A Finsler geometric Attack on Wildfires

Geodesic Sprays

First Principles

Steen Markvorsen

DTU Compute

Synopsis

1. Matsumoto's cross country metric

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2. Finsler geometric analysis

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3. Wildfires

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4. Huygens' principle

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4. Huygens' principle
5. Richards' equations solved

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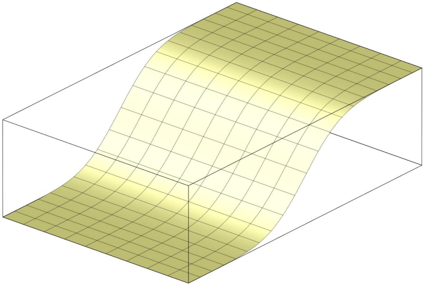
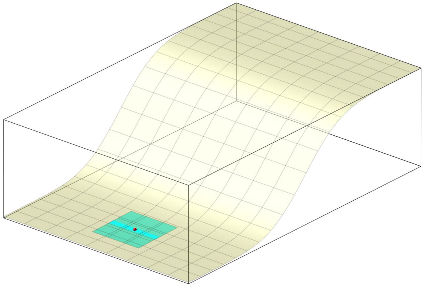
3. Wildfires
4. Huygens' principle
5. Richards' equations solved
6. Further work

Cross country training on a hillside

Running on a hillside



Running on a hillside



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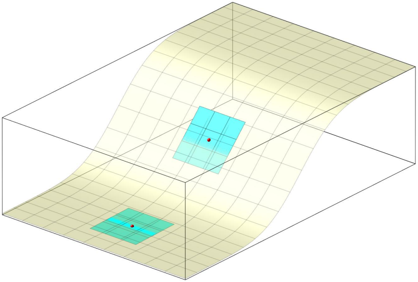

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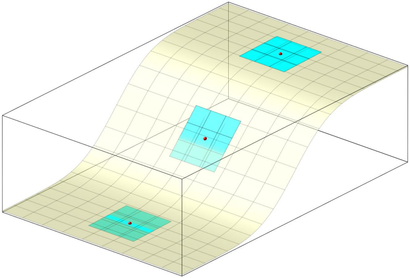

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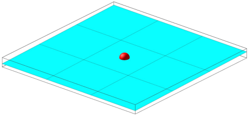

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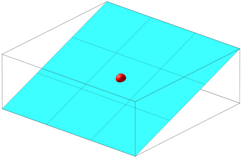

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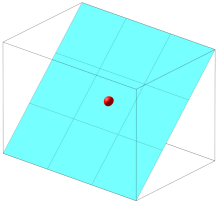

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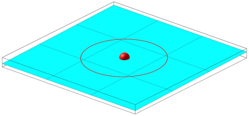

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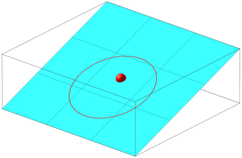

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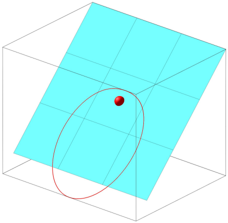

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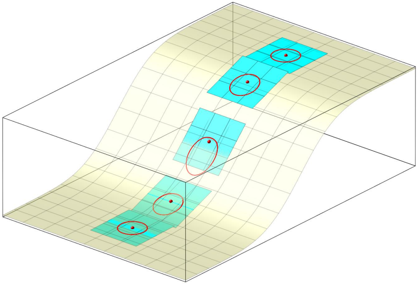

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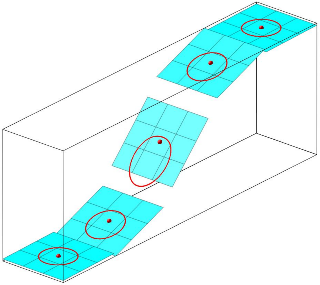

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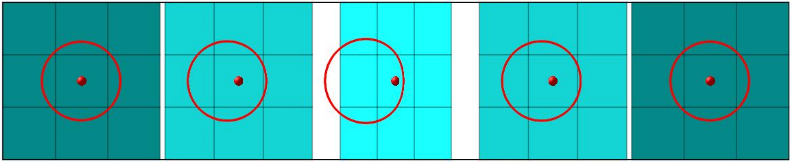

Finsler manifolds

Definition

A Finsler manifold is a pair (M^n, F) consisting of a smooth manifold M and a real valued nonnegative function $F(p, y)$ on the tangent bundle TM, $p \in M$, $y \in T_pM$, such that

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1. F is C^∞ on $TM - \{0\}$
2. For each $p \in M$ the function $F_{|_{T_pM}}$ is a *Minkowski norm* on the tangent space T_pM

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1. F is C^∞ on $V - \{0\}$
2. $F(\lambda \cdot y) = \lambda \cdot F(y)$ for all $\lambda > 0$ and for all $y \in V$
3. For every $y \in V - \{0\}$ the following symmetric bilinear form $g_y(u, v)$ is positive definite:

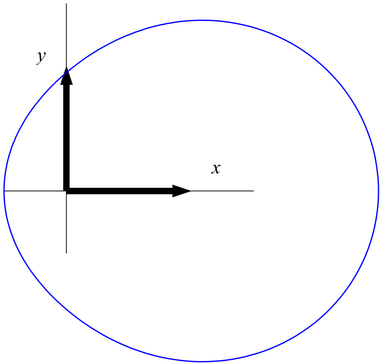
$$
\begin{aligned}
g_y(u, v) &= u \cdot \left(\frac{1}{2} \operatorname{Hess}_y F^2(y) \right) \cdot v^\top \\
&= \frac{1}{2} \frac{\partial^2}{\partial s \partial t}_{|s=0,t=0} F^2\left(y + s \cdot u + t \cdot v\right)
\end{aligned}
$$

The ovaloid indicatrix construction of Minkowski norms

Theorem

If a given ovaloid containing $0 \in V$ has positive sectional curvature in V, then the corresponding F is a Minkowski norm on V.

The ovaloid indicatrix construction of Minkowski norms



Oval indicatrix for Minkowski norm in $V = \mathbb{R}^2$

The ovaloid indicatrix construction of Minkowski norms

Oval indicatrix construction of F and F^2 extensions in $V = \mathbb{R}^2$

The Finsler length functional

Definition (Finsler length)

A curve $\gamma(t)$, $t \in [0, T]$, has direction dependent $F-$length defined directly by

$$\mathcal{L}(\gamma) = \int_0^T F(\gamma(t), \dot{\gamma}(t))\, dt \quad .$$

The fundamental tensor and some sputniks

Definition

The positive definite symmetric bilinear form g_y considered as a function on TM is called the *fundamental tensor* on (M, F):

$$
\begin{aligned}
g_{ij}(p, y) &= \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j} F^2(p, y) \quad , \quad \text{(fundamental tensor)} \\
C_{ijk}(p, y) &= \frac{1}{4}\frac{\partial^3}{\partial y^i \partial y^j \partial y^k} F^2(p, y) \quad , \quad \text{(Cartan tensor)} \\
g^{ij}(p, y) &= \left(g_{ij}(p, y)\right)^{-1} \\
\gamma^k_{ij}(p, y) &= \frac{1}{2} g^{km}(p, y) \cdot \left(\frac{\partial g_{mj}}{\partial p^i} + \frac{\partial g_{im}}{\partial p^j} - \frac{\partial g_{ij}}{\partial p^m} \right) \\
N^i_j(p, y) &= \gamma^i_{jk} \cdot y^k - C^i_{jk} \cdot \gamma^k_{rs} \cdot y^r \cdot y^s \quad .
\end{aligned}
$$

More sputniks

Definition

The Chern connection is defined by the following Christoffel symbols:

$$\Gamma^i_{jk} = \frac{\partial^2 G^i}{\partial y^j \partial y^k} - g^{il} \cdot B_{jkl} \quad ,$$

where

$$\begin{aligned} B_{ijk}(p, y) = \frac{\partial C_{ijk}}{\partial p^l} \cdot y^l - 2\frac{\partial C_{ijk}}{\partial y^l} \cdot G^l \\ - C_{ljk} \cdot N^l_i - C_{ilk} \cdot N^l_j - C_{ijl} \cdot N^l_k \quad . \end{aligned}$$

The flag curvature

Definition (Intermediate)

$$R^i_k(p,y) = 2\cdot\frac{\partial G^i}{\partial p^k} - y^j\cdot\frac{\partial^2 G^i}{\partial p^j \partial y^k} + 2G^j\cdot\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\cdot\frac{\partial G^j}{\partial y^k}$$

$$R_y|_p = R^i_k(p,y)\cdot\frac{\partial}{\partial p^i}\otimes dp^k\,|_p \quad : \quad T_pM \to T_pM$$

$$g_y|_p = g_{ij}(p,y)\cdot dp^i\,|_p\otimes dp^j\,|_p \quad : \quad T_pM\times T_pM \to \mathbb{R}$$

The flag curvature

Definition (Flag curvature)

$$K(\sigma, y) = \frac{g_y(R_y(u),\ u)}{g_y(y,y) \cdot g_y(u,u) - (g_y(y,u))^2} \quad , \quad \text{(the flag curvature)} \quad ,$$

where $\sigma = \mathrm{span}\{y, u\} \in T_pM$ is a $2D$ y-flag in the tangent space at p.

Geodesic equation

Definition (Geodesic spray)

The geodesic spray coefficients are

$$G^i(p, y) = \gamma^i_{jk}(p, y) \cdot y^j \cdot y^k \quad ,$$

and the geodesic equations thereby

$$\ddot{c}^j(t) + 2G^j(c(t), \dot{c}(t)) = 0 \quad .$$

Geodesic equation

The geodesic equations stem directly from the first variation formula:

$$\mathcal{L}'(0) = -\int_a^b \frac{1}{F(c(t),\dot{c}(t))} \cdot g_{jk}(c(t),\dot{c}(t)) \cdot \left(\ddot{c}^j(t) + 2G^j(c(t),\dot{c}(t))\right) \cdot V^k \; dt \; .$$

Geodesic equation

$$\ddot{c}^j(t) + 2G^j(c(t), \dot{c}(t)) = 0$$

Wildfires

Wildfire templates

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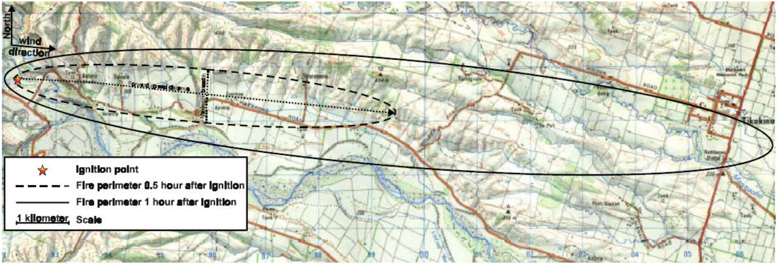
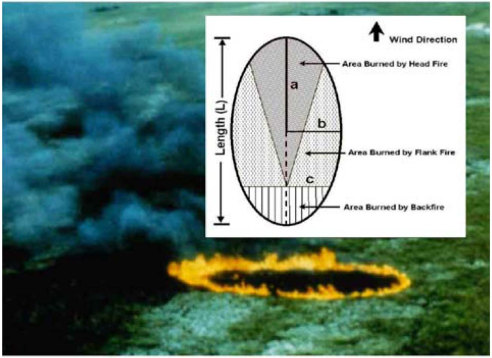
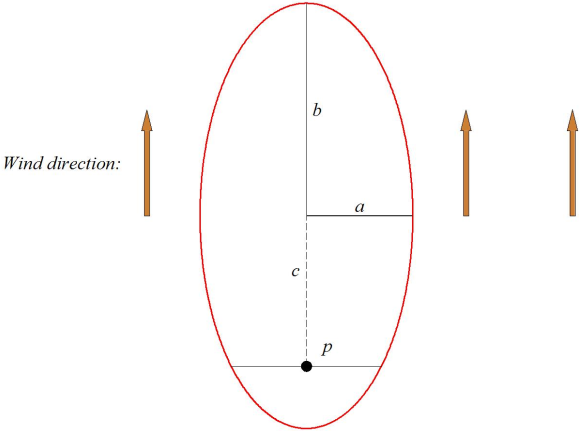


Figure 4. Free-burning fire growth projected for conditions similar to those experienced during the 1991 Tikokino Fire using the *Field Guide*.

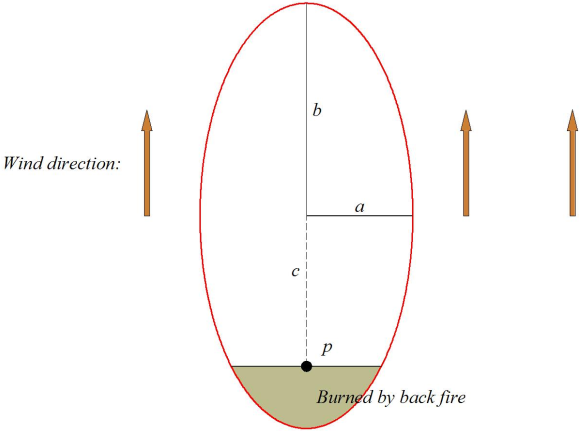
Wildfire templates



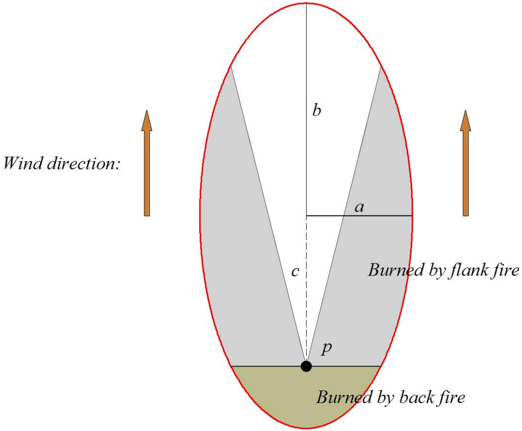
Wildfire templates



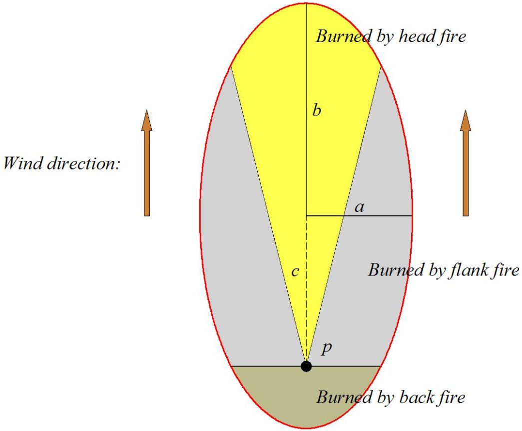
Wildfire templates



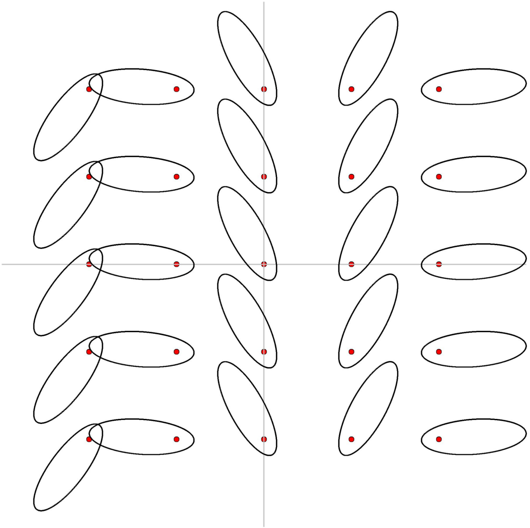
Wildfire templates



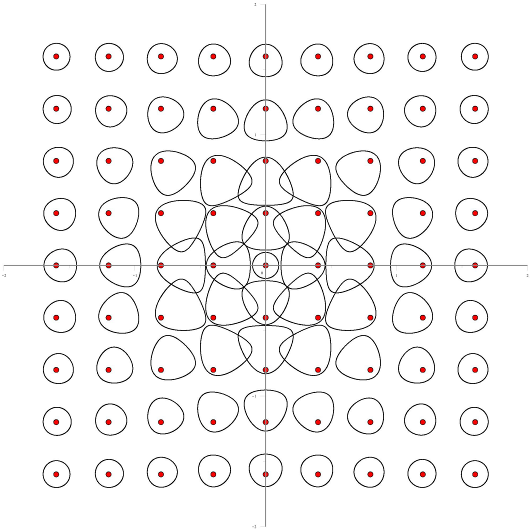
Wildfire templates



Non-constant ellipse field



Non-constant oval field



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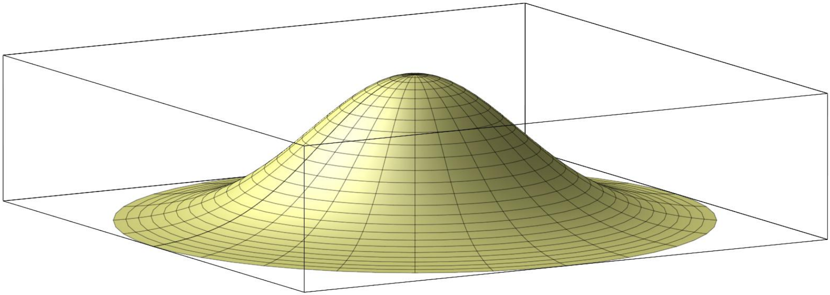

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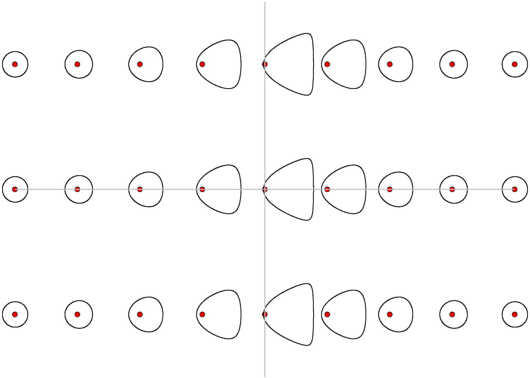
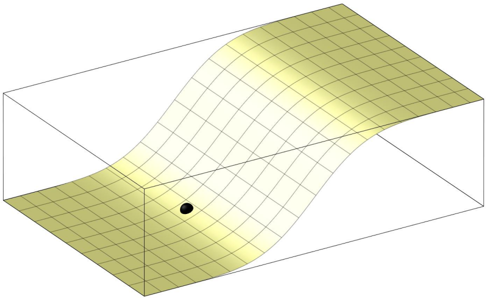
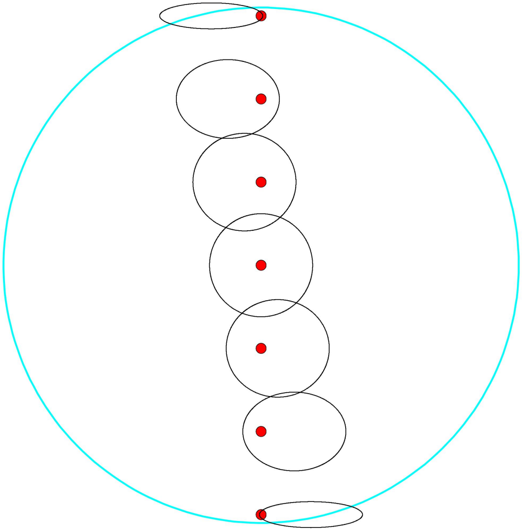


Non-constant oval field



Non-constant ellipse field

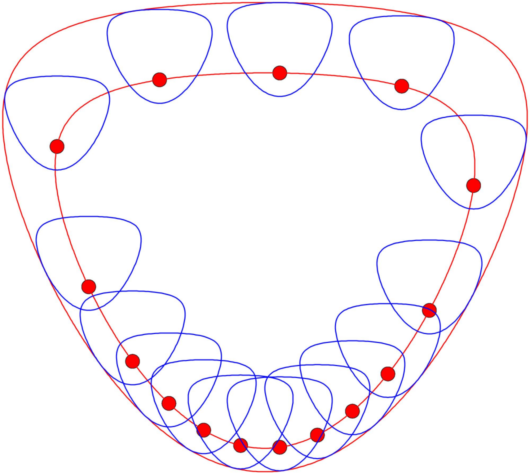


Constant oval field

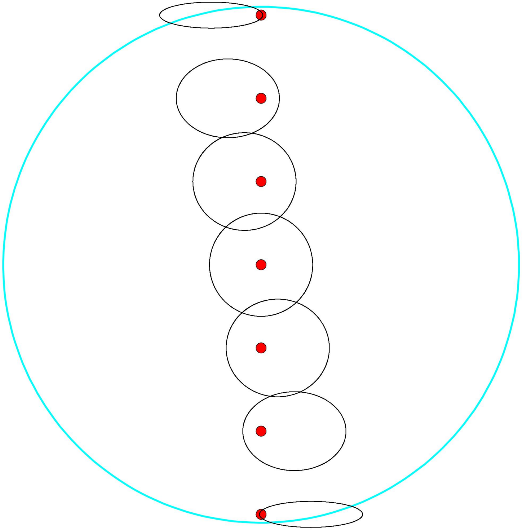
Huygens' Principle

Any fire front $\eta_{t_1}(s)$ at time t_1 is the envelope of the point-ignited wildfires of duration $t_1 - t_0$ from the points on the previous fire front $\eta_{t_0}(s)$ from time t_0.

Constant oval field



Non-constant ellipse field



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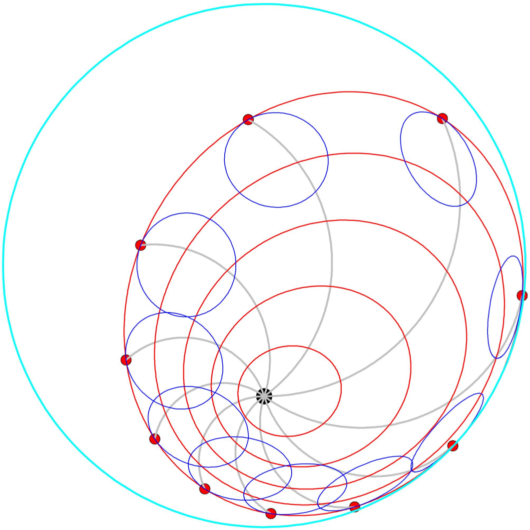

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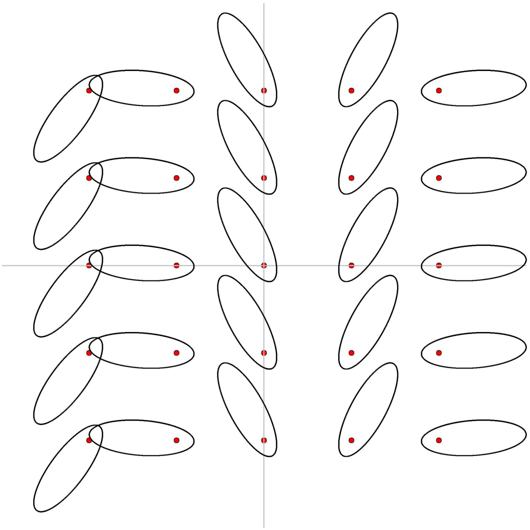

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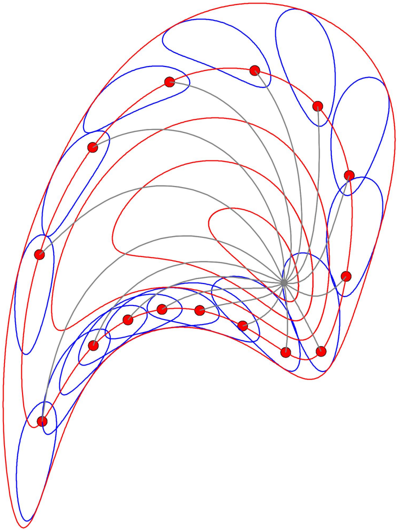


For (non-contant) ellipse fields

Theorem (G.D. Richards, 1990)

An elliptic wildfire $\gamma(s,t) = (u(s,t), v(s,t))$ with ellipse data $a(u,v)$, $b(u,v)$, $C(u,v) = (c_1(u,v), c_2(u,v))$, and $\theta(u,v)$ is determined by the following equations:

$$u'_t = \frac{a^2 \cos(\theta)\left(u'_s \sin(\theta) + v'_s \cos(\theta)\right) - b^2 \sin(\theta)\left(u'_s \sin(\theta) + v'_s \cos(\theta)\right)}{\sqrt{a^2\left(u'_s \sin(\theta) + v'_s \cos(\theta)\right)^2 + b^2\left(u'_s \cos(\theta) - v'_s \sin(\theta)\right)^2}} + c_1 \cos(\theta) + c_2 \sin(\theta)$$

$$v'_t = \frac{-a^2 \sin(\theta)(u'_s \sin(\theta) + v'_s \cos(\theta)) - b^2 \cos(\theta)(u'_s \sin(\theta) + v'_s \cos(\theta))}{\sqrt{a^2(u'_s \sin(\theta) + v'_s \cos(\theta))^2 + b^2(u'_s \cos(\theta) - v'_s \sin(\theta))^2}} - c_1 \sin(\theta) + c_2 \cos(\theta) \quad .$$

For (non-contant) ellipse fields

Theorem

Richards' equations are equivalent to the following two Hamilton conditions for the wildfire $\gamma(s,t) = (u(s,t), v(s,t))$ with respect to the Finsler metric F whose indicatrix field is the given ellipse field:

$$\begin{aligned} \|\gamma_t'(s,t)\|_F &= 1 && (1) \\ \gamma_s'(s,t) \perp_F &\; \gamma_t'(s,t) \quad , && (2) \end{aligned}$$

where $V \perp_F W$ is defined in each tangent space T_pM as follows:

$$g_{p,V}(V,W) = \frac{1}{2}\frac{\partial}{\partial t}[F^2(V+tW)]_{|t=0} = 0 \quad .$$

For every template field

Theorem

Huygens' principle for any (smooth, strongly convex) template field is equivalent to the Hamilton conditions for the wildfire $\gamma(s,t) = (u(s,t), v(s,t))$ with respect to the Finsler metric F whose indicatrix field is the given field.

For every template field

Theorem

The wildfire Hamilton conditions are solved by the geodesic sprays of the corresponding Finsler metric.

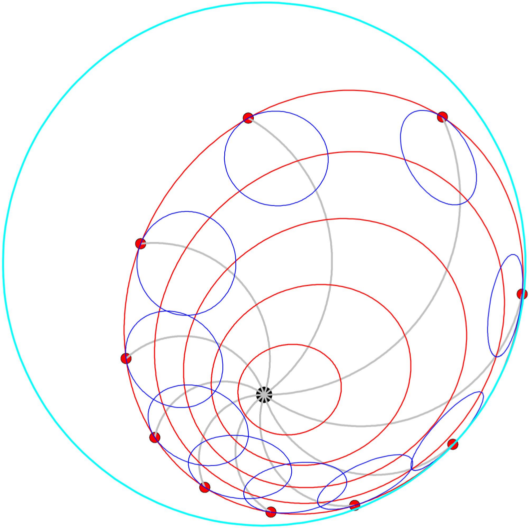
For every time-dependent template field

Theorem

A regular $\gamma(s,t)$ in $\mathcal{U}$ is a Hamilton conditioned wildfire in the rheonomic Lagrange manifold (M,F) if and only if each ray $\gamma(t) = \gamma(s_0, t)$ is a forced geodesic in the following sense:

$$\|\dot{\gamma}(t)\|_F = 1 \quad , \quad \text{and}$$
$$\sum_j \Big(\ddot{\gamma}^j(t) + 2G^j(t, \gamma(t), \dot{\gamma}(t)) + 2N_0^j(t, \gamma(t), \dot{\gamma}(t)) \Big) \partial_j \quad \propto \dot{\gamma}(t) \quad .$$

The Crampin–Mestdag example, 2014



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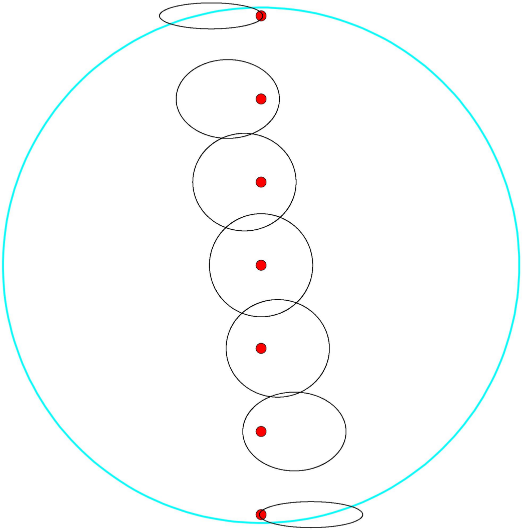

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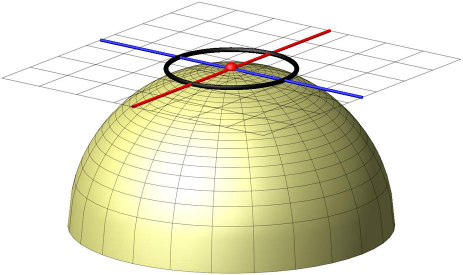

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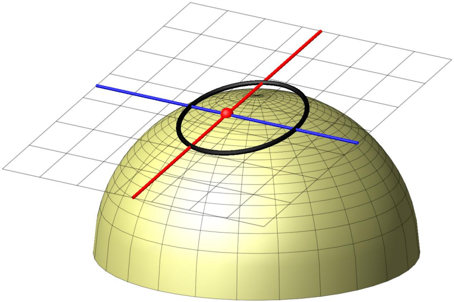

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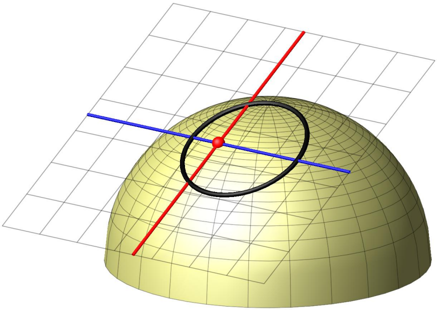

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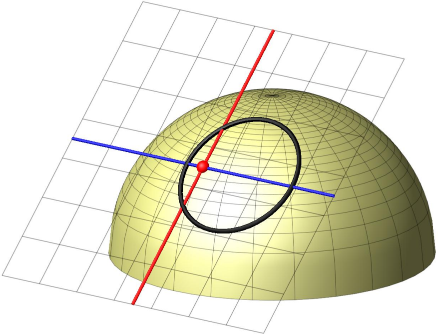

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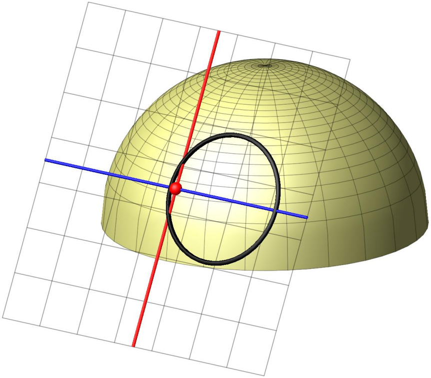

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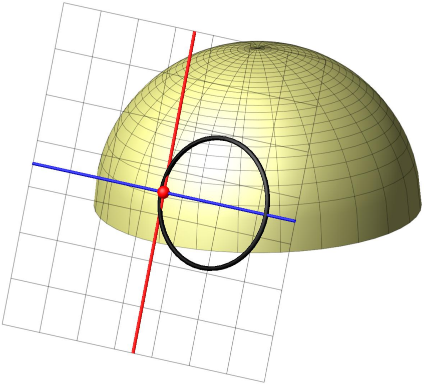

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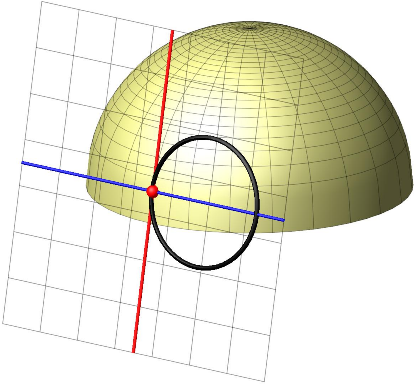

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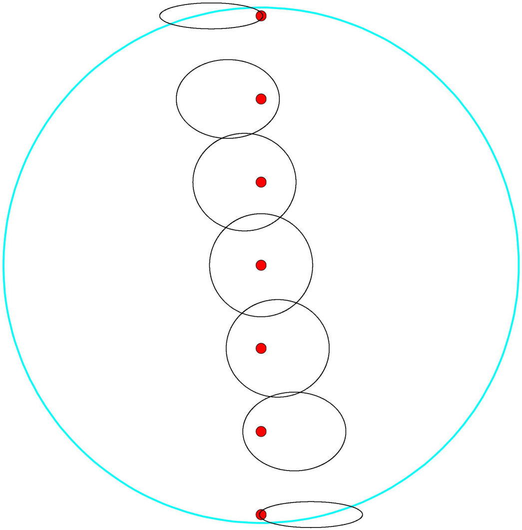

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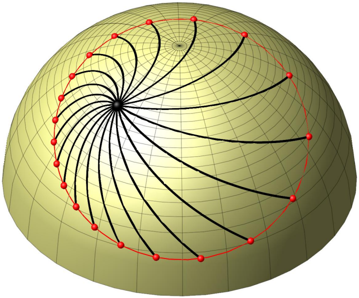

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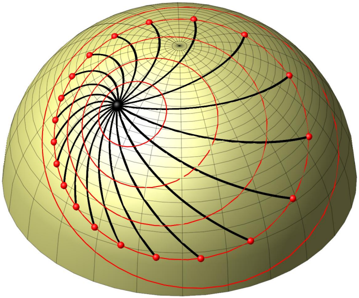
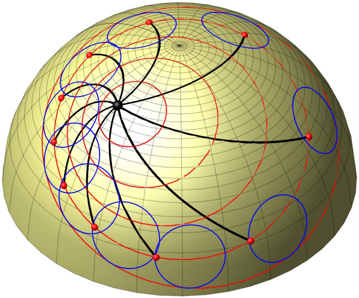
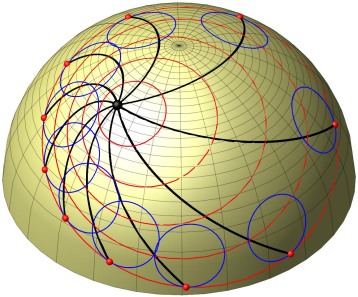


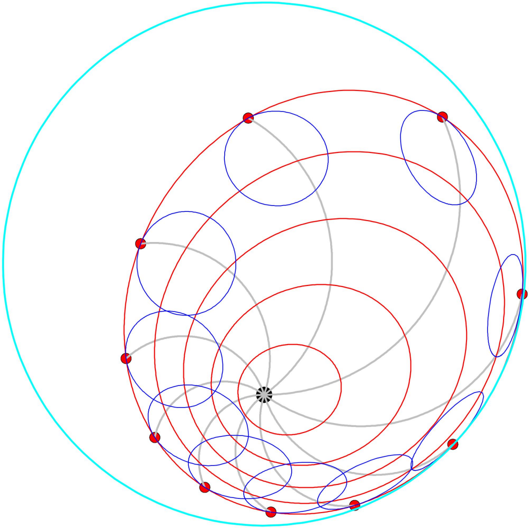
The Crampin–Mestdag example, 2014



The Crampin–Mestdag example, 2014



The Crampin–Mestdag example, 2014



Work in progress

1. Hybrid model for fire front straightening

Work in progress

1. Hybrid model for fire front straightening
2. Cut locus issues

Work in progress

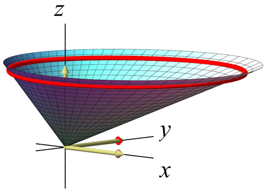
1. Hybrid model for fire front straightening
2. Cut locus issues
3. Higher dimensional examples

Work in progress

1. Hybrid model for fire front straightening
2. Cut locus issues
3. Higher dimensional examples
4. Benchmarking

Summing up

Thank you for your attention!



Summing up

Thank you for your attention!

Further questions?
Comments?