered systems has already been discussed in Section 3.5.

EXERCISE. Investigate the Hopf bifurcations in the system

$$\begin{aligned}
u_1'(t) &= u_1(1 - u_1) - p_4 u_1 u_2, \\
u_2'(t) &= -\tfrac{1}{4}u_2 + p_4 u_1 u_2 - 3u_2 u_3 - p_1(1 - e^{-5u_2}), \\
u_3'(t) &= -\tfrac{1}{2}u_3 + 3u_2 u_3.
\end{aligned}$$

(See AUTO demos pp3 and ppp.) This is a simple predator-prey model where, say, $u_1 \sim$ plankton, $u_2 \sim$ fish, $u_3 \sim$ sharks, $p_1 \sim$ fishing quota. The factor $(1 - e^{-5u_2})$ models that the quota cannot be met if the fish population is small.

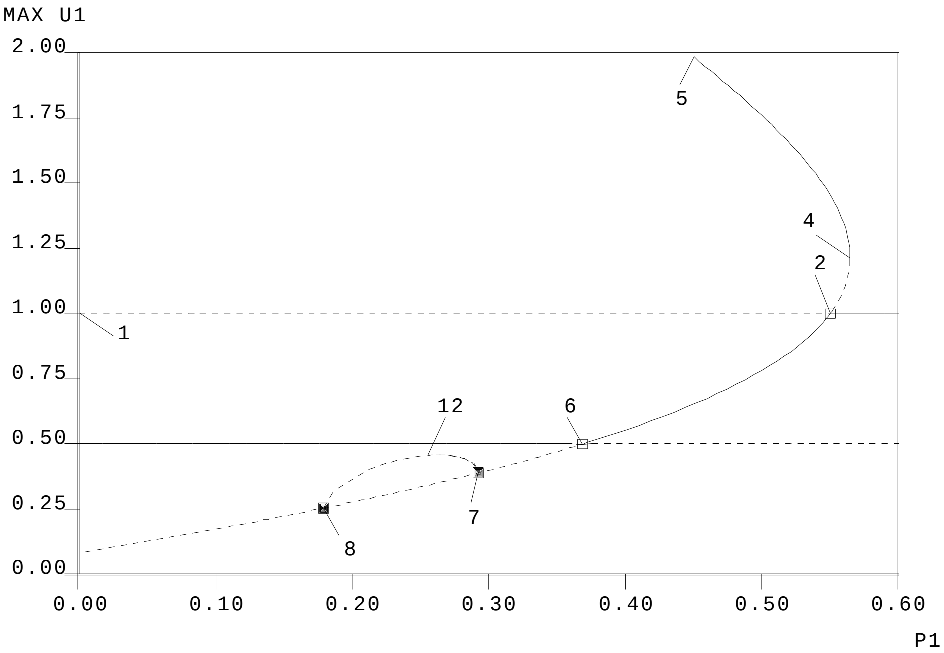

Figure 10.6: A bifurcation diagram for the 3-species model.

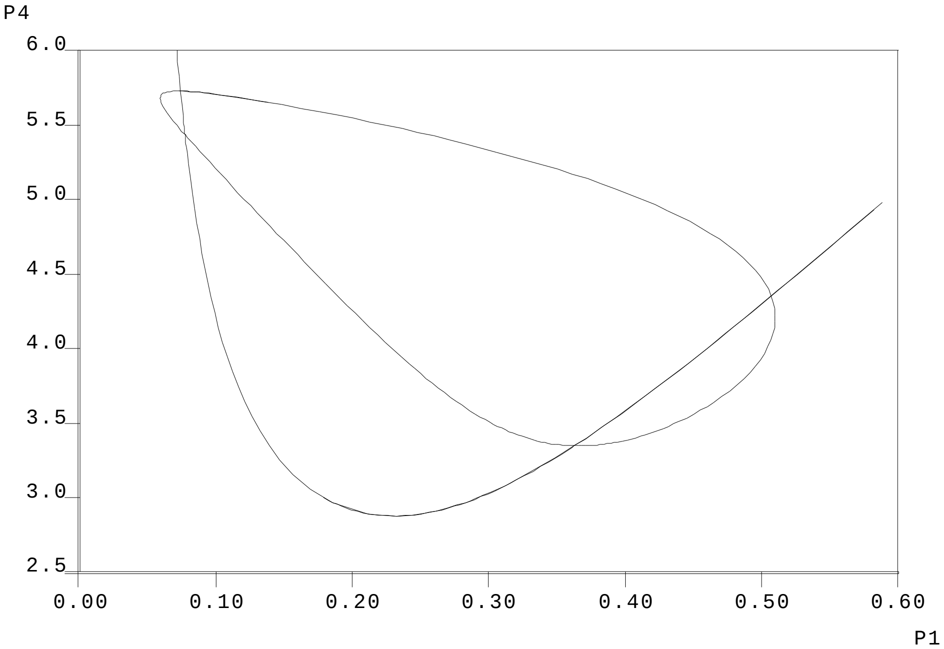

Figure 10.7: Loci of Hopf bifurcations for the 3-species model.

Chapter 11

Periodic Solution Theory

11.1 Poincaré Continuation

Consider the periodic solution problem

$$\begin{aligned} \frac{dv}{d\tau} &= f(v(\tau),\lambda), \\ v(T) &= v(0), \end{aligned}$$

where T is the (minimal) period. Scale time : $\tau = Tt, \quad u(t) = v(\tau(t)) \;=\; v(Tt)$. Then

$$\begin{aligned} \frac{du}{dt} &= \frac{dv}{d\tau}\frac{d\tau}{dt} = \frac{dv}{d\tau}\,T \;=\; T\ f(v(\tau(t)),\lambda) \;=\; T\ f(u(t),\lambda), \\ u(1)-u(0) &= v(\tau(1)) - v(\tau(0)) \;=\; v(T)-v(0) \;=\; 0. \end{aligned}$$

Thus $u(t)$ satisfies

$$\begin{cases} u'(t) = T\ f(u(t),\lambda), \\ u(1)-u(0) = 0. \end{cases}$$

Let $u_0(t)$ be a periodic solution of "period" T_0 for $\lambda = \lambda_0$:

$$\begin{cases} u_0'(t) \;=\; T_0\ f(u_0(t),\lambda_0), \\ u_0(1) \;=\; u_0(0), \end{cases}$$

where $u_0(t) \;\neq\;$ constant, and $T_0 \;>\; 0$.

Do there exist solutions $u(t)$, of period T near T_0, near $\lambda = \lambda_0$? Note that such $u(t)$ is not uniquely defined : If $u(t)$ is a periodic solution then so is $u(t+\sigma)$, for any $\sigma \in$ R. This is the non-uniqueness due to phase shift invariance. To fix the phase we add a constraint, for example,

$$u(0) - u_0(0) \quad \perp \quad u_0'(0).$$

See Figure 11.1 for a graphical interpretation. (See Section 12.3 for a numerically more suitable phase condition.)

Thus for λ near λ_0, we want to solve

$$\left.\begin{array}{l} u'(t) = T\ f(u(t),\lambda), \\ u(1) - u(0) = 0, \\ (u(0) - u_0(0))^*\ u_0'(0) = 0. \end{array}\right\} \tag{11.1}$$

Consider the related initial value problem

$$\left.\begin{array}{r} u'(t) = T\ f(u(t),\lambda), \\ u(0) = p, \end{array}\right\} \ \Rightarrow\ u = u(t;p,T,\lambda).$$

Let $p_0 \equiv u_0(0)$, $q_0 \equiv u_0'(0)$. Then (11.1) is equivalent to

$$\left.\begin{array}{rcl} u(1;p,T,\lambda) - p & = & 0, \\ (p-p_0)^*\ q_0 & = & 0, \end{array}\right\} \ \sim\ G(\underbrace{p,T}_{\equiv w},\lambda) = 0, \tag{11.2}$$

where

$$G\ :\ \mathrm{R}^{n+1} \times\ \mathrm{R}\ \rightarrow\ \mathrm{R}^n\ \times\ \mathrm{R}.$$

Let $V(t)$ be the fundamental solution matrix of

$$v'(t)\ =\ T_0\ f_u(u_0(t),\lambda_0)\ v(t),$$

that is,

$$\left\{\begin{array}{ll} V'(t) & = T_0\ f_u(u_0(t),\lambda_0)\ V(t), \\ V'(0) & = I. \end{array}\right.$$

LEMMA. V(1) always has 1 as an eigenvalue.

PROOF. We have

$$(u_0'(t))'\ =\ T_0\ f_u(u_0(t),\lambda_0)\ u_0'(t),$$

so that, by the variation of parameter formula,

$$u_0'(t)\ =\ V(t)\ u_0'(0). \tag{11.3}$$

Note that

$$q_0\ =\ u_0'(0)\ =\ u_0'(1)\ =\ V(1)\ u_0'(0)\ =\ V(1)\ q_0,$$

so that

$$(V(1) - I)\ q_0\ =\ 0, \qquad ,\ \text{i.e.},\quad q_0 \in \mathcal{N}(V(1) - I).$$

It follows that V(1) always has 1 as an eigenvalue. $\bullet$

THEOREM.

Let 1 be an algebraically simple eigenvalue of V(1). Then there is a unique solution branch to $G(p,T,\lambda)=0$ near (p_0,T_0,λ_0), i.e., there is a unique branch of periodic solutions near $(u_0(t),T_0,\lambda_0)$.

PROOF.

From (11.2), using the definition of $V(t)$, we have

$$G_w^0 \;=\; G_{pT}^0 \;=\; \begin{pmatrix} V(1)-I & u_T^0(1) \\ q_0^* & 0 \end{pmatrix},$$

where u_T^0 satisfies

$$u_T'^0(t) \;=\; T_0\, f_u(u_0(t),\lambda_0)\; u_T^0(t) \;+\; \overbrace{f(u_0(t),\lambda_0)}^{=u_0'(t)/T_0}, \qquad u_T^0(0)=0,$$

so that, by the variation of parameter formula,

$$u_T^0(1) \;=\; V(1)\int_0^1 V(s)^{-1}\frac{u_0'(s)}{T_0}\;ds.$$

Hence, using (11.3),

$$u_T^0(1)=V(1)\int_0^1 V(s)^{-1}V(s)\;\frac{u_0'(0)}{T_0}\;ds \;=\; V(1)\;\frac{u_0'(0)}{T_0} \;=\; \frac{u_0'(1)}{T_0} \;=\; \frac{u_0'(0)}{T_0} \;=\; \frac{q_0}{T_0}.$$

Thus

$$G_w^0=\begin{pmatrix} V(1)-I & q_0/T_0 \\ q_0^* & 0 \end{pmatrix}.$$

Now 1 is assumed to be an algebraically simple eigenvalue of $V(1)$ i.e., 0 is an algebraically simple eigenvalue of $V(1)-I$. This implies, in particular, that $\mathcal{N}(V(1)-I)$ is one-dimensional. In fact, $\mathcal{N}(V(1)-I))=\mathrm{Span}\{q_0\}$. By the Bordering Lemma (Section 3.4) G_w^0 is nonsingular if $q_0\notin\mathcal{R}(V(1)-I)$ and $q_0^*\notin\mathcal{R}((V(1)-I)^*)$. Since $q_0\in\mathcal{N}(V(1)-I))$ we cannot have $q_0^*\in\mathcal{R}((V(1)-I)^*)$. Also, since 1 is algebraically simple, we cannot have both $q_0\in\mathcal{N}(V(1)-I))$ and $q_0\in\mathcal{R}(V(1)-I)$. Hence G_w^0 is nonsingular. $\bullet$

EXAMPLE. Consider the linear problem

$$\left\{\begin{array}{l} u_1'=-\lambda u_1-u_2 \\ u_2'=u_1 \end{array}\right. \quad\sim\quad \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}=\begin{pmatrix} -\lambda & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad\sim\quad u'(t) \;=\; A(\lambda)\;u(t).$$

When $\lambda=0$, $u_1(t)=\cos(t)$, $u_2(t)=\sin(t)$, is a solution with period $T=2\pi$. In this problem

$$f(u,\lambda)=A(\lambda)u, \qquad f_u=A(\lambda), \qquad f_u^0=A(0)\equiv A_0.$$

$$\left.\begin{array}{r} V'(t) \;=\; 2\pi A_0 V(t), \\ V(0) \;=\; I, \end{array}\right\} \;\Rightarrow\; V(t) \;=\; e^{2\pi t\; A_0}\;V(0) \;=\; e^{2\pi t A_0}.$$

Here

$$V(1) \;=\; e^{2\pi A_0} \;=\; e^{\begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix}} \;=\; I,$$

has two eigenvalues that equal 1. Hence the persistence condition is not satisfied. Indeed, the branch is vertical, and cannot be parametrized in λ. (See Figure 10.1.)

11.2 Folds

Recall that (w_0, λ_0) is a simple fold of $G(w, \lambda) = 0$ if

$$\mathcal{N}(G_w^0) \text{ is one-dimensional} \qquad \text{and} \qquad G_\lambda^0 \notin \mathcal{R}(G_w^0).$$

In the case of periodic solutions we have

$$w = (p, T) \qquad \text{and } G_w^0 = \begin{pmatrix} V(1) - I & q_0/T_0 \\ q_0^* & 0 \end{pmatrix},$$

where

$$q_0 \in \mathcal{N}(V(1) - I), \qquad q_0 \neq 0, \qquad q_0 \equiv u_0'(0).$$

When does G_w^0 have a $1 - D$ null space ? Consider

$$\begin{pmatrix} V(1) - I & q_0/T_0 \\ q_0^* & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e.,

$$(V(1) - I)\ q_1 = -s\ q_0/T_0, \qquad q_0^*\ q_1 = 0.$$

11.2.1 The case of geometric multiplicity 2

Assume $q_0 \notin \mathcal{R}(V(1) - I)$. Then $s = 0$, $q_1 \in \mathcal{N}(V(1) - I)$, $q_1 \perp q_0$. Thus $\mathcal{N}(V(1) - I)$ is 2-dimensional,

$$\mathcal{N}(V(1) - I) = \text{Span}\{q_0, q_1\}, \qquad q_1 \perp q_0.$$

To be a simple fold we also need $G_\lambda^0 \notin \mathcal{R}(G_w^0)$, i.e.,

$$\begin{pmatrix} u_\lambda^0 \\ 0 \end{pmatrix} \notin \mathcal{R} \begin{pmatrix} V(1) - I & q_0/T_0 \\ q_0^* & 0 \end{pmatrix},$$

where $u_\lambda^0 \equiv u_\lambda(1; p_0, T_0, \lambda_0)$, i.e., we need $u_\lambda^0 \notin \mathcal{R}\Big(V(1) - I \;\Big|\; \frac{q_0}{T_0}\Big)$.

In this case the geometric multiplicity of the Floquet multiplier $z = 1$ is 2, and the algebraic multiplicity is ≥ 2.

11.2.2 The case of geometric multiplicity 1

Assume $q_0 \in \mathcal{R}(V(1) - I)$, and $\mathcal{N}(V(1) - I) = \text{Span}\{q_0\}$ (one-dimensional). Then

$$(V(1) - I)\ q \;=\; -q_0/T_0,$$

has solution

$$q = q_p + \alpha\ q_0, \qquad q_p \perp q_0,$$

and the equation

$$\begin{pmatrix} V(1)-I & q_0/T_0 \\ q_0^* & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., the equation

$$\begin{cases} (V(1)-I)\ q_1 = -s\ q_0/T_0, \\ \qquad\qquad q_0^*\ q_1 = 0, \end{cases}$$

has solution $q_1 = s\ q_p + \alpha\ q_0$. But $q_0^*\ q_1 = 0 \ \Rightarrow\ \alpha = 0$. Thus

$$\begin{pmatrix} q_1 \\ s \end{pmatrix} \ = \ s \begin{pmatrix} q_p \\ 1 \end{pmatrix} \ \text{(1-D null space)}.$$

To be a simple fold, we also need $G_\lambda^0 \notin \mathcal{R}(G_w^0)$, i.e.,

$$u_\lambda^0 \notin \mathcal{R}\Big(V(1)-I \ \Big|\ \frac{q_0}{T_0}\Big), \qquad \text{i.e., } u_\lambda^0 \notin \mathcal{R}(V(1)-I).$$

Thus in this case

$$\mathcal{N}(V(1)-I) \ = \ \mathrm{Span}\{q_0\}, \qquad q_0 \in \mathcal{R}(V(1)-I).$$

This happens, for example, if

$$(V(1)-I) \sim \left(\begin{array}{c|cc} \mathrm{J} & 0 & 0 \\ \hline 0^* & 0 & 1 \\ 0^* & 0 & 0 \end{array}\right), \qquad q_0 \sim \left(\begin{array}{c} 0 \\ \hline 1 \\ 0 \end{array}\right), \qquad \det J \neq 0.$$

Thus, there are two Floquet multipliers that are equal to 1, so the algebraic multiplicity is 2. The geometric multiplicity is 1.

11.3 Bifurcation Points

Consider

$$G(\underbrace{\overbrace{p, T}^{\equiv w}, \lambda}_{\equiv x}) = 0.$$

Recall from Section 6.1 that a simple singular point (p_0, T_0, λ_0) we have that $\mathcal{N}(G_x^0)$ is two-dimensional and $\mathcal{N}(G_x^{0^*})$ is one-dimensional. In the case of periodic solutions we have

$$G_x^0 = \begin{pmatrix} V(1)-I & q_0/T_0 & u_\lambda^0 \\ q_0^* & 0 & 0 \end{pmatrix}.$$

Since $q_0 \in \mathcal{N}(V(1)-I)$, $q_0 \neq 0$, we have $q_0 \in \mathcal{R}(V(1)-I)^{*^\perp}$, and $q_0 \notin \mathcal{R}(V(1)-I)^*$. Therefore

$$\mathrm{Rank} \begin{pmatrix} V(1)-I \\ q_0^* \end{pmatrix} = \mathrm{Rank}\ (V(1)-I)+1,$$

and

$$\text{Rank }(G_x^0) \;=\; \text{Rank}\Big(V(1)-I \;\Big|\; \frac{q_0}{T_0} \;\Big|\; u_\lambda^0\Big) + 1.$$

Thus $\mathcal{N}(G_x^0)$ is two-dimensional if

$$\text{Rank }\Big(V(1)-I \;\Big|\; \frac{q_0}{T_0} \;\Big|\; u_\lambda^0\Big) \;=\; n-1.$$

EXERCISE. Consider the 3-species predator-prey model

$$\begin{cases} u'(t) &= u(1-u) - c_{uv}uv, \\ v'(t) &= -c_v v + c_{uv}uv - c_{vw} - \lambda(1-e^{-c_e v}), \\ w'(t) &= -c_{ww} + -c_{vw}vw, \end{cases}$$

where $u \sim$ plankton, $v \sim$ fish, $w \sim$ sharks, $\lambda \sim$ fishing quota. (See AUTO demo ppp, and Figures 10.6 and 10.7.) Take $c_r = \frac{1}{4}$, $c_w = \frac{1}{2}$, $c_{uv} = 4$, $c_{vw} = 3$, $c_e = 5$.

- Compute the stationary solution bifurcation diagram.

- Compute the periodic solutions, and describe the behavior of the Floquet multipliers.

- Describe the long time behavior of the system for all $\lambda \geq 0$.

11.3.1 The case of geometric multiplicity 2

Assume Rank $(V(1)-I) = n-2$, i.e.,

$$\mathcal{N}(V(1)-I) \;=\; \text{Span}\{q_0, q_1\}, \qquad q_1 \;\perp\; q_0,$$

and

$$q_0 \notin \mathcal{R}(V(1)-I), \qquad u_\lambda^0 \in \mathcal{R}\Big((V(1)-I)\Big|\frac{q_0}{T_0}\Big).$$

Here the geometric multiplicity is 2 and the algebraic multiplicity is ≥ 2.

11.3.2 The case of geometric multiplicity 1

Assume Rank$(V(1)-I) = n-1$, i.e.,

$$\mathcal{N}(V(1)-I) \;= \text{1-D} \;=\; \text{Span}\{q_0\},$$

and

$$q_0 \in \mathcal{R}(V(1)-I), \qquad u_\lambda^0 \in \mathcal{R}((V(1)-I).$$

Here the geometric multiplicity is 1 and the algebraic multiplicity is ≥ 1.

11.4 Period-Doubling Bifurcations

Suppose $(u_0(t), T_0, \lambda_0)$ is a solution of

$$\left.\begin{array}{l} u'(t) \;=\; T\; f(u(t),\lambda), \\ u(1)-u(0)=0, \\ (u(0)-u_0(0)^*\; u_0'(0) \;=\; 0, \end{array}\right\} \qquad (11.4)$$

with $p_0 \equiv u_0(0)$, $q_0 \equiv u_0'(0) \neq 0$. Consider again the related initial value problem

$$\left.\begin{array}{ll} u'(t) & = T f(u(t),\lambda), \\ u(0) & = \; p, \end{array}\right\} \;\Rightarrow\; u = u(t; p, T, \lambda).$$

Then (11.4) is equivalent to

$$\left.\begin{array}{ll} u(1; p, T, \lambda) - p & = 0, \\ (p-p_0)^* q_0 & = 0, \end{array}\right\} \sim\; G(\underbrace{\overbrace{p, T}^{\equiv w}, \lambda}_{\equiv x}) = 0.$$

We have a solution p_0, T_0, λ_0. We found earlier that

$$G_x^0 \;=\; G_{pT\lambda}^0 \;=\; \begin{pmatrix} u_p^0 - I & u_T^0 & u_\lambda^0 \\ q_0^* & 0 & 0 \end{pmatrix} = \begin{pmatrix} V(1)-I & q_0/T_0 & u_\lambda^0 \\ q_0^* & 0 & 0 \end{pmatrix},$$

where V(t) is the fundamental solution matrix for $u_0(t)$. The Floquet multipliers are the eigenvalues of V(1). There is always a Floquet multipliers equal to 1. Assume that 1 is an algebraically simple eigenvalue, and that -1 is also an algebraically simple eigenvalue. Thus Poincaré continuation applies :

$$\begin{pmatrix} V(1)-I & q_0/T_0 \\ q_0^* & 0 \end{pmatrix} \qquad \text{is nonsingular,}$$

and locally there exists a unique branch $(p(\lambda), T(\lambda))$ satisfying $p(\lambda_0) = p_0, T(\lambda_0) = T_0$. Note that

$$u_2(t) = u_0(2t), \qquad T_2 = 2T_0, \qquad \lambda_2 = \lambda_0,$$

is also a solution of Equation 11.4, i.e.

$$G(p_0, 2T_0, \lambda_0) \;=\; 0.$$

For this solution we have

$$G_{x2}^0 \equiv \begin{pmatrix} V_2(1)-I & u_{T_2}(1; p_0, \overbrace{2T_0}^{T_2}, \lambda_0) & u_{\lambda 2}(1; p_0, \overbrace{2T_0}^{T_2}, \lambda_0) \\ q_0^* & 0 & 0 \end{pmatrix},$$

where

$$\left\{\begin{array}{l} V_2'(t) = T_2\; f_u(u_2(t),\lambda_0)\; V_2(t), \\ V_2'(0) = I, \end{array}\right.$$

$$\left\{\begin{array}{ll} u_{T2}'(t) & = T_2\; f_u(u_2(t),\lambda_0)\; u_{T2}(t) + f(u_2(t),\lambda_0), \\ u_{T2}(0) & = 0, \end{array}\right.$$

$$\left\{\begin{array}{ll} u_{\lambda 2}'(t) & = T_2\; f_u(u_2(t),\lambda_0)\; u_{\lambda 2}(t) + T_2 f_\lambda(u_2(t),\lambda_0), \\ u_{\lambda 2}(0) & = 0. \end{array}\right.$$

FACT 1 : V(1+t) = V(t) V(1).

PROOF.

$$\begin{cases} V'(t) &= T_0\ f_u(u_0(t),\lambda_0)\ V(t),\\ V(0) &= I,\end{cases}$$

$$V'(1+t) = T_0\ f_u(u_0(1+t),\lambda_0)V(1+t) = T_0\ f_u(u_0(t),\lambda_0)V(1+t)$$

$$\Rightarrow V(1+t)\ =\ V(t)\ V(1). \quad \bullet$$

FACT 2 : $V_2(t)\ =\ V(2t)$.

PROOF.

$$\begin{cases} V_2'(t) &= 2T_0\ f_u(u_0(2t),\lambda_0)\ V_2(t),\\ V_2(0) &= I.\end{cases}$$

Let $V_1(t) = V_2(\frac{1}{2}t)$, i.e., $V_1(2t) = V_2(t)$. Then $V_2'(t) = 2V_1'(2t) = 2T_0 f_u(u_0(2t),\lambda_0)V_1(2t)$.

$$\Rightarrow \begin{cases} V_1'(t) &= T_0\ f_u(u_0(t),\lambda_0)\ V_1(t),\\ V_1(0) &= V_2(0) = I,\end{cases}$$

i.e., $V_1 \equiv V$. Thus $V_2(t) = V(2t)$. $\bullet$

FACT 3 : $V_2(1) = V(1)^2$.

PROOF. Using Fact 3 and Fact 1 : $V_2(1)\ =\ V(2)\ =\ V(1+1)\ =\ V(1)\ V(1)$. $\bullet$

FACT 4 : $u_{T2}(1;p_0,2T_0,\lambda_0)\ =\ q_0/T_0$.

PROOF.

$$\begin{cases} u_{T2}'(t) &= T_2\ f_u(u_2(t),\lambda_0)\ u_{T2}(t) + f(u_2(t),\lambda_0),\\ u_{T2}(0) &= 0.\end{cases}$$

Here $f(u_2(t),\lambda_0) = u_2'(t)/T_2 = V_2(t)\frac{u_2'(0)}{T_2}$. By the variation of parameters formula,

$$u_{T2}(1)\ =\ V_2(1)\int_0^1 V_2(s)^{-1}V_2(s)\frac{u_2'(0)}{T_2}\ ds\ =\ V_2(1)\frac{u_2'(0)}{T_2}\ =\ \frac{u_2'(1)}{T_2}\ =\ \frac{u_2'(0)}{T_2}\ =\ \frac{2u_0'(0)}{2T_0}\ =\ \frac{q_0}{T_0}. \quad \bullet$$

FACT 5 : $V_2(1) - I$ has a zero eigenvalue with algebraic and geometric multiplicity equal to 2.

PROOF. We assumed that $V(1)$ has an algebraically simple eigenvalue 1, and an algebraically simple eigenvalue -1. Hence $V_2(1) = V(1)^2$ has 1 as algebraically double eigenvalue.

$$(V(1)-I)q_0 = 0, \qquad V(1)q_0 = q_0, \qquad V(1)q_1 = -q_1,$$

where $q_1,\ q_2\ \neq\ 0$, are linearly independent,

$$V_2(1)q_0 = V(1)V(1)q_0 = V(1)q_0 = q_0,$$

$$V_2(1)q_1 = V(1)V(1)q_1 = -V(1)q_1 = q_1.$$

Thus $(V_2(1)-I)q_0 = 0$, $(V_2(1)-I)q_1 = 0$. Hence the geometric multiplicity also equals 2. $\bullet$

FACT 6 : $u_{\lambda 2}(1; p_0, 2T_0, \lambda_0) \in \mathcal{R}\Big((V_2(1)-I)\Big|q_0/T_0\Big)$.

PROOF. Since V(1) has 1 as an algebraically simple eigenvalue,

$$G_w^0 \equiv \begin{pmatrix} V(1)-I & q_0/T_0 \\ q_0^* & 0 \end{pmatrix} \text{ is nonsingular,}$$

$$\Rightarrow \quad \begin{pmatrix} u_\lambda(1; p_0, T_0, \lambda_0 \\ 0 \end{pmatrix} \in \mathcal{R}(G_w^0),$$

where u_λ satisfies

$$\begin{cases} u_\lambda'(t) & = T_0\ f_u(u_0(t), \lambda_0)\ u_\lambda(t)\ +\ f_\lambda(u_0(t), \lambda_0), \\ u_\lambda(0) & = 0. \end{cases}$$

By the variation of parameters formula

$$u_\lambda(1) \ = \ V(1) \int_0^1 V(s)^{-1} f_\lambda(u_0(s), \lambda_0)\ ds.$$

Thus $u_\lambda(1) \in \mathcal{R}(V(1))$, so that $u_\lambda(1) = (V(1)-I)r + sq_0$. Now $u_{\lambda 2}$ satisfies

$$\begin{cases} u_{\lambda 2}'(t) & = T_2\ f_u(u_2(t), \lambda_0) u_{\lambda 2}(t) + f_\lambda(u_2(t), \lambda_0), \\ u_{\lambda 2}(0) & = 0. \end{cases}$$

Thus

$$u_{\lambda 2}(1) \ = \ V_2(1) \textstyle\int_0^1 V_2(s)^{-1} f_\lambda(u_2(s), \lambda_0)\ ds \ = \ V(1)^2 \int_0^1 V(2s)^{-1} f_\lambda(u_0(2s), \lambda_0)\ ds$$

$$= \tfrac{1}{2} V(1)^2 \textstyle\int_0^2 V(t)^{-1} f_\lambda(u_0(t), \lambda_0)\ dt \ = \ \tfrac{1}{2} V(1)^2 \Big\{ \int_0^1 \ \cdots \ dt + \int_1^2 \ \cdots \ dt \Big\}$$

$$= \tfrac{1}{2} V(1)^2 \Big\{ \textstyle\int_0^1 \ \cdots \ dt + \int_0^1 V(1+t)^{-1} f_\lambda(u_0(1+t), \lambda_0)\ dt \Big\}$$

$$= \tfrac{1}{2} V(1)^2 \Big\{ \textstyle\int_0^1 V(t)^{-1} f_\lambda(u_0(t), \lambda_0)\ dt + \int_0^1 [V(t)V(1)]^{-1} f_\lambda(u_0(t), \lambda_0)\ dt \Big\}$$

$$= \tfrac{1}{2} V(1) \Big\{ V(1) \textstyle\int_0^1 V(t)^{-1} f_\lambda(u_0(t), \lambda_0)\ dt + \int_0^1 V(t)^{-1} f_\lambda(u_0(t), \lambda_0)\ dt \Big\}$$

$$= \tfrac{1}{2} V(1) u_\lambda(1) + \tfrac{1}{2} u_\lambda(1) = \tfrac{1}{2}(V(1)+I)\ u_\lambda(1)$$

$$= \tfrac{1}{2}(V(1)+I)\ [(V(1)-I)r + sq_0] \ = \ \tfrac{1}{2}(V(1)^2 - I)\ r \ + \ \tfrac{1}{2} s (V(1)+I) q_0$$

$$= \tfrac{1}{2}(V(1)^2 - I)\ r \ + \ \tfrac{1}{2} s (V(1) q_0 + q_0)$$

$$= \ \tfrac{1}{2}(V(1)^2 - I)\ r \ + \ \tfrac{1}{2} s (q_0 + q_0) \ = \ \tfrac{1}{2}(V(1)^2 - I)\ r \ + \ s q_0,$$

where the first term lies obviously in $\mathcal{R}(V(1)^2 - I)$, and hence

$$(V(1)^2 - I)\ r \ \in \ \mathcal{R}(V_2(1) - I). \quad \bullet$$

THEOREM. If 1 and -1 are algebraically simple eigenvalues of $V(1)$ then $(p_0, 2T_0, \lambda_0)$ is a simple singular point of $G(p, T, \lambda) = 0$.

PROOF.

We must show that $\mathcal{N}(G^0_{x2})$ is two-dimensional, i.e., Rank $(G^0_{x2}) = n$. We have shown that

$$G^0_{x2} \;=\; \left(\begin{array}{ccc} V_2(1) - I & u_{T2}(1; p_0, 2T_0, \lambda_0) & u_{\lambda 2}(\overbrace{1; p_0, 2T_0, \lambda_0}^{\equiv u^0_{\lambda 2}(1)} \\ q_0^* & 0 & 0 \end{array}\right) \;=\; \left(\begin{array}{ccc} V(1)^2 - I & q_0/T_0 & u^0_{\lambda 2}(1) \\ q_0^* & 0 & 0 \end{array}\right),$$

where

$$\mathcal{N}(V(1)^2 - I) \;=\; \text{Span}\{q_0, q_1\},$$

with algebraic multiplicity 2. We also have

$$q_0 \notin \mathcal{R}(V(1)^2 - I), \quad u^0_{\lambda 2} \in \mathcal{R}\Big((V(1)^2 - I)\Big| q_0/T_0\Big), \quad q_0 \notin \mathcal{R}\Big(V(1)^2 - I)^*\Big).$$

Hence Rank $G^0_{x2} = n$. $\bullet$

EXERCISE. Use AUTO to compute brances of stationary solutions and periodic solutions of the transformed Lorenz equation :

$$\left\{\begin{array}{ll} x' & = -y - z, \\ y' & = x, \\ z' & = ay(1-y) - bz, \end{array}\right.$$

with $a = 0.375$, $b =$ bifurcation parameter. Start at $b = -5$, $x = y = z = 0$. You should find a stationary bifurcation, a Hopf bifurcation, a torus bifurcation, and a period-doubling bifurcation. Compute some of the bifurcating period-doubling branches. Determine the stability properties of the computed solutions. Plot some orbits in 3-D.

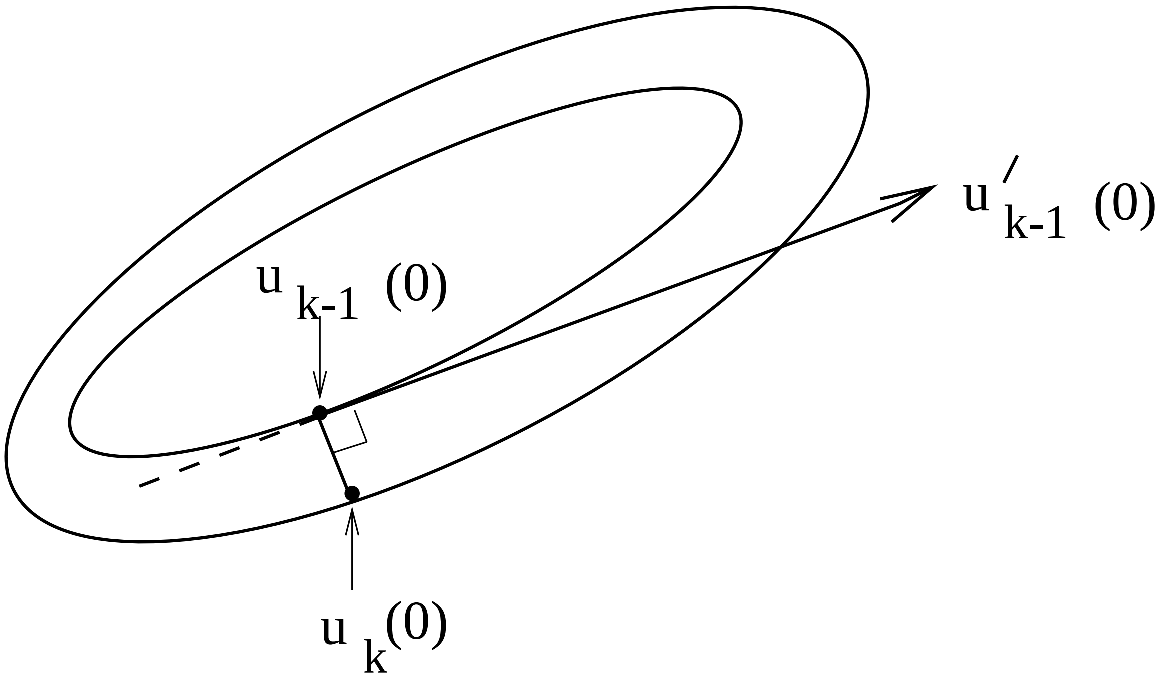

Figure 11.1: Graphical interpretation of the Poincaré phase condition.

Chapter 12

Computation of Periodic Solutions

12.1 Boundary Value Problems.

Consider the first order system of ordinary differential equations

$$u'(t) - f(u(t), \mu, \lambda) = 0, \qquad t \in [0, 1],$$

where

$$u(\cdot), f(\cdot) \in \mathrm{R}^n, \qquad \lambda \in \mathrm{R}, \qquad \mu \in \mathrm{R}^{n_\mu},$$

subject to general boundary conditions

$$b(u(0), u(1), \mu, \lambda) = 0, \qquad b(\cdot) \in \mathrm{R}^{n_b},$$

and integral constraints

$$\int_0^1 q(u(s), \mu, \lambda) ds = 0, \qquad q(\cdot) \in \mathrm{R}^{n_q}.$$

We want to solve for $u(\cdot)$ and μ. In order for problem to be formally well posed we require that

$$n_\mu = n_b + n_q - n \geq 0.$$

We think of λ as the parameter in which the solution (u, μ) may be continued. A simple case is $n_q = 0$ and $n_b = n$, where $n_\mu = 0$. Another simple case is the computation of periodic solution, as shown in the following section.

12.2 Numerical Computation

Consider

$$u'(t) = f(u(t), \lambda), \qquad u(\cdot), f(\cdot) \in \mathrm{R}^n, \qquad \lambda \in \mathrm{R}\ .$$

Fix the interval of periodicity by the transformation $t \to \frac{t}{T}$. Then the equation becomes

$$\boxed{u'(t) = T f(u(t), \lambda)}\ , \qquad u(\cdot), f(\cdot) \in \mathrm{R}^n, \qquad T, \lambda \in \mathrm{R}. \tag{12.1}$$

and we seek solutions of period 1, i.e.,

$$\boxed{u(0) = u(1)}\,. \tag{12.2}$$

Note that the period T is one of the unknowns.

Assume that we have computed $(u_{k-1}(\cdot), T_{k-1}, \lambda_{k-1})$ and we want to compute

$$(u_k(\cdot), T_k, \lambda_k) \equiv (u(\cdot), T, \lambda).$$

The above equations do not uniquely specify u and T, since $u(t)$ can be translated freely in time, i.e., if $u(t)$ is a periodic solution then so is $u(t+\sigma)$, for any σ. Thus, a "*phase condition*" is needed. An example is the Poincaré orthogonality condition

$$(u(0) - u_{k-1}(0))^* \ u'_{k-1}(0) = 0.$$

However, below we derive a numerically more suitable phase condition.

12.3 Phase Condition

If $\tilde{u}(t)$ is a solution then so is $\tilde{u}(t+\sigma)$, for any σ. We want the solution that minimizes

$$D(\sigma) \equiv \int_0^1 \| \ \tilde{u}(t+\sigma) - u_{k-1}(t) \ \|_2^2 \ dt.$$

The optimal solution $\tilde{u}(t+\hat{\sigma})$ satisfies the necessary condition $D'(\hat{\sigma}) = 0$, i.e.,

$$\int_0^1 (\tilde{u}(t+\hat{\sigma}) - u_{k-1}(t))^* \ \tilde{u}'(t+\hat{\sigma}) \ dt = 0.$$

Writing $u(t) \equiv \tilde{u}(t+\hat{\sigma})$ gives

$$\int_0^1 (u(t) - u_{k-1}(t))^* \ u'(t) \ dt = 0.$$

Integration by parts, using periodicity, gives

$$\boxed{\textstyle\int_0^1 u(t)^* \ u'_{k-1}(t) \ dt = 0}\,. \tag{12.3}$$

12.4 Pseudo-Arclength Continuation

In practice we use pseudo-arclength continuation (see Section 3.3) to trace out a branch of periodic solutions. In particular, this allows calculation past folds along a branch of periodic solutions. In this application, the pseudo-arclength equation becomes

$$\boxed{\textstyle\int_0^1 (u(t) - u_{k-1}(t))^* \dot{u}_{k-1}(t) \ dt + (T - T_{k-1})\dot{T}_{k-1} + (\lambda - \lambda_{k-1})\dot{\lambda}_{k-1} = \Delta s}\,. \tag{12.4}$$

Equations (12.1-12.4) are the equations used in AUTO for the continuation of periodic solutions.

EXERCISE. Compute and plot the periodic solutions that bifurcate from a Hopf bifurcation in

$$\begin{aligned} u_1' &= \lambda u_1 - u_2, \\ u_2' &= u_1(1-u_1). \end{aligned}$$

(AUTO demo phs.) This equation has a "vertical" Hopf bifurcation from the trivial solution at $\lambda = 0$. The branch terminates in a homoclinic orbit through $(u_1, u_2) = (1, 0)$. Plot some orbits versus time to see how the phase condition keeps the "peak" in almost exactly the same place. This is very advantageous for discretization methods.

12.5 Hopf Bifurcation

Let (u_0, λ_0) be a Hopf bifurcation point. Assume $f_u(u_0, \lambda_0)$ has a simple conjugate pair of purely imaginary eigenvalues $\pm i\omega_0$, $\omega_0 \neq 0$, and no other eigenvalues on the imaginary axis. Also assume the conjugate pair crosses the imaginary axis transversally with respect to λ. By the Hopf Bifurcation Theorem, these conditions ensure the existence of a bifurcating branch of periodic solutions.

The following asymptotic estimates are valid for periodic solutions near the Hopf bifurcation :

$$\begin{aligned} u(t;\epsilon) &= u_0 + \epsilon\phi(t) + \mathcal{O}(\epsilon^2), \\ T(\epsilon) &= T_0 + \mathcal{O}(\epsilon^2), \\ \lambda(\epsilon) &= \lambda_0 + \mathcal{O}(\epsilon^2). \end{aligned}$$

Here ϵ locally parametrizes the branch of periodic solutions. $T(\epsilon)$ denotes the period, and $T_0 = 2\pi/\omega_0$. The function $\phi(t)$ is the normalized nonzero periodic solution of the linearized, constant coefficient problem

$$\phi'(t) = f_u(u_0, \lambda_0)\ \phi(t).$$

To compute a first periodic solution

$$(u_1, T_1, \lambda_1) \equiv (u, T, \lambda),$$

near a Hopf bifurcation (u_0, λ_0), we still have

$$\boxed{u'(t) = T\ f(u(t), \lambda)}\ , \tag{12.5}$$

$$\boxed{u(0) = u(1)}\ . \tag{12.6}$$

Initial estimates for Newton's method are

$$u^{(0)}(t) = u_0 + \Delta s\ \phi(t), \qquad T^{(0)} = T_0, \qquad \lambda^{(0)} = \lambda_0,$$

where $\phi(t)$ is a nonzero solution of the time-scaled, linearized equations

$$\phi'(t) = T_0\ f_u(u_0, \lambda_0)\ \phi(t), \qquad \phi(0) = \phi(1),$$

namely,

$$\phi(t) = sin(2\pi t)w_s + cos(2\pi t)w_c,$$

where (w_s, w_c) is a null vector in

$$\begin{pmatrix} -\omega_0 I & f_u(u_0, \lambda_0) \\ f_u(u_0, \lambda_0) & \omega_0 I \end{pmatrix} \begin{pmatrix} w_s \\ w_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \omega_0 = \frac{2\pi}{T_0}.$$

The nullspace is two-dimensional since $(-w_c, w_s)$ is also a solution.

For the phase equation we "align" $u = u_1$ with $\phi(t)$, i.e.,

$$\boxed{\int_0^1 u(t)^* \ \phi'(t) \ dt = 0}\,. \tag{12.7}$$

Since $\dot{\lambda}_0 = \dot{T}_0 = 0$, the pseudo-arclength equation for the first step reduces to

$$\boxed{\int_0^1 (u(t) - u_0(t))^* \ \phi(t) \ dt = \Delta s}\,. \tag{12.8}$$

The equations (12.5)-(12.8) are used to compute the first orbit on the branch.

12.6 Periodically Forced Systems

Here we illustrate the computation of periodic solutions to a periodically forced system. In AUTO this can be done by adding a nonlinear oscillator with the desired periodic forcing as one of its solution components. An example of such an oscillator is

$$\begin{aligned} x' &= x + \beta y - x(x^2 + y^2), \\ y' &= -\beta x + y - y(x^2 + y^2), \end{aligned}$$

which has the asymptotically stable solution

$$x = \sin(\beta t), \qquad y = \cos(\beta t).$$

EXERCISE. Couple the above oscillator to the Fitzhugh-Nagumo equations :

$$\begin{aligned} v' &= [v(v-a)(1-v) - w]/\epsilon, \\ w' &= v - dw - (b + r\ x). \end{aligned}$$

If $r = 0$ then there is an exact analytical solution. First continue this solution from $r = 0$ to, say, $r = 0.2$, and then fix $r = 0.2$ and continue the solution in β. (See AUTO demo frc.)

12.7 General Non-Autonomous Systems

If the forcing is not periodic or is difficult to model by an autonomous oscillator then the equations can be rewritten in autonomous form as follows. The system

$$u'(t) = f(t, u(t)), \quad u(\cdot), f(\cdot) \in \mathrm{R}^n, \quad t \in [0,1],$$

$$b(u(0), u(1)) = 0, \qquad b(\cdot) \in \mathrm{R}^n,$$

can be transformed into

$$u'(t) = f(v(t), u(t)), \quad u(\cdot), f(\cdot) \in \mathrm{R}^n, \quad t \in [0,1],$$

$$v'(t) = 1, \qquad v(\cdot) \in \mathrm{R}, \quad t \in [0,1],$$

$$b(u(0), u(1)) = 0, \qquad b(\cdot) \in \mathrm{R}^n,$$

$$v(0) = 0,$$

which is an autonomous system of $n+1$ first order differential equations with $n+1$ boundary conditions.

Chapter 13

Computation of Connecting Orbits

Orbits that connect fixed points of a vector field are important in various applications. A basic algorithm consists of continuation of solutions to the following equations :

$$u'(t) = T f(u(t), \lambda), \qquad u(\cdot), f(\cdot,\cdot) \in \mathrm{R}^n, \qquad \lambda \in \mathrm{R}^{n_\lambda}, \tag{13.1}$$

$$\begin{array}{ll} a) & f(w_0, \lambda) = 0, \\ b) & f(w_1, \lambda) = 0, \end{array} \tag{13.2}$$

$$\begin{array}{lll} a) & f_u(w_0, \lambda) v_{0i} = \mu_{0i} v_{0i}, & i = 1, \cdots, n_0, \\ b) & f_u(w_1, \lambda) v_{1i} = \mu_{1i} v_{1i}, & i = 1, \cdots, n_1, \end{array} \tag{13.3}$$

$$\begin{array}{lll} a) & v_{0i}^* \ v_{0i} = 1, & i = 1, \cdots, n_0, \\ b) & v_{1i}^* \ v_{1i} = 1, & i = 1, \cdots, n_1, \end{array} \tag{13.4}$$

$$\textstyle\int_0^1 \ (f(u,\lambda) - f(\hat{u}, \hat{\lambda}))^* \ f_u(\hat{u}, \hat{\lambda}) \ f(\hat{u}, \hat{\lambda}) \ dt = 0, \tag{13.5}$$

$$\begin{array}{lll} a) & u(0) = w_0 + \epsilon_0 \sum_{i=1}^{n_0} c_{0i} v_{0i}, & \sum_{i=1}^{n_0} c_{0i}^2 = 1, \\ b) & u(1) = w_1 + \epsilon_1 \sum_{i=1}^{n_1} c_{1i} v_{1i}, & \sum_{i=1}^{n_1} c_{1i}^2 = 1. \end{array} \tag{13.6}$$

We now explain the above equations :

Equation (13.1) : This is the ODE with independent variable t scaled to $[0, 1]$.

Equation (13.2) defines two fixed points w_0 and w_1.

Equation (13.3) : We assume $f_u(w_0, \lambda)$ has n_0 distinct real positive eigenvalues μ_{0i} with eigenvectors v_{0i}, and $f_u(w_1, \lambda)$ has n_1 distinct real negative eigenvalues μ_{1i} with eigenvectors v_{1i}.

Equation (13.4) normalizes the eigenvectors.

Equation (13.5) : This *phase condition*, with *reference orbit* $\hat{u}(t)$, is a necessary condition for

$$D(\sigma) \equiv \int_0^1 \parallel u'(t+\sigma) - \hat{u}'(t) \parallel^2 \, dt$$

to be minimized over σ, using $u''(t) = f_u(u, \lambda) u'(t) = f_u(u, \lambda) f(u, \lambda)$.

Equation (13.6) requires $u(0)$ to lie in the tangent manifold U_0 at “distance” ϵ_0 from w_0. Similarly $u(1)$ must lie in S_1 at distance ϵ_1 from w_1.

Using the last equation we can eliminate w_0 and w_1, to be left with n coupled differential equations subject to

$$n_c \equiv 2n + (n+1)(n_0+n_1) + 3$$

constraints. In addition to $u(t) \in \mathrm{R}^n$ we have scalar variables

$$\lambda \in \mathrm{R}^{n_\lambda}, \quad \epsilon_0, \epsilon_1 \in \mathrm{R},$$

$$\mu_{0i}, c_{0i} \in \mathrm{R}, \quad v_{0i} \in \mathrm{R}^n, \quad i = 1, \cdots, n_0,$$

$$\mu_{1i}, c_{1i} \in \mathrm{R}, \quad v_{1i} \in \mathrm{R}^n, \quad i = 1, \cdots, n_1.$$

The total number of scalar variables equals

$$n_v \equiv n_\lambda + (n+2)(n_0+n_1) + 2.$$

Formally we need

$$n_v = n_c - n,$$

for a single heteroclinic connection. This gives

$$n_\lambda \equiv n - (n_0+n_1) + 1.$$

For a branch of orbits, use $n-(n_0+n_1)+2$ free parameters. T is large and fixed in this continuation.

Consider the special case $n=2$, $n_0=n_1=1$. (See Figure 13.1.) Then $n_\lambda+1=2$, i.e., a branch of heteroclinic orbits requires two free problem parameters (λ_1, λ_2). Consider λ_2 as fixed here. For $\lambda_1 = \lambda_1^*$ we assume the existence of the heteroclinic connection in **(b)**. Generically, perturbation of λ_1 will produce either **(a)** or **(c)**, depending on the sign of the perturbation. If ϵ_0 and ϵ_1 are sufficiently small, then there exists a λ_1 close to λ_1^* for which the equations (13.1-13.4) (and (13.6)) can be satisfied (Here in case **(a)**). Furthermore, the radii ϵ_0 and ϵ_1 can be chosen such that the period of the orbit equals a given large value T, and such that the phase condition (13.5) is satisfied.

Some more particular cases :

(1) The connection of a saddle to a node in R^2. Here $n=2$, $n_0=1$, $n_1=2$, so $n_\lambda=0$. A branch of connections requires one problem parameter. Example : Traveling waves in Fisher's equation.

(2) If $n=3$, $n_0=3$, $n_1=2$. Then $n_\lambda=-1$, which means that a 2-D manifold of connecting orbits is already possible for fixed problem parameters.

(3) The *homoclinic orbit.* In this case $w_0=w_1$ and $n_0+n_1=n$, so that $n_\lambda=1$. Such orbits can also be computed as the limit of periodic orbits as the period $T\to\infty$.

EXERCISE. Compute a branch of traveling wave front solutions to the Nagumo equation (AUTO demo nag) :

$$w_t = w_{xx} + f(w,a), \qquad -\infty < x < \infty, \quad t > 0,$$

$$f(w,a) \equiv w(1-w)(w-a), \qquad 0 < a < 1.$$

Look for solutions to of the form

$$w(x,t) = u(x+ct),$$

where c is the *wave speed.* This gives the first order system

$$\begin{aligned} u_1'(z) &= u_2(z), \\ u_2'(z) &= cu_2(z) - f(u_1(z),a), \end{aligned}$$

where $z = x + ct$, and $' = d/dz$. If $a = 0.5$ and $c = 0$ then there is a heteroclinic cycle connecting $(0,0)$ and $(1,0)$. One of the two heteroclinic orbits has the exact representation

$$u_1(z) = \frac{e^{\frac{1}{2}\sqrt{2}z}}{1+e^{\frac{1}{2}\sqrt{2}z}}, \qquad u_2(z) = u_1'(z), \qquad -\infty < z < \infty.$$

The other heteroclinic connection is obtained by symmetry. These solutions can be used as starting points in the 2-parameter (here a and c) continuation. The orbits actually remain unchanged along the two branches, which have analytical representation $c = \pm\sqrt{2}(a - \frac{1}{2})$.

EXERCISE. Compute a branch of traveling wave front solutions to the Fisher equation (AUTO demo fsh) :

$$w_t = w_{xx} + f(w), \qquad -\infty < x < \infty, \quad t > 0,$$

$$f(w) \equiv w(1-w),$$

Look for solutions of the form

$$w(x,t) = u(x+ct),$$

where c is the *wave speed.* This gives the first order system

$$\begin{aligned} u_1'(z) &= u_2(z), \\ u_2'(z) &= cu_2(z) - f(u_1(z)). \end{aligned}$$

The fixed point $(0,0)$ has two positive eigenvalues when $c > 2$. The other fixed point at $(1,0)$ is a saddle point. Thus here we have $n = 2, \quad n_0 = 1, \quad n_1 = 2, \quad n_\lambda = 0$. A branch of connecting orbits requires one free parameter (c).

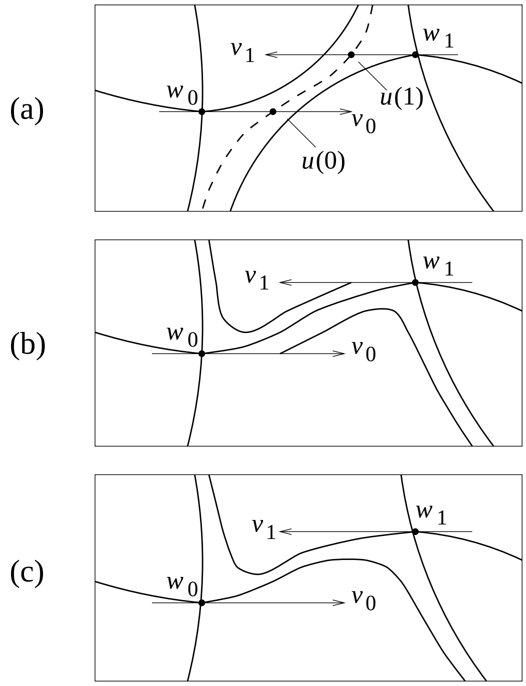

Figure 13.1: Geometric interpretation of the equations for for computing heteroclinic orbits in the special case $n = 2$, $n_0 = n_1 = 1$.

Chapter 14

Discretization

14.1 Orthogonal Collocation

Recall the ODE problem introduced in Section 12.1 :

$$u'(t) - f(u(t), \mu, \lambda) = 0, \qquad t \in [0,1],$$

where

$$u(\cdot), f(\cdot) \in \mathrm{R}^n, \qquad \lambda \in \mathrm{R}, \qquad \mu \in \mathrm{R}^{n_\mu},$$

subject to boundary conditions

$$b(u(0), u(1), \mu, \lambda) = 0, \qquad b(\cdot) \in \mathrm{R}^{n_b},$$

and integral constraints

$$\int_0^1 q(u(s), \mu, \lambda)\ ds = 0, \qquad q(\cdot) \in \mathrm{R}^{n_q},$$

with

$$n_\mu = n_b + n_q - n \geq 0.$$

The "extra" parameter λ will be freed in continuation. Introduce a mesh

$$\{0 = t_0 < t_1 < \cdots < t_N = 1\}, \qquad \Delta t_j \equiv t_j - t_{j-1}, \quad (1 \leq j \leq N),$$

Define

$$\mathcal{P}_h^m = \{p_h \in C[0,1] \ :\ p_h\Big|_{[t_{j-1},t_j]} \in \mathcal{P}^m\},$$

where $\mathcal{P}^m$ is the space of polynomials of degree $\leq m$.

The collocation method consists of finding $p_h \in \mathcal{P}_h^m$, $\mu \in \mathrm{R}^{n_\mu}$, such that

$$p_h'(z_{j,i}) = f(p_h(z_{j,i}), \mu, \lambda)), \quad j = 1, \cdots, N, \quad i = 1, \cdots, m,$$

and such that p_h satisfies the boundary and integral conditions.

In each suninterval $[t_{j-1}, t_j]$ the *collocation points* $z_{j,i}$ are the (transformed) roots of the mth orthogonal polynomial (*Gauss points*). See Figure 14.1 for a graphical interpretation.

Since each local polynomial is determined by $(m+1)n$ coefficients, the total number of degrees of freedom is $(m+1)nN + n_\mu$. This is matched by the total number of equations :

$$\begin{aligned} \text{collocation :} \quad & mnN, \\ \text{continuity :} \quad & (N-1)n, \\ \text{constraints :} \quad & n_b + n_q (= n + n_\mu). \end{aligned}$$

If the solution $u(t)$ is sufficiently smooth then the global accuracy of this method is known to be of order m, i.e.,

$$\| p_h - u \|_\infty = \mathcal{O}(h^m).$$

At the main meshpoints t_j we have *superconvergence* :

$$max_j \mid p_h(t_j) - u(t_j) \mid = \mathcal{O}(h^{2m}).$$

The scalar variables μ are also superconvergent.

14.2 Implementation in AUTO

For each subinterval $[t_{j-1}, t_j]$ of the mesh introduce the Lagrange basis polynomials

$$\{\ell_{j,i}(t)\}, \quad j = 1, \cdots, N, \qquad i = 0, 1, \cdots, m,$$

defined by

$$\ell_{j,i}(t) = \prod_{k=0, k\neq i}^{m} \frac{t - t_{j-\frac{k}{m}}}{t_{j-\frac{i}{m}} - t_{j-\frac{k}{m}}}, \qquad t_{j-\frac{i}{m}} \equiv t_j - \frac{i}{m}\Delta t_j.$$

The local polynomials can then be written

$$p_j(t) = \sum_{i=0}^{m} \ell_{j,i}(t) u_{j-\frac{i}{m}},$$

and the collocation equations are

$$p_j'(z_{j,i}) = f(p_j(z_{j,i}), \mu, \lambda), \qquad i = 1, \cdots, m, \quad j = 1, \cdots, N.$$

With the above choice of basis

$$u_j \sim u(t_j) \qquad \text{and} \qquad u_{j-\frac{i}{m}} \sim u(t_{j-\frac{i}{m}})$$

where $u(t)$ is the solution of the continuous problem.

The discrete boundary conditions are

$$b_i(u_0, u_N, \mu, \lambda) = 0, \quad i = 1, \cdots, n_b.$$

The integrals can be discretized as

$$\sum_{j=1}^{N} \sum_{i=0}^{m} \omega_{j,i} q_k(u_{j-\frac{i}{m}}, \mu, \lambda) = 0, \qquad k = 1, \cdots, n_q,$$

where the $\omega_{j,i}$ are the Lagrange quadrature coefficients.

The pseudo-arclength equation is

$$\int_0^1 (u(t) - u_0(t))^* \dot{u}_0(t) dt + (\mu - \mu_0)^* \dot{\mu}_0 + (\lambda - \lambda_0)\dot{\lambda}_0 - \Delta s = 0,$$

where

$$(u_0, \mu_0, \lambda_0),$$

is the previously computed point on the solution branch and

$$(\dot{u}_0, \dot{\mu}_0, \dot{\lambda}_0),$$

is the normalized direction of the branch at that point.

The discretized pseudo-arclength equation is

$$\sum_{j=1}^{N} \sum_{i=0}^{m} \omega_{j,i} (u_{j-\frac{i}{m}} - (u_0)_{j-\frac{i}{m}})^* (\dot{u}_0)_{j-\frac{i}{m}} + (\mu - \mu_0)^* \dot{\mu}_0 + (\lambda - \lambda_0)\dot{\lambda}_0 - \Delta s = 0.$$

14.3 Numerical Linear Algebra

The complete discretization consists of

$$mnN + n_b + n_q + 1$$

nonlinear equations in the unknowns

$$\{u_{j-\frac{i}{m}}\} \in \mathrm{R}^{mnN+n}, \qquad \mu \in \mathrm{R}^{n_\mu}, \qquad \lambda \in \mathrm{R}.$$

These equations can be solved by a Newton-Chord iteration.

The structure and solution method of the linear systems is illustrated in Figures 14.2-14.6. The indicated operations are also carried out on the right hand side, which is not shown in the Figures.

14.4 Numerical Examples

EXERCISE. A simple accuracy test is to treat the linear equation

$$\begin{aligned} u_1'(t) &= (1-\lambda)u_1 - u_2, \\ u_2'(t) &= u_1, \end{aligned}$$

as a bifurcation problem. This equation has a Hopf bifurcation point at $\lambda = 1$ from the zero solution branch. The bifurcating branch of periodic solutions is vertical and along it the period remains constant, namely, $T = 2\pi$. Compute a selected periodic solution, for different choices of the number of mesh points and the number of collocation points, and determine the error in the period for the selected solution. Typical results are shown in Table 14.1.

ntst	ncol = 2		ncol = 3		ncol = 4	
4	0.47e-1	(5.5)	0.85e-3	(7.9)	0.85e-5	(8.9)
8	0.32e-2	(4.0)	0.14e-4	(6.0)	0.35e-7	(8.0)
16	0.20e-3	(4.0)	0.22e-6	(6.0)	0.14e-9	(8.0)
32	0.13e-4		0.35e-8		0.54e-11	

Table 14.1: Accuracy of T for the linear problem

EXERCISE. Consider the equations

$$\begin{aligned} u'(t) &= \lambda \sin(u) - v, \\ v'(t) &= 2u - v, \end{aligned}$$

which has a Hopf bifurcation point at $\lambda = 1$. Compute a selected periodic solution, for different choices of the number of mesh points and the number of collocation points, and determine the error in the Floquet multiplier $z = 1$. Typical results are shown in Table 14.2.

ntst	ncol = 2		ncol = 3		ncol = 4	
4	0.12e+0	(6.5)	0.14e-1	(9.1)	0.10e-2	(11.9)
8	0.13e-2	(4.8)	0.25e-4	(5.4)	0.26e-6	(7.0)
16	0.45e-4	(4.2)	0.58e-6	(6.1)	0.21e-8	(8.0)
32	0.25e-5		0.84e-8		0.80e-11	

Table 14.2: Accuracy of the Floquet multiplier at $z = 1$

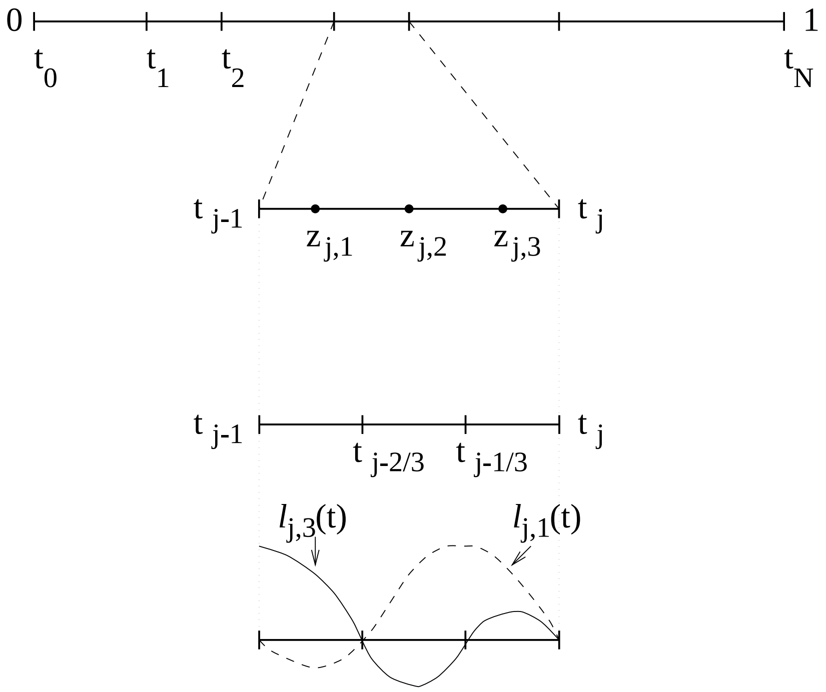

Figure 14.1: The mesh $\{0 = t_0 < t_1 < \cdots < t_N = 1\}$. Collocation points and "extended-mesh"-points are shown for the case $m = 3$ in the jth mesh interval. Also shown are two of the four local Lagrange basis polynomials.

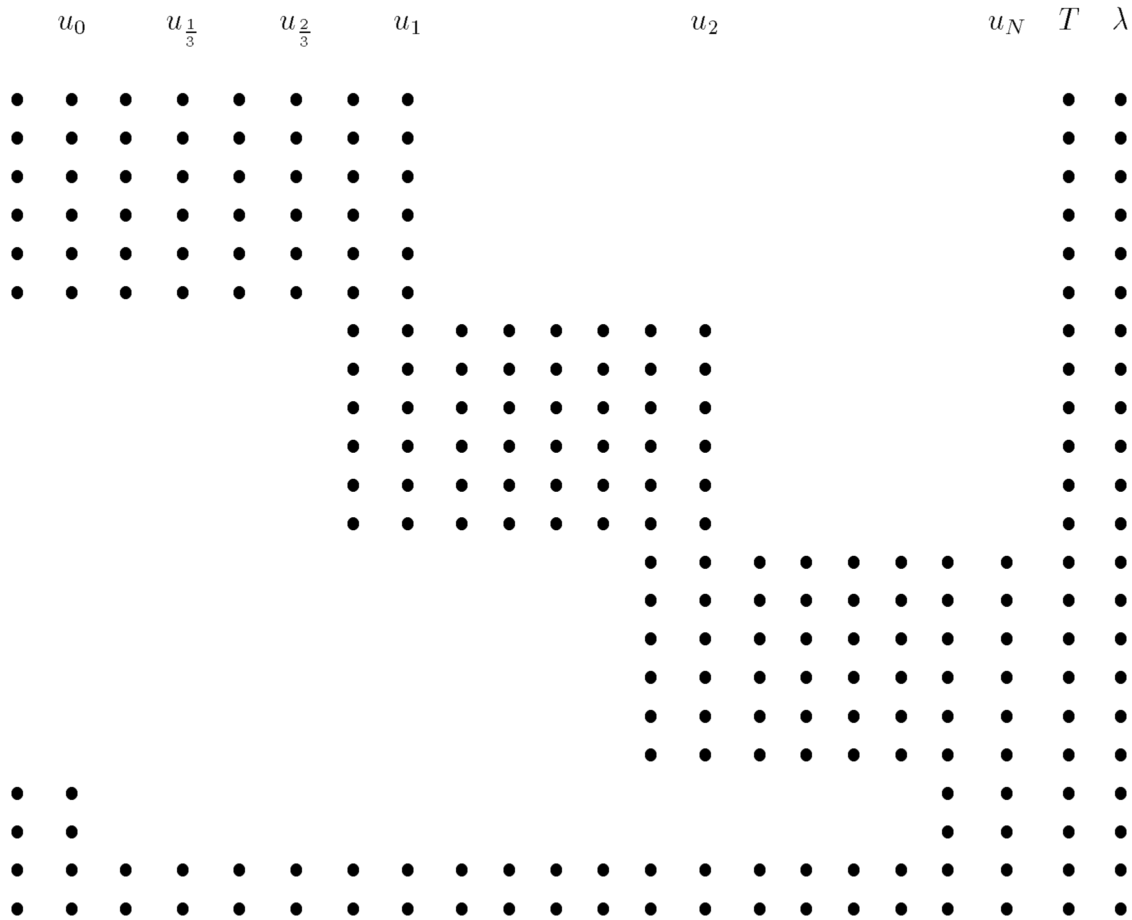

Figure 14.2: Structure of the Jacobian for the case of $n = 2$ differential equations, number of mesh intervals $N = 3$, number of collocation points per mesh interval $m = 3$, number of boundary conditions $n_b = 2$ and number of integral constraints $n_q = 1$. The last row corresponds to the pseudo-arclength equation, which is not included in the $n_q = 1$ count. In a typical problem N will be larger, say, $N = 5$ for "very easy" problems, and $N = 200$ for "very difficult" problems.

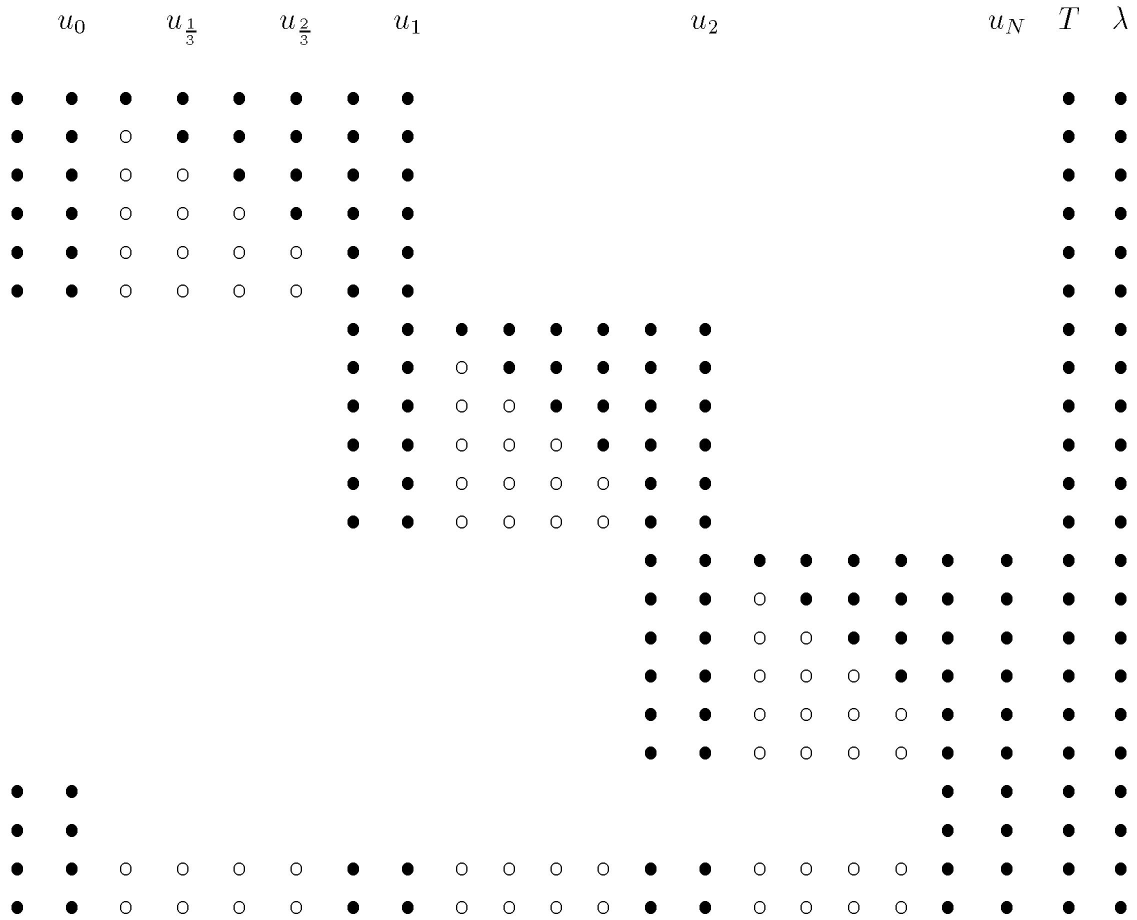

Figure 14.3: The system after "condensation of parameters". The entries ∘ have been eliminated by Gauss elimination. These operations can be done in parallel.

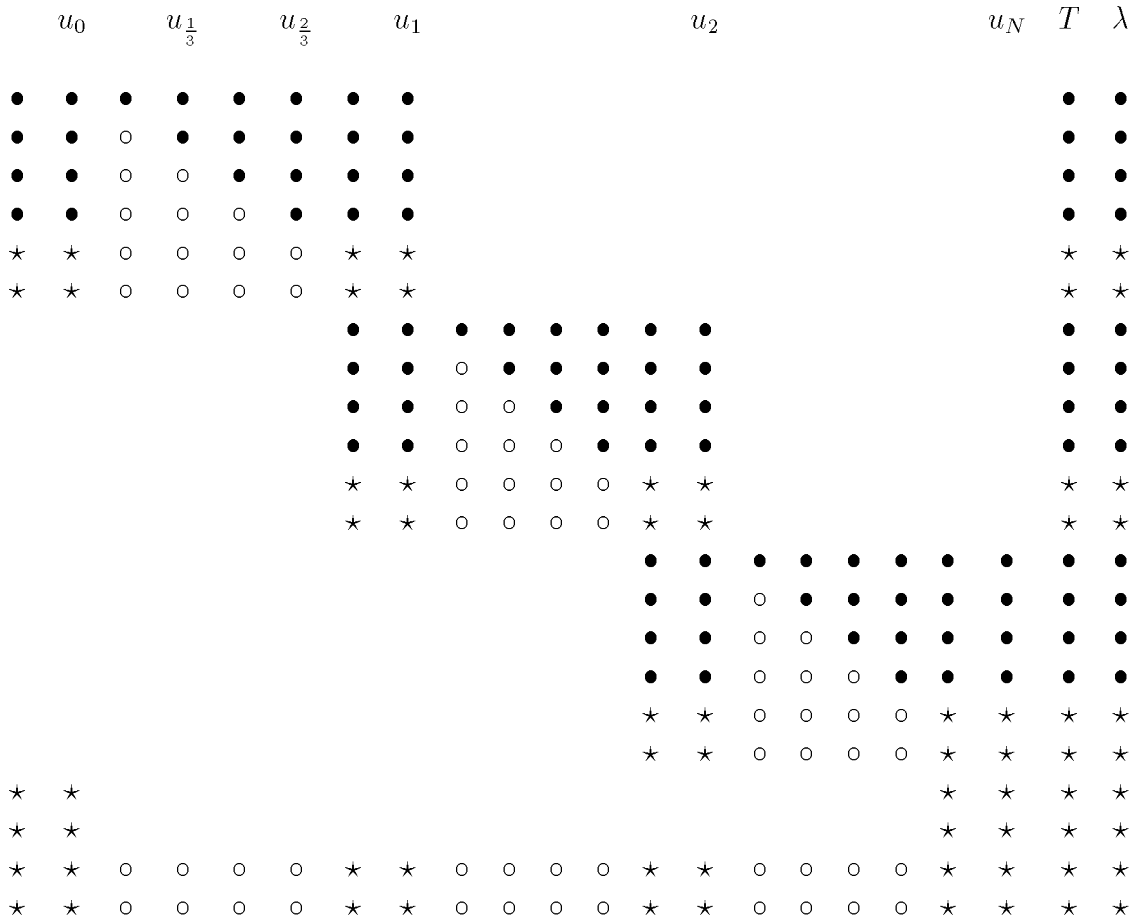

Figure 14.4: This is the same matrix as in the preceding Figure, except that some entries are now marked by a $\star$. Note that the $\star$ sub-system is fully decoupled from the remaining equations and can therefore be solved separately. This is known as "condensation of parameters".

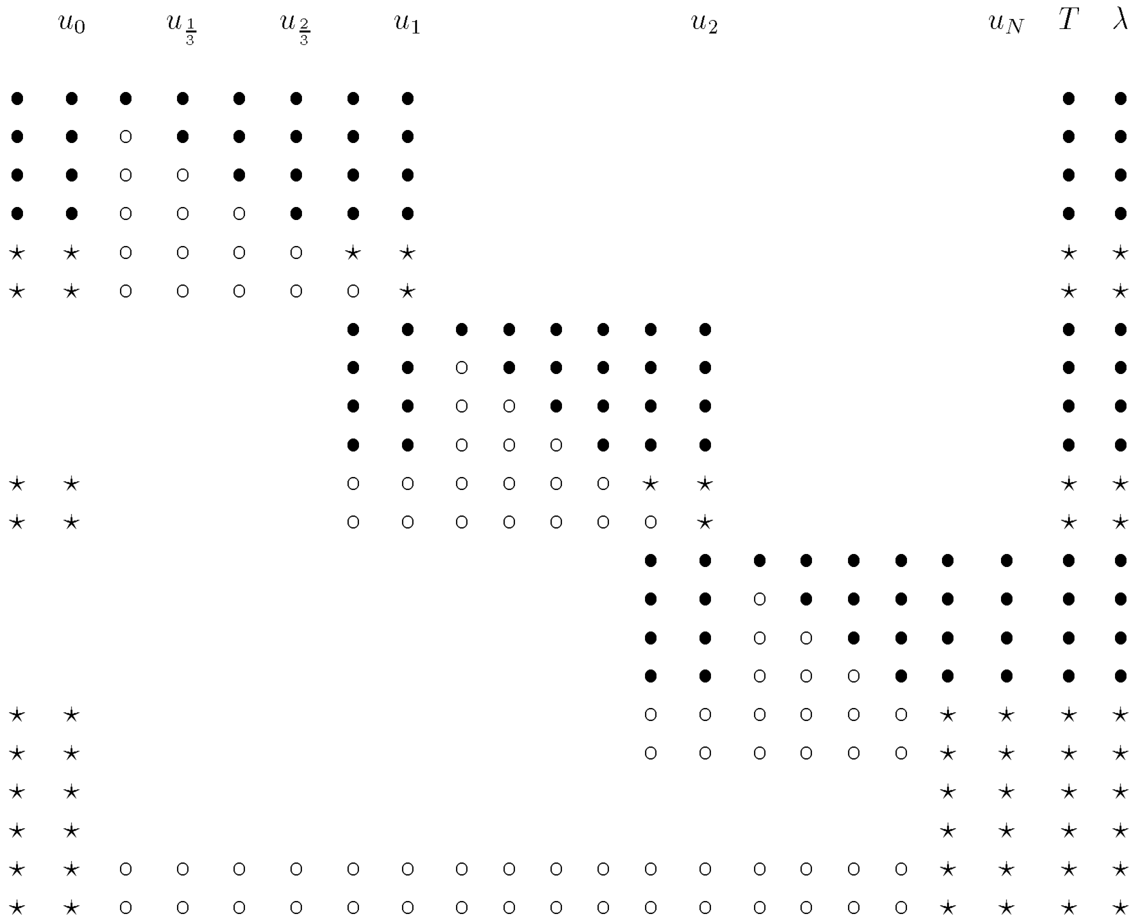

Figure 14.5: The decoupled $\star$ system can be solved, for example, by nested dissection. This eliminates some of the $\star$, but also introduces some new ones, due to fill-in.

$$
\begin{array}{cccccccccccccccccccccc}
 & u_0 & & u_{\frac{1}{3}} & & u_{\frac{2}{3}} & & u_1 & & & & & & u_2 & & & & & & u_N & T & \lambda \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & & & & & & & & & & & & \bullet & \bullet \\
\bullet & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & & & & & & & & & & & & & \bullet & \bullet \\
\bullet & \bullet & \circ & \circ & \bullet & \bullet & \bullet & \bullet & & & & & & & & & & & & & \bullet & \bullet \\
\bullet & \bullet & \circ & \circ & \circ & \bullet & \bullet & \bullet & & & & & & & & & & & & & \bullet & \bullet \\
\star & \star & \circ & \circ & \circ & \circ & \star & \star & & & & & \cdot & \cdot & & & & & & & \star & \star \\
\star & \star & \circ & \circ & \circ & \circ & \circ & \star & & & & & \cdot & \cdot & & & & & & & \star & \star \\
 & & & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & & & & & & \bullet & \bullet \\
 & & & & & & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & & & & & & & \bullet & \bullet \\
 & & & & & & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \bullet & \bullet & & & & & & & \bullet & \bullet \\
 & & & & & & \bullet & \bullet & \circ & \circ & \circ & \bullet & \bullet & \bullet & & & & & & & \bullet & \bullet \\
\star & \star & & & & & \circ & \circ & \circ & \circ & \circ & \circ & \star & \star & & & & & \cdot & \cdot & \star & \star \\
\star & \star & & & & & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \star & & & & & \cdot & \cdot & \star & \star \\
 & & & & & & & & & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & & & & & & & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & & & & & & & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & A_0 & & & & & & & & & & & \bullet & \bullet & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\
+ & + & & & & & & & & & & & \circ & \circ & \circ & \circ & \circ & \circ & + & + & + & + \\
+ & + & & & & & & & & & & & \circ & \circ & \circ & \circ & \circ & \circ & + & + & + & + \\
+ & + & & & & & & & & & & & & & & & & A_1 & + & + & + & + \\
+ & + & & & & & & & & & & & & & & & & & + & + & + & + \\
+ & + & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & + & + & + \\
+ & + & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & + & + & +
\end{array}
$$

Figure 14.6: This is the same system as in the preceding Figure, except that some entries are now marked by a $+$. Note that the $+$ sub-system is decoupled from the other equations and can therefore be solved separately. For periodic solutions the Floquet Multipliers are the eigenvalues of $-A_1^{-1}A_0$.

Chapter 15

Numerical Stability and Convergence

15.1 An Introductory Example

$$Lu \equiv u''(t) - a(t)u(t) = f(t), \qquad a(t) \geq a_* > 0, \qquad u(0) = u(1) = 0.$$

Finite difference approximation :

$$(\frac{1}{h^2})u_{j-1} - (a_j + \frac{2}{h^2})u_j + (\frac{1}{h^2})u_{j+1} = f(t_j), \qquad j = 1, 2, ..., N-1,$$

In matrix form

$$L_h u_h = f_h,$$

i.e.,

$$\begin{pmatrix} -a_1 - \frac{2}{h^2} & \frac{1}{h^2} & & & \\ \frac{1}{h^2} & -a_2 - \frac{2}{h^2} & \frac{1}{h^2} & & \\ & \cdot & \cdot & \cdot & \\ & & \frac{1}{h^2} & -a_{N-1} - \frac{2}{h^2} & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f(t_1) \\ f(t_2) \\ \cdot \\ f(t_{N-1}) \end{pmatrix}.$$

It can easily be shown by Taylor expansion that

$$\tau_j \equiv (u(t_{j-1}) - 2u(t_j) + u(t_{j+1}))/h^2 - u''(t_j), \qquad j = 1, 2, ..., N-1,$$

satisfies

$$| \tau_j | \leq K_T h^2, \text{ for some constant } K_T = K_T[u].$$

Note that

$$\begin{aligned} \frac{1}{h^2}u(t_{j-1}) \quad &- \quad (a_j + \frac{2}{h^2})u(t_j) + \frac{1}{h^2}u(t_{j+1}) \\ &= \quad \frac{u(t_{j-1}) - 2u(t_j) + u(t_{j+1})}{h^2} - a_j u(t_j) \\ &= \quad u''(t_j) + \tau_j - a_j u(t_j) \quad = \quad \tau_j + f(t_j). \end{aligned}$$

Thus, if we define

$$\tilde{u}_h \equiv (u(t_1), u(t_2), ..., u(t_{N-1}))^T, \qquad \tau_h \equiv (\tau_1, \tau_2, ..., \tau_{N-1})^T,$$

then

$$L_h \tilde{u}_h = \tau_h + f_h.$$

Since

$$L_h u_h = f_h,$$

it follows that

$$L_h(\tilde{u}_h - u_h) = \tau_h.$$

Therefore, if L_h has an inverse and if

$$\| L_h^{-1} \|_\infty \leq K_S,$$

for some constant K that does not depend on the mesh h, then

$$\| \tilde{u}_h - u_h \|_\infty = \| L_h^{-1} \tau_h \|_\infty \leq \| L_h^{-1} \|_\infty \| \tau_h \|_\infty \leq \ K_S K_T h^2.$$

Now

$$L_h = \begin{pmatrix} -\frac{2}{h^2} - a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & -\frac{2}{h^2} - a_{N-1} \end{pmatrix} + \frac{1}{h^2} \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \cdot & \cdot & \cdot \\ & & 1 & 0 \end{pmatrix}$$

$$\equiv D_h + \frac{1}{h^2} E_h \ = \ D_h(I_h + D_h^{-1} \frac{1}{h^2} E_h) \ \equiv \ D_h(I_h + B_h),$$

where I_h is the identity matrix and

$$B_h \equiv D_h^{-1} \frac{1}{h^2} E_h.$$

Since

$$\| B_h \|_\infty \leq \| D_h^{-1} \|_\infty \ \frac{1}{h^2} \ \| E_h \|_\infty = \ \frac{1}{\frac{2}{h^2} + a_*} \frac{1}{h^2} \, 2 = \frac{1}{1 + \frac{h^2}{2} a_*} \ < 1,$$

it follows from the Banach Lemma (Section 1.3) that $(I_h + B_h)^{-1}$ exists and

$$\| (I_h + B_h)^{-1} \|_\infty \leq \ \frac{1}{1 - \| B_h \|_\infty}.$$

Therefore L_h^{-1} also exists,

$$L_h^{-1} = (I_h + B_h)^{-1} D_h^{-1},$$

and

$$\| L_h^{-1} \|_\infty \leq \| (I_h + B_h)^{-1} \|_\infty \| D_h^{-1} \|_\infty \leq \ \frac{1}{1 - \frac{1}{1 + \frac{h^2}{2} a_*}} \ \frac{1}{\frac{2}{h^2} + a_*} \ = \ \frac{1}{a_*}.$$

Thus the stability constant K_S equals $1/a_*$, and therefore

$$\| \tilde{u}_h - u_h \|_\infty \leq \ \frac{K_T}{a_*} \ h^2.$$

REMARK. Note that the "*diagonal dominance*" property of the discretization matrix L_h was used in an essential way. In fact, the above method of proof has very limited applicability. For example, it can not be carried out if $a(x)$ is not strictly positive, even if the continuous problem has a unique solution.

In general, to prove numerical stability, one needs a compactness argument plus bounding of certain interpolation errors. Below we give a sketch of the theory, leaving out some technical details.

15.2 Numerical Stability for Linear Problems

Let

$$Lu = f, \qquad L \ : \ \mathcal{B}_1 \to \mathcal{B}_2,$$

denote a general linear ODE subject to boundary- and other side conditions. Here $\mathcal{B}_1$ and $\mathcal{B}_2$ are appropriate function spaces.

The discretization is written as

$$L_h u_h = f_h, \qquad L_h \ : \ \mathcal{B}_1^h \to \mathcal{B}_2^h,$$

where

$$dim(\mathcal{B}_1^h) = dim(\mathcal{B}_2^h) < \infty.$$

Let $\tilde{u}_h$ denote the interpolant of $u \in \mathcal{B}_1$ in $\mathcal{B}_1^h$.

THEOREM.

Assume that $Lv = 0$ implies $v = 0$, and that all coefficient functions in L are sufficiently smooth. Also assume that the approximation scheme is consistent, i.e.,

$$\tau_h \ \equiv \ L_h \tilde{u}_h \ - \ f_h \ \to \ 0 \quad as \quad h \ \to \ 0.$$

Finally assume that the approximation scheme is as compact as possible. Then for all small enough h (i.e., for all sufficiently fine meshes) we have that L_h^{-1} exists, and

$$\| L_h^{-1} \| \leq K_s, \qquad (numerical \ stability),$$

and hence convergence :

$$\| u_h - \tilde{u}_h \| \leq \| L_h^{-1} \| \, \| \tau_h \| \to 0 \quad as \quad h \to 0. \qquad \bullet$$

REMARK. For a difference scheme to be "as compact as possible", it must use the smallest possible difference stencil, for example, a two-point approximation for a second order problem. For piecewise polynomial collocation schemes, the requirement means that minimal global continuity must be required, for example C^1 for a second order problem.

15.3 Existence and Convergence for Nonlinear Problems

Denote the nonlinear ordinary differential equation by

$$N(u) = 0, \qquad N \ : \ \mathcal{B}_1 \to \mathcal{B}_2.$$

Write the discretization as

$$N_h(u_h) = 0, \qquad N_h \ : \ \mathcal{B}_1^h \to \mathcal{B}_2^h,$$

where $\mathcal{B}_1^h$ and $\mathcal{B}_2^h$ have the same (finite) dimension.

Let $\tilde{u}_h \in \mathcal{B}_1^h$ be the interpolant of a solution $u \in \mathcal{B}_1$ of $N(u) = 0$.

Let $L[u]$ denote the linearization of

$$N(u) \equiv u' - f(u)$$

about $u \in \mathcal{B}_1$, i.e.,

$$L[u] \ v = v' - f'(u) \ v.$$

Similarly

$$L_h[u_h] \text{ is the linearization of } N_h \text{ about } u_h,$$

and

$$L_h[\tilde{u}_h] \text{ is the linearization of } N_h \text{ about } \tilde{u}_h.$$

REMARK. For simplicity of presentation we consider a first order system and we assume that boundary and integral conditions are satisfied in $\mathcal{B}_1$ and $\mathcal{B}_1^h$ and that there are no scalar unknowns.

THEOREM.

Let u be any solution of $N(u) = 0$ satisfying

(i) $L \ [u] \ v = 0$ implies $v = 0$,

(ii) $\parallel L_h[u_h] - L_h[v_h] \parallel \leq K_L \parallel u_h - v_h \parallel$ for all $u_h, v_h \in B_{R_0}(\tilde{u}_h)$ and all $h \leq H_0$, (i.e., L_h is Lipschitz continuous near $\tilde{u}_h$),

(iii) $L_h[\tilde{u}_h]^{-1}$ exists and $\parallel L_h^{-1}(\tilde{u}_h) \parallel \leq K_S$ for all $h \leq H_0$, where K_S is independent of the mesh h, (i.e., the discretization is stable for linear problems),

(iv) $\tau_h \equiv N_h(\tilde{u}_h)$ satisfies $\parallel \tau_h \parallel \to 0$ as $h \to 0$, (consistency).

Then for any sufficiently small r there exists an $h_0 > 0$ such that

$$N_h(u_h) = 0$$

has an isolated solution in $B_r(\tilde{u}_h)$ for all $h \leq h_0$. •

REMARK. In the conclusion of the theorem the term "isolated" means that $L_h^{-1}[u_h]$ exists and that its norm is uniformly bounded.

NOTATION. We use h to denote the entire mesh as well as the maximum mesh size.

The proof of the theorem uses :

(a) The Banach Lemma (see Section 1.3) :

$$\| B_h \| < 1 \ \Rightarrow \ \begin{cases} (I_h + B_h)^{-1} \text{ exists,} \\ \\ \| (I_h + B_h)^{-1} \| \leq 1/(1 - \| B_h \|). \end{cases}$$

(b) The Mean Value Theorem :

$$N_h(u_h) - N_h(v_h) = \hat{L}_h[u_h, v_h](u_h - v_h),$$

where

$$\hat{L}_h[u_h, v_h] \equiv \int_0^1 L_h[tu_h + (1-t)v_h] \ dt.$$

(c) The Brouwer Fixed Point Theorem :

$$F_h \ : \ B_r(\tilde{u}_h) \to B_r(\tilde{u}_h)$$

has a fixed point.

PROOF. First choose r such that

$$r < R_0 \quad \text{and} \quad K_S K_L r < \frac{1}{2}.$$

Then choose h such that

$$K_S \| N_h(\tilde{u}_h) \| < \frac{r}{2} \quad \text{and} \quad h \leq H_0.$$

Let $v_h \in B_r(\tilde{u}_h)$. Consider the fixed point iteration

$$u_h^{(k+1)} = F_h(u_h^{(k)}), \qquad k = 0, 1, 2, \cdots$$

where

$$F_h(v_h) \equiv v_h - L_h^{-1}[\tilde{u}_h] \ N_h(v_h).$$

(Note that this is the Chord Method centered at $\tilde{u}_h$ for solving $N_h(u_h) = 0$.)

Let $v_h \in B_r(\tilde{u}_h)$. Then

$$\begin{aligned}
&\| F_h(v_h) - \tilde{u}_h \| \\
&= \| v_h - \tilde{u}_h \ - \ L_h^{-1}[\tilde{u}_h] \ N_h(v_h) \| \\
&= \| v_h - \tilde{u}_h \ - \ L_h^{-1}[\tilde{u}_h] \ \Big(N_h(\tilde{u}_h) + \hat{L}_h[v_h, \tilde{u}_h](v_h - \tilde{u}_h) \ \Big) \| \\
&= \| L_h^{-1}[\tilde{u}_h] \ N_h(\tilde{u}_h) \| \ + \ \| L_h^{-1}[\tilde{u}_h] \ \Big(L_h[\tilde{u}_h] \ - \ \hat{L}_h[v_h, \tilde{u}_h] \Big) (v_h - \tilde{u}_h) \| \\
&\leq K_S \| N_h(\tilde{u}_h) \| \ + K_S \| L_h[\tilde{u}_h] - \hat{L}_h[v_h, \tilde{u}_h] \| \ r \\
&\leq \tfrac{r}{2} \ + \ K_S \| \textstyle\int_0^1 L_h[t\tilde{u}_h + (1-t)\tilde{u}_h] - L_h[tv_h + (1-t)\tilde{u}_h] \ dt \| \ r \\
&\leq \tfrac{r}{2} \ + \ K_S K_L r^2 \ < \ r.
\end{aligned}$$

Thus F_h maps $B_r(\tilde{u}_h)$ into itself and hence F_h has a fixed point, i.e., there exist u_h such that

$$u_h = F_h(u_h),$$

i.e.,

$$u_h = u_h - L_h^{-1}(\tilde{u}_h)N_h(u_h),$$

i.e.,

$$N_h(u_h) = 0.$$

That the solution u_h is isolated can be seen as follows.

$$\begin{aligned} L_h[u_h] &= L_h[\tilde{u}_h] \ + \ L_h[u_h] \ - \ L_h[\tilde{u}_h] \\ &= L_h[\tilde{u}_h] \ \{I_h \ + \ L_h^{-1}[\tilde{u}_h] \ (L_h[u_h] - L_h[\tilde{u}_h]) \ \} \\ &\equiv L_h[\tilde{u}_h] \ \{I_h + B_h\}, \end{aligned}$$

where

$$\parallel B_h \parallel \leq K_S K_L r < \frac{1}{2}.$$

Hence, by the Banach Lemma, $L_h^{-1}[u_h]$ exists, and

$$\parallel L_h^{-1}[u_h] \parallel \leq \frac{K_S}{1 - \frac{1}{2}} = 2K_S. \quad \bullet$$

15.4 Multiplicity of Solutions

THEOREM.

Assume that the continuous problem

$$N(u) \equiv u' - f(u),$$

has exactly n_u solutions

$$u^{[k]}, \qquad k = 1, \cdots, n_u, \qquad n_u < \infty.$$

Assume each $\tilde{u}^{[k]}$ satisfies the conditions of the previous Theorem.

Then for every R with

$$R > \max_{1 \leq k \leq n_u} \parallel u^{[k]} \parallel,$$

there exists a $\delta_R > 0$ such that the discrete problem

$$N_h(u_h) = 0,$$

has exactly n_u solutions $u_h^{[k]}$ with

$$\| u_h^{[k]} \| < R,$$

for all meshes h with $h < \delta_R$. $\bullet$

Under rather restrictive assumptions on the differential operator it is sometimes possible to derive a priori bounds of the form

$$\| u_h \| \leq K$$

for all solutions of

$$N_h(u_h) = 0,$$

and for all meshes h with h sufficiently small. It then follows from the Theorem that extraneous solutions disappear as $h \to 0$. This is the case, for example, in the following class of problems :

COROLLARY. Assume that the conditions of the Theorem are satisfied and that for all sufficiently small h we have

(i) $N_h = L_h + G_h$, L_h linear, invertible , $\| L_h^{-1} \| \leq K_S^*$,

(ii) For each $\epsilon > 0$ there is a positive number $K(\epsilon)$ such that

$$\| G_h(u_h) \| \leq K(\epsilon) + \epsilon \| u_h \|,$$

for all solutions u_h of $N_h(u_h) = 0$ ("sublinearity").

Here K_S^*, $K(\epsilon)$, and ϵ are independent of h.

Then for all sufficiently small h the discrete problem

$$N_h(u_h) = 0,$$

has the same number of solutions as the continuous problem

$$N(u) = 0. \qquad \bullet$$

PROOF. We need only establish that all solutions of $N_h(u_h) = 0$ are uniformly bounded :

$$\| u_h \| \leq \| L_h^{-1} G_h(u_h) \| \leq K_S^* \| G_h(u_h) \| \leq K_S^* [K(\epsilon) + \epsilon \| u_h \|].$$

Thus if we choose $\epsilon < 1/K_S^*$ then

$$\| u_h \| < \frac{K_S^* K(\epsilon)}{1 - \epsilon K_S^*}. \qquad \bullet$$

REMARK.

Condition (i) of the Corollary is usually satisfied.

Condition (ii) is very restrictive. It is satisfied, for example, if the nonlinearity is bounded.

EXERCISE. Consider the first order system

$$u_1' - u_2 = 0,$$

$$u_2' + (\lambda\pi)^2 sin(u_1 + u_1^2 + u_1^3) = 0, \qquad t \in [0,1],$$

$$u_1(0) = 0, \quad u_1(1) = 0,$$

with λ as bifurcation parameter. This is a nonlinear eigenvalue problem, which has a countably infinite number of bifurcations from the trivial solution branch. For any fixed λ the number of solution is finite. Note that the nonlinearity is bounded here, so that the Corollary applies for any λ for which the continuous problem has isolated solutions only. Compute the first few bifurcating branches using a very crude *fixed* mesh, to see whether there are spurious solutions, and if so, how fine the mesh must be for these to disappear. (See AUTO demo ext.)