



Dijkstra meets Finsler

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Technical University of Denmark

Joint work with Lucas Damkvist
(who just finished his master's at DTU)

And inspired by joint work with Enrique P.-R.

WDG-UGR 2026 at IMAG, Granada University

20 years ago

Motivation



Motivation for a Calculus 1 exercise

Skovbrand

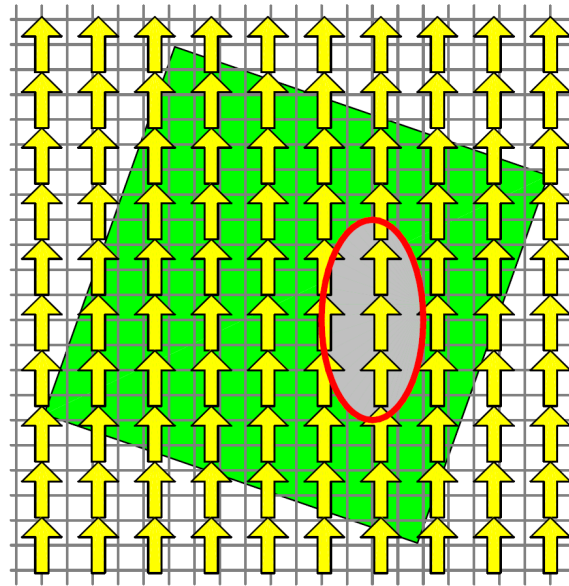
Tema-opgave om vektoranalyse

01005 Matematik 1 - FORÅR 2006 - Semesteruge 13



Figur 1: Brand!!

Motivation for a Calculus 1 exercise



Figur 3: Øjebliksbillede af elliptisk udbredt brandzone.

"Under constant conditions for homogeneous, non-spotting fuels it is generally accepted that a fire ignited at a point will expand, at a constant rate, as an ellipse of the form:

$$\begin{aligned} x(t, \phi) &= at \cos(\phi) \\ y(t, \phi) &= bt \sin(\phi) + ct \end{aligned} \quad (10)$$

Motivation: The Tikokino fire, 1991

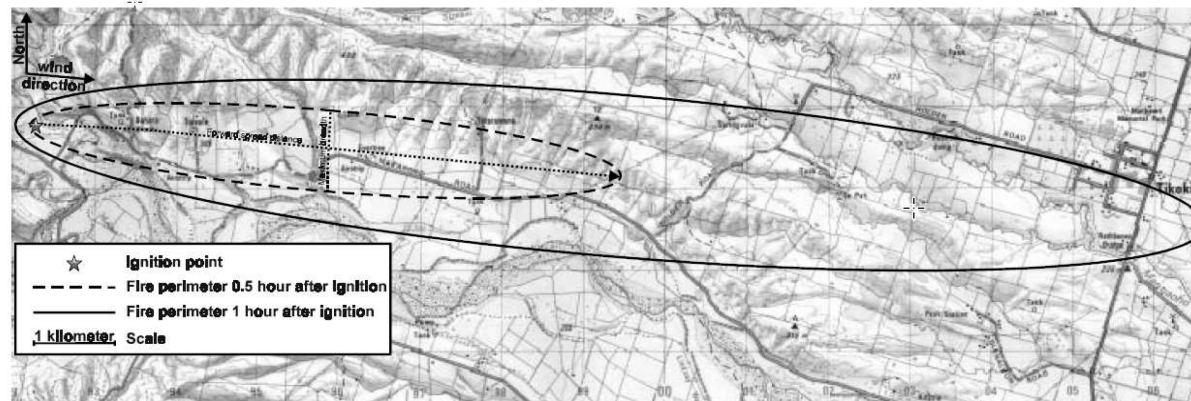
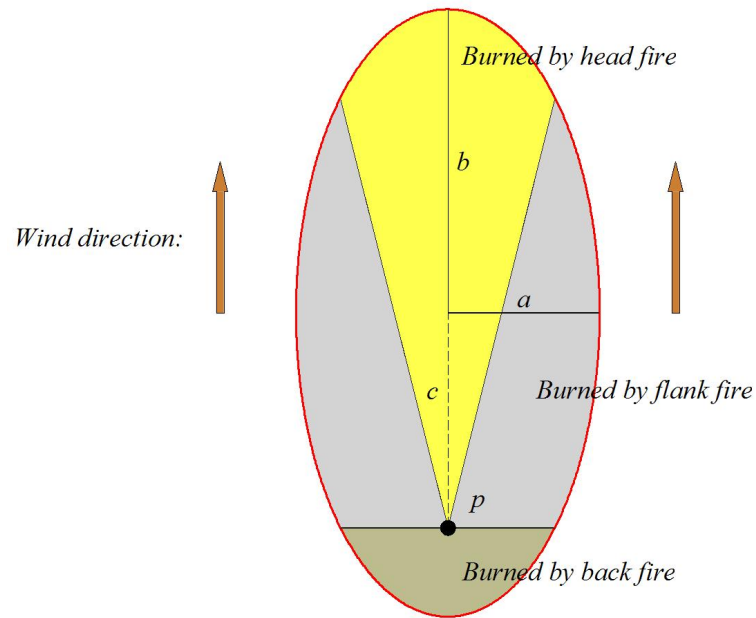
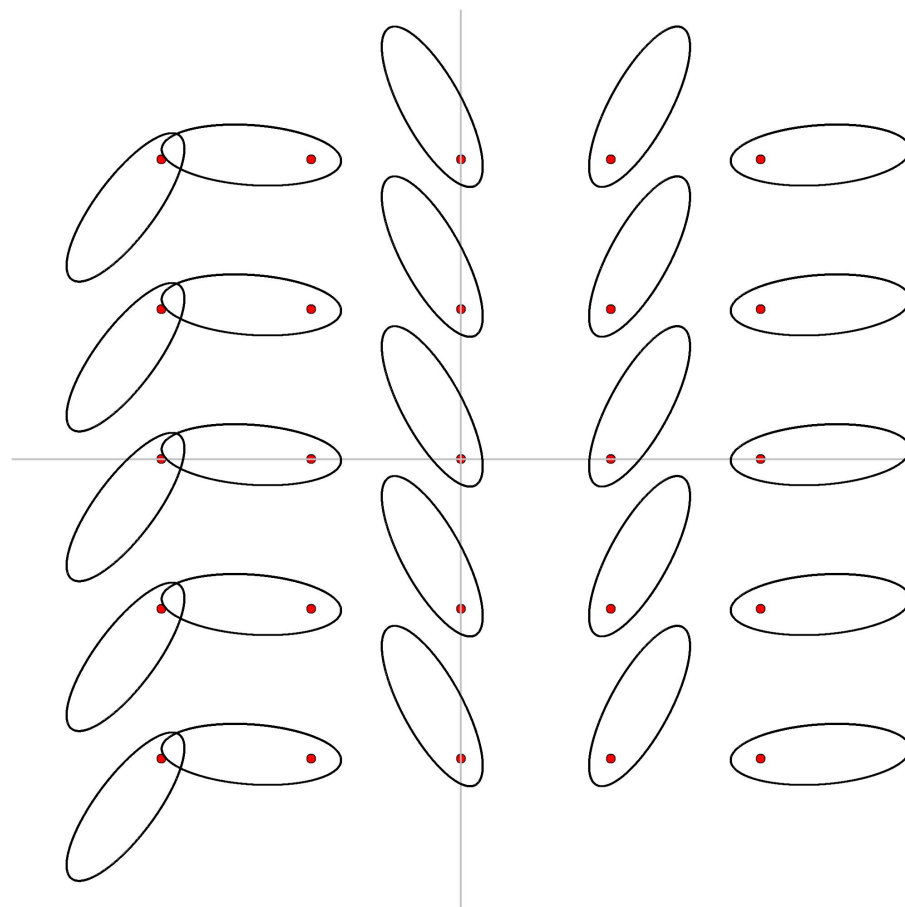


Figure 4. Free-burning fire growth projected for conditions similar to those experienced during the 1991 Tikokino Fire using the *Field Guide*.

Australian firefighter's pocket book fire ellipse



A field of pocket book fire ellipses



Huygens' principle

We ignite the fire at a given point and integrate the indicatrix data along each front to **envelope the next front via Huyghens' principle**:

A later front is obtained by **igniting a fire at each point** on the given front, let them **burn for a small time**, and then **take the outer envelope** of the ensuing fire domains.

Richards' PDE system for elliptical growth models

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 30, 1163–1179 (1990)

AN ELLIPTICAL GROWTH MODEL OF FOREST FIRE FRONTS AND ITS NUMERICAL SOLUTION

GWYNFOR D. RICHARDS

Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba, Canada R7A6A9

SUMMARY

A system of differential equations that describe the growth of a forest fire front in time for variable fuel, weather and topographical conditions is derived. The system is first order, non-linear and contains parameters that can be obtained from forestry data. A finite difference solution is presented, together with its inherent problems and their solution. Results are presented for a variety of situations that include variable fuel, wind and fire breaks. It is found that the equations and their solution efficiently produce simulations for complex problems.

Richards' PDE system for elliptical growth models

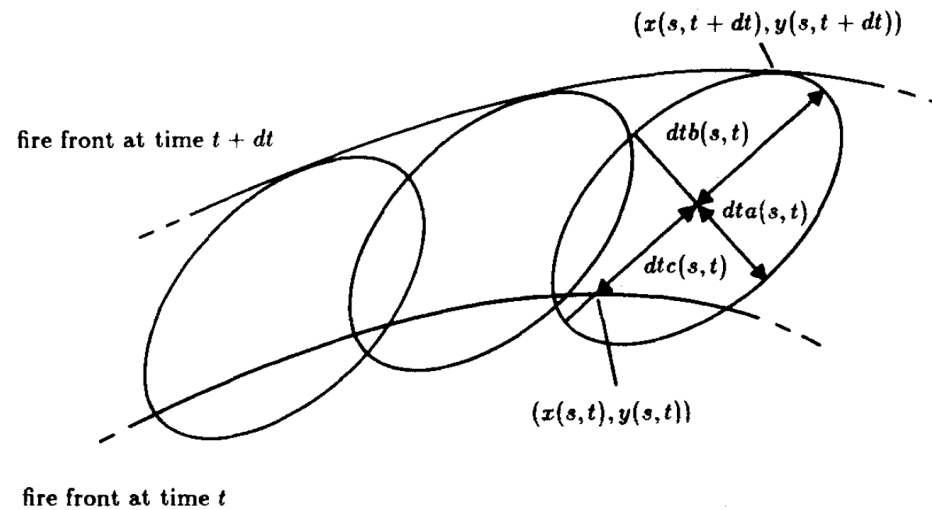


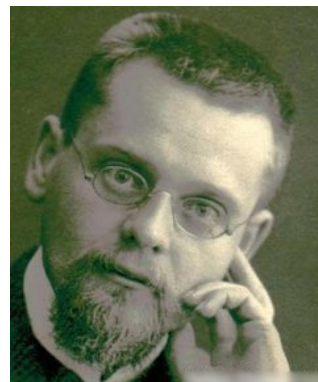
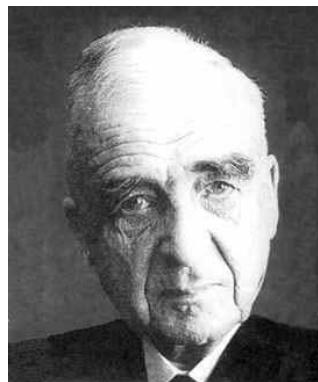
Figure 2. The envelope of ellipses forming the fire front at time $t + dt$

Richards' PDE system for elliptical growth models

$$x_t = \frac{a^2 \cos \theta (x_s \sin \theta + y_s \cos \theta) - b^2 \sin \theta (x_s \cos \theta - y_s \sin \theta)}{(b^2 (x_s \cos \theta - y_s \sin \theta)^2 + a^2 (x_s \sin \theta + y_s \cos \theta)^2)^{1/2}} + c \sin \theta \quad (10)$$

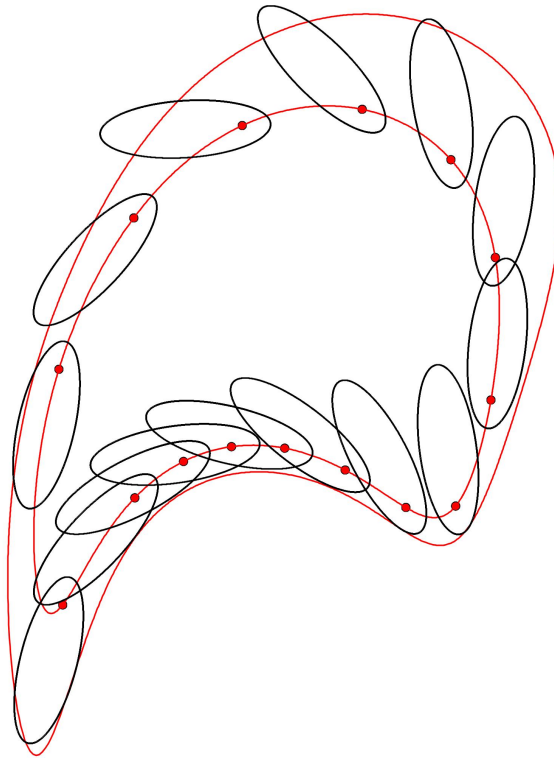
$$y_t = \frac{-a^2 \sin \theta (x_s \sin \theta + y_s \cos \theta) - b^2 \cos \theta (x_s \cos \theta - y_s \sin \theta)}{(b^2 (x_s \cos \theta - y_s \sin \theta)^2 + a^2 (x_s \sin \theta + y_s \cos \theta)^2)^{1/2}} + c \cos \theta \quad (11)$$

The Finsler moment



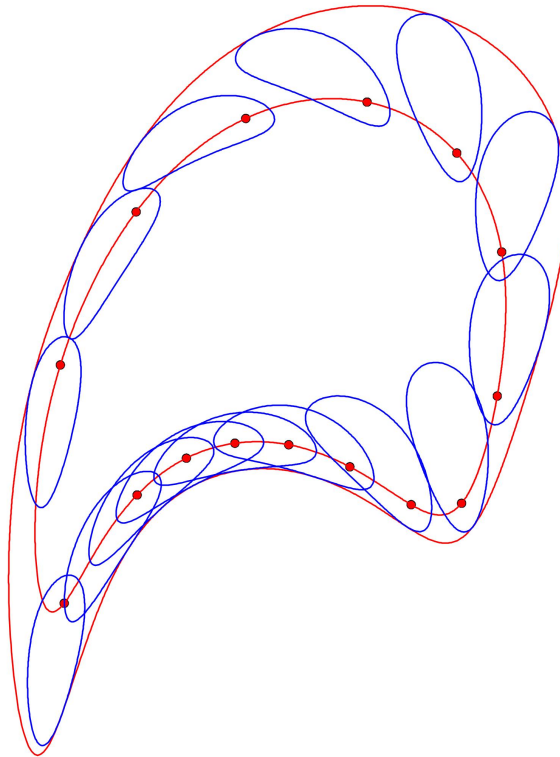
Huyghens' principle

The indicatrices are objects in the tangent bundle – not in the fire-domain:



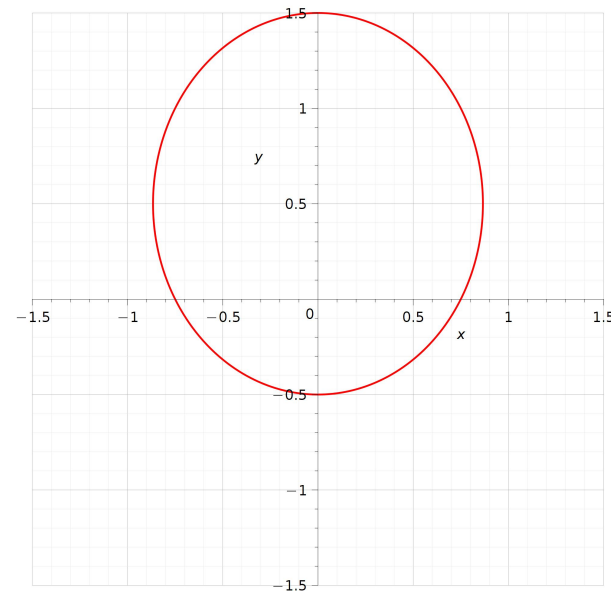
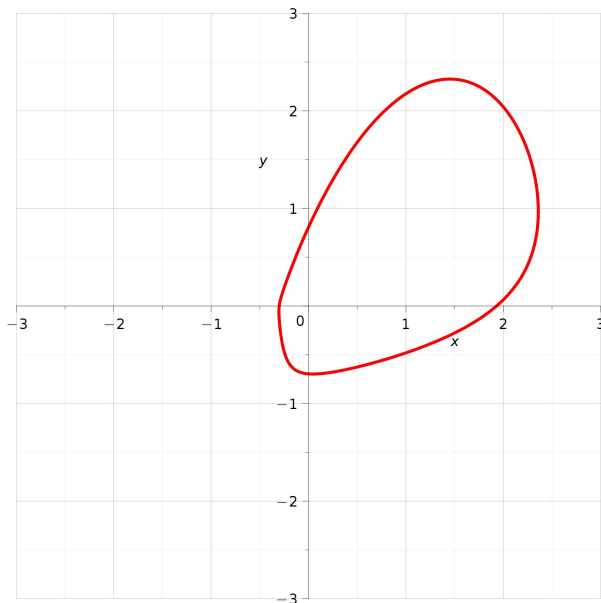
Huyghens' principle

The local indicatrix field should itself be *integrated* from the respective points of ignition:



Finsler geometry

Every strongly convex oval around the origin in a given tangent space T_pM counts as a metric indicatrix – equivalent to a Minkowski norm:



Finsler geometry

Minkowski indicatrices are determined by Finsler metric functions F :

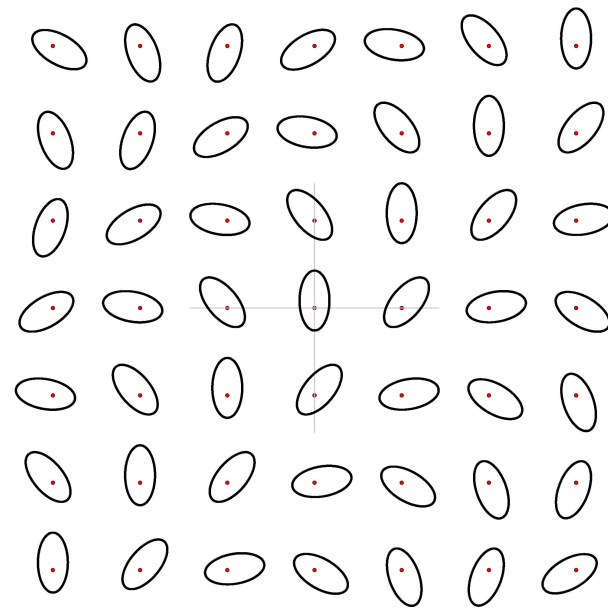
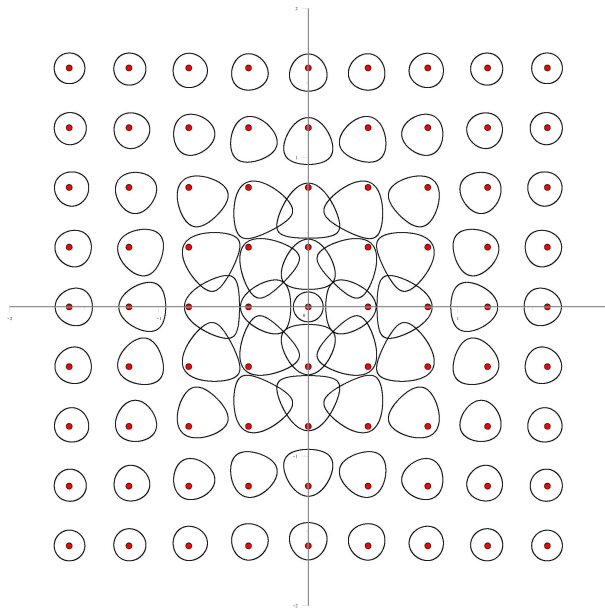
$$\mathcal{I} = \{w \in T_p M \mid F(w) = 1\} \quad ,$$

where

- ⊙ F is C^∞ on $T_p M - \{0\}$
- ⊙ $F(\lambda \cdot V) = \lambda \cdot F(V)$ for all $\lambda > 0$ and for all $V \in T_p M$

Finsler geometry

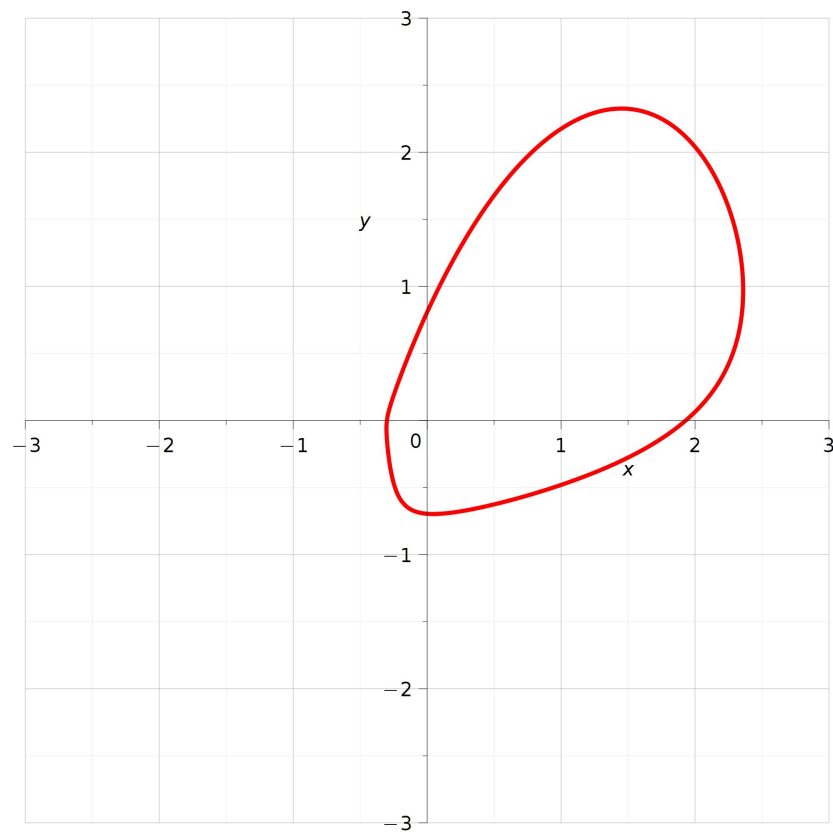
Smooth fields of Finsler indicatrices
mold the Finsler metrics:



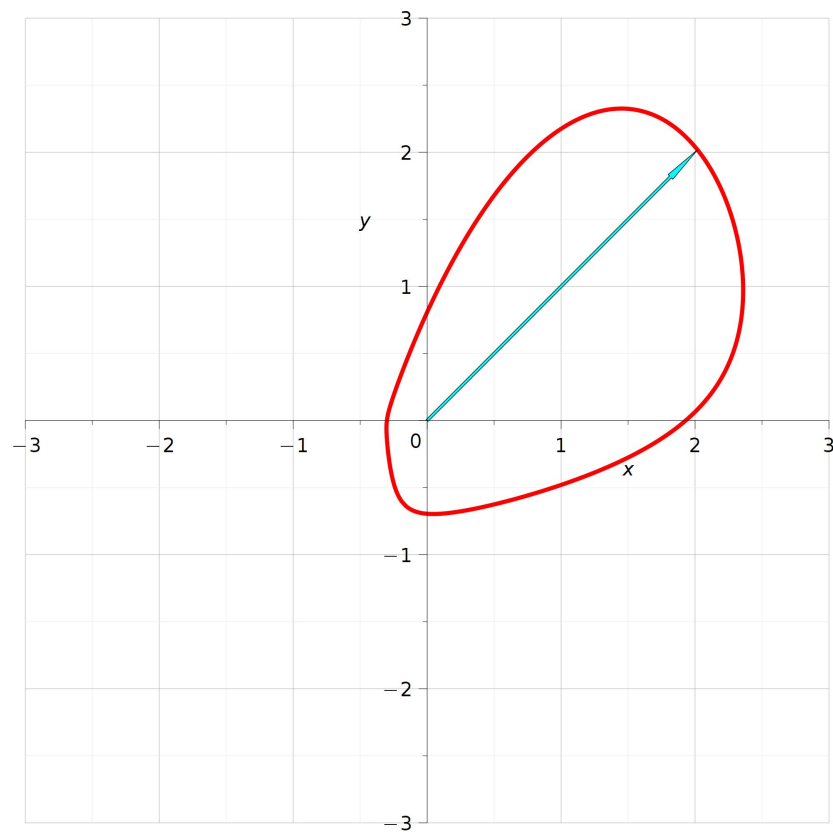
Finsler geometry

Each single Finsler indicatrix is equivalent to a *family* of approximating Riemann indicatrices:

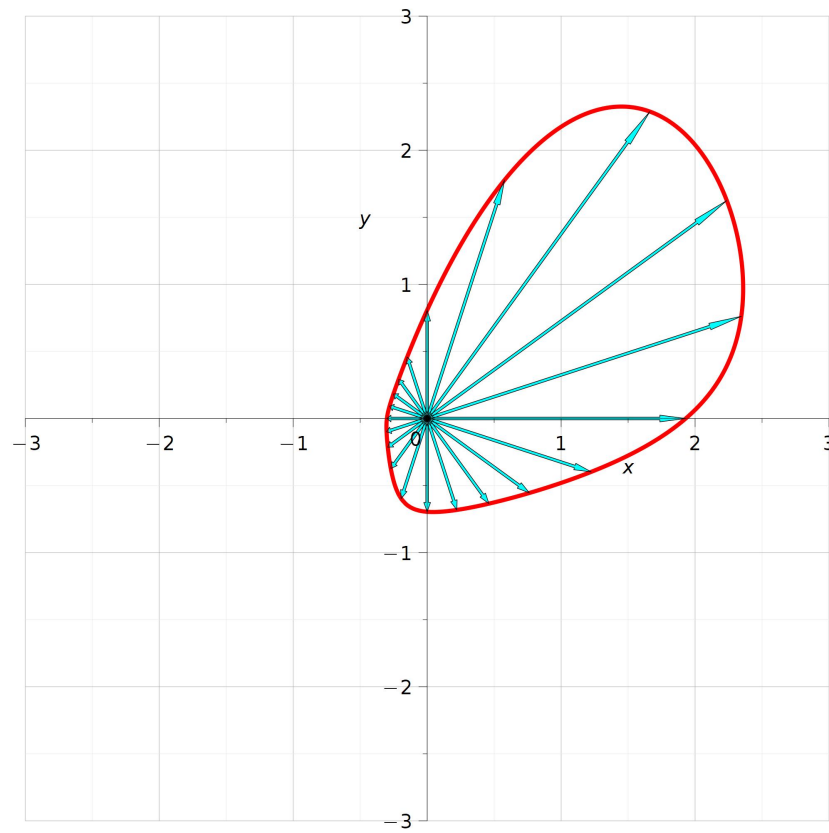
Finsler geometry



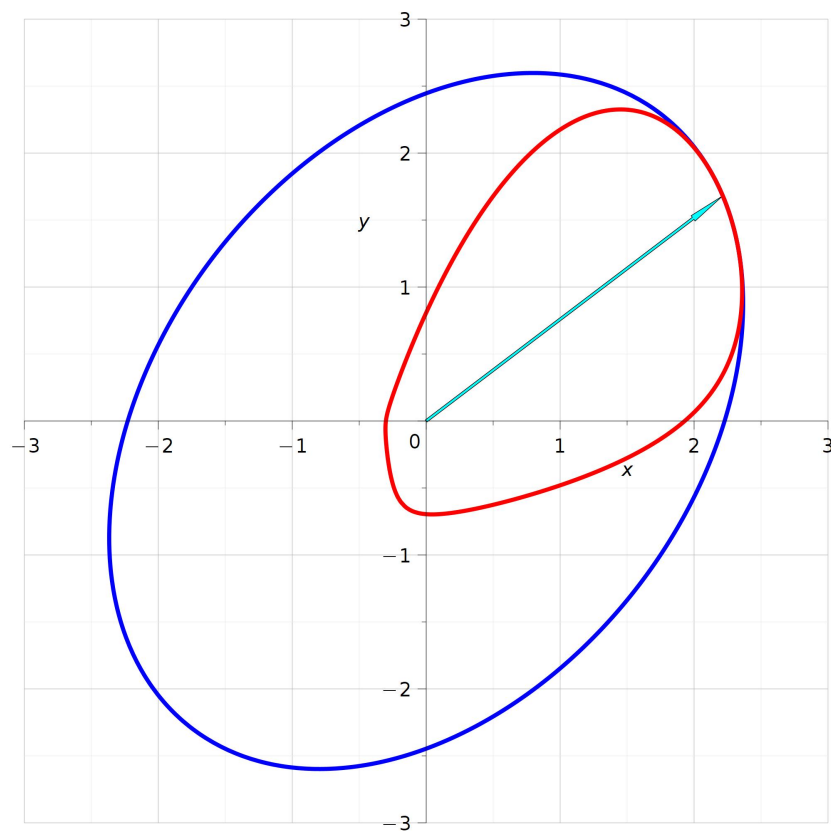
Finsler geometry



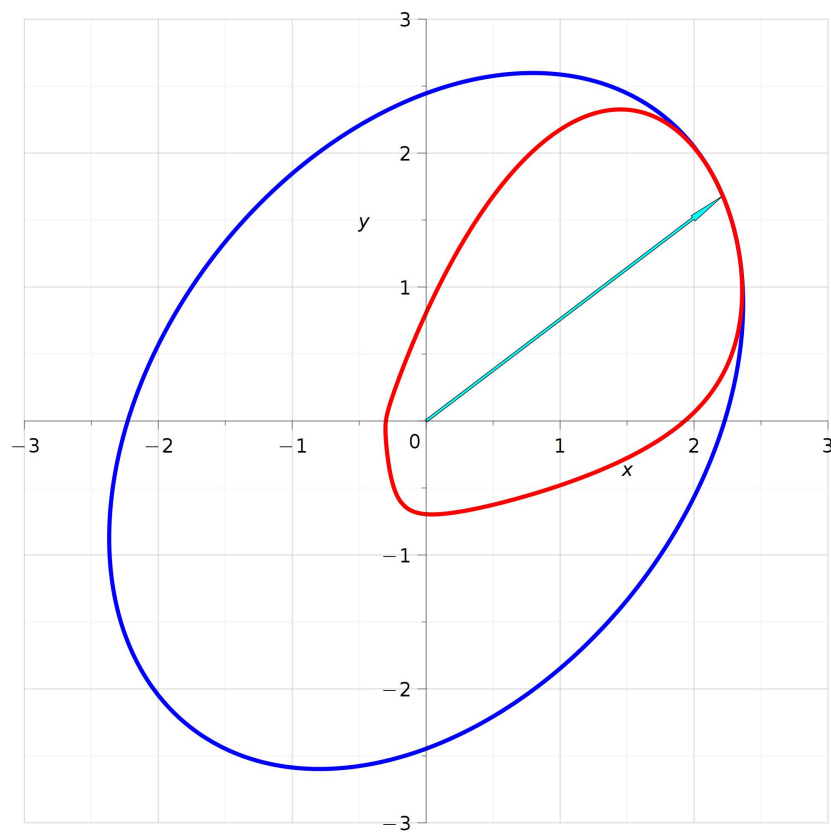
Finsler geometry



Finsler geometry



Finsler geometry



Finsler geometry

The V -dependent Riemannian approximating indicatrices correspond to the metrics:

$$g_V(x, y) = \frac{1}{2} \cdot \frac{\partial^2}{\partial s \partial t} \left(F^2 (V + s \cdot x + t \cdot y) \right) \Big|_{s=t=0}$$

Finsler geometry

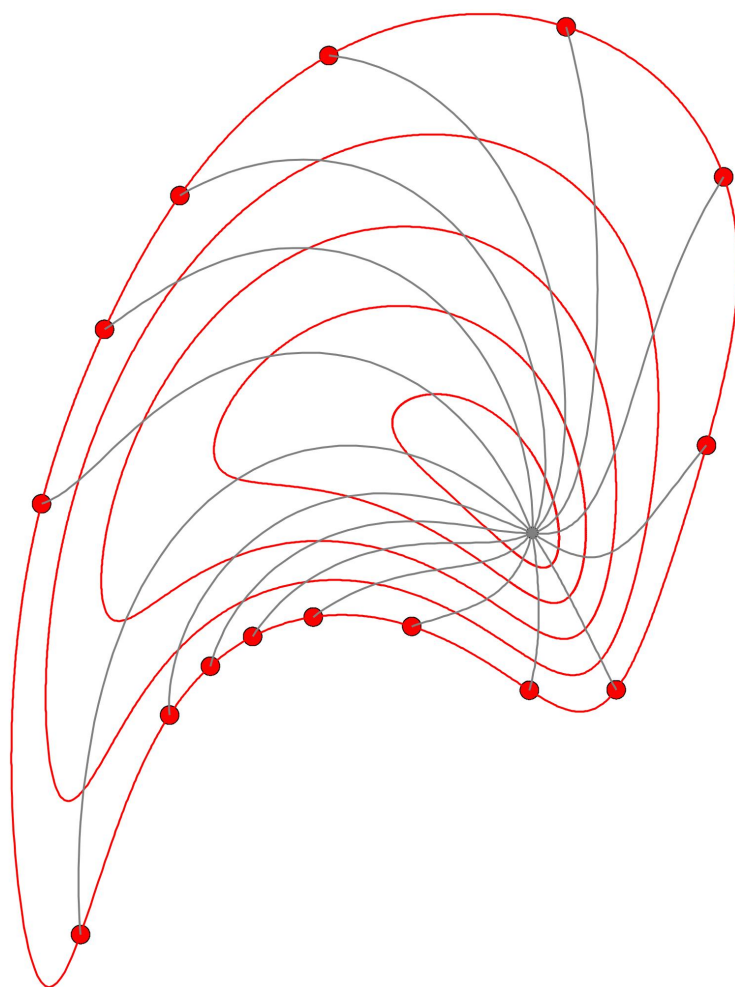
Theorem 1. *The unit speed F -energy extremals γ are obtained via the following Euler-Lagrange equations:*

$$\ddot{\gamma}_t^i(t) + 2G^i(\dot{\gamma}_t(t)) = 0$$

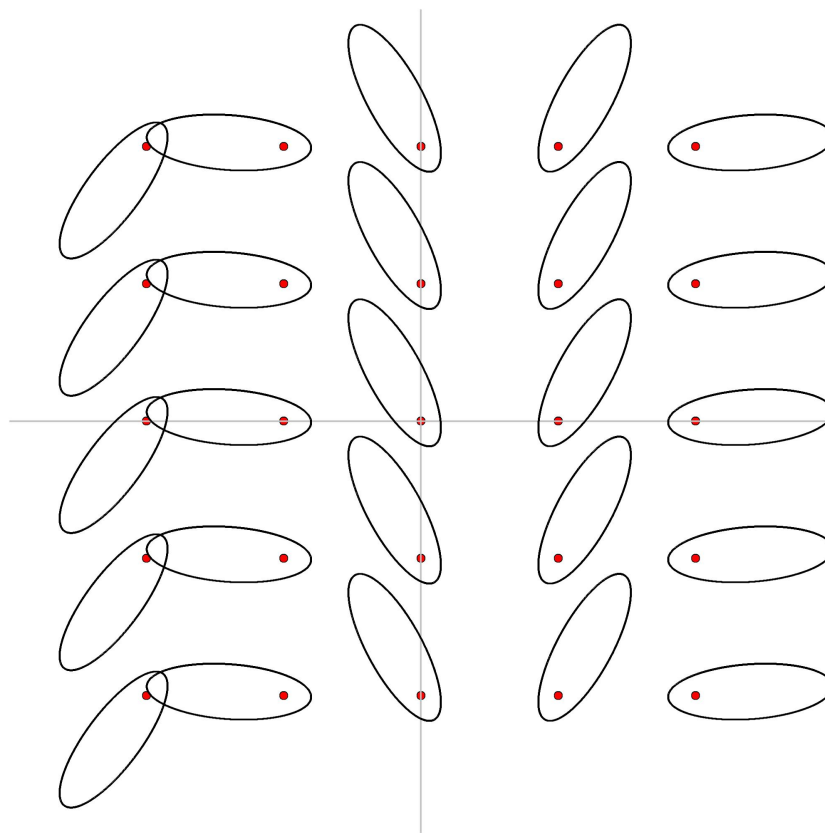
where

$$G^i(V) = \frac{1}{4} \cdot g_V^{ij} \cdot \left(F_{x^k v^j}^2(V) \cdot v^k - F_{x^j}^2(V) \right) \quad .$$

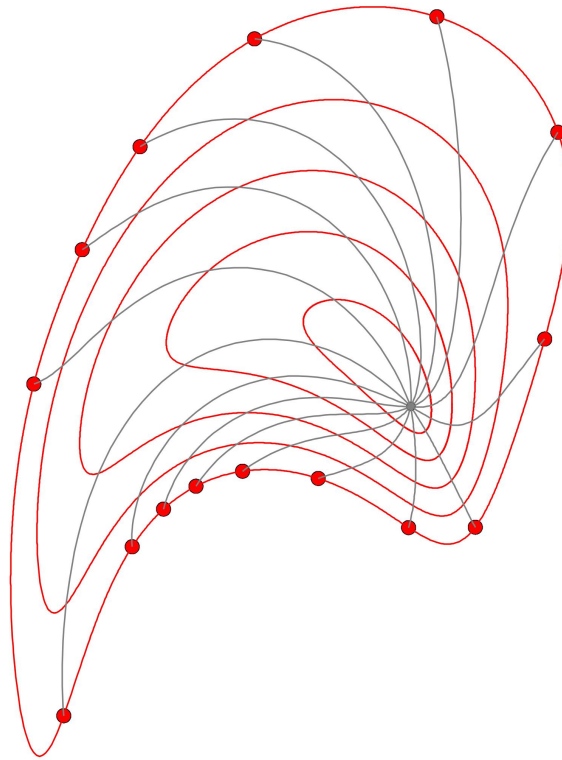
Finsler geodesic circle frontals



From a given indicatrix field

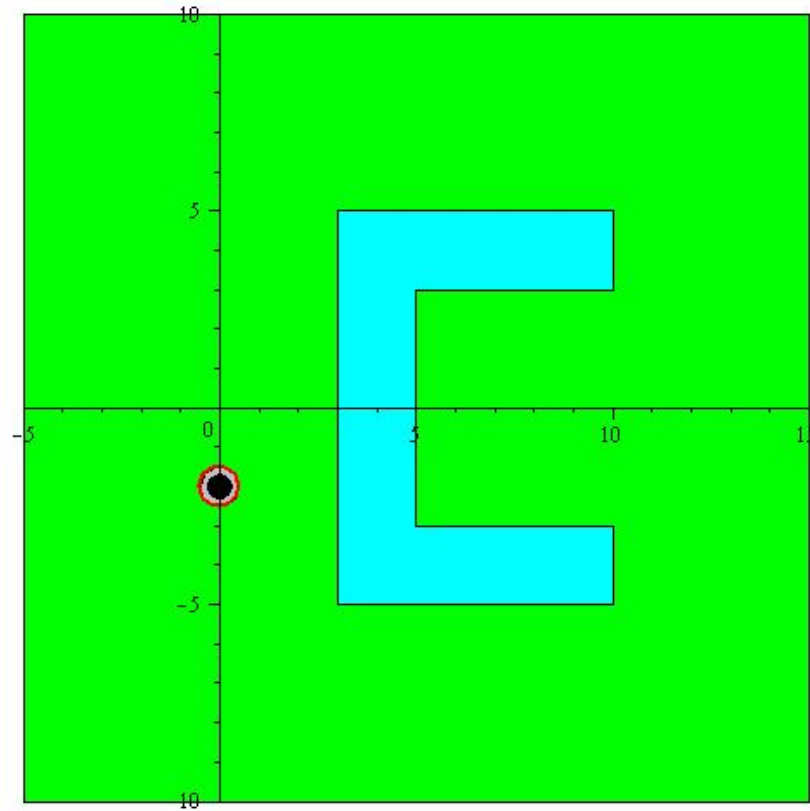


Finsler geodesic circle frontals

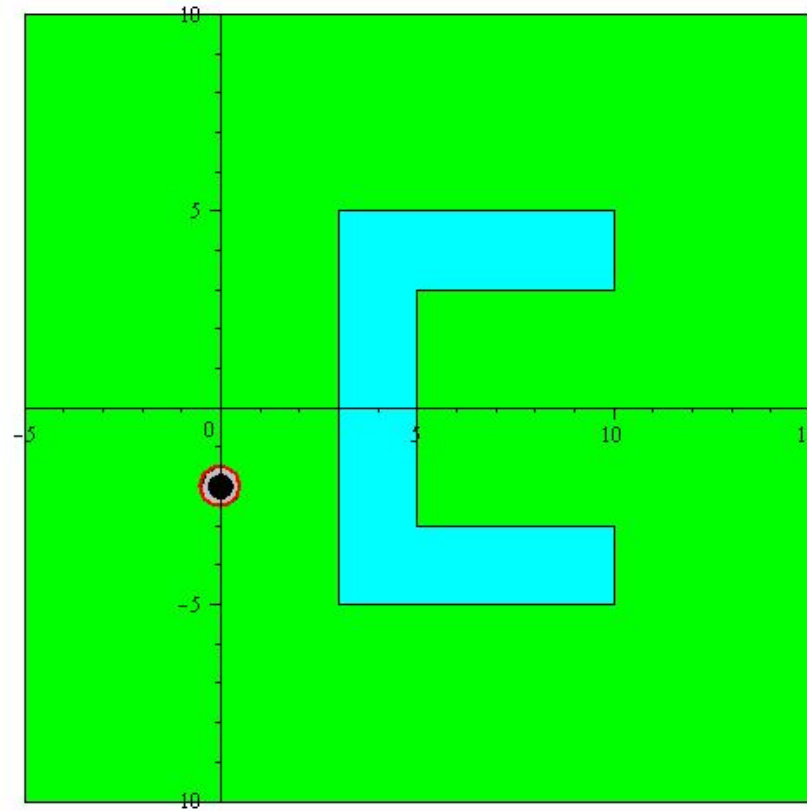


When the background Finsler metric is itself time-dependent:
See M. A. Javaloyes, E. Pendás-Recondo, and M. Sanchez:
A general model for wildfire propagation with wind and slope,
(2021)

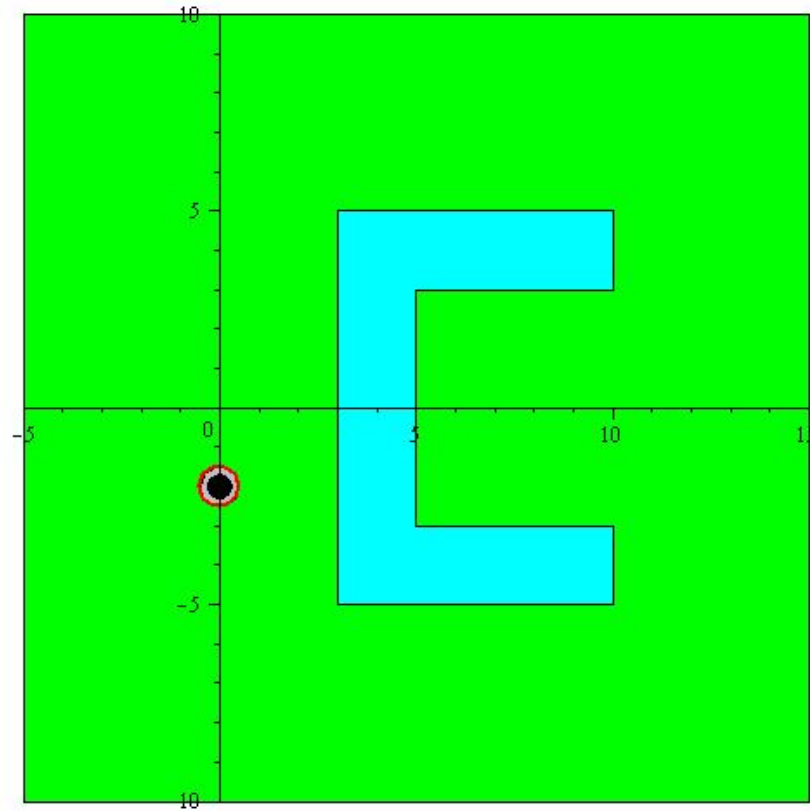
First experiments;
Showing cut locus; No wind



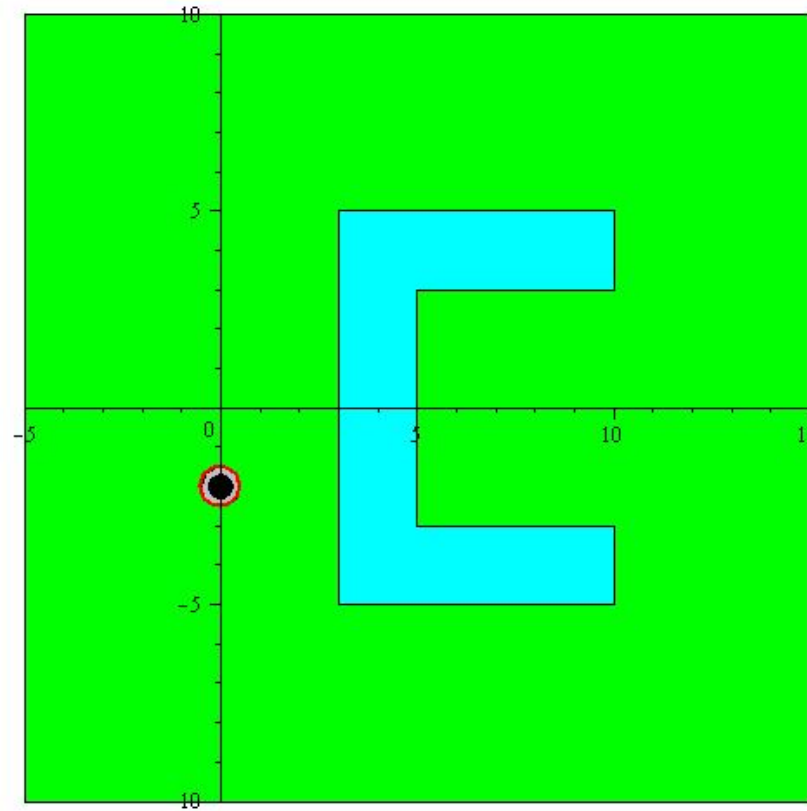
First experiments;
Showing cut locus; No wind



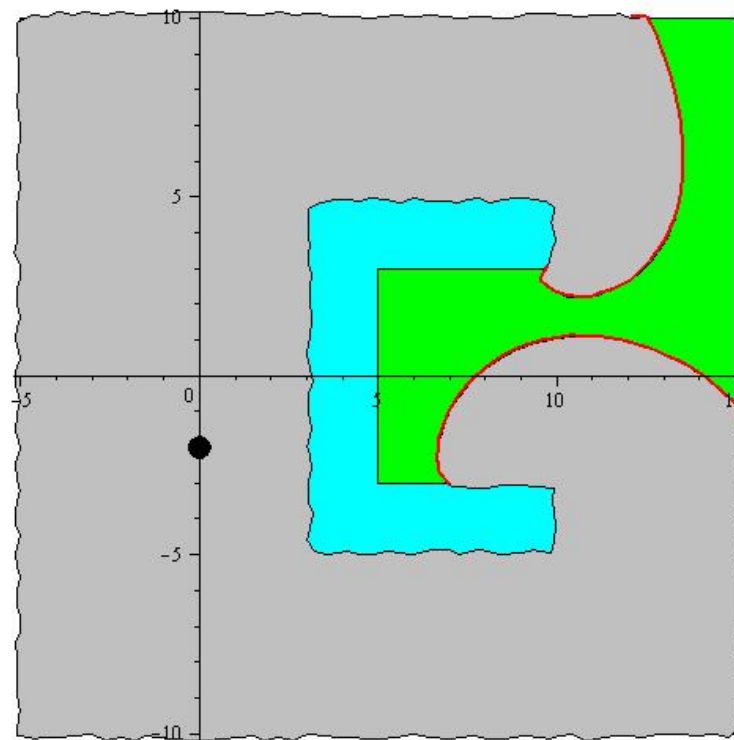
First experiments;
Showing cut locus; With wind



First experiments;
Showing cut locus; With wind



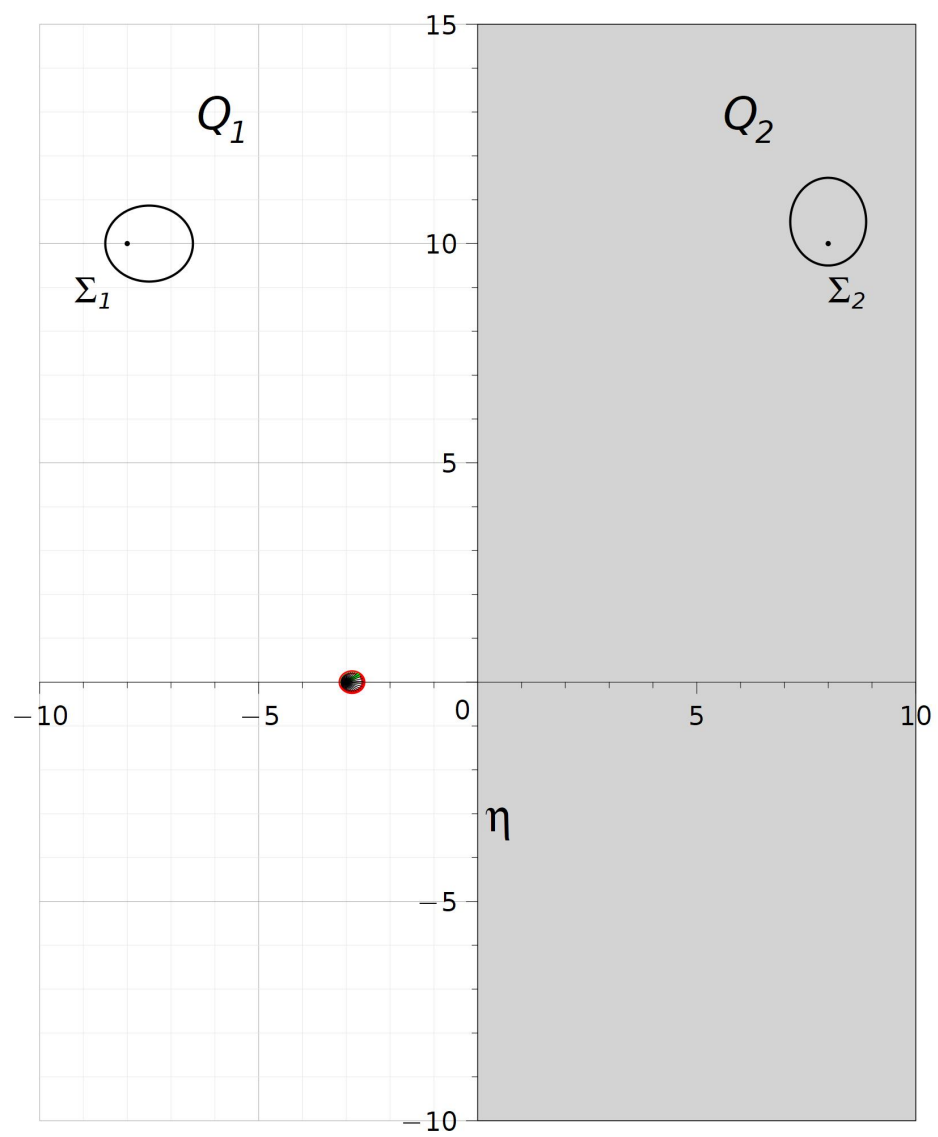
How to cope with
physically interacting frontals
before the formation of cut loci?

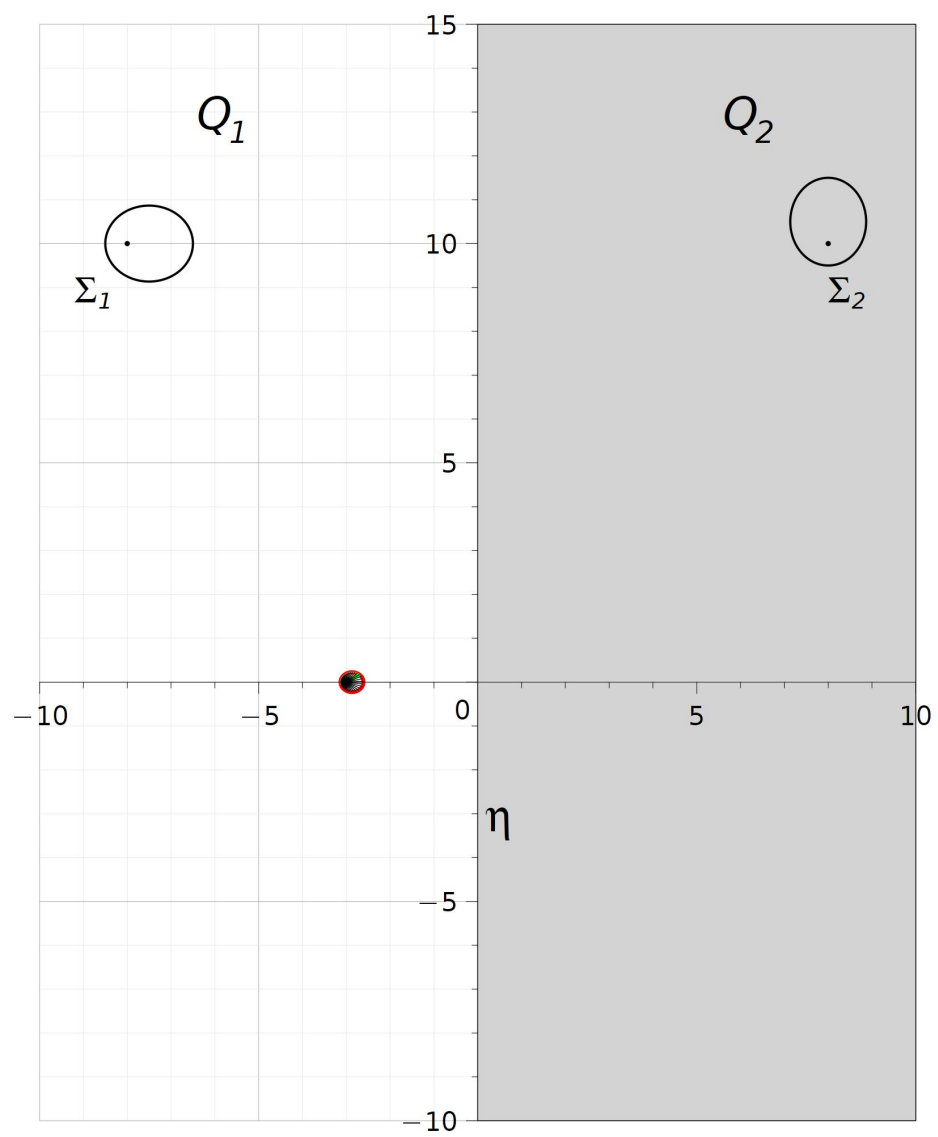


Snells law

Joint with Enrique Pendás-Recondo:
*Snell's law revisited and generalized
via Finsler geometry*, 2022.

Has recently been developed into a time-dependent Lorentz-Finsler setting via application of Fermat's principle in work with Miguel Ángel Javaloyes, Miguel Sanchez, and Enrique Pendás-Recondo.

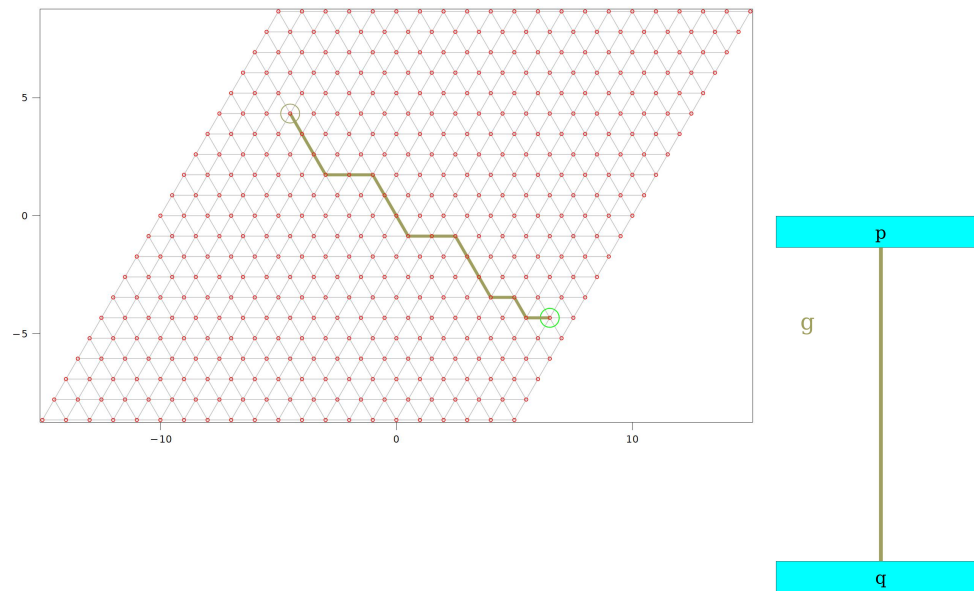




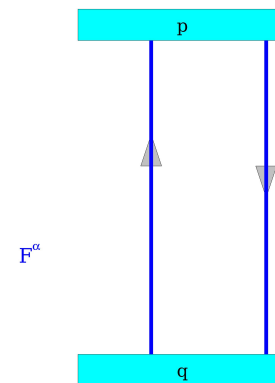
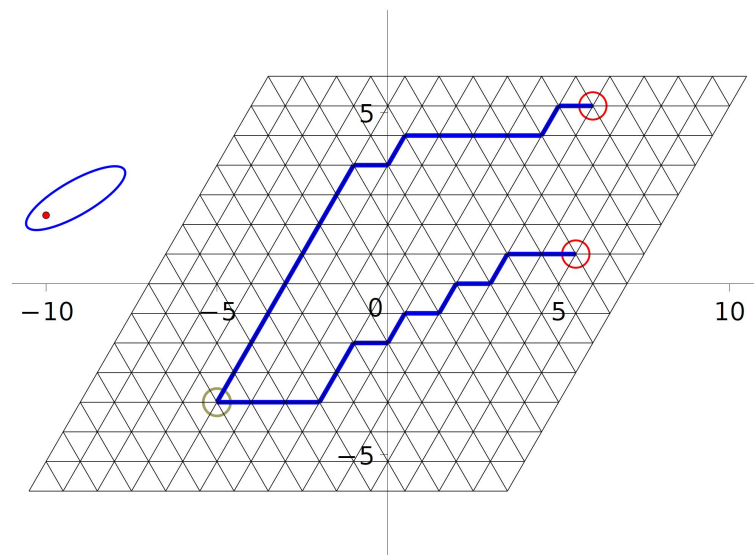
Finsler weighted di-graphs

Dijkstra geodesic in a Euclidean weighted graph

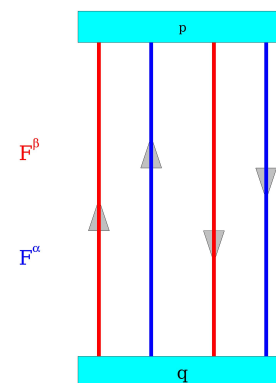
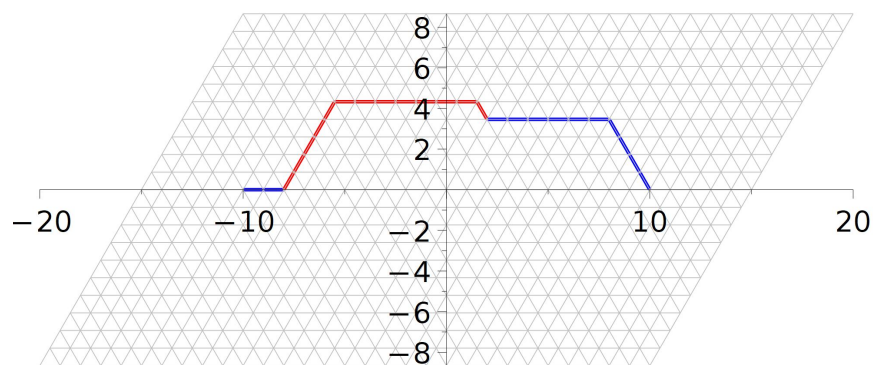
Rough isometry constant: $\sqrt{3}/2$:



Dijkstra geodesics in a Finsler-weighted digraph



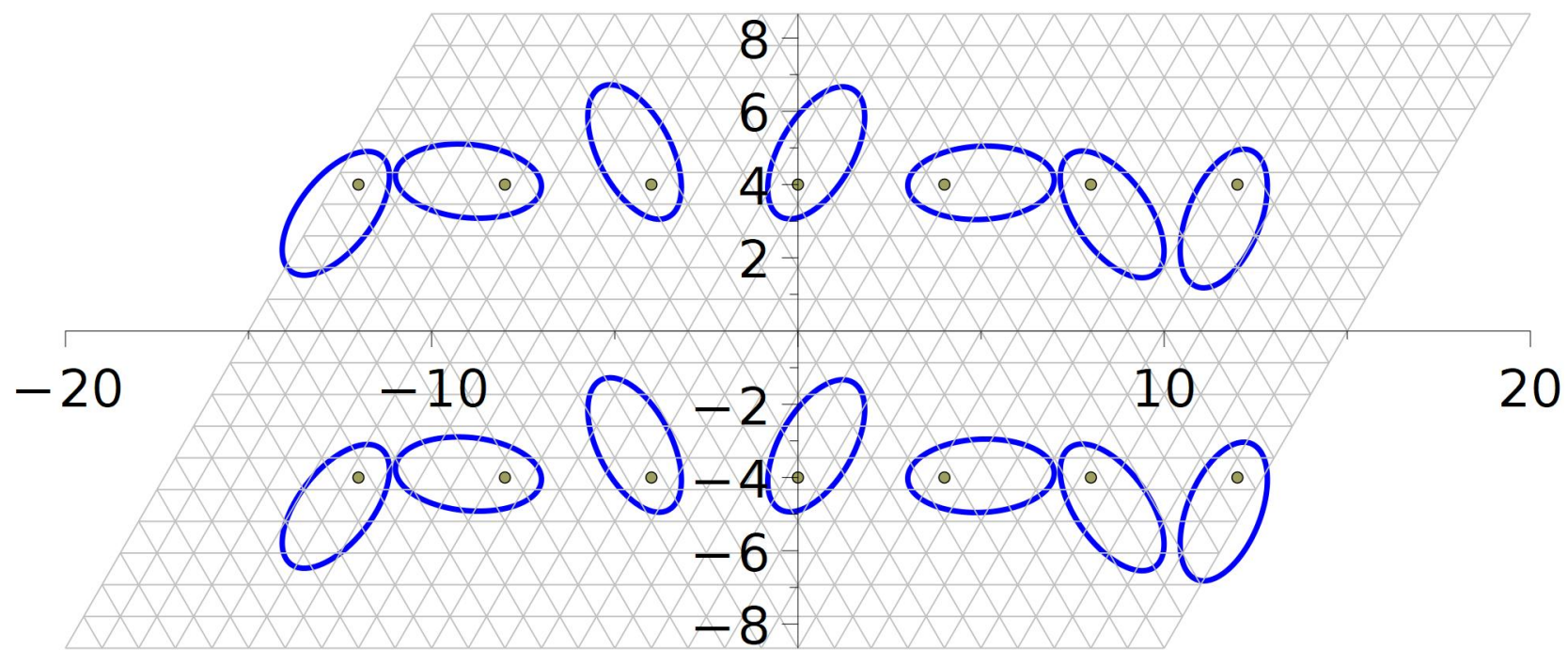
Colored Dijkstra geodesic in a bi-metric Finsler-weighted digraph



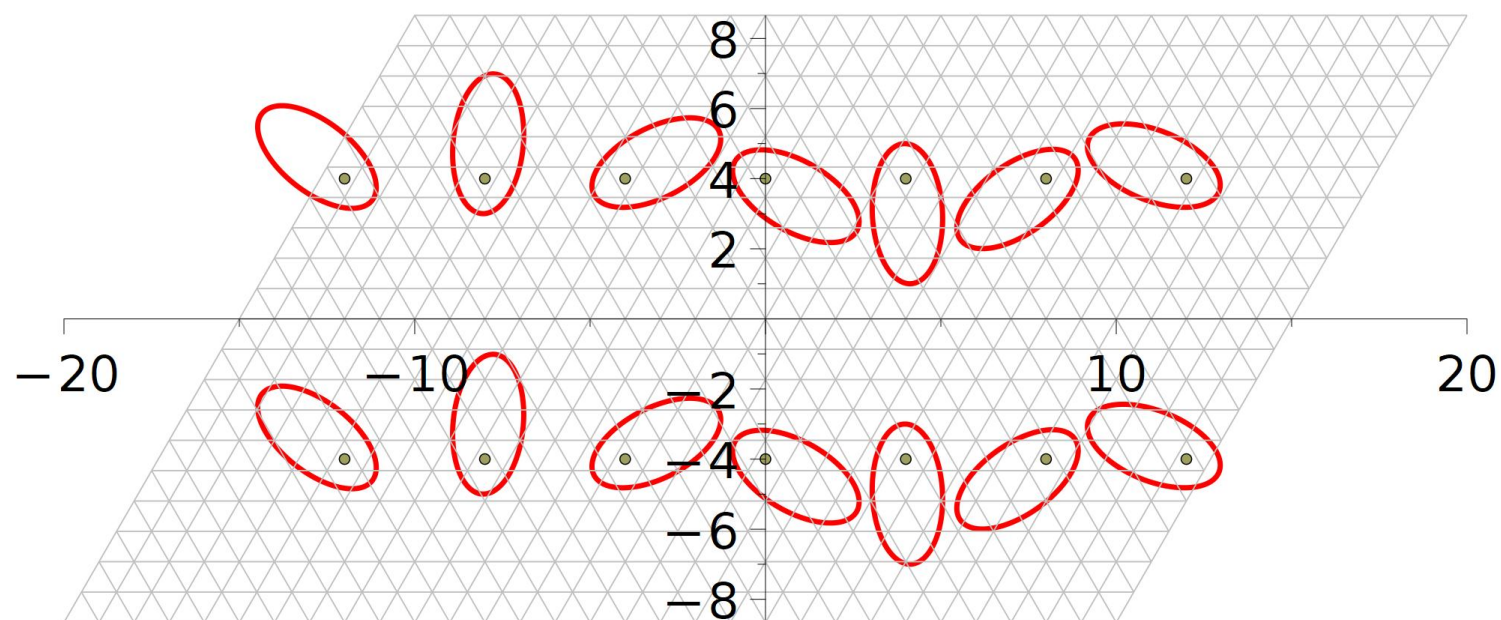
Shortest paths between two points in the net

- ⊙ With one given metric F^α (blue)
- ⊙ With another one given metric F^β (red)
- ⊙ With an everywhere free choice of metric among the two (i.e. with full tackning freedom in a given bi-metric restricted to the net)

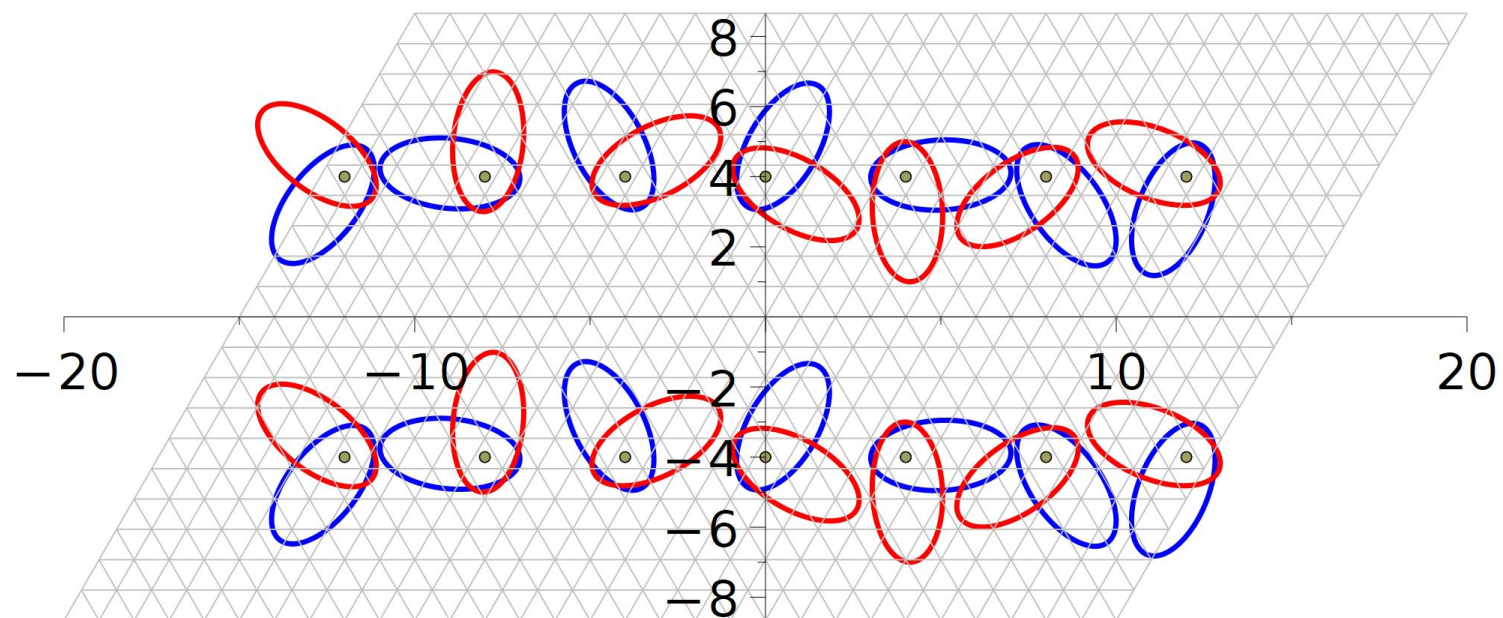
An example



An example

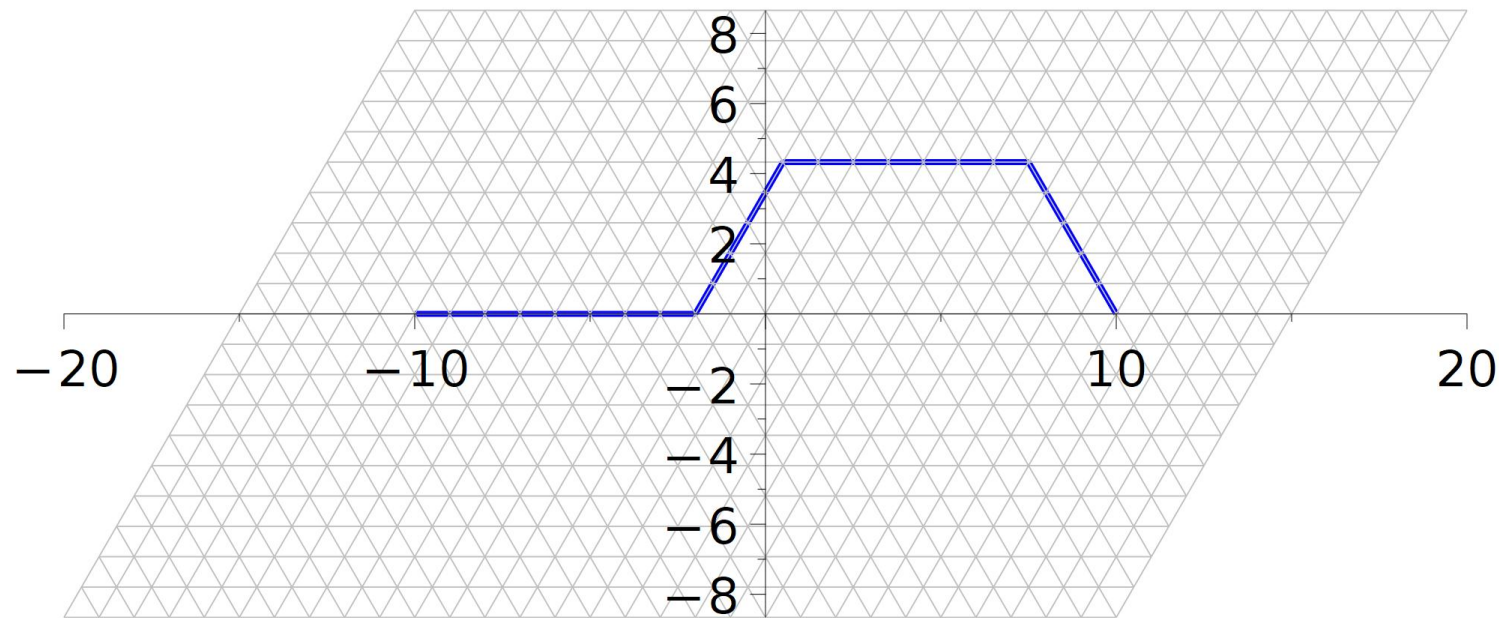


An example



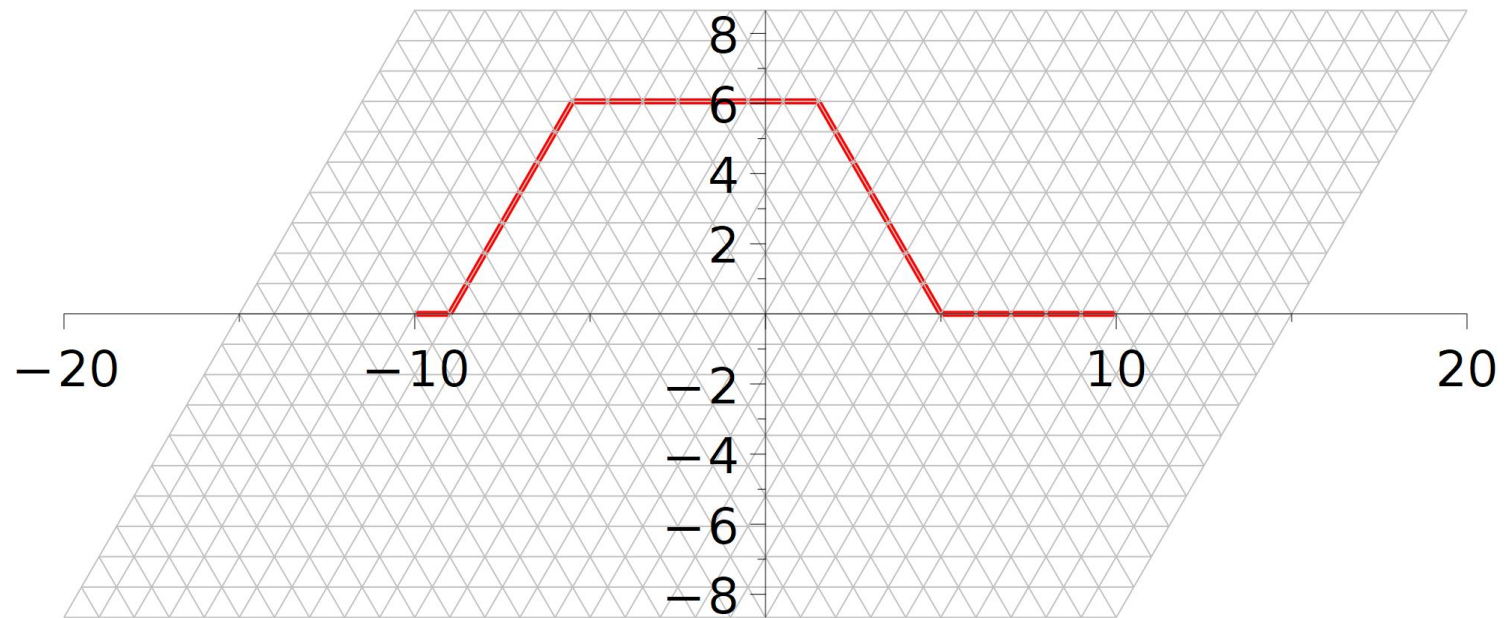
An example

Dijkstra blue path length: $L = 32.6$:



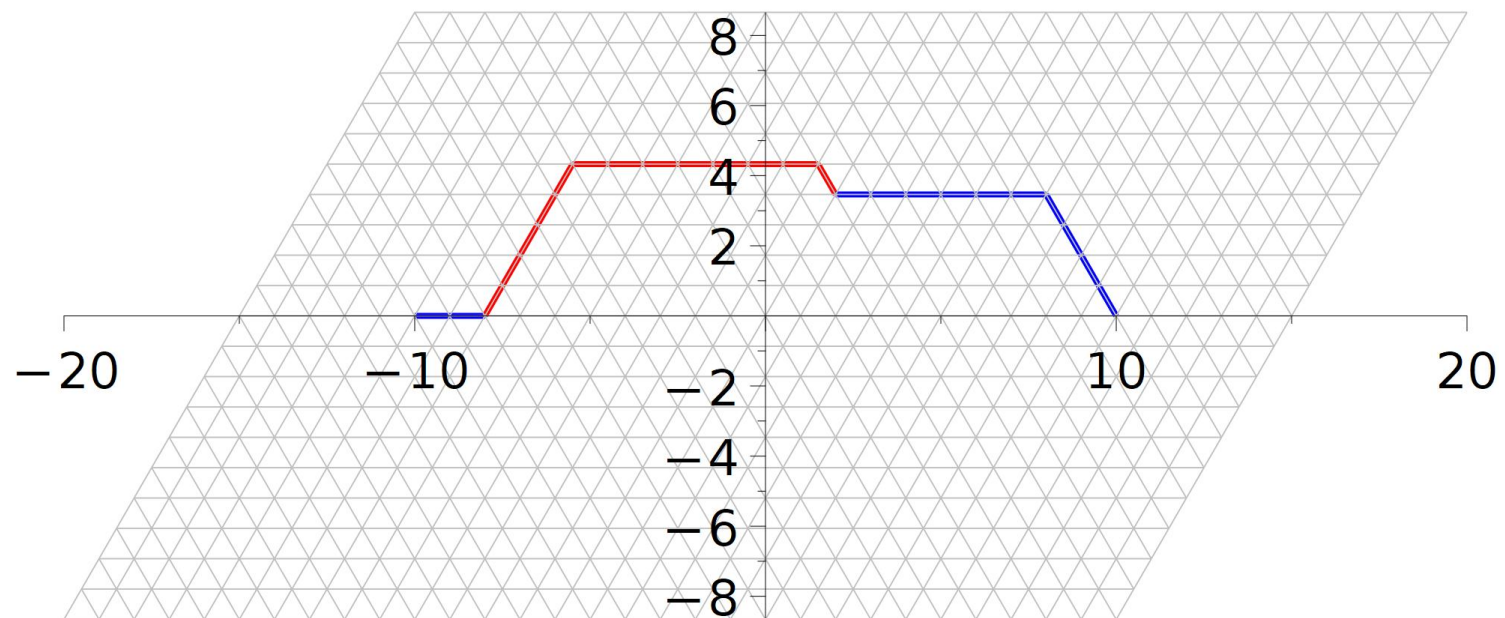
An example

Dijkstra red path length: $L = 33.1$:



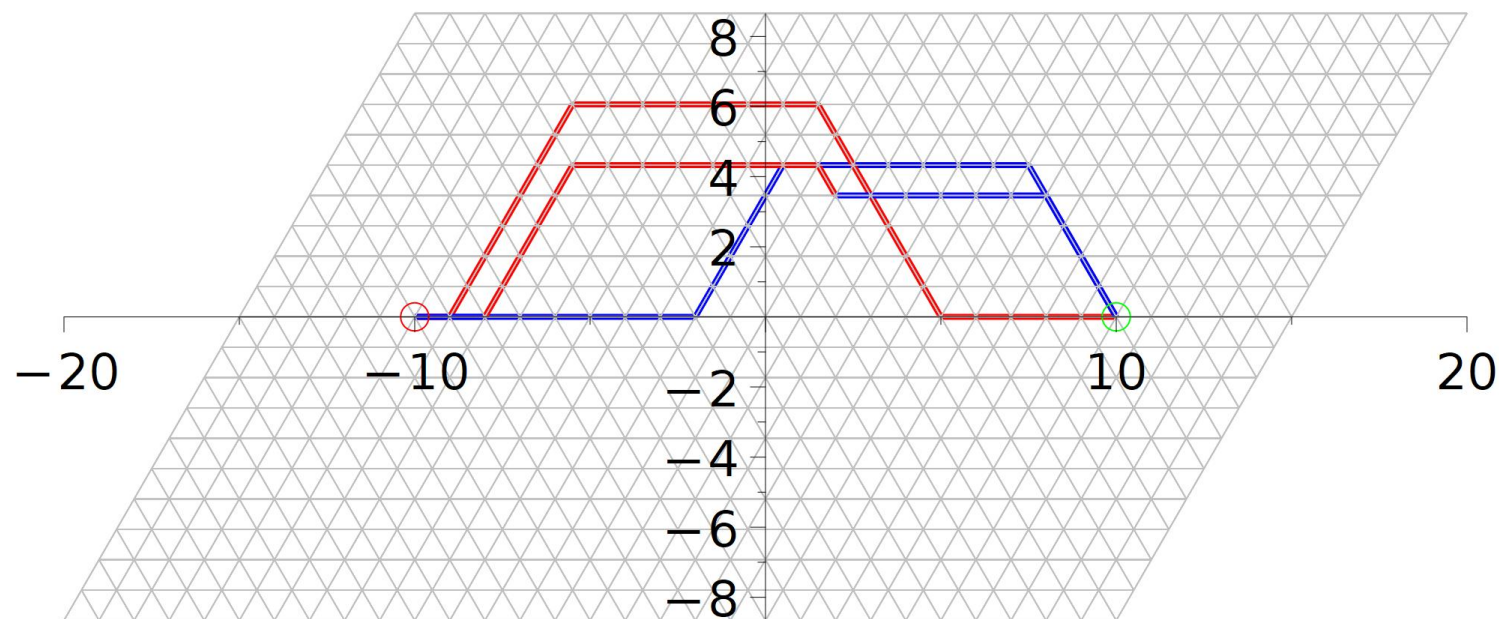
An example

Dijkstra shortest path length: $L = 24.1$ obtained by tacking between blue and red (and within blue and red):

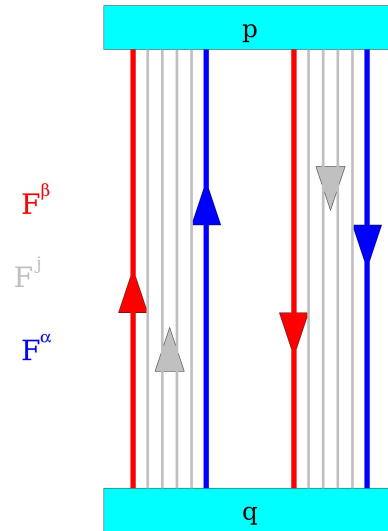


An example

All together:



Multi-metric Finsler weights



Definition 2. The *colored (effective)* length of a vector $V \in T_p M$ with respect to a Finsler multi-metric:

$$\|V\|_{\Lambda} = \inf_{\alpha \in \Lambda} F^{\alpha}(p, V) \quad , \quad (1)$$

where Λ is the index set for the set of Finsler metrics under consideration.

Multi-metric Finsler weights

Definition 3. The *colored (effective)* length of a curve γ with respect to a Finsler multi-metric is

$$L(\gamma) = \int_0^T \|V\|_{\Lambda} dt, \quad (2)$$

and the corresponding definition of geodesics (if they exist) is then clear.

For $\Lambda = \{\text{blue}, \text{red}\}$ the above discrete method gives colored Dijkstra-geodesics, i.e. shortest colored paths in the given net. It solves the Zermelo problem with as many net-tacking points as needed to find the shortest colored path in the net from A to B .

Multi-metric Finsler weights

The problem is then clearly:

Quest 4. What is (are) the colored geodesics in (M, Λ) from A to B ?

Multi-metric Riemannian weights

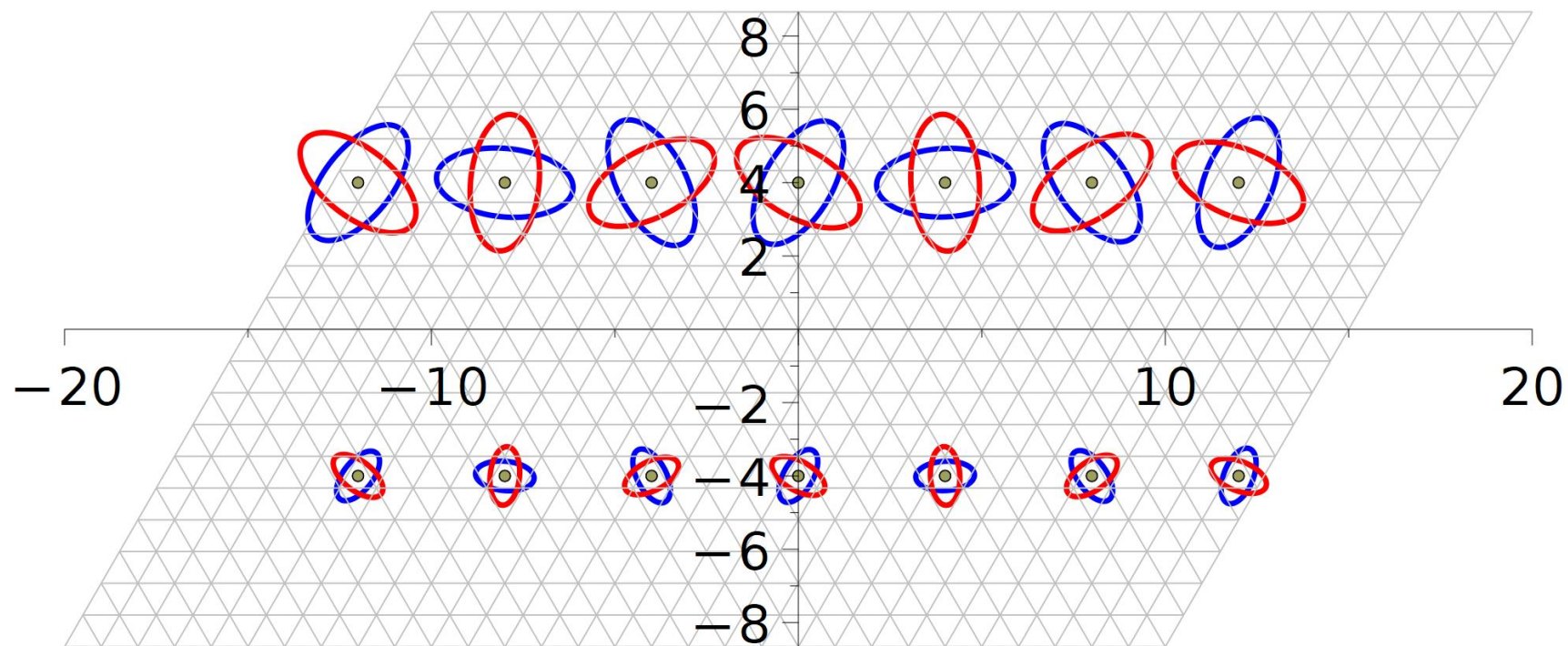
That problem makes sense already in the Riemannian setting:

An example with $\Lambda = \{\text{blue}, \text{red}\}$,
 F^α blue Riemannian and
 F^β red Riemannian:

Bi-metric Riemannian weights

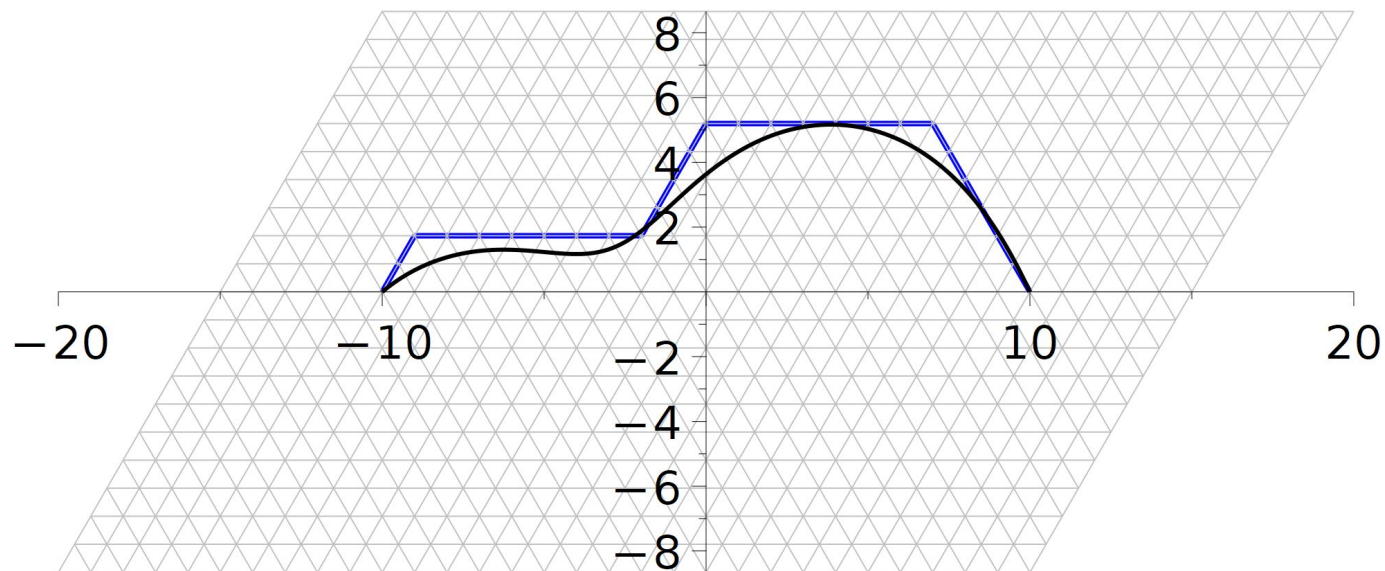
We choose a simple bi-metric which is scaled in the u^2 direction and rotated in the u^1 -direction. The two metrics are themselves identical except for a fixed angle between the half-axes of the corresponding indicatrices:

A Riemannian bi-metric



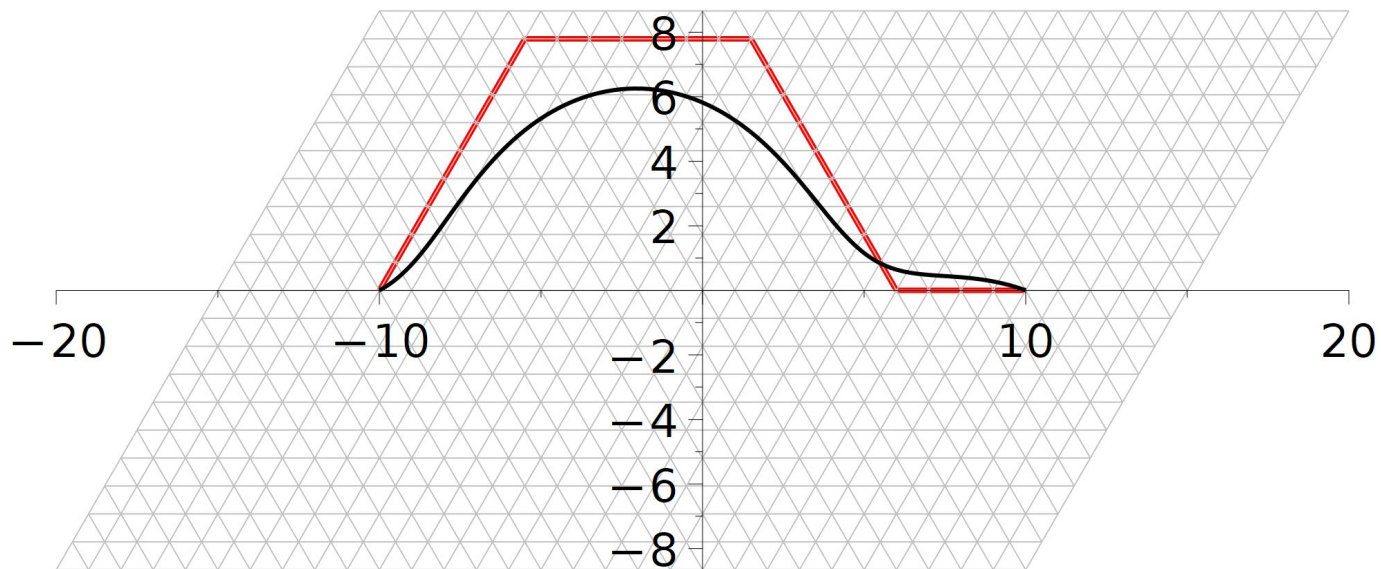
The blue Riemannian metric

The blue Dijkstra geodesic and the 'blue' correct geodesic: $L_D = 6.6$, $L_G = 6.5$



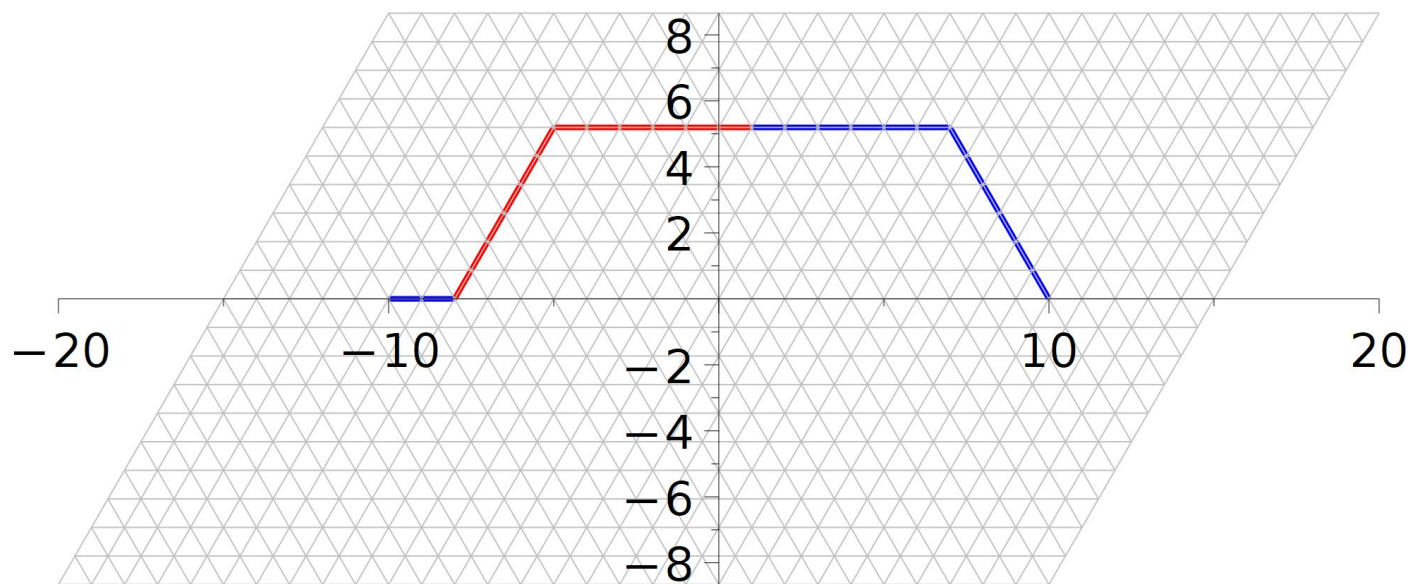
The red Riemannian metric

The red Dijkstra geodesic and the 'red' correct geodesic: $L_D = 7.0$, $L_G = 6.5$



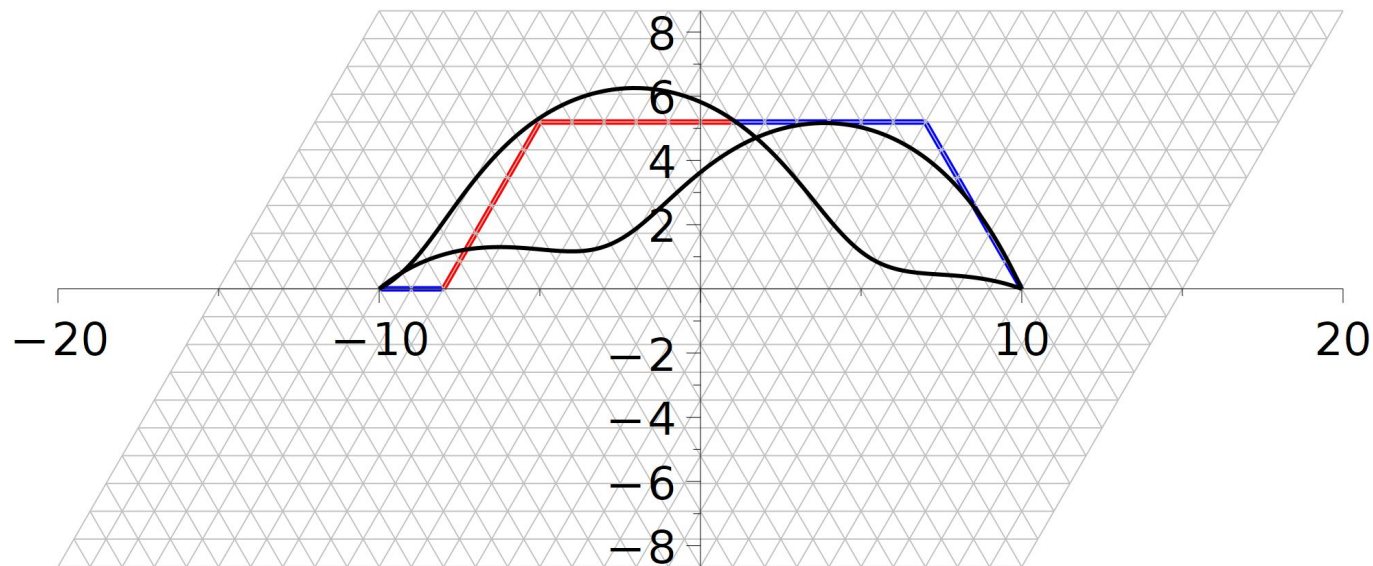
The Riemannian bi-metric

The blue-red colored Dijkstra geodesic with $L_D = 5.5$

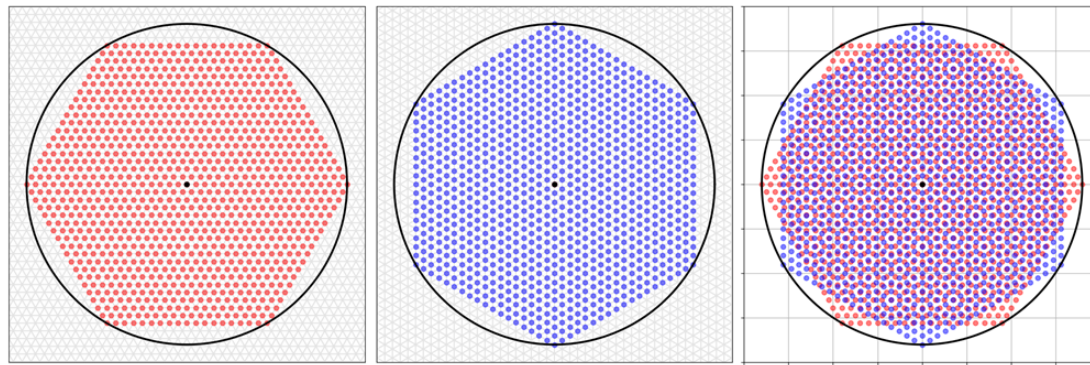


A Riemannian bi-metric

The blue-red colored Dijkstra geodesic and the two correct geodesics: $L_D = 5.5$, $L_G = 6.5$

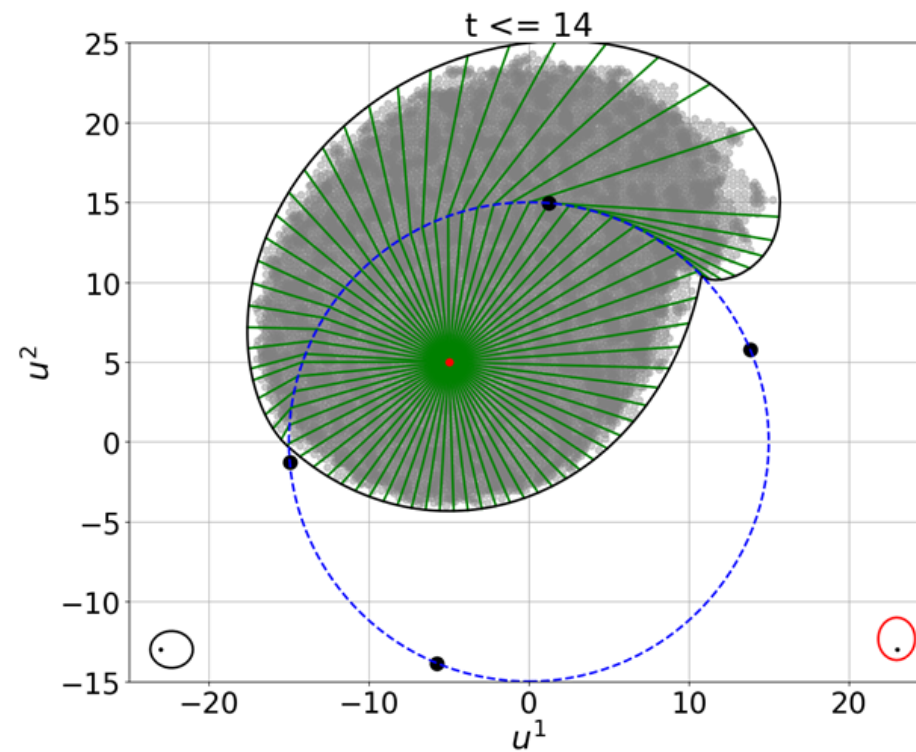


Covering wildfires by rotation of Dijkstra nets



Credit: Lucas Damkvist

Showing Snell phenomena at a circular interface



Credit: Lucas Damkvist



Thank you for your attention!