Double Series Representation of Bounded Signals

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Abstract **—Series representations of the form**

$$f(t) \sim \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{n,k} v(t-n) e^{2\pi i k t}$$

for bounded signals $f(t)$ are studied, as are conditions on the unit function $v(t)$, such that the coefficients $a_{n,k}$ reveal the energy content of $f(t)$ in the time interval $n-(1/2) \le t \le n+(1/2)$ and frequency interval $2\pi(k-(1/2)) \le \omega \le 2\pi(k+(1/2))$. These conditions turn out to be 1) orthogonality, i.e.,

$$\int_{-\infty}^{\infty} v(t) v(t-n) e^{ik2\pi t}\, dt = \begin{cases} 1, & \text{if } n=k=0 \\ 0, & \text{otherwise} \end{cases}$$

and 2) integrability,

$$v(t) \in L^1(R) \qquad V(\omega) \in L^1(R).$$

Based on these conditions a number of properties of the expansion are derived, including summability of the double series and energy and power estimations. A unit function $v(t)$ is constructed which is optimal, within a restricted class, with respect to the duration of $V(\omega)$, and also a unit function which is optimal with respect to the duration of $v(t)$. Finally, some examples of the expansion are presented.

I. Introduction

The problem we consider is closely related to aspects of information theory and signal analysis. We shall briefly sketch its origins.

Many real signals that exhibit essentially time-varying properties nonetheless have properties that are described best in the frequency domain. Important examples are speech and signals obtained from pulse or carrier modulation schemes. Several authors have suggested representations which use both time and frequency as independent variables. In Gabor's historic attempt to construct a theory of information [1], a representation of the form

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{n,k} v(t-n) e^{2\pi i k t} \qquad (1.1)$$

with

$$v(t) = e^{-t^2/2}$$

appears as an essential step in the definition of a unit of information. From the point of view of information theory, this definition suffered primarily from the lack of appreciation of noise as a limiting factor. From a signal analysis point of view, the major defect in (1.1) was that the basis functions $v(t-n)e^{2\pi i k t}$ were not mutually orthogonal. Gabor correctly pointed out the significance of the uncertainty relations as a limitation on the resolution of time–frequency representations. Shannon [2] gave the correct definition of the unit of information, but in his analysis of continuous channels he relied on the sampling expansion. This approach amounts to using

$$v(t) = \frac{\sin \pi t}{\pi t}$$

in (1.1). Although this makes the system of basis functions orthogonal, it is unsatisfactory in that the sampling expansion has poor convergence properties. Lerner [3] gave a valuable discussion of time–frequency series expansions and suggested a method for orthogonalizing a given system of functions. However, this approach does not lead to a function $v(t)$ with the required properties. Lerner also rejected the uncertainty relations in this context by pointing out that the frequency of a harmonic signal may be determined from an arbitrarily small segment. The correct statement of the uncertainty relations for time series expansions was derived by Landau and Pollak [4], who determined the dimension of the space and time and frequency limited functions. The set of orthogonal functions used in their analysis solves a number of problems related to signal analysis, but they do not appear to be useful in the present context. Recently, a number of papers have appeared which discuss Gabor and related series. In connection with problems in optics and quantum optics [5]–[11] the completeness of the system $\{v(t-n)e^{i2\pi k t}\}$, $n \in \mathbf{Z}$, $k \in \mathbf{Z}$, as well as the uniqueness of the coefficients $a_{n,k}$, is discussed, and the convergence properties of the double series are also treated in [12].

In the literature on theoretical problems in signal analysis, attention shifted to integral transforms defining a power distribution in time and frequency. This shift is noticeable in the papers by Helstrom [13] and Rihaaczek [14]. The energy distribution may be expressed by the Wigner distribution

$$W_f(t,\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} f(t+\tau/2) f(t-\tau/2)\, d\tau. \quad (1.2)$$

Many properties of this representation are discussed by Claasen and Mecklenbraüker [15]. An important aspect of

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(1.2) is that, if f is time-limited to $T_1 \le t \le T_2$ or frequency-limited to $\Omega_1 \le \omega \le \Omega_2$, these properties are inherited by W_f. However, W_f has the disturbing property that it may take on negative values. Indeed, Janssen [16] has proved that, for any compact set $S \subset R^2$, a function f exists such that

$$\int_S W_f(t,\omega)\,dt\,d\omega < 0.$$

For practical computations, some window function has to be introduced into (1.2), and in [17] Claasen and Meklenbraüker discuss how the Wigner distribution may be computed as the squared magnitude of the short-time Fourier transform,

$$F_t(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau) h(\tau - t)\,d\tau \tag{1.3}$$

where $h(t)$ is a time-limited window. In [18] Janssen compares the Wigner distribution with the classical Gabor series from a practical signal analysis point of view.

The short-time Fourier transform has been widely used in the analysis of speech [19], usually without much mathematical justification. The rationale has been that many sounds of speech have rather well-defined power spectra and last long enough to allow the calculation of several spectra by (1.3). As pointed out by de Bruijn [20], this situation is even more obvious in music, where a time–frequency notation has been used for centuries. In this context, the short-time transform is also used for purposes of filtering and other types of signal processing. The emphasis is on the possibility of reconstructing the signal rather than on analysis [21]. For example, no attention is paid to the orthogonality, or the lack of orthogonality, of the time-shifted windows.

Presently, instruments known as real-time spectrum analyzers are finding widespread application in the analysis of many different types of signals. A real-time measurement implies that the signal is segmented, a window function is applied, and the result is transformed by a short-time Fourier transform. Usually, the processing is carried out by digital circuits. The results of such processing seem to be described best by a series of the form (1.1). Thus it appears desirable to resolve some of the problems encountered in the early history of such representations; this is the subject of this paper.

Our main concern is to construct unit functions $v(t)$ such that a series representation of the form (1.1) holds for almost all bounded measurable functions $f(t)$, with *infinite energy*, and for which the coefficients $a_{n,k}$ reflect essential time–frequency energy properties of $f(t)$. Since, for this large class of functions, no unique mathematical setup is available for discussing spectral properties of $f(t)$, the double series representation could be used as a definition of the time–frequency spectrum of any bounded function $f(t)$. To be more specific, the definition or interpretation we have in mind is that $|a_{n,k}|^2$ gives the energy of $f(t)$ in the time interval $n-(1/2) \le t \le n+(1/2)$ and frequency interval $2\pi(k-(1/2)) \le \omega \le 2\pi(k+(1/2))$. Of course, this definition should correspond in a reasonable way to the results obtained by using Fourier series or Fourier transform on periodic and L^2 functions, where the spectrum is well-defined. This implies some conditions on the unit function $v(t)$.

In Section II we derive two such unavoidable conditions, which turn out to be 1) orthogonality, i.e.,

$$\int_{-\infty}^{\infty} v(t) v(t-n) e^{ik2\pi t}\,dt = \begin{cases} 1, & \text{if } n = k = 0 \\ 0, & \text{otherwise} \end{cases}$$

and 2) the unit function $v(t)$ and its Fourier transform

$$V(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\,dt$$

both belong to $L^1(\boldsymbol{R})$. The coefficients $a_{n,k}$ are given by

$$a_{n,k} = \int_{-\infty}^{\infty} f(t) v(t-n) e^{2\pi i k t}\,dt. \tag{1.4}$$

Based on 1) and 2) and the definition (1.4), we investigate the series representation (1.1).

In Section III we derive some consequences of the conditions on the unit function and prove that the system $\{v(t-n)e^{i2\pi kt}\}, n \in \boldsymbol{Z}, k \in \boldsymbol{Z}$ is a complete orthonormal system in $L^2(\boldsymbol{R})$. We also discuss convergence properties of the expansion and mention that, if $f(t)$ is periodic with period 1, then (1.1) is the ordinary Fourier series for $f(t)$.

In Section IV we consider energy and power estimations and show that

$$\lim_{N\to\infty} \left| \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{k=-\infty}^{\infty} |a_{n,k}|^2 - \frac{1}{2N+1} \int_{-N-(1/2)}^{N+(1/2)} f^2(t)\,dt \right| = 0,$$

which in the aforementioned case reduces to the ordinary Parseval equation.

To prove results it is sometimes necessary to impose further conditions on the unit function $v(t)$, and it is not obvious that functions exist satisfying the conditions. In Section V we prove that 1) and 2) imply that either the duration of $v(t)$ or the duration of $V(\omega)$ must be infinite, and we construct unit functions which are optimal with respect to the duration either of $v(t)$ or of $V(\omega)$, within a restricted class of possible unit functions. Finally, Section VI presents some examples of the expansion.

II. Conditions on the Unit Function $v(t)$

Unless otherwise stated, throughout the paper $v(t)$ is assumed to be a real function. Let $f(t)$ be a bounded function for which a series representation (1.1) holds. According to the interpretation stated in the introduction this implies that for each n, the sum

$$f_n(t) = \sum_{k=-\infty}^{\infty} a_{n,k} v(t-n) e^{ik2\pi t}$$

shall give an approximate restriction of $f(t)$ to the time interval $[n-(1/2), n+(1/2)]$, and for each k, the sum

$$f^k(t) = \sum_{n=-\infty}^{\infty} a_{n,k} v(t-n) e^{i2\pi kt}$$

shall be interpreted as an approximately bandlimited component of $f(t)$.

If $f(t) \in L^2(\boldsymbol{R})$, then $f(t)$ has a well-defined spectrum using the ordinary Fourier transform, and in this case the energy in a certain time interval, as well as the energy in a certain frequency band, has an *a priori* meaning. Therefore, if our expansion and its interpretation are to have any practical meaning, a reasonable connection must exist between the two ways of calculating energy.

To this end, we shall formulate three requirements to the double series expansion, from which we derive three *necessary* conditions on $v(t)$. Then later we return to the three requirements and prove that these are satisfied for a large class of functions $v(t)$. Our first requirement is as follows.

1) For each $f(t) \in L^2(\boldsymbol{R})$, an expansion of the form (1.1) holds in the sense of L^2 convergence, and furthermore,

$$\int_{-\infty}^{\infty} f^2(t)\,dt = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |a_{n,k}|^2. \tag{2.1}$$

Equation (2.1) states that the total energy of $f(t)$ equals the energy which the double series expansion, with the stated interpretation, assigns to $f(t)$.

It is well-known that if requirement 1) holds, then the system $\{v(t-n)e^{ik2\pi t}\}$ is an orthogonal normalized system in $L^2(\boldsymbol{R})$ and

$$a_{n,k} = \int_{-\infty}^{\infty} f(t)v(t-n)e^{-ik2\pi t}\,dt. \tag{2.2}$$

Hence our first condition on $v(t)$ is

$$\int_{-\infty}^{\infty} v(t)v(t-n)e^{ik2\pi t}\,dt = \begin{cases} 1, & \text{if } n=k=0 \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

We shall refer to (2.3) as the *orthogonality condition* on $v(t)$. Let us now suppose that requirement 1) is satisfied, and again let $f(t) \in L^2(\boldsymbol{R})$. In a specific time interval $[n-(1/2), n+(1/2)]$, $f(t)$ has a well-defined energy, namely,

$$\int_{n-(1/2)}^{n+(1/2)} f^2(t)\,dt, \tag{2.4}$$

whereas the energy assigned to that time interval by (1.1) is

$$\sum_{k=-\infty}^{\infty} |a_{n,k}|^2 \tag{2.5}$$

where $a_{n,k}$ is given by (2.2). The optimal situation would, of course, be that these two quantities and corresponding quantities in the frequency domain were always equal. This turns out to be impossible, but we shall add the following requirements.

2) For each $n \in \boldsymbol{Z}$ and $A > 0$, a $K > 0$ exists such that

$$\left|\int_{n-(1/2)}^{n+(1/2)} f^2(t)\,dt - \sum_{k=-\infty}^{\infty} |a_{n,k}|^2\right| < K \tag{2.6}$$

for all $f(t) \in L^2(\boldsymbol{R})$ with $|f(t)| \le A$, $t \in \boldsymbol{R}$.

3) For each $k \in \boldsymbol{Z}$ and $A_1 > 0$, a $K_1 > 0$ exists such that

$$\left|\int_{2\pi(k-(1/2))}^{2\pi(k+(1/2))} |F(\omega)|^2\,d\omega - \sum_{n=-\infty}^{\infty} |a_{n,k}|^2\right| < K_1 \tag{2.7}$$

for all $f(t) \in L^2(\boldsymbol{R})$ for which the Fourier transform $F(\omega)$ satisfies $|F(\omega)| \le A_1$, $\omega \in \boldsymbol{R}$.

Lemma 2.1: If requirements 2) and 3) hold, then both $v(t)$ and the Fourier-transform $V(\omega)$ are absolutely integrable functions.

Proof: Consider for $l \in \boldsymbol{N}$ the function $f_l(t) = Ag(t)\cdot 1_{[-l,l]}$ where $g(t) = \operatorname{sgn} v(t)$. For $n = k = 0$, we have from (2.2)

$$a_{0,0} = A\int_{-l}^{l} |v(t)|\,dt.$$

From (2.6) it follows that $|a_{0,0}|^2 < K + A$ for all such functions $f_l(t)$, and hence $v(t)$ is absolutely integrable. That $V(\omega)$ is absolutely integrable follows in the same way by considering functions $f_l(t)$ with Fourier transform $F_l(\omega) = A_1 V(\omega)/|V(\omega)|\cdot 1_{[-l,l]}$ for $V(\omega) \ne 0$ and zero for $V(\omega) = 0$. In this case

$$a_{0,0} = \int_{-\infty}^{\infty} f_l(t)v(t)\,dt$$

$$= \int_{-\infty}^{\infty} F_l(\omega)\overline{V(\omega)}\,d\omega = A_1\int_{-l}^{l} |V(\omega)|\,d\omega.$$

Hence our second condition on $v(t)$ is that both $v(t)$ and $V(\omega)$ belong to $L^1(\boldsymbol{R})$. We shall refer to this as the *integrability condition* on $v(t)$. This implies that both $v(t)$ and $V(\omega)$ are continuous.

We have thus seen that if the expansion (1.1) is to have a chance to give a reasonable picture of spectral properties of $f(t)$—which for L^2 functions we have formulated in requirements 1)–3)—then $v(t)$ must satisfy both the orthogonality and the integrability conditions. While it is not difficult to find unit functions $v(t)$ which have one of these two properties, we have not been able to find any standard function that has both properties, and we should emphasize that it is by no means obvious *a priori* that such a function exists and that 1)–3) can be obtained. Moreover, it is even less clear whether these conditions are sufficient for the existence of the expansion (1.1) for a wide class of merely *bounded functions* and are sufficient to ensure that (1.1) reflects spectral properties of $f(t)$ in a reasonable way. These problems are discussed in the following sections.

III. Consequences of the Conditions on the Unit Function

We shall first obtain an equivalent formulation of the orthogonality condition (see also [7]). Suppose that $v(t) \in L^2(\boldsymbol{R})$. We have

$$\int_{-\infty}^{\infty} v(t)v(t-n)e^{ik2\pi t}\,dt$$

$$= \sum_{j=-\infty}^{\infty}\int_0^1 v(j+\tau)v(j+\tau-n)e^{ik2\pi\tau}\,d\tau$$

$$= \int_0^1\left[\sum_{j=-\infty}^{\infty} v(j+\tau)v(j+\tau-n)\right]e^{ik2\pi\tau}\,d\tau$$

which shows that $v(t)$ satisfies the orthogonality condition (2.3) if and only if

$$\sum_{j=-\infty}^{\infty} v(j+\tau)v(j+\tau-n)=\begin{cases}1, & \text{if } n=0\\ 0, & \text{if } n\neq 0\end{cases} \tag{3.1}$$

for almost all $\tau\in[0,1]$. Next we consider the Zak transform of $v(t)$, that is

$$V_\tau(\omega)=\sum_{n=-\infty}^{\infty} v(n+\tau)e^{in\omega}. \tag{3.2}$$

Theorem 3.1: If $v(t)\in L^1(\boldsymbol{R})\cap L^2(\boldsymbol{R})$, then $v(t)$ satisfies the orthogonality condition if and only if

$$|V_\tau(\omega)|^2=\left|\sum_{n=-\infty}^{\infty} v(n+\tau)e^{in\omega}\right|^2=1$$

for almost all $\tau\in[0,1]$ and almost all $\omega\in[-\pi,\pi]$.

Proof: For $v(t)\in L^1(\boldsymbol{R})$ the series $\sum_{n=-\infty}^{\infty}|v(\tau+n)|$ is convergent for almost all $\tau\in[0,1]$. Hence we can change the order of summation in calculating $|V_\tau(\omega)|^2$ from (3.2) and get

$$|V_\tau(\omega)|^2=\sum_{n=-\infty}^{\infty}\left[\sum_{j=-\infty}^{\infty} V(j+\tau)V(j+\tau-n)\right]e^{in\omega}.$$

The result then follows from (3.1).

This result will be our key to finding unit functions with the desired properties. At this point, we shall use (3.1) to prove a principal consequence of our main conditions.

Theorem 3.2: Suppose that $v(t)$ satisfies both the orthogonality and the integrability condition. Then $v(t)$ is neither time- nor band-limited.

Proof: Suppose that $v(t)$ is time-limited: that is, it equals zero outside a finite interval. Let $K\in\boldsymbol{Z}$ and $\tau\in[0,1]$ be chosen in such a way that $v(K+\tau)\neq 0$ and $v(n+\tau)=0$ for $n>K$. There is a number $K_1\in\boldsymbol{Z}$ such that $v(n+\tau)=0$ for $n<K_1$. Now, for $n=K-K_1$ the sum in (3.1) has only one term, $v(K+\tau)v(K_1+\tau)$, and hence $v(K_1+\tau)=0$ if $K_1<K$. Continuing in this way we conclude that $v(n+\tau)=0$ for $n\neq K$ and, again from (3.1), $|v(K+\tau)|=1$. Since this holds for any K and τ chosen as above, we have a contradiction because $v(t)$ is continuous. Hence $v(t)$ is not time-limited.

Since $\{v(t-n)e^{ik2\pi t}\}$ is a normalized orthogonal system, the Fourier transformed system $\{V(\omega-k2\pi)e^{-in\omega}\}$ has the same properties, and therefore an equation analogous to (3.1) holds for $V(\omega)$. As before, we conclude that $v(t)$ is not band-limited.

Next we consider the completeness of the system $\{v(t-n)e^{ik2\pi t}\}$. A related problem is discussed in [7]. Some of the steps in our proof will be used in other contexts.

In the following theorem, $v(t)$ is a unit function satisfying the orthogonality and the integrability conditions, $f(t)$ is a function in $L^\infty(\boldsymbol{R})$ and the coefficients $a_{n,k}$ are defined by (2.2). Further, we deal with two functions

$$h(t)=\sum_{n=-\infty}^{\infty}|v(t-n)| \tag{3.3}$$

$$w_n(t)=\sum_{p=-\infty}^{\infty} f(t+p)v(t+p-n) \tag{3.4}$$

where $n\in\boldsymbol{Z}$.

Theorem 3.3: Suppose that $h(t)\in L^2([0,1])$ or $f(t)\in L^2(\boldsymbol{R})$. Then

$$f_n(t)=\sum_{k=-\infty}^{\infty} a_{n,k}v(t-n)e^{ik2\pi t} \tag{3.5}$$

exists as an L^2 function. Further,

$$f_n(t)=v(t-n)w_n(t) \tag{3.6}$$

$$f(t)=\sum_{n=-\infty}^{\infty} f_n(t) \tag{3.7}$$

for almost all $t\in\boldsymbol{R}$, where the convergence in the last sum is pointwise.

Proof: Notice first that $h(t)$ and $w_n(t)$ are finite almost everywhere and both belong to $L^1([0,1])$, because $v(t)\in L^1(\boldsymbol{R})$ and $f(t)$ is bounded. We have

$$\begin{aligned}&\int_0^1 w_n(t)e^{-ik2\pi t}\,dt\\ &=\sum_{p=-\infty}^{\infty}\int_0^1 f(t+p)v(t+p-n)e^{-ik2\pi t}\,dt\\ &=\int_{-\infty}^{\infty} f(t)v(t-n)e^{-ik2\pi t}\,dt=a_{n,k}\end{aligned}$$

where the summation and integration can be interchanged by the bounded convergence theorem since $h(t)\in L^1([0,1])$, Both assumptions imply that $w_n(t)\in L^2([0,1])$. This gives (3.5), and moreover, if

$$s_p(t)=\sum_{k=-p}^{p} a_{n,k}e^{ik2\pi t},$$

then $s_p(t)v(t-n)$ converges to $w_n(t)v(t-n)$ in L^2-norm on $\boldsymbol{R}$, because

$$\begin{aligned}&\int_{-\infty}^{\infty}|s_p(t)v(t-n)-w_n(t)v(t-n)|^2\,dt\\ &=\int_0^1\sum_{l=-\infty}^{\infty}|v(t+l-n)|^2|s_p(t)-w_n(t)|^2\,dt\\ &=\int_0^1|s_p(t)-w_n(t)|^2\,dt\end{aligned}$$

where we have used (3.1). Hence (3.6) follows, and again using (3.1) we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} f_n(t)&=\sum_{n=-\infty}^{\infty} w_n(t)v(t-n)\\ &=\sum_{p=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(t+p)v(t+p-n)v(t-n)\\ &=f(t).\end{aligned}$$

from (3.5) and (3.7) we get the following.

Corollary 3.4: If $v(t)$ satisfies the orthogonality and integrability conditions, then $\{v(t-n)e^{ik2\pi t}\}$ is a complete, orthogonal system.

This means that requirement 1) from Section II holds. Requirement 2) holds if and only if the functions $w_n(t)$ in (3.4), considered for all $f(t)$ in question, has L^2 norms below some common bound. This in turn is satisfied, if and only if $h(t) \in L^2([0,1])$. Requirement 3) can be treated in the same way because, for any $f(t) \in L^2(\boldsymbol{R})$ with Fourier transform $F(\omega)$, we have for the coefficients (2.2)

$$a_{n,k} = \int_{-\infty}^{\infty} F(\omega)\bar{V}(\omega - 2\pi k)e^{-in\omega}\,d\omega.$$

Therefore, if

$$h_1(\omega) = \sum_{k=-\infty}^{\infty} |V(\omega - 2\pi k)|, \tag{3.8}$$

we have the following.

Corollary 3.5: Suppose that $v(t)$ satisfies the orthogonality and the integrability condition. Then requirements 2) and 3) hold if and only if the function (3.3) belongs to $L^2([0,1])$ and the function (3.8) belongs to $L^2([-\pi,\pi])$.

Let us just suppose that $f(t) \in L^\infty(\boldsymbol{R})$. Still, $v(t)$ satisfies our main conditions and

$$a_{n,k} = \int_{-\infty}^{\infty} f(t)v(t-n)e^{-ik2\pi t}\,dt. \tag{3.9}$$

We will consider the double sums

$$\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{n,k}v(t-n)e^{ik2\pi t} \tag{3.10}$$

$$\sum_{k=-\infty}^{\infty}\sum_{n-\infty}^{\infty} a_{n,k}v(t-n)e^{ik2\pi t}. \tag{3.11}$$

It is important for our whole setup and the interpretation of the coefficients that these two sums in some reasonable sense "equal" the function $f(t)$.

Let us suppose that $h(t)$ in (3.3) belongs to $L^2([0,1])$. Then, according to Theorem 3.3 the double sum (3.10) equals $f(t)$ almost everywhere, if the inner sum is L^2-convergent on $\boldsymbol{R}$ and the outer sum is pointwise convergent. A similar procedure does not seem possible for (3.11). However, one can prove that both (3.10) and (3.11) are summable with sum $\frac{1}{2}(f(t^+)+f(t^-))$ for each $t \in \boldsymbol{R}\setminus\boldsymbol{Z}$, if $f(t)$ is piecewise continuous. A short proof of this is given in Appendix I. However, here we shall note that if we suppose that $\sum_{n=-\infty}^{\infty} v(t+n)=1$ and consider a function f which is periodic with period 1, then (3.9) becomes

$$a_{n,k} = \int_0^1 f(t)e^{-ik2\pi t}\,dt,$$

and hence $a_{n,k}$ equals the k'th Fourier coefficient of $f(t)$ for all $n \in \boldsymbol{Z}$, so (3.11) is reduced to the ordinary Fourier series for $f(t)$.

IV. Energy and Power Estimations

In this section, we only suppose that the considered functions $f(t)$ belongs to $L^\infty(\boldsymbol{R})$. The integrability and orthogonality conditions are supposed to hold, the function $h(t)$ in (3.3) is supposed to be in $L^2([0,1])$, and the notation is as before.

If, for $N \in \boldsymbol{N}$, we put

$$s_N(t) = \sum_{n=-N}^{N} f_n(t) \tag{4.1}$$

where $f_n(t)$ is given by (3.5), then we can think of $s_N(t)$ as an approximated reconstruction of $f(t)$ from the coefficients $a_{n,k}$. The problem we shall consider in the following is, how "good" this approximation is in terms of the concept of energy. More precisely, we will estimate the energy of $f(t)-s_N(t)$ in the interval $[-N-(1/2), N+(1/2)]$, cf., the interpretation of the coefficients $a_{n,k}$. We have from (3.7)

$$\int_{-N-(1/2)}^{N+(1/2)} (f(t)-s_N(t))^2\,dt = \int_{-N-(1/2)}^{N+(1/2)} \left(\sum_{|n|>N} f_n(t)\right)^2 dt. \tag{4.2}$$

To estimate the right side of (4.2), we shall assume that a constant K exists such that

$$\sum_{n=-\infty}^{\infty} |v(t+n)| \le K, \qquad t \in \boldsymbol{R}. \tag{4.3}$$

If $|f(t)| \le A$, it follows from (3.4) and (3.6), that $|f_n(t)| \le AK|v(t-n)|$. Consequently, we have from (4.2)

$$\begin{aligned}\int_{-N-(1/2)}^{N+(1/2)} (f(t)-s_N(t))^2\,dt &\le \int_{-N-(1/2)}^{N+(1/2)} A^2K^2\left(\sum_{|n|>N}|v(t-n)|\right)^2 dt\\ &= A^2K^2\int_{-1/2}^{1/2}\left(\sum_{p=-N}^{N}\left[\sum_{|n|>N}|v(t+p-n)|\right]^2\right)dt.\end{aligned} \tag{4.4}$$

To make the following calculations more clear, let us for some fixed $t \in [-1/2,1/2]$ put $b_q = |v(t-q)|$. For the integrand on the right side of (4.4), we then have

$$\begin{aligned}\sum_{p=-N}^{N}\left(\sum_{|n|>N} b_{n-p}\right)^2 &\le 2\left[\sum_{p=-N}^{N}\left(\sum_{n=N+1}^{\infty} b_{n-p}\right)^2 + \sum_{p=-N}^{N}\left(\sum_{n=-\infty}^{-N-1} b_{n-p}\right)^2\right]\\ &= 2\sum_{k=1}^{2N+1}\left(\sum_{n=k}^{\infty} b_n\right)^2 + 2\sum_{k=-2N-1}^{-1}\left(\sum_{n=-\infty}^{k} b_n\right)^2.\end{aligned} \tag{4.5}$$

From the inequality [22, p. 246] it follows

$$\sum_{k=1}^{2N+1}\left(\sum_{n=k}^{\infty} b_n\right)^2 \le \sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} b_n\right)^2 \le 4\sum_{k=1}^{\infty}(kb_k)^2 \tag{4.6}$$

and in the same way

$$\sum_{k=-2N-1}^{-1}\left(\sum_{n=-\infty}^{k} b_n\right)^2 \le \sum_{k=-\infty}^{-1}\left(\sum_{n=-\infty}^{k} b_n\right)^2 \le 4\sum_{k=-\infty}^{-1}(kb_k)^2. \quad (4.7)$$

Using (4.4)–(4.7), we have thus proved the following.

Theorem 4.1: If $|f(t)| \le A$, $t \in \boldsymbol{R}$, and $v(t)$ satisfies (4.3), then

$$\int_{-N-(1/2)}^{N+(1/2)} (f(t)-s_N(t))^2\,dt \le 8A^2K^2 \sum_{k=-\infty}^{\infty} k^2 \int_{-1/2}^{1/2} v(t-k)^2\,dt \quad (4.8)$$

for all $N \in \boldsymbol{N}$.

The interesting thing about this result is that, in terms of $v(t)$ and some constants, it gives an upper bound for $N \to \infty$ for the energy error (4.2). Concerning the "sharpness" of (4.8), we should add that in the last inequality in (4.6) the constant 4 is the best possible [22], and the two sums in this inequality are, roughly speaking, of the same order.

Unfortunately, (4.8) and the condition $v(t) \in L^1(\boldsymbol{R})$ do not ensure—which would be preferable—that (4.2) stays bounded for $N \to \infty$ since the right side of (4.8) might be infinite for $v(t) \in L^1(\boldsymbol{R})$. Actually, it converges if and only if $\int_{\infty}^{\infty} t^2 v(t)^2\,dt < \infty$, and a later result, Theorem 5.2, will then show that it is not possible to obtain "good" estimates of the type (4.8) in both time and frequency domains. However, in terms of power—which to some extent is a more natural concept to use when considering functions with infinite energy—the situation is completely satisfactory, as the following result shows.

Parseval's Equation: If $|f(d(t))| \le A$, $t \in \boldsymbol{R}$ and $v(t)$ satisfies (4.3) we have

$$\lim_{N\to\infty}\left|\frac{1}{2N+1}\sum_{n=-N}^{N}\sum_{k=-\infty}^{\infty}|a_{n,k}|^2 - \frac{1}{2N+1}\int_{-N-(1/2)}^{N+(1/2)} f^2(t)\,dt\right| = 0. \quad (4.9)$$

Proof: We use the notation as above. We have

$$\left|\sum_{n=-N}^{N}\sum_{k=-\infty}^{\infty}|a_{n,k}|^2 - \int_{-N-(1/2)}^{N+(1/2)} f^2(t)\,dt\right| = \left|\int_{-\infty}^{\infty} s_N^2(t)\,dt - \int_{-N-(1/2)}^{N+(1/2)} f^2(t)\,dt\right|$$
$$\le \int_{-N-(1/2)}^{N+(1/2)} |f^2(t)-s_N^2(t)|\,dt + \int_{|t|>N+(1/2)} s_N^2(t)\,dt. \quad (4.10)$$

Let us first consider the last sum. We have from the assumptions $|f_n(t)| = |w_n(t)v(t-n)| \le AK|v(t-n)|$ and hence $|s_N(t)| \le AK^2$. Therefore,

$$\int_{|t|>N+1/2} s_N^2(t)\,dt \le A^2K^3 \sum_{n=-N}^{N}\int_{|t|>N+1/2}|v(t-n)|\,dt. \quad (4.11)$$

We put $b_q = \int_{-1/2}^{1/2}|v(t-q)|\,dt$, $q \in \boldsymbol{Z}$. Then by rewriting we get

$$\sum_{n=-N}^{N}\int_{|t|>N+1/2}|v(t-n)|\,dt = \sum_{|k|=1}^{2N+1}|k|b_k + (2N+1)\cdot\sum_{|k|=2N+2}^{\infty} b_k.$$

Since the series $\sum_{q=-\infty}^{\infty} b_q$ is convergent, we conclude from this and (4.11), that

$$\frac{1}{2N+1}\int_{|t|>N+1/2} s_N^2(t)\,dt \to 0, \qquad \text{for } N\to\infty. \quad (4.12)$$

Moreover, using the Cauchy–Schwarz inequality, we obtain

$$\int_{-N-(1/2)}^{N+(1/2)} |f^2(t)-s_N^2(t)|\,dt = \int_{-N-(1/2)}^{N+(1/2)}\left|(s_N(t)-f(t))^2 - 2f(t)(f(t)-s_N(t))\right|dt$$
$$\le \int_{-N-(1/2)}^{N+(1/2)}|s_N(t)-f(t)|^2\,dt + 2\left[\int_{-N-(1/2)}^{N+(1/2)} f^2(t)\,dt\right]^{1/2}\cdot\left[\int_{-N-(1/2)}^{N+(1/2)}|f(t)-s_N(t)|^2\,dt\right]^{1/2}.$$

Now, $f(t)$ is bounded and generally, if c_k is a sequence of positive numbers with $c_k \to 0$ for $k \to \infty$, then $(1/N)\sum_{k=1}^{N} c_k \to 0$ for $N \to \infty$. Using this we conclude from the above inequality and (4.4) and (4.5) that

$$\frac{1}{2N+1}\int_{-N-(1/2)}^{N+(1/2)}|f^2(t)-s_N^2(t)|\,dt \to 0, \qquad \text{for } N\to\infty, \quad (4.13)$$

which concludes the proof of the theorem.

We shall make two comments on this result. First, in the case where $f(t)$ is periodic with period 1, it follows—using the last remarks in Section III—that (4.9) is the ordinary Parseval equation. Second, we can say that the validity of Parseval's equation is equivalent to the orthogonality condition on $v(t)$ *and* the condition $v(t) \in L^1(\boldsymbol{R})$. It follows from the discussion in Section II that (4.9) can never hold if $v(t)$ is not in $L^1(\boldsymbol{R})$. On the other hand, we have proved (4.9) for all $f(t) \in L^\infty(\boldsymbol{R})$, if (4.3) holds. Of course, (4.3) is not the same as $v(t) \in L^1(\boldsymbol{R})$, but we consider the difference as a more or less "proof-technical" issue, where we have stated the assumption (4.3) to make proofs and arguments reasonable to handle in this context.

V. The Choice of the Unit Function

Up to now we have derived various results concerning the double sum expansion assuming the orthogonality and the integrability condition on the unit function $v(t)$. We have actually not proved that those conditions can be satisfied simultaneously, nor have we discussed what possibilities there are for the choice of $v(t)$.

We shall here mainly restrict ourselves to considering a special class of functions, the choice of which is motivated by the following considerations. Recall, with the previous notation, that $v(t)$ satisfies the orthogonality condition if and only if

$$|V_\tau(\omega)|=1 \tag{5.1}$$

for almost all $\tau\in[0,1]$ and $\omega\in[-\pi,\pi]$.

To decide whether functions satisfying the integrability and orthogonality conditions exist, we consider unit functions such that $V_\tau(\omega)$ is a rational function of $e^{i\omega}$. The condition (5.1) implies that $V_\tau(\omega)$ may be interpreted as the transfer function of a not necessarily causal discrete stable all-pass filter. It is well-known [23] that for such a filter, the poles must be located within the unit circle, and the zeros must be the reciprocals of the poles. Consider

$$V_\tau(\omega)=\frac{e^{-ik\omega}P_\tau(e^{i\omega})}{e^{ik\omega}P_\tau(e^{-i\omega})}$$

where $P_\tau(z)$ is a polynomial of degree $2k+1$. If for $\tau=0$, $P_0(e^{i\omega})=e^{ik\omega}$, we have $V_0(\omega)=1$. Thus in the limit $\tau\to1$, we should—provided continuity of $V_\tau(\omega)$ as a function of τ—get $V_\tau(\omega)=e^{i\omega}$, which is satisfied if $P_1(z)=z^{2k+1}P_1(z^{-1})$. For τ approaching 1, the poles and zeros converge towards the unit circle and cancel for $\tau=1$.

A function constructed in this way can be continuous in both domains, but the sidelobes will decrease slowly for τ close to 1. This property appears to be a consequence of the orthogonality condition, and the situation is not improved by taking a polynomial of higher degree.

We prefer the simple case,

$$V_\tau(\omega)=\frac{1+\alpha(\tau)e^{-i\omega}}{1+\alpha(\tau)e^{i\omega}} \tag{5.2}$$

where $\alpha(\tau)$ is a function mapping $[0,1]$ onto itself. From (5.2) and [24, p. 40], it follows that

$$v(n+\tau)=\begin{cases}0, & \text{if } n<-1\\ \alpha(\tau), & \text{if } n=-1,\\ (1-\alpha^2(\tau))(-\alpha(\tau))^n, & \text{if } n>-1\end{cases}$$

$$\tau\in[0,1[\tag{5.3}$$

and

$$\hat{V}(\omega)=\int_0^1\frac{1+\alpha(\tau)e^{i\omega}}{1+\alpha(\tau)e^{-i\omega}}e^{-i\omega\tau}\,d\tau. \tag{5.4}$$

A direct calculation gives

$$\sum_{n=-\infty}^{\infty}|v(n+\tau)|=\alpha(\tau)+\sum_{n=0}^{\infty}(1-\alpha^2(\tau))(\alpha(\tau))^n$$
$$=\begin{cases}1+2\alpha(\tau), & \text{if } \alpha(\tau)<1\\ 1, & \text{if } \alpha(\tau)=1\end{cases}. \tag{5.5}$$

Hence, if we choose a *continuous increasing function* $\alpha(\tau)$ *mapping* $[0,1]$ *into itself*, then $v(t)$ is continuous, satisfies the orthogonality condition, and $v(t)\in L^1(\boldsymbol{R})$.

What is left to consider is therefore the condition $V(\omega)\in L^1(\boldsymbol{R})$. First, we remark that, for $\alpha(\tau)=\tau$, a direct calculation will show—evaluating (5.4)—that $V(\omega)$ is not in $L^1(\boldsymbol{R})$. Hence the continuity of $v(t)$ is not sufficient. A well-known and unsolved problem is how to give necessary and sufficient conditions in the time domain for the Fourier transform $V(\omega)$ of a function $v(t)$ for it to be absolutely integrable. However, we shall use the following result, which follows from [25, theorem 68].

Lemma 5.1: Suppose that $v(t)$ is a continuous piecewise C^1 function for which $v(t)\in L^2(\boldsymbol{R})\cap L^1(\boldsymbol{R})$ and $v'(t)\in L^2(\boldsymbol{R})$. Then $V(\omega)\in L^1(\boldsymbol{R})$.

By direct calculation, we find from (5.3)

$$\int_{-\infty}^{\infty}v'(t)^2\,dt$$
$$=\int_0^1\alpha'(\tau)^2\,d\tau+\int_0^1\sum_{n=0}^{\infty}\Big[-2\alpha(\tau)\alpha'(\tau)(-\alpha(\tau))^n$$
$$-n(1-\alpha^2(\tau))(-\alpha(\tau))^{n-1}\alpha'(\tau)\Big]^2d\tau$$
$$=\int_0^1\alpha'(\tau)^2\left[1+4\alpha^2(\tau)+\sum_{n=1}^{\infty}(\alpha^2(\tau))^{n-1}\right.$$
$$\left.\cdot(n-(n+2)\alpha^2(\tau))^2\right]d\tau. \tag{5.6}$$

From ordinary power-series expansions, we next conclude that

$$\int_{-\infty}^{\infty}v'(t)^2\,dt=2\int_0^1\frac{\alpha'(\tau)^2}{1-\alpha^2(\tau)}\,d\tau. \tag{5.7}$$

Consequently, we shall choose $\alpha(\tau)$ in such a way that also the integral (5.7) is convergent. Doing this, $v(t)$ satisfies the orthogonality and integrability condition, (4.3) holds, and requirements 1)–3) from Section II hold. The last claim follows from Corollaries 3.4 and 3.5, if we can prove that the function (3.8), that is,

$$h_1(\omega)=\sum_{k=-\infty}^{\infty}|V(\omega-2\pi k)|$$

belongs to $L^2([-\pi,\pi])$. Now for some constants K_1, K_2 we have

$$\int_{-\pi}^{\pi}h_1^2(\omega)\,d\omega\le K_1\sum_{k=-\infty}^{\infty}k^2\int_{-\pi}^{\pi}|V(\omega-2\pi k)|^2\,d\omega+K_2$$

which follows from (4.6) and (4.7) with just one term in the outer sum. However, when $v'(t)\in L^2(\boldsymbol{R})$, then $\omega V(\omega)\in L^2(\boldsymbol{R})$, and therefore the right side is finite.

Indeed, it is possible to choose $\alpha(\tau)$ such that (5.7) is convergent. For instance, direct calculation shows that any $\alpha(\tau)=1-(1-\tau)^p$, $p>1$ will do. We shall therefore ask which other conditions would be preferable.

It is fairly obvious that to have a completely satisfactory double series expansion, the unit function $v(t)$ should be both time- and band-limited. This is impossible, as is well-known, and by Theorem 3.2 $v(t)$ cannot have any of these properties in our situation. However, $v(t)$ and $V(\omega)$ should both be of what is called "short duration." In connection with the problem considered here, the obvious way to interpret the term short duration is deduced from the inequality (4.8) from Section IV. Here we proved that the expression

$$8A^2K^2\sum_{k=-\infty}^{\infty}k^2\int_{-1/2}^{1/2}v(t-k)^2\,dt \tag{5.8}$$

gives an upper bound for the energy error over arbitrary large intervals between signals bounded by A and reconstructions from the double sum expansion. Therefore, as a measure of the duration of $v(t)$ we shall use

$$\int_{-\infty}^{\infty}t^2v(t)^2\,dt. \tag{5.9}$$

Of course, (5.8) and (5.9)—besides the constant $8A^2K^2$—are in general not equal, but both expressions measure the same behavior of $v(t)$, and (5.9) is a more familiar term in problems concerning spectral estimations.

A discussion parallel to the one in Section IV can be carried out in the frequency domain for bounded L^2 signals. Hence

$$8K_1^2A_1^2\sum_{p=-\infty}^{\infty}p^2\int_{-\pi}^{\pi}|V(\omega-2\pi p)|^2\,d\omega \tag{5.10}$$

gives an upper bound—over all L^2 signals $f(t)$ for which the Fourier transform $F(\omega)$ is bounded by A_1—for the energy error over arbitrary large intervals between $F(\omega)$ and the reconstruction from the double series expansion. Hence we shall use

$$\int_{-\infty}^{\infty}\omega^2|V(\omega)|^2\,d\omega \tag{5.11}$$

which is usually called the energy moment of $v(t)$ as the measure of the duration of $V(\omega)$.

The natural optimization problem then consists of minimizing the sum of (5.9) and (5.11) or, as in connection with the uncertainty relations [20], the square root of the product of those two integrals. This problem, however, has no solution, since one by direct calculation and use of the Cauchy–Schwartz inequality can show, that the sum of (5.9) and (5.11) is infinite for unit functions of the form (5.3). One could be tempted from this to conclude that the class of functions of the form (5.3) is, not representative of the actual possibilities. It turns out that the result is completely general, as stated in the next theorem.

Theorem 5.2: If a unit function $v(t)$ satisfies both the orthogonality and the integrability condition, then the sum of (5.9) and (5.11) is infinite.

This result can be found in [26], where the proof, however, contains a gap. This gap has recently been corrected by Coifman and Semmes. Since we consider the result striking and since the proof does not appear in the literature, we shall give a sketch of the proof in Appendix II. We would like to thank Coifman and Semmes for their courtesy in letting us include this as yet unpublished proof.

The above theorem gives useful information about how rapidly a function can decrease in both time and frequency domains, when the orthogonality condition must be satisfied. This, of course, has influence on the sharpness which is possible in time–frequency analysis of signals with infinite energy. We shall point out two possibilities.

First, one can choose the function of the form (5.3), which minimizes the integral (5.11). Doing this, we obtain the unit function which, in the class considered, gives the best spectrum. Both the orthogonality and integrability conditions are satisfied and hence also the previously stated consequences of this.

Minimizing (5.11) is by (5.7), the same as minimizing

$$\int_0^1\frac{\alpha'(\tau)^2}{1-\alpha^2(\tau)}\,d\tau, \tag{5.12}$$

and here we shall use the calculus of variation. The Euler differential equation for the problem is

$$\frac{\alpha'(\tau)^2}{1-\alpha(\tau)^2}-\frac{2\alpha'(\tau)}{1-\alpha^2(\tau)}\alpha'(\tau)=c$$

where c is a constant. This gives

$$\alpha'(\tau)=c_1\sqrt{1-\alpha(\tau)^2}$$

which has the solutions

$$\arcsin\alpha(\tau)=c_1\tau+c_2.$$

The initial conditions $\alpha(0)=0$, $\alpha(1)=1$ give $c_2=0$, $c_1=\pi/2$, and hence $\alpha(\tau)=\sin(\tau\pi/2)$.

Therefore, our first choice of unit function is

$$v(n+\tau)=\begin{cases}0, & \text{if } n<-1\\ \sin(\tau\pi/2), & \text{if } n=-1,\\ \cos^2(\tau\pi/2)(-\sin(\tau\pi/2))^n, & \text{if } n>-1\end{cases}\qquad \tau\in[0,1[. \tag{5.13}$$

The graph of $v(t)$ is shown in Fig. 1. The function $v(t)$ is continuous and differentiable except for $t=\pm1$.

Next, let us make the following remark. Up to now we have only considered real functions $v(t)$. However, there are no principal difficulties in carrying out the discussion for complex $v(t)$. The same results and estimates will hold. Therefore, we can turn things around and consider a class of functions whose Fourier transforms are determined as in (5.3), and among these functions choose the one, which minimizes the integral (5.9). The result of doing this—we omit the details—is the unit function

$$v(t)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1+\sin\frac{\omega}{4}e^{i2\pi t}}{1+\sin\frac{\omega}{4}e^{-i2\pi t}}e^{-it\omega}\,d\omega. \tag{5.14}$$

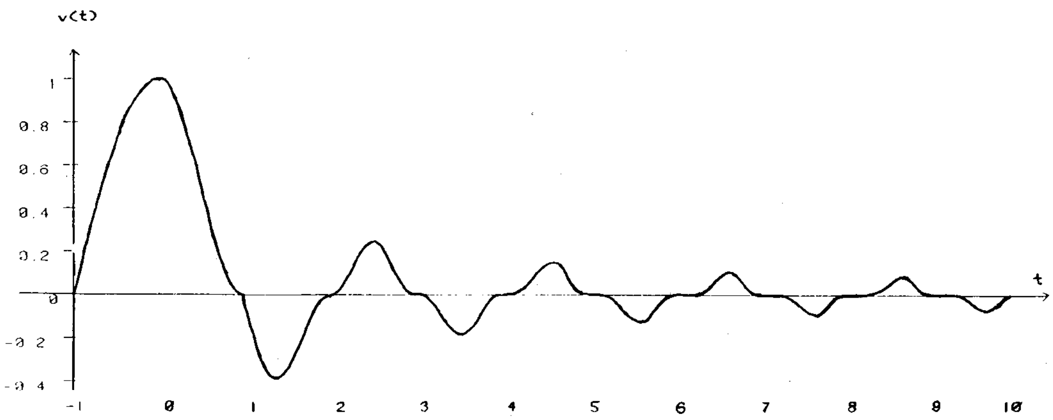

Fig. 1. Graph of unit function.

This function has complex values. It satisfies both the orthogonality and integrability conditions and the consequences of this, and using this unit function, it is certain that the left side of (4.8) is bounded for $N \to \infty$ with one and the same bound for *any* bounded signal, especially those with infinite energy which are the main objects of this investigation.

Finally, it should be remarked that in many respects it would be preferable with a unit function which, besides satisfying the orthogonality and the integrability condition, is real and even. However, we have not yet been able to decide whether such a function exists or not.

VI. Examples and Applications

As discussed in the Introduction, the orthonormal system of functions studied in the previous sections was motivated by an application to real-time spectrum analysis of time-varying signals. We shall present several examples of expansions of finite energy signals in series of the form (1.1). The coefficients were computed by sampling $f(t)v(t-n)$ and using the fast Fourier transform.

To give an illustration of the results, we have collected the terms corresponding to the same real frequency $2\pi k$ and time n into the function

$$g'_{n,k}(t) = v(t-n)\left(a_{n,k}e^{2\pi ikt} + a_{n,-k}e^{-2\pi ikt}\right)$$

which is always real.

In the figures $g'_{n,k}$ is represented by

$$g_{n,0} = a_{n,0}$$

$$g_{n,k} = a_{n,k}e^{2\pi ikt} + a_{n,-k}e^{-2\pi ikt}, \qquad k > 0,\ n - \tfrac{1}{2} \le t \le n + \tfrac{1}{2}$$

which is plotted in the rectangle (n, k) for $0 \le k \le K$ and $0 \le n \le N$. Thus $g_{n,k}$ provides information about the frequency, magnitude, and phase of the corresponding term in the series expansion.

Fig. 2 illustrates the series expansion of a segment of a square wave with frequency 4.2π. It may be noted that most of the energy is found in the terms $k = 2$ and $k = 6$ as should be expected from the Fourier series of the square wave. No disturbing transients exist at the endpoints of the segment. This series was calculated for a number of values of the frequency to confirm that no critical changes in the coefficients occurred.

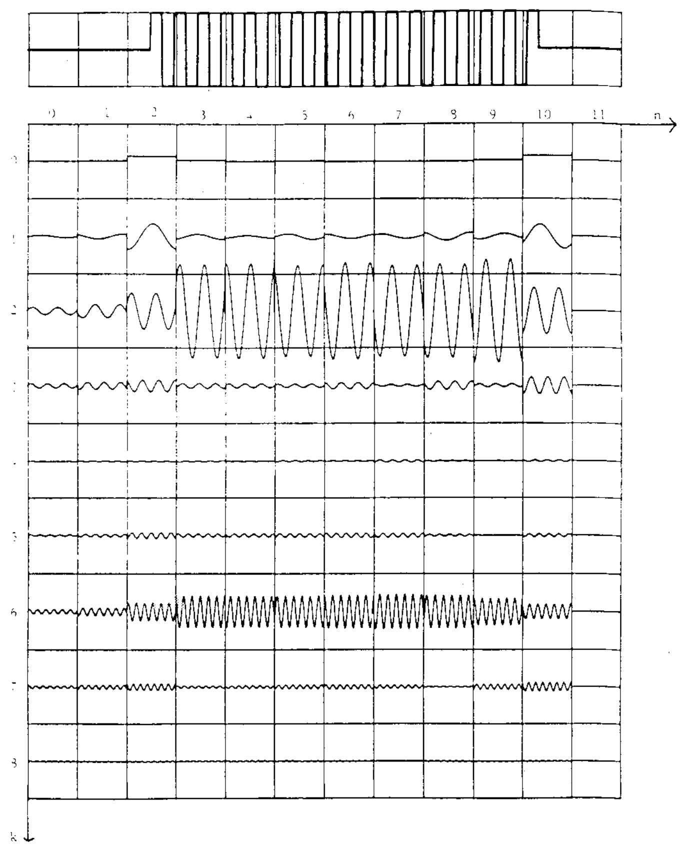

Fig. 2. Series expansion of square wave.

Fig. 3 similarly illustrates the series obtained by expanding a segment of a harmonic function which contains a 180° phase shift. The result indicates a narrow spectrum except for interval containing the phase shift where higher frequency terms appear. Fig. 4 represents a series obtained from a band-limited function with cutoff frequency 5π and approximately constant energy spectrum at lower frequencies. Again, the expected properties of the signal is brought out by the coefficients without significant artificial effects introduced by the signal processing.

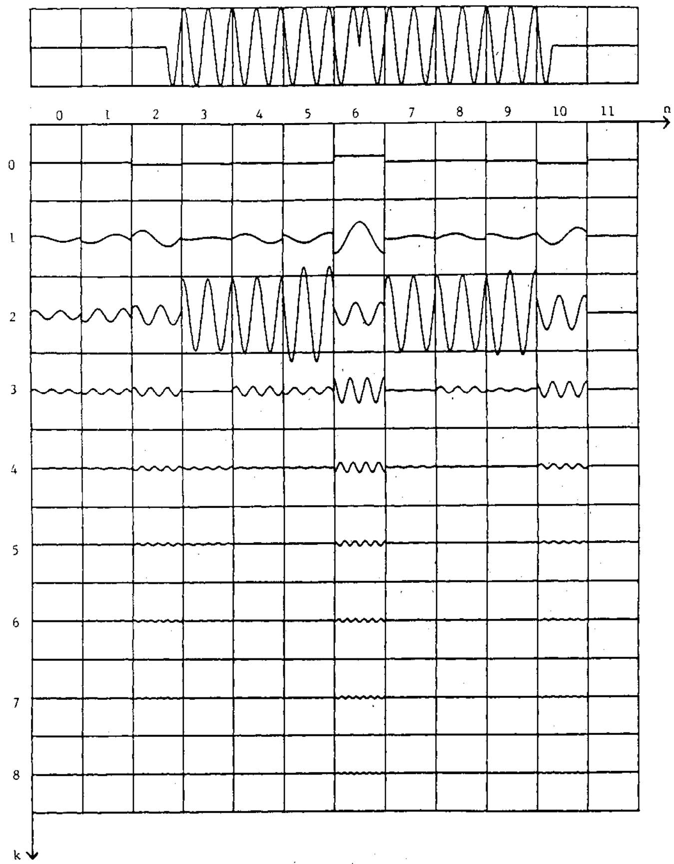

Fig. 3. Series expansion of phase shift signal.

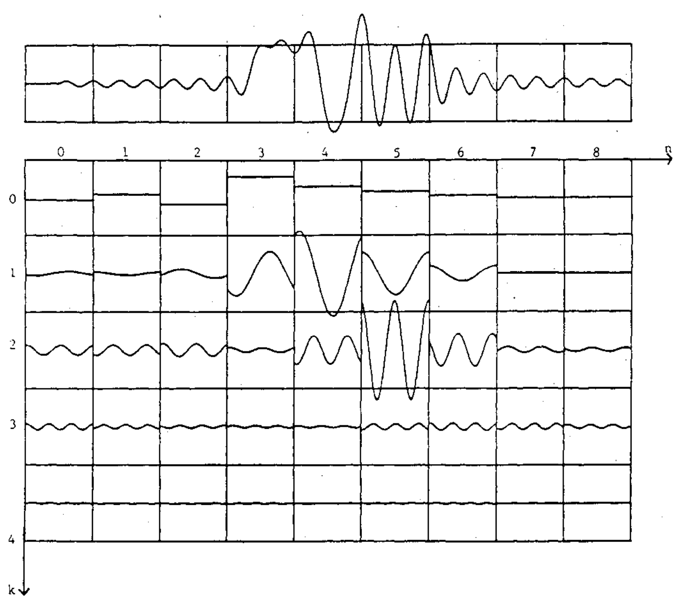

Fig. 4. Series expansion of band-limited signal.

If $v(t)$ is interpreted as a pulse shape used in pulse position modulation, the condition

$$\int_{-\infty}^{\infty} v(t)v(t+n)\,dt = 0$$

indicates the absence of intersymbol interference. This condition is commonly used in the design of signals for data communication. However, the orthogonality of pulses modulated to other frequency bands suggests that such pulse shapes would be useful when several mutually synchronized channels are frequency multiplexed. The usual approach in frequency division multiplexing is to use sharp filters for separating the channels, but this method requires a significant amount of bandwidth in excess of the theoretical minimum. Our results indicate that the interference from adjacent channels can be controlled by the use of pulses which satisfy the orthogonality conditions studied here.

Acknowledgment

The examples were programmed and run by Klaus Grue. The authors want to thank the referees for pointing out inaccuracies and errors in an earlier version of the paper. Further, we thank R. Coifman and S. Semmes for letting us use their proof of Theorem 5.2, and the referee for transmitting it to us.

Appendix I
Proof of the Summability Results

Let $v(t)$ satisfy the orthogonality and integrability conditions, and suppose furthermore that the series $\Sigma_{n=-\infty}^{\infty} v(t+n)$ in a neighborhood of each $t_0 \in]0,1[$ has a convergent majorant. Let $a_{n,k}$ be given by (3.9), where $f(t)$ is bounded and piecewise continuous.

Let us consider the sum,

$$w_n(t) = \sum_{p=-\infty}^{\infty} f(t+p)v(t-n+p). \tag{I.1}$$

From the above assumption it follows that the limits $w_n(t_0^+)$ and $w_n(t_0^-)$ both exist, and we have

$$w_n(t_0^+) = \sum_{p=-\infty}^{\infty} f\big((t_0+p)^+\big)v(t_0-n+p)$$

and the analogous expression for $w_n(t_0^-)$. Now it can be seen that $w_n(t) \in L^1([0,1])$, and therefore it follows from the Fejér theorem that the Fourier series for $w_n(t)$ is summable at t_0 with sum $\frac{1}{2}(w_n(t_0^+) + w_n(t_0^-))$. Therefore, in the sense of summability we have

$$\sum_{k=-\infty}^{\infty} a_{n,k} v(t_0-n)e^{i2\pi k t_0} = \frac{1}{2}v(t_0-n)\left[w_n(t_0^+) + w_n(t_0^-)\right]$$

and hence

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} & a_{n,k} v(t_0-n)e^{i2\pi k t_0} \\ &= \sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} v(t_0-n)v(t_0-n+p) \\ &\quad \cdot \frac{1}{2}\left[f\big((t_0+p)^+\big) + f\big((t_0+p)^-\big)\right] \\ &= \frac{1}{2}\left[f(t_0^+) + f(t_0^-)\right]\end{aligned}$$

since the integrability condition allows changing the order of summation, and then using the orthogonality.

For the double sum

$$\sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{n,k} v(t-n) e^{ik2\pi t},$$

one first considers

$$g(s) = \sum_{n=-\infty}^{\infty} f(s) v(s-n) v(t-n), \qquad s \in \boldsymbol{R}$$

and prove that $g(s) \in L^1(\boldsymbol{R})$ and therefore

$$\sum_{n=-\infty}^{\infty} a_{n,k} v(t-n) e^{ik2\pi t} = \int_{-\infty}^{\infty} g(s+t) e^{-ik2\pi s} \, ds.$$

Moreover, the function $h(s)$ defined by

$$h(s) = \sum_{m=-\infty}^{\infty} g(s+t+m)$$

can be seen to belong to $L^1(\boldsymbol{R})$ and to be periodic with period 1.

From the fact that an easy calculation shows that $h(0^+)$ and $h(0^-)$ both exist it then follows from the Fejér theorem applied to $h(s)$ that in the sense of summability we have

$$\sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} a_{n,k} v(t-n) e^{i2\pi kt} = \frac{1}{2}(f(t^+) + f(t^-)).$$

Appendix II
Proof of Theorem 5.2 (Due to Coifman and Semmes)

Suppose, that the sum of (5.9) and (5.11) is finite. If

$$V(\tau, \omega) = \sum v(n+\tau) e^{in\omega}, \tag{II.1}$$

this implies that $\partial_\tau V$ and $\partial_\omega V$ exist and are square integrable over any bounded, measurable set in $\boldsymbol{R}^2$. Hence there is a number r_0 such that, for $0 \le r \le r_0$ and $(\tau_0, \omega_0) \in [-1,2] \times [-3\pi, 3\pi]$, we have

$$\left[\iint_{\substack{|\tau-\tau_0| \le 2r \\ |\omega-\omega_0| \le 2r}} d\tau \, d\omega \left| (\partial_\tau V)(\tau,\omega) \right|^2 + \left| (\partial_\omega V)(\tau,\omega) \right|^2 \right) \right]^{1/2} \le \frac{1}{32\sqrt{2}}. \tag{II.2}$$

Define the function V_r by

$$V_r(\tau, \omega) = \frac{1}{4r^2} \iint_{\substack{|\tau'-\tau| \le r \\ |\omega'-\omega| \le r}} d\tau' \, d\omega' V(\tau', \omega').$$

Then V_r is continuous, and by the Cauchy–Schwarz inequality and (II.2) one gets

$$\begin{aligned} &\frac{16}{r^2} \iint_{\substack{|\tau-\tau_0| \le r/8 \\ |\omega-\omega_0| \le r/8}} d\tau \, d\omega \left| V(\tau,\omega) - V_r(\tau,\omega) \right| \\ &\le \frac{4}{r} \left[\int_{|\tau''-\tau_0| \le 2r} d\tau'' \int_{|\omega-\omega_0| \le 2r} d\omega \left| (\partial_\tau V)(\tau'', \omega) \right| \right. \\ &\quad \left. + \int_{|\tau'-\tau_0| \le 2r} d\tau' \int_{|\omega''-\omega_0| \le 2r} d\omega'' \left| (\partial_\omega V)(\tau', \omega'') \right| \right] \\ &\le 8\sqrt{2} \left[\iint_{\substack{|\tau-\tau_0| \le 2r \\ |\omega-\omega_0| \le 2r}} d\tau \, d\omega \left((\partial_\tau V)(\tau,\omega)^2 \right. \right. \\ &\quad \left. \left. + (\partial_\omega V)(\tau,\omega)^2 \right) \right] \le 1/4. \end{aligned}$$

The continuity of V_r and the condition $V(\tau, \omega) = 1$ almost everywhere (the orthogonality condition) then implies that $|V_r(\tau, \omega)| \ge 1/2$ for all $(\tau, \omega) \in [-1,2] \times [-3\pi, 3\pi]$.

The next important step is that, since V_r is continuous and bounded below by a constant greater than zero, a *continuous* function $\varphi_r(\tau, \omega)$ exists [27, ch. I] such that

$$V_r(\tau, \omega) = e^{\varphi_r(\tau,\omega)}, \qquad \text{for all } (\tau, \omega) \in [1,2] \times [-3\pi, 3\pi]. \tag{II.3}$$

We have $V(\tau, \omega + 2\pi) = V(\tau, \omega)$, and therefore $V_r(\tau, \omega + 2\pi) = V_r(\tau, \omega)$, which gives

$$\varphi_r(\tau, \omega + 2\pi) = \varphi_r(\tau, \omega) + i2\pi k(\tau, \omega) \tag{II.4}$$

where $k(\tau, \omega)$ is constant, because φ_r is continuous.

Moreover, we have $V(\tau + 1, \omega) = e^{-i\omega} V(\tau, \omega)$, from which, by the definition of V_r, one derives $|V_r(\tau+1, \omega) - e^{-i\omega} V_r(\tau, \omega)| \le r$. By choosing r small enough, one can then prove that

$$\varphi_r(\tau + 1, \omega) = \varphi_r(\tau, \omega) - i\omega + i2\pi l(\tau, \omega) + \psi(\tau, \omega) \tag{II.5}$$

where, by continuity of φ_r, the function $l(\tau, \omega)$ is constant, and

$$|\psi(\tau, \omega)| \le \pi/2. \tag{II.6}$$

The statements (II.4)–(II.6) all hold in the rectangle $[-1,2] \times [-3\pi, 3\pi]$ and therefore especially in the boundary of $[0,1] \times [-\pi, \pi]$. From (II.4) and (II.5) follows

$$\varphi_r(1, \pi) = \varphi_r(0, -\pi) + i2\pi k + i\pi + i2\pi l + \psi(0, -\pi)$$

$$\varphi_r(1, \pi) = \varphi_r(0, -\pi) + i2\pi k - i\pi + i2\pi l + \psi(0, \pi),$$

but this contradicts (II.6). This contradiction proves the theorem.

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