[8] T. S. Lee, "Large sample identification and spectral estimation of noisy multivariate autoregressive processes," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-31, pp. 76–82, Feb. 1983.
[9] J. B. Moore and H. Weiss, "Recursive prediction error methods for adaptive estimation," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-9, pp. 197–206, Apr. 1979.
[10] L. Ljung and T. Söderström, *Theory and Practice of Recursive Identification*. Cambridge, MA: MIT Press, 1983.
[11] L. Ljung, "Analysis of a general recursive prediction error identification algorithm," *Automatica*, vol. 17, pp. 89–99, Jan. 1981.
[12] B. Friedlander, "System identification techniques for adaptive signal processing," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-30, pp. 240–246, Apr. 1982.
[13] L. Ljung, "Asymptotic gain and search direction for recursive identification algorithm," in *Proc. 19th IEEE Conf. Decision and Control*, Dec. 1980, pp. 981–985.
[14] J. J. Kormylo and J. M. Mendel, "Maximum-likelihood seismic deconvolution," *IEEE Trans. Geosci. Remote Sensing*, vol. GE-21, pp. 72–82, Jan. 1983.
[15] S. A. Tretter, *Introduction to Discrete Signal Processing*. New York: Wiley, 1976.
[16] H. Akaike, "A new look at the statistical model identification," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 716–723, Dec. 1974.
[17] R. L. Kashyap, "Inconsistency of the AIC rule for estimating the order of autoregressive models," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 996–998, Oct. 1980.
[18] A. C. Atkinson, "Likelihood ratios, porterior odds, and information criteria," *J. Econometrics, Annals of Applied Econometrics 1981-2*, a supplement to *J. Econometrics*, vol. 16, pp. 15–20, May 1981.
[19] D. F. Findley, "Introduction to the papers," in *Applied Time Series Analysis II*, D. F. Findley, Ed. New York: Academic, 1981.
[20] B. Friedlander, "System identification techniques for adaptive noise cancelling," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 699–709, Oct. 1982.
[21] A. Benveniste and G. Ruget, "A measure of the tracking capability of recursive stochastic algorithms with constant gains," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 639–649, June 1982.

Autocorrelation Properties of a Class of Infinite Binary Sequences

TOM HØHOLDT, HELGE E. JENSEN AND JØRN JUSTESEN

***Abstract*—A class of infinite ± 1 sequences is presented whose autocorrelation function is zero for all nonzero shifts.**

I. Introduction

Many problems in information theory call for the construction of an infinite binary (± 1) sequence, with a known or prescribed autocorrelation function. De Carvalho and Clark [1] characterized the autocorrelations of binary sequences, and Yarlagadda and Hersey [2] calculated the autocorrelations and the power spectral density $S(\omega)$ of the Thue–Morse sequence, and proved that $S(\omega)$ was everywhere discontinuous and consisted of an infinite number of delta functions. In this correspondence we show that the sequences considered in [3] can be used to generate a class of infinite binary sequences, whose autocorrelation is zero for all nonzero shifts and whose power spectral density $S(\omega)$ therefore is equal to one for all ω.

Manuscript received March 5, 1985; revised September 16, 1985.
T. Høholdt and H. E. Jensen are with the Mathematical Institute, The Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark.
J. Justesen is with the Institute of Circuit Theory and Telecommunication, The Technical University of Denmark, Building 343, DK-2800, Lyngby, Denmark.
IEEE Log Number 8406954.

II. The Sequences and Their Autocorrelations

The sequences considered in [3] are defined recursively by

$$x_0 = 1$$

$$x_{2^i+j} = (-1)^{j+f(i)} x_{2^i-j-1}, \qquad 0 \le j \le 2^i - 1, \quad i = 0,1,\cdots, \tag{2.1}$$

where f is any function mapping the set of natural numbers into $\{0,1\}$. We shall need the following fact from [3]:

$$x_{2^m+i} = -x_i(-1)^{f(m-1)+f(m-2)}, \qquad \text{if } i < 2^{m-1}, \quad m \ge 1. \tag{2.2}$$

We define

$$C(m,k) = \sum_{i=0}^{2^m-k-1} x_i x_{i+k}, \tag{2.3}$$

then $C(m,k) = 0$ if k is even and

$$|C(m,k)| \le A \cdot (2^m)^{0.9}, \tag{2.4}$$

where A is a constant. Based on these, we will prove that the autocorrelations, that is,

$$C_k = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} x_i x_{i+k}, \tag{2.5}$$

for the infinite sequences defined by (2.1) are zero if $k \ne 0$. To see this we define for fixed k

$$A_k = \sum_{i=0}^{N} x_i x_{i+k} \tag{2.6}$$

and then prove the following theorem.

Theorem: For the sequences defined by (2.1) we have

$$|A_k| \le |C(m,k)| + |C(m-1,k)| + \cdots + |C(j,k)| + 2mk + 1,$$

where $m+1 = \lceil \log_2 (N+k+1) \rceil$ and $j = \lceil \log_2 k \rceil$.

Proof: The proof is by induction on m, and the start is obvious. So, suppose we have proved the theorem for all N, where $\lceil \log_2 (N+k+1) \rceil \le p$, and let us consider the case where $\lceil \log_2 (N+k+1) \rceil = p+1$.

We have to consider the four cases

1) $k > 2^p - 1$;
2) $k \le 2^p - 1$, $N + k < 2^p + 2^{p-1}$;
3) $2^{p-1} \le k \le 2^p - 1$, $N + k \ge 2^p + 2^{p-1}$;
4) $k \le 2^{p-1} - 1$, $N + k \ge 2^p + 2^{p-1}$.

In case 1) we must have $N < 2^p - 1 < k$ and the theorem is obvious. In case 2) we have

$$A_k = C(p,k) + \sum_{i=0}^{N-2^p+k} x_i + 2^p - kx_i + 2^p,$$

which, for $N + k < 2^p + 2^{p-1}$ and $N \ge 2^p$, using (2.2), yields

$$A_k = C(p,k) + \sum_{i=0}^{k-1} x_i + 2^p - kx_i + 2^p + \sum_{i=0}^{N-2^p} x_i x_{i+k}.$$

This proves the statement by the induction hypothesis. If $N < 2^p$ and $k \le 2^p - 1$ the statement is trivial. Case 3) follows immediately from the fact that $N - 2^p + k + 1 \le 2k$.

0018-9448/86/0500-0430$01.00

Case 4), which is the only difficult one, can be seen as follows:

$$A_k = C(p,k) + \sum_{i=0}^{k-1} x_i + 2^p - kx_i + 2^p + \sum_{i=0}^{2^{p-1}-k-1} x_i + 2^p x_i + 2^p + k + \sum_{i=0}^{N+k-2^p-2^{p-1}} x_i + 2^p + 2^{p-1} - kx_i + 2^p + 2^{p-1}$$

$$= C(p,k) + \sum_{i=0}^{k-1} x_i + 2^p - kx_i + 2^p + C(p-1,k) + (-1)^k \sum_{j=2^{p+1}-(N+k)-1}^{2^{p-1}-1} x_j x_{j+k},$$

where we have used (2.2) in the third sum, and in the last sum the definition (2.1) and the substitution j: $2^{p-1} - i - 1$. We then get

$$A_k = C(p,k) + C(p-1,k)$$
$$+ \sum_{i=0}^{k-1} x_i + 2^p - kx_i + 2^p + (-1)^k C(p-1,k)$$
$$+(-1)^k \sum_{j=2^{p-1}-k}^{2^{p-1}-1} x_j x_{j+k} - (-1)^k \sum_{j=0}^{2^{p+1}-(N+k)} x_j x_{j+k}.$$

The sums in the third and fifth term on the right side have at most k terms and $\lceil \log_2 (2^{p+1} - (N+k) + k + 1) \rceil \leq p$. So, if k is odd the statement follows, and if k is even we use (2.3) and the proof of the theorem is finished.

Based on the theorem it is straightforward to see that for the sequences defined by (2.1) we have

$$C_k = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} x_i x_{i+k} = 0, \qquad \text{if } k \neq 0.$$

Moreover, since the power spectral density $S(\omega)$ can be defined as

$$S(\omega) = \sum_{k=-\infty}^{\infty} C_{|k|} e^{ik\omega},$$

we have $S(\omega) = 1$.

References

[1] J. L. Martins de Carvalho and J. M. C. Clark, "Characterizing the autocorrelations of binary sequences," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 502–508, July 1983.

[2] R. Yarlagadda and J. E. Hersey, "Spectral properties of the Thue–Morse sequence," *IEEE Trans. Commun.*, vol. COM-32, pp. 974–977, Aug. 1984.

[3] T. Høholdt, H. E. Jensen, and J. Justesen, "Aperiodic correlations and the merit factor of a class of binary sequences," *IEEE Trans. Inform. Theory*, vol. IT-31, pp. 549–552, July 1985.

The Coding Capacity of Mismatched Gaussian Channels

IAN W. McKEAGUE AND CHARLES R. BAKER, MEMBER, IEEE

Abstract—**Bounds on the coding capacity of Gaussian channels are obtained when the power constraint on the signal is mismatched to the channel noise. In the case of some feedback channels in which the noise has a Cramér–Hida representation of finite multiplicity, an exact expression for the coding capacity is given.**

Manuscript received January 18, 1985; revised September 10, 1985. This work was supported by ARO under Contract DAAG 29-82-K-0168 and by ONR under Contracts N00014-81-K-0373 and N00014-84-C-0212.

I. W. McKeague is with the Department of Statistics, Florida State University, Tallahassee, FL 32306-3033.

C. R. Baker is with the Department of Statistics, University of North Carolina, Chapel Hill, NC 27514.

IEEE Log Number 8406956.

I. Introduction

Consider a message $X = (X_t, 0 \leq t \leq T)$ transmitted through a noisy channel with output $Y = (Y_t, 0 \leq t \leq T)$ given by

$$dY_t = X_t\,dt + a(t)\,dW_t, \qquad 0 \leq t \leq T, \tag{1.1}$$

where $a(\cdot)$ is a nonrandom continuous function and is nonconstant and bounded away from zero, W is standard Wiener process, and X is adapted to Y. X is subject to the power constraint

$$\int_0^T X_t^2\,dt \leq P_0 T, \qquad \text{a.s.} \tag{1.2}$$

where P_0 is a positive constant. This channel is an example of a mismatched time-continuous Gaussian feedback channel (MCGF). It is said to be mismatched because the power constraint is not expressed in terms of the covariance of the noise process $N_t = \int_0^t a(s)\,dW_s$, but in terms of the covariance of some other noise process, namely, $\bar{N}_t = W_t$.

Mismatched channels can arise in jamming situations, in problems where there is insufficient knowledge of the environment, or where one prefers to use a constraint not expressed in terms of the channel noise. Various special cases have been treated (see, for example, Fano [5], Gallager [6], Ihara [9], and Glonti [7]). General results on the information capacity for such channels have been obtained by Baker [2], [3]. Here we are interested in evaluating the *coding* capacity of mismatched Gaussian channels. The main result of this correspondence is Theorem 1, which represents an extension of the coding capacity results of McKeague [11], [12] in two directions: to mismatched Gaussian channels and to nonwhite Gaussian channels, respectively. In Sections II and III we deal with mismatched Gaussian channels without feedback. In Section IV we treat the feedback case under the assumption that the Gaussian noise process has a Cramér–Hida representation of finite multiplicity satisfying some assumptions on the representation. As an illustration of Theorem 1 (see Example 2 in Section IV) we note that the coding capacity of the channel in the example is given by

$$C_0 = \frac{P_0}{2 \inf_{t\geq 0} a^2(t)}.$$

II. The Mismatched Channel Without Feedback for Finite Time Intervals

Let $N = (N_t, 0 \leq t \leq T)$ be a Gaussian noise process with reproducing kernel Hilbert space (RKHS) denoted $(H, \|\cdot\|)$. Throughout this section T remains fixed. The distribution induced by N on the cylindrical σ-algebra B_T of $\mathbb{R}^{[0,T]}$ is denoted μ_N. For a message $X \in \mathbb{R}^{[0,T]}$, the received signal is given by

$$Y_t = X_t + N_t, \qquad 0 \leq t \leq T.$$

The distribution of Y on B_T is given by

$$\mu_{Y|x}(D) = \mu_N\{y: x + y \in D\}, \qquad D \in B_T.$$