Time-dependent Finsler geometry for wildfire spread modelling

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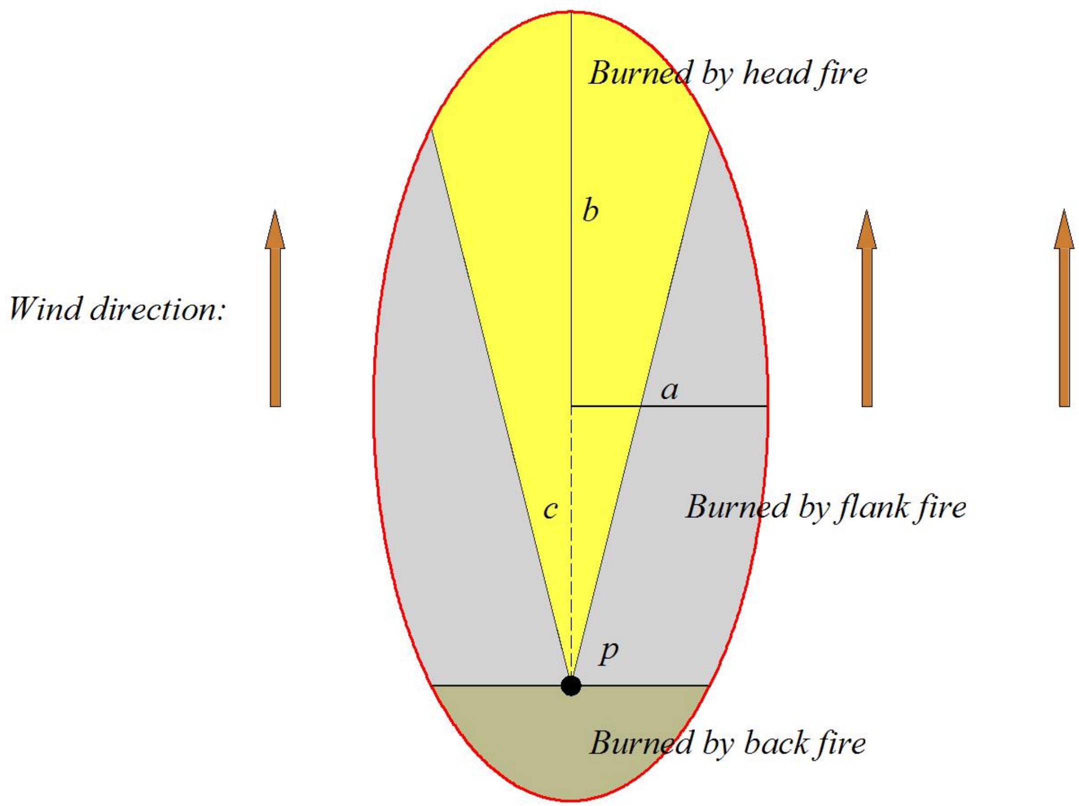
Motivation


Burned by head fire
b
Wind direction:
a
c
Burned by flank fire
p
Burned by back fire

Motivation

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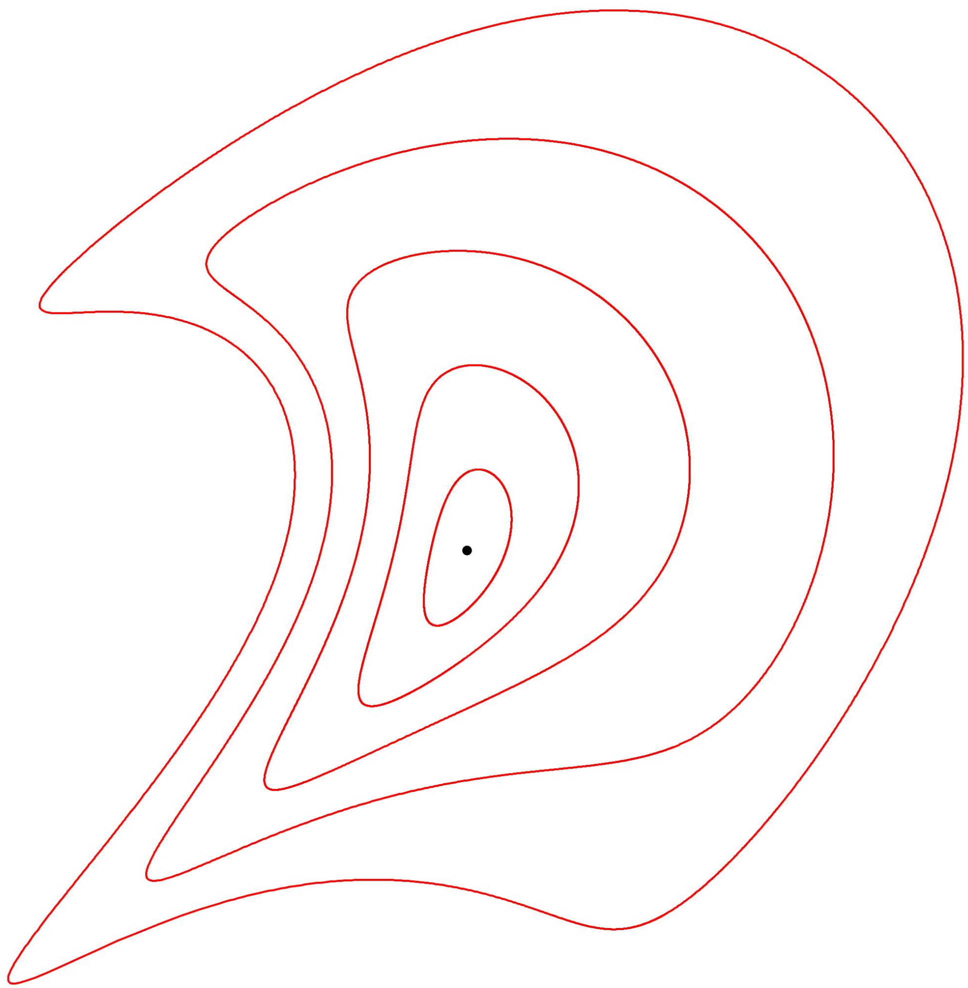

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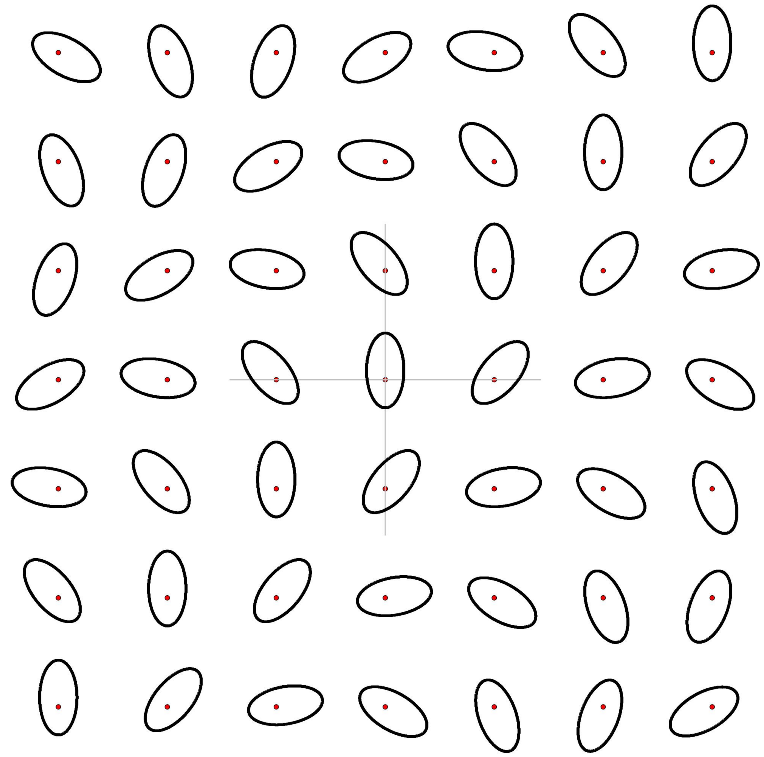

Motivation

Zermelo data for field of ellipses in (tangent bundle of) $\mathbb{R}^2$, considered by G. W. Richards (1990) – equivalent to a Randers Finsler metric on $\mathbb{R}^2$:

$$Z = (a, b, C, \theta)(u, v, t) \quad , \text{ where}$$

- a and b are the half-axes,
- C is the (wind-)shift vector, and
- θ is the (clockwise) rotation angle of the ellipse

– all are functions of position (u, v) and time t.

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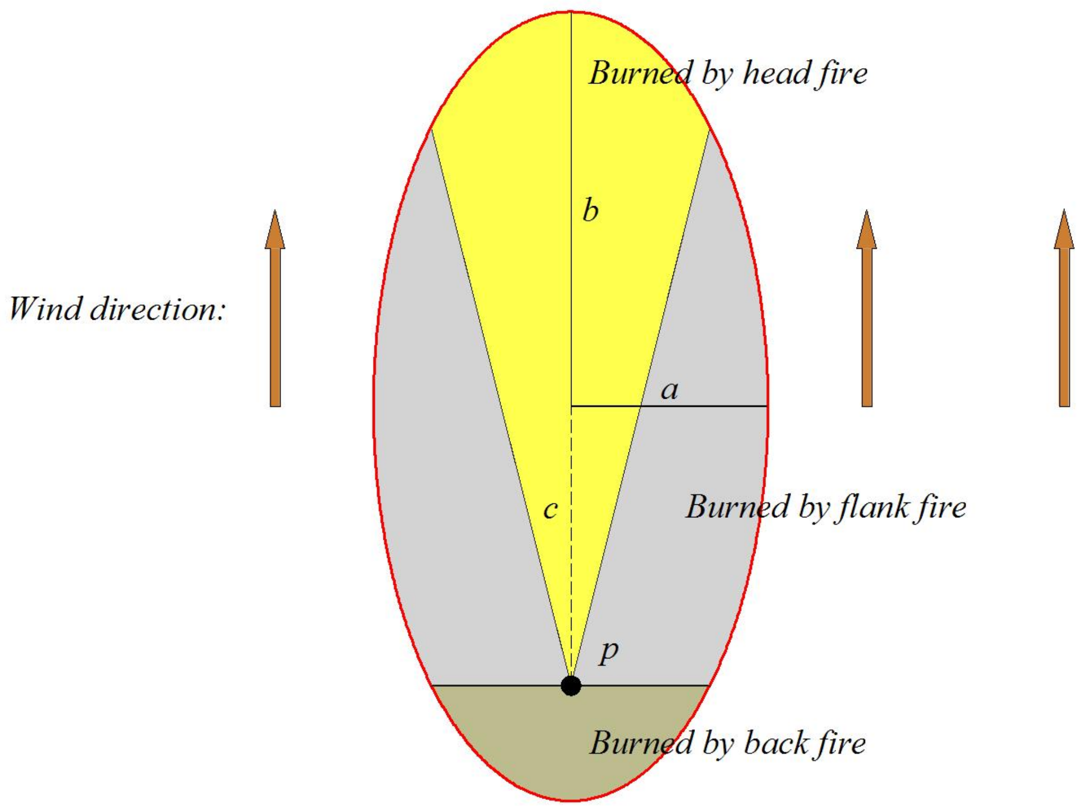

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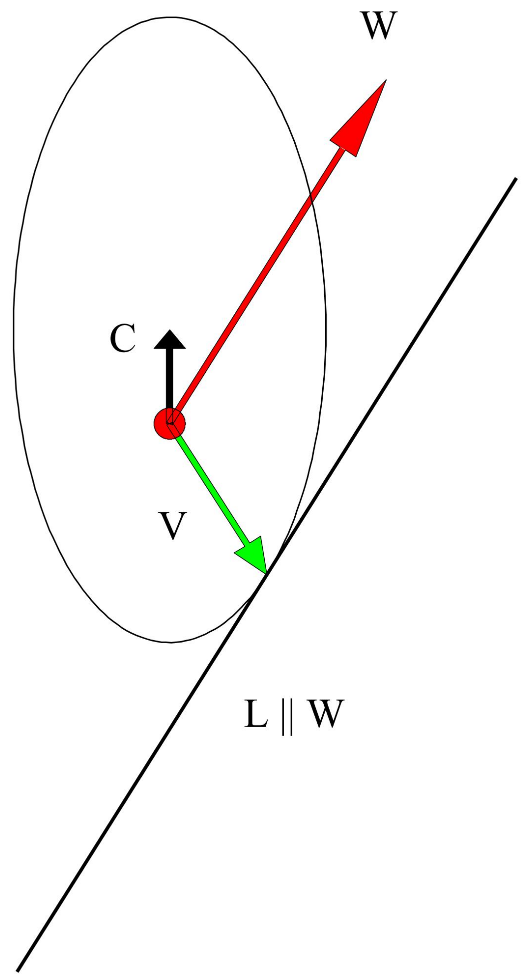


Motivation, en passant

Suppose that we are given Zermelo field data a, b, C, and θ with a, b, and θ defining the Riemannian metric h. Then the corresponding Finsler metric F is determined by the following expression (after [Bao, Robles, and Shen, 2004]):

$$F(t,p,V) = F(t,u,v,x,y)$$

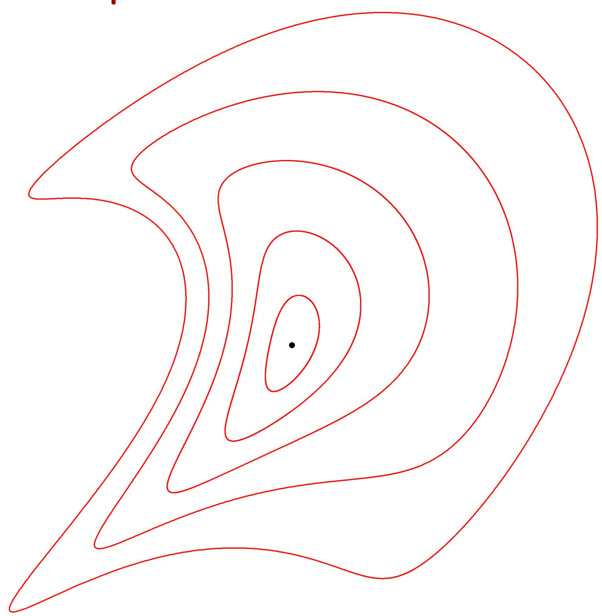
$$= \left(\frac{\sqrt{\lambda\, h(V,V) + h^2(V,C)}}{\lambda}\right) - \left(\frac{h(V,C)}{\lambda}\right),$$

where we assume generally that

$$\lambda = 1 - h(C,C) > 0 \quad . \tag{1}$$

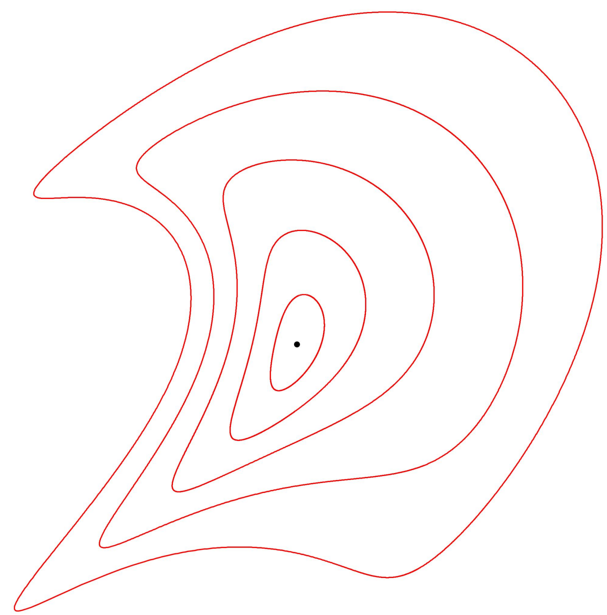
Motivation

Now ignite a large time fire at some point and use the Zermelo data at each front to envelope the next front via Huyghens' principle:



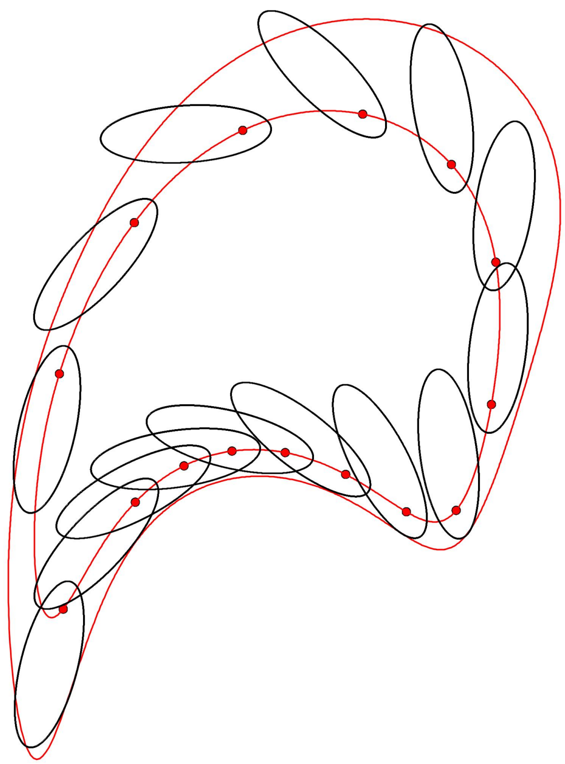
Huyghens' principle, intuitively

A later front is obtained by igniting a fire at each point on a given front, let them burn for a small time, and then take the outer envelope of the ensuing fire domains:



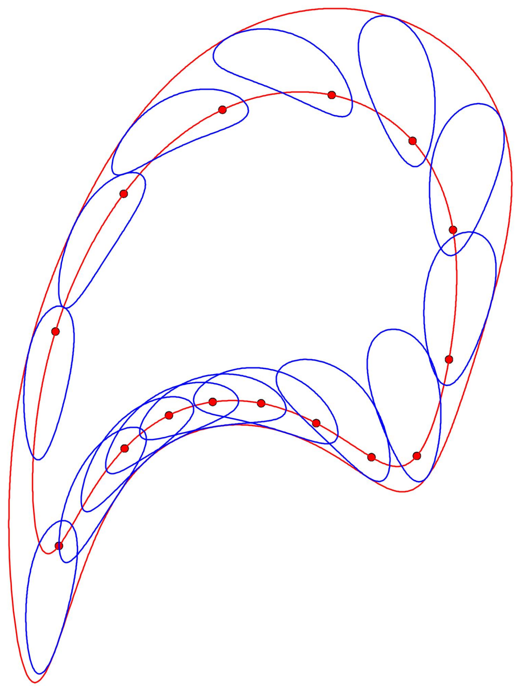
Huyghens' principle, intuitively

The ellipses cannot be used for this in a direct way (since they are linearized elements in the tangent bundle):

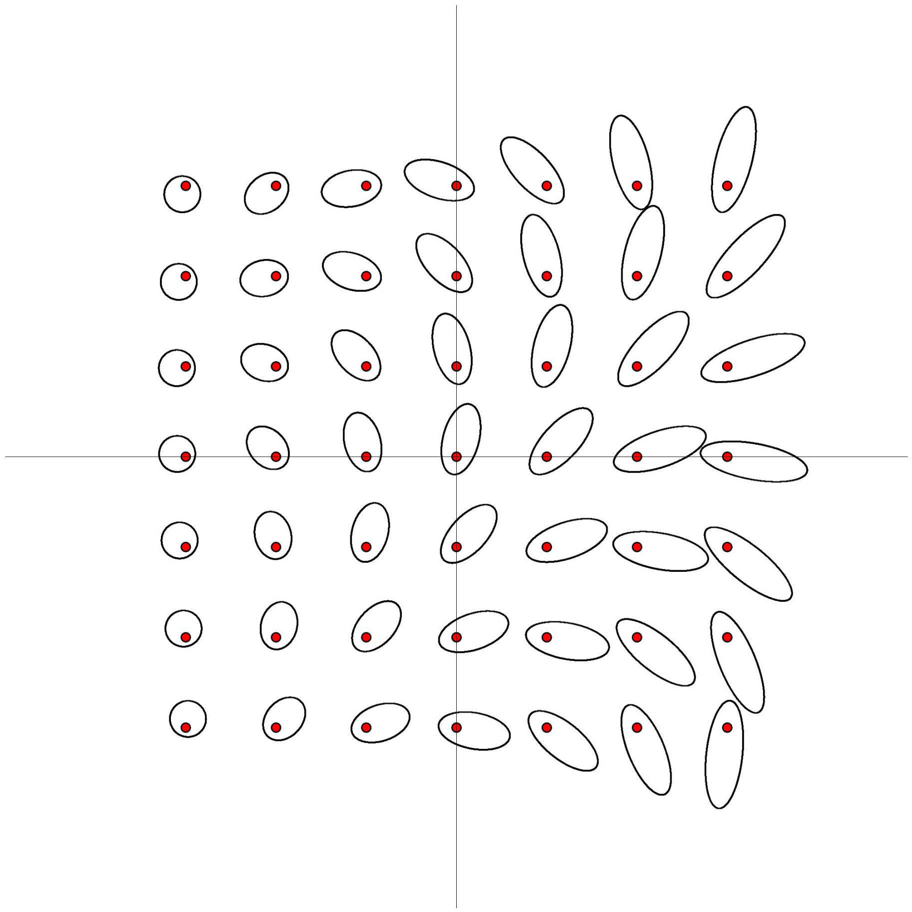


Huyghens' principle, intuitively

The correct fire domains do envelope the next frontal (via 'integration' of the ellipse field):



Motivation: Time-variation



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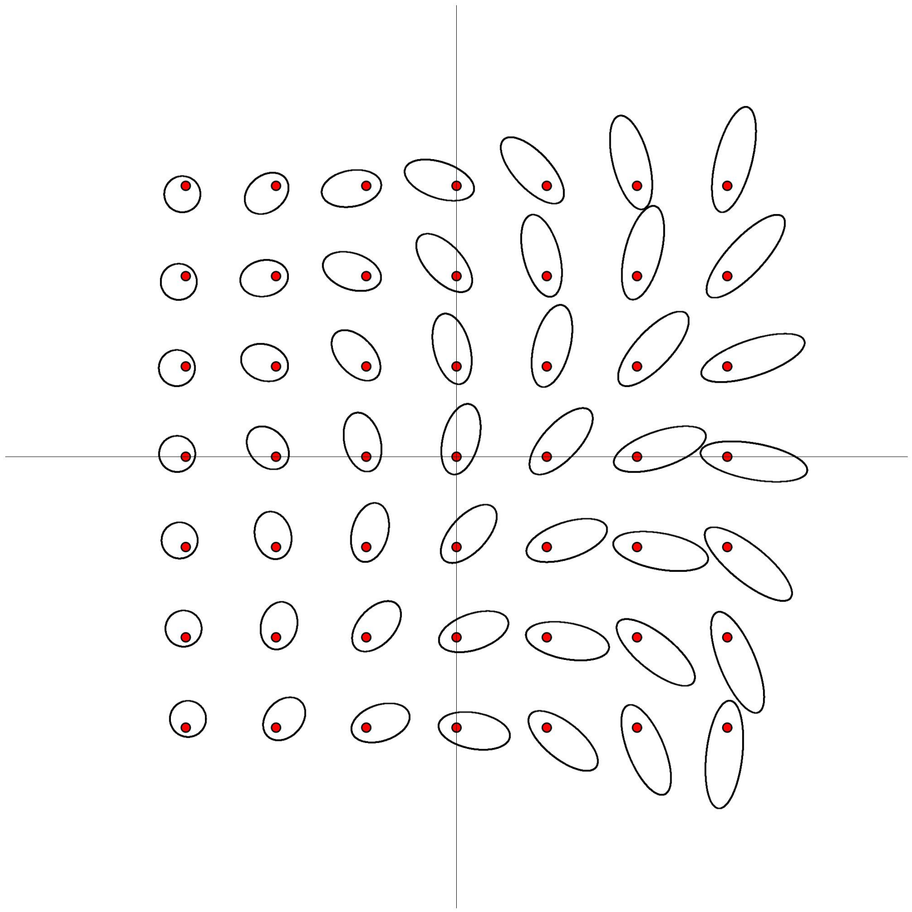

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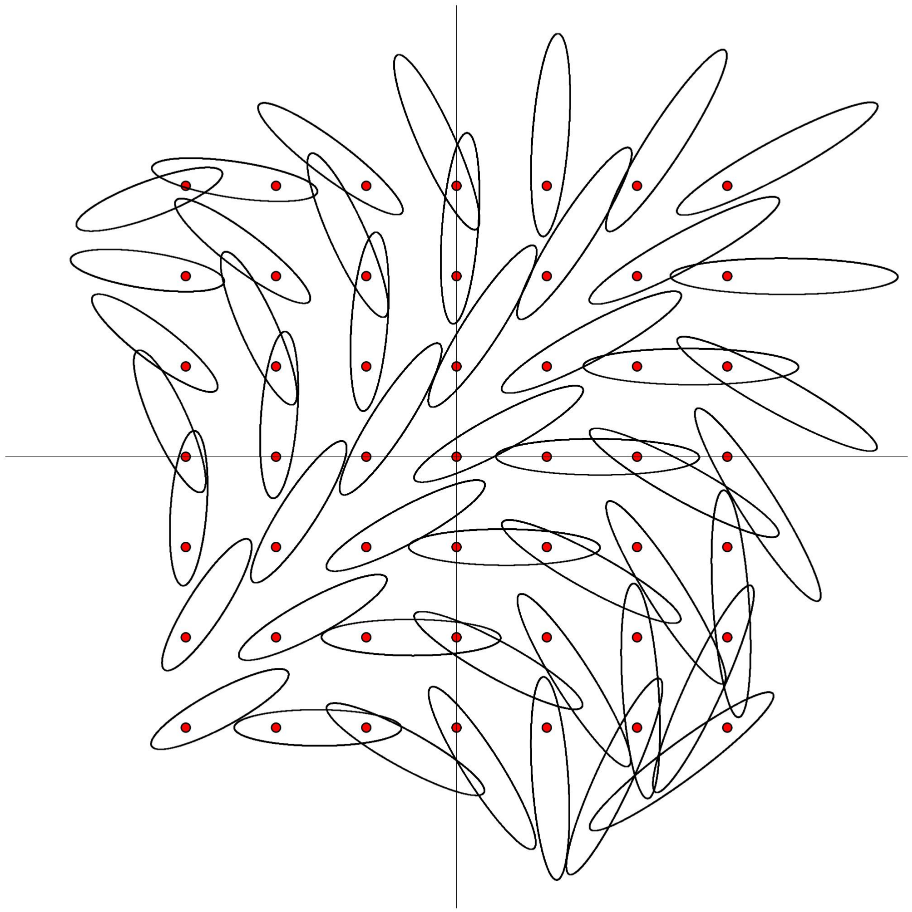
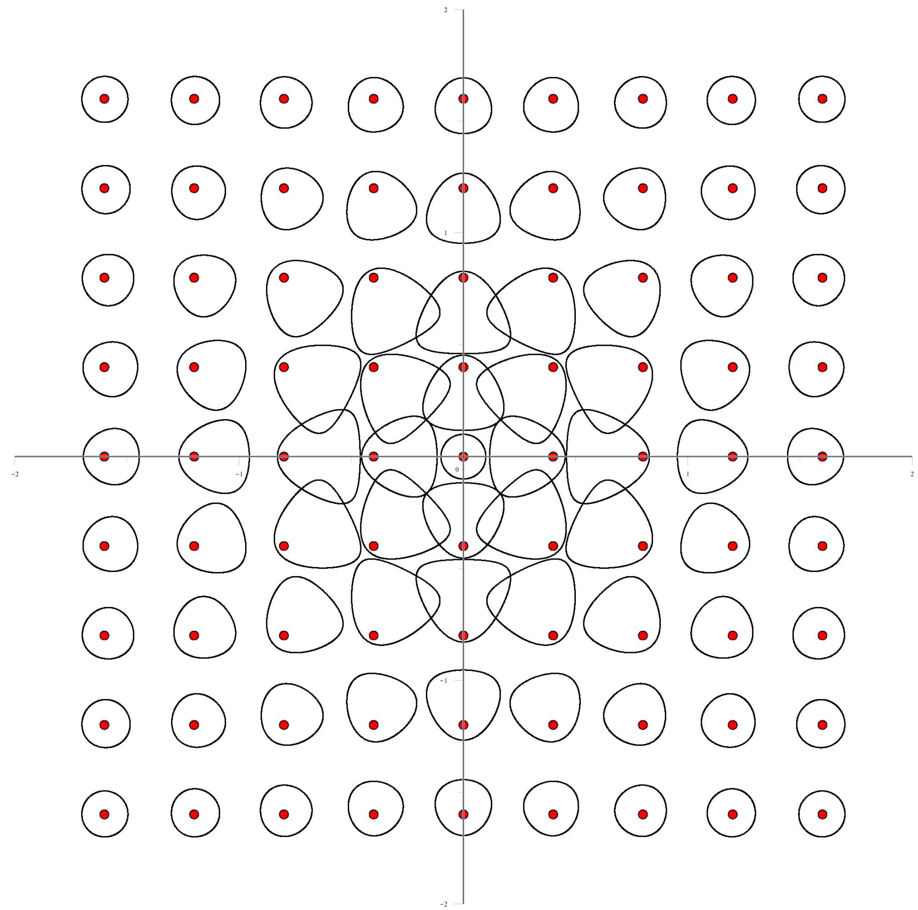
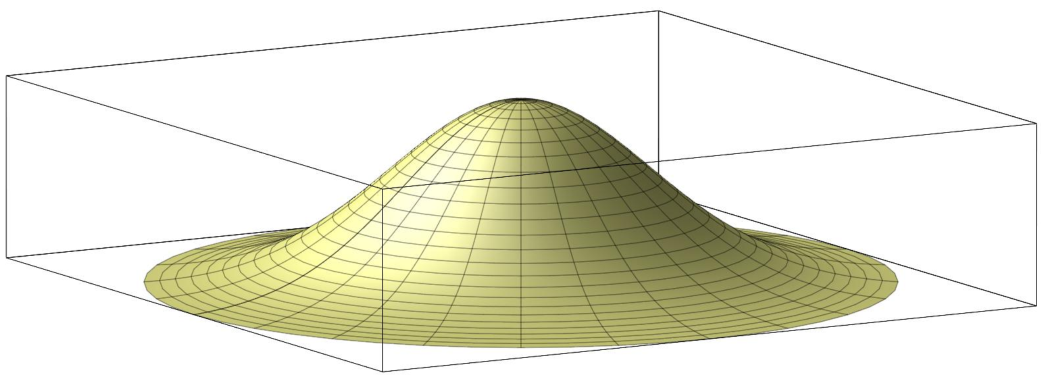


Motivation: Non-Zermelo, Matsumoto, ...



Motivation: Non-Zermelo, Matsumoto, ...



Motivation: Quest

- How to find the frontal spread by 'integration'?
- How to prove that it satisfies Huyghens' principle?

Initially, we do both via the differential inclusions defined by the field of linearized fire-lets (now to be called $\mathcal{I}_t$):

Time-dependent differential inclusions

Definition 1. A time-dependent differential inclusion – denoted by shorthand $\mathcal{I}_t$ – in $\mathcal{U}$ is a control-geometric extension of the notion of a first order differential equation:

$$\dot{\xi}(t) \in \mathcal{I}_t(\xi(t)) \quad ,$$

where $\mathcal{I}_t$ is a multimap with strongly convex graphs:

$$\mathcal{I}_t \; : \; \mathcal{U} \rightrightarrows T\mathcal{U} \quad ,$$

so that for each time t and for each $p \in \mathcal{U}$ the value $\mathcal{I}_t(p)$ is a compact strongly convex set in $T_p\mathcal{U}$ containing the origin of $T_p\mathcal{U}$ in its interior. The inclusion field $\mathcal{I}_t(p)$ is assumed to be smooth in the variables t and p.

Reachable sets

Definition 2. The reachable set $R_N(T) \in \mathcal{U}$ for the differential inclusion $\mathcal{I}$ until time T is then defined by solution curves as follows:

$$R_N(T) = \{\xi(T) : \xi(t) \text{ is an absolutely continuous curve in } \mathcal{U} , \\ \xi(0) \in N, \dot{\xi}(t) \in \mathcal{I}_t(\xi(t)) \text{ for a.e. } t \in [0,T]\} \quad ,$$

where N denotes the *ignition hypersurface* for the differential inclusion.

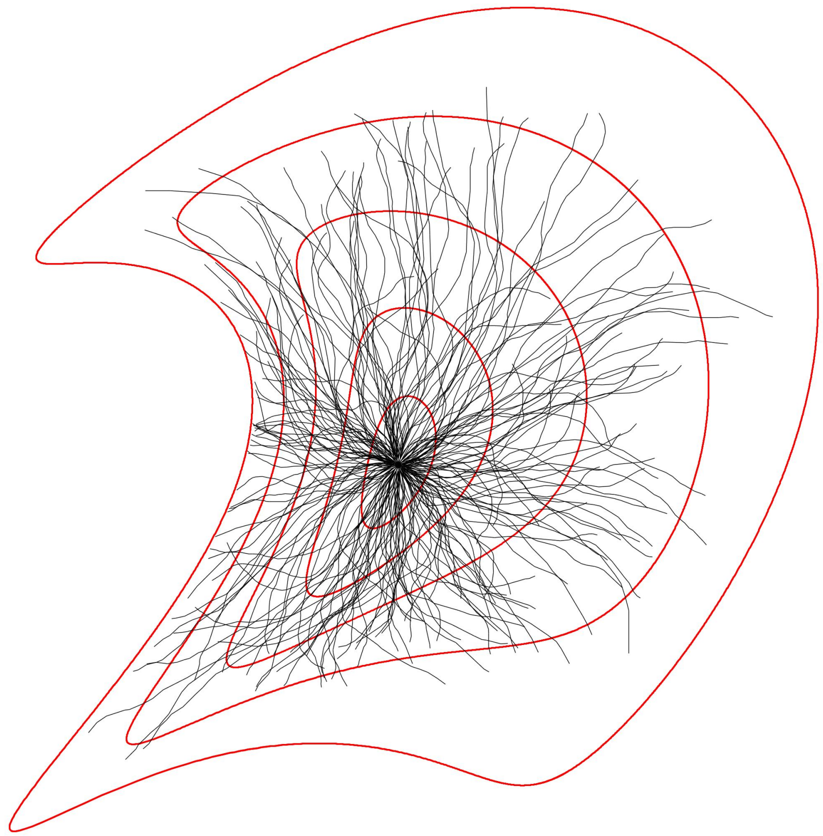
Frontals

Definition 3. The frontal $\mathcal{F}_N(T)$ at time T of an N-ignited wildfire is the boundary of the reachable set defined by the multimap $\mathcal{I}_t$:

$$\mathcal{F}_N(T) = \partial R_N(T) \quad .$$

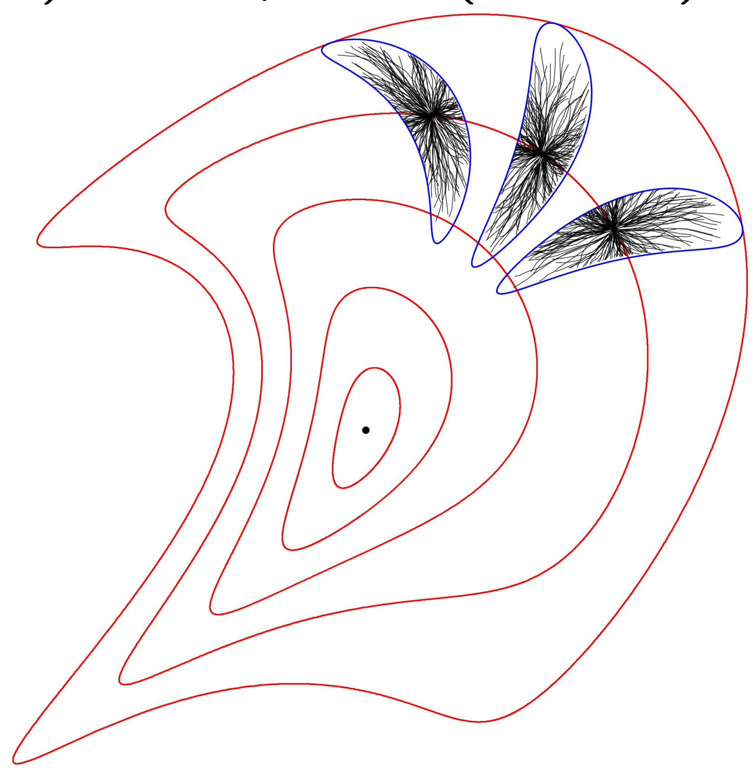
Frontals

Trying to reach the boundary of a reachable set:



Huyghens' principle I

The point-ignited wildfires of duration δ from a given front $\mathcal{F}_N(T-\delta)$ envelope the (forward) frontal $\mathcal{F}_N(T)$:



Huyghens' principle I

Proof:

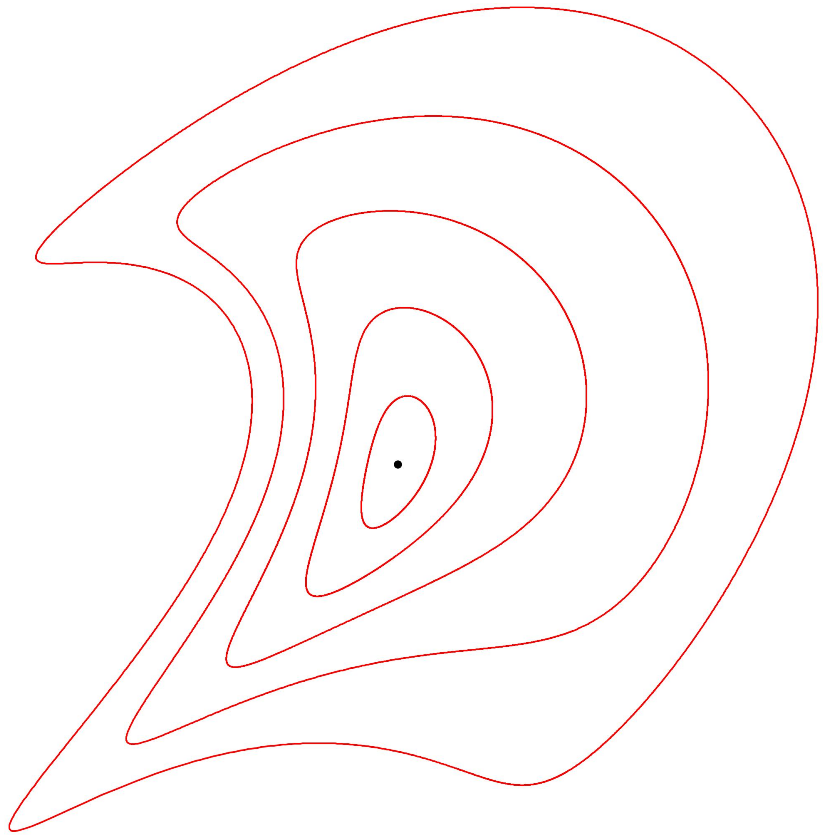
$$R_N(T) = R_N(T-\delta) \bigcup_{q \in \mathcal{F}_N(T-\delta)} R_q(\delta) \quad ,$$

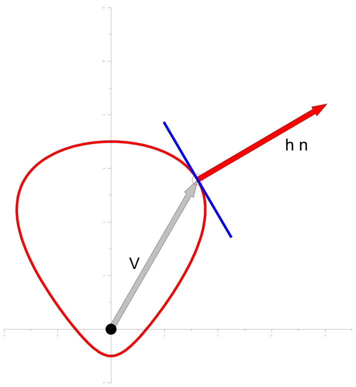
$$\mathcal{F}_N(T) = \partial^+ \left(\bigcup_{q \in \mathcal{F}_N(T-\delta)} \mathcal{F}_q(\delta) \right) \quad .$$

□

Huyghens' principle II

Local foliation consequence:





Huyghens' principle II

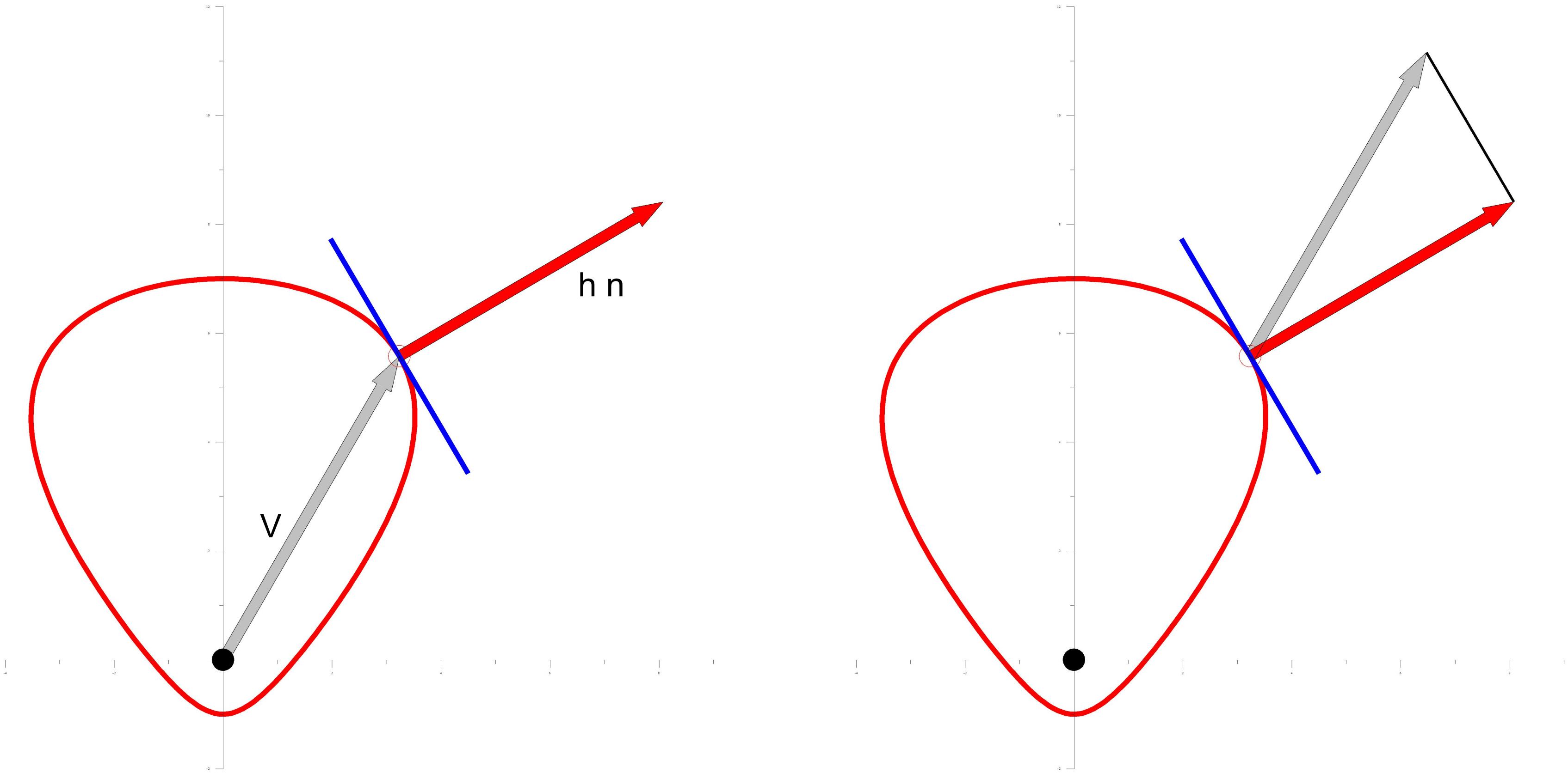
Proof:

At the point w at time t_0: Using Huyghens' principle I, the ovaloid $\mathcal{I}_{t_0}(w)$ and the shown tangent appear in the scaled limit of the foliation by frontals $\mathcal{F}_N(t_0 + \varepsilon)$ for $\varepsilon \to 0$.

$\square$

Huyghens' principle II

Local foliation consequence:



Huyghens' principle II

In Minkowski geometric terms this local correspondence is expressed by:

$$V = J^*(h \cdot n) \quad ,$$

where n is the Euclidean unit normal to the ovaloid, h is the support function, and J is the Legendre-Fenchel transform (generated by the ovaloid) and J^* its inverse.

Wildfire parametrization

We want to find a parametrization $\gamma(s,t)$ of the (burnt out) domain so that this relation holds everywhere:

$$\dot{\gamma}_t = J^*(h \cdot n) \quad ,$$

where then n is the Euclidean unit vector field normal to the frontal at $\gamma(s,t)$.

Wildfire parametrization

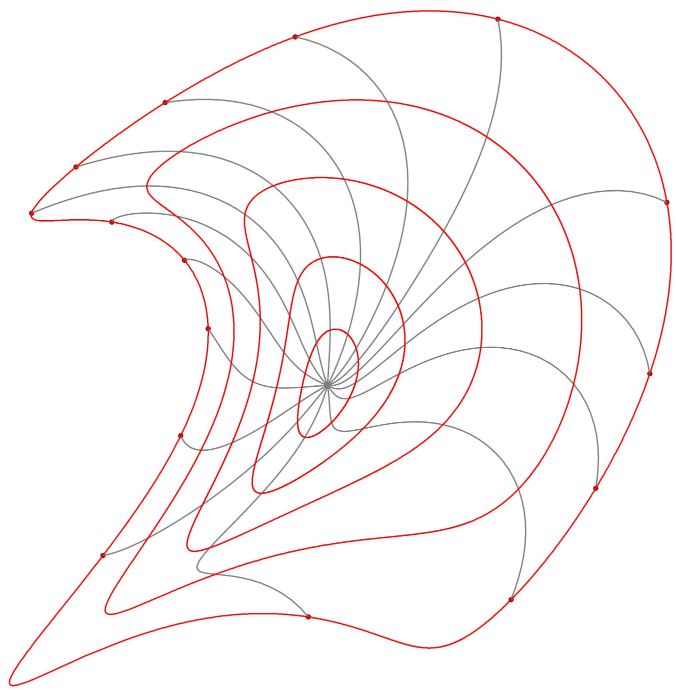
In dimension 2 we want to find $\gamma(s,t)$ so that:

$$\dot{\gamma}_t(s,t) = \frac{J^*(h \cdot \widehat{\dot{\gamma}}_s(s,t))}{\|\dot{\gamma}_s(s,t)\|_E} \quad ,$$

where $\widehat{\dot{\gamma}}_s / \|\dot{\gamma}_s\|_E$ is now the Euclidean unit vector normal to the frontal at $\gamma(s,t)$.

Wildfire parametrization

I.e. we want to find the parameter curves of the wildfire parametrization $\gamma(s,t)$:



Wildfire parametrization

In dimension 2 the wildfire equation is a first order coupled partial differential equation system:

$$\dot{\gamma}_t(s,t) = \frac{J^*(h \cdot \widehat{\dot{\gamma}}_s(s,t))}{\|\dot{\gamma}_s(s,t)\|_E} \quad .$$

We can spell out this equation in coordinates using:

$$\dot{\gamma}_t(s,t) = (\dot{u}_t, \dot{v}_t) \quad \text{and} \quad \dot{\gamma}_s(s,t) = (\dot{u}_s, \dot{v}_s) \quad .$$

The Zermelo examples

Zermelo data for field of ellipses in (tangent bundle of) $\mathbb{R}^2$, considered by G. W. Richards (1990) – equivalent to a Randers Finsler metric on $\mathbb{R}^2$:

$$Z = (a, b, C, \theta)(u, v, t) \quad , \text{ where}$$

- a and b are the half-axes,
- C is the (wind-)shift vector, and
- θ is the (clockwise) rotation angle of the ellipse

– all are functions of position (u, v) and time t.

G. D. Richards' equations for Zermelo fires

Theorem 4 (Richards, 1990). *In dimension 2, if the oval field is an ellipse field with Zermelo data* a, b, $C = (c_1, c_2)$, *and* θ *(all depending on position and time), then the wildfire equation is:*

$$\dot{u}_t = \frac{a^2 \cos(\theta)\left(\dot{u}_s \sin(\theta) + \dot{v}_s \cos(\theta)\right) - b^2 \sin(\theta)\left(\dot{u}_s \cos(\theta) - \dot{v}_s \sin(\theta)\right)}{\sqrt{a^2 \left(\dot{u}_s \sin(\theta) + \dot{v}_s \cos(\theta)\right)^2 + b^2 \left(\dot{u}_s \cos(\theta) - \dot{v}_s \sin(\theta)\right)^2}} + c_1 \cos(\theta) + c_2 \sin(\theta) \quad , \quad \textit{and}$$

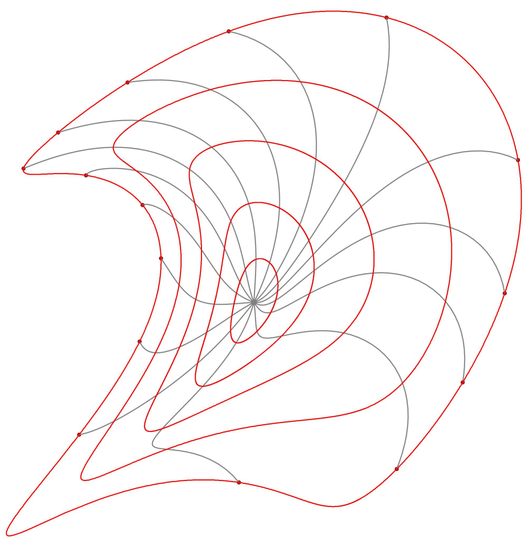
$$\dot{v}_t = \frac{-a^2 \sin(\theta)\left(\dot{u}_s \sin(\theta) + \dot{v}_s \cos(\theta)\right) - b^2 \cos(\theta)\left(\dot{u}_s \cos(\theta) - \dot{v}_s \sin(\theta)\right)}{\sqrt{a^2 \left(\dot{u}_s \sin(\theta) + \dot{v}_s \cos(\theta)\right)^2 + b^2 \left(\dot{u}_s \cos(\theta) - \dot{v}_s \sin(\theta)\right)^2}} - c_1 \sin(\theta) + c_2 \cos(\theta) \quad .$$

Main theorem

Theorem 5. *The general first order wildfire equation*

$$\dot{\gamma}_t = J^*(h \cdot n) \quad ,$$

has individual solution rays (fire particles) $\gamma(s_0, t)$ that are unit speed energy pre-extremals for the unique Finsler metric F on $\mathcal{U}$ that stems directly from the local Minkowski interpretation above.



Main challenge

Challenge 6. The general second order wildfire equation incorporates the curvature H of the instantaneous frontal curve:

$$\dot{\gamma}_t = f(H) \cdot J^*(h \cdot n) \quad ,$$

where $f(H)$ denotes a physically reasonable function of a (physically reasonable choice of) curvature H.

Does this second order wildfire equation also have individual solutions $\gamma(s_0, t)$ that are solutions to an F-induced ordinary differential equation system?

Main theorem

Theorem 7. *The unit speed energy pre-extremal ray solutions to the first order wildfire equation are obtained via the following conditioned rheonomic Euler-Lagrange equations in terms of the Finsler function F:*

$$\rho(s,t)\cdot\dot{\gamma}_t(s,t) = \sum_j \left(\ddot{\gamma}^j_{t\,t}(s,t) + 2G^j(\dot{\gamma}_t(s,t)) + N^j_0(\dot{\gamma}_t(s,t))\right)\partial_j$$

with the F-unit speed condition:

$$\|\dot{\gamma}_t\|_F = 1 \quad .$$

The governing functions G^j and N^j_0, $j = 1, \cdots, n$, are the well-known derivatives of the Finsler function F:

Main theorem ingredient

$$G^i(y) = \left(\frac{1}{4}\right) g^{i\,l}(y) \left(\left[F_t^2\right]_{x^k\,y^l}(y) y^k - \left[F_t^2\right]_{x^l}(y) \right)$$

$$N_0^i(y) = \left(\frac{1}{2}\right) g^{i\,l}(y) \left[F_t^2\right]_{t\,y^l}(y) \quad .$$

Main theorem

Proof:

Reverse Gauss' lemma for time-dependent Finsler metrics via first variation of F-unit speed curves issuing F-orthogonally from N.

$\square$

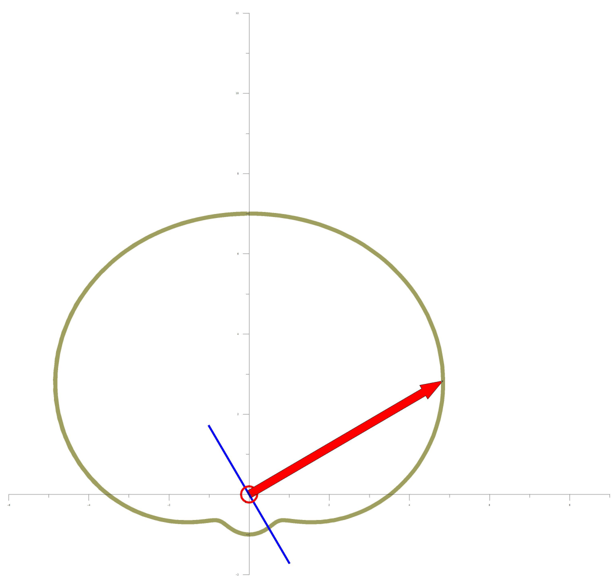
Constructions

We finally address the two obvious questions:

- How to set up (experimentally) the ovaloid field $\mathcal{I}$.
- How to set up the ensuing time- and space-dependent Finsler metric F from a given ovaloid field $\mathcal{I}_t$.

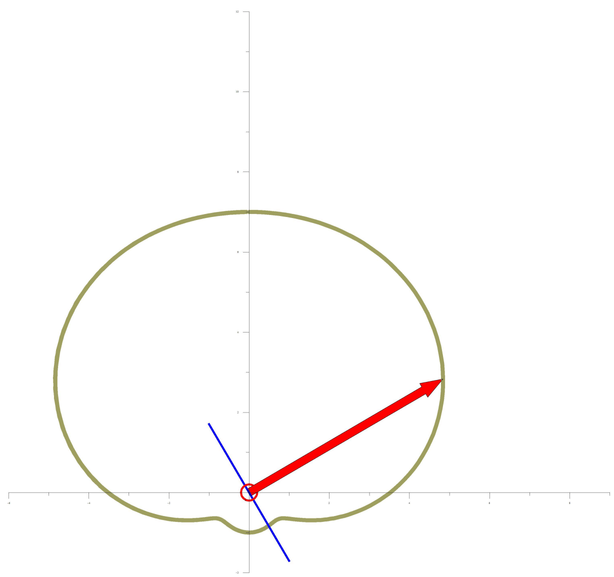
Frontal tangent speed profile

An instantaneous tangent speed profile at a point in dimension 2 is obtained from line-ignitions with homogeneous constant fuel data, i.e. the data at that point at that time:



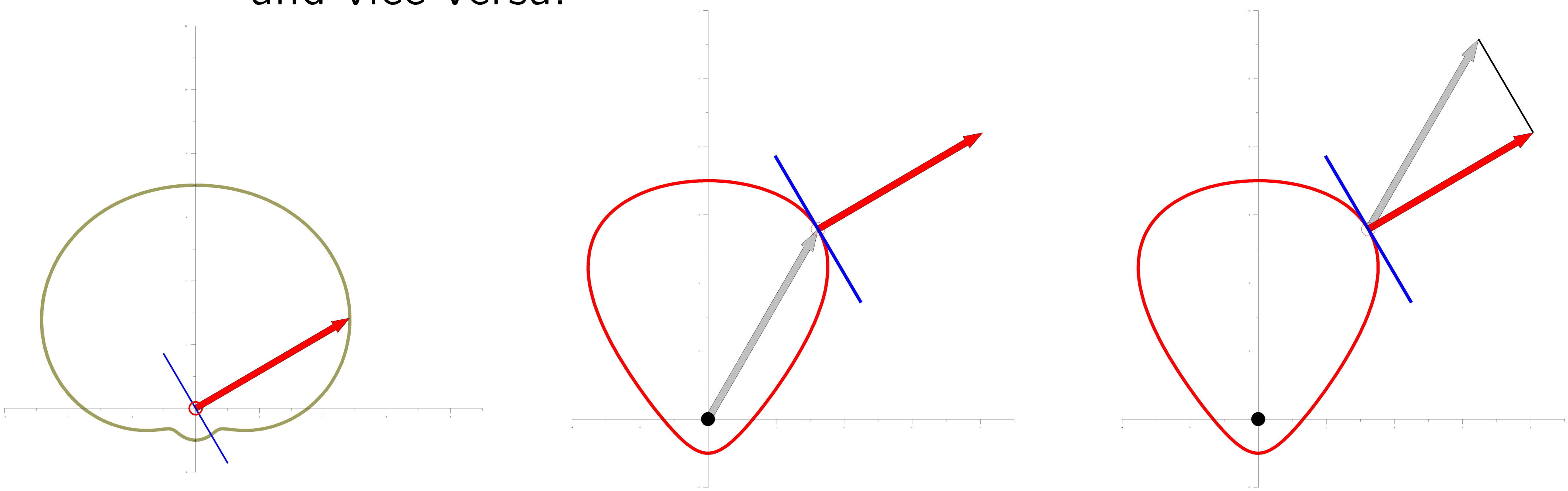
Frontal tangent speed profile

$$p(v) = (h(v)\,\mathrm{cos}(v)\,,\; h(v)\,\mathrm{sin}(v)) \quad , \quad v \in \; [0, 2\pi] \quad .$$



Frontal point speed indicatrix

Given the function h we can build the equivalent point speed ovaloid indicatrix from the tangent speed profile – and vice versa:



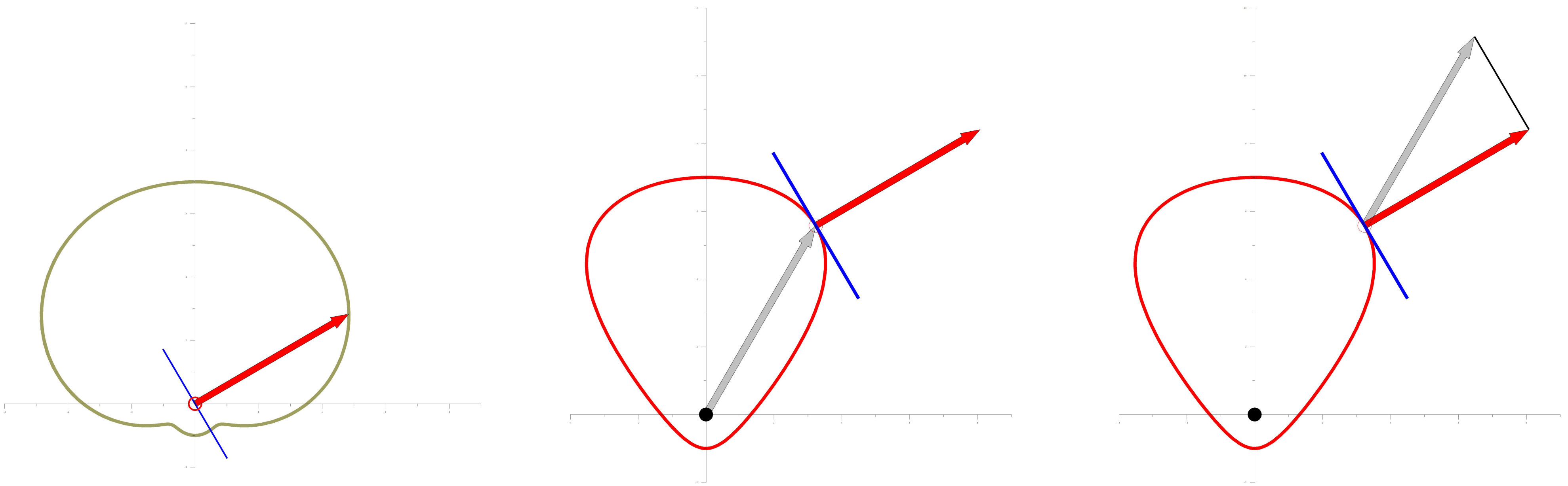
Frontal point speed indicatrix

In dimension $n = 2$ the point speed indicatrix $r(v)$ is constructed from the tangent speed profile $p(v)$, $v \in [0, 2\pi]$, as follows:

$$p(v) = (h(v)\cos(v)\,,\ h(v)\sin(v))$$

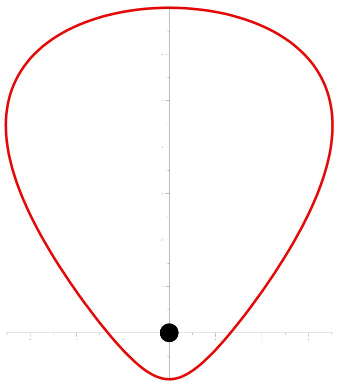
$$r(v) = \Big(h'(v)\cos(v) + h(v)\sin(v)\,,\ h'(v)\sin(v) - h(v)\cos(v)\Big) \quad .$$

Frontal point speed indicatrix



Frontal point speed indicatrix

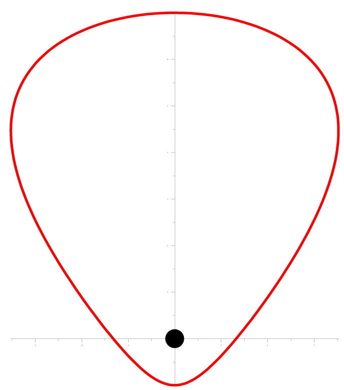
The resulting local point speed indicatrix:



Frontal point speed indicatrix

Observation 8. The curvature of the indicatrix curve $r(v)$ is positive (it is a strongly convex oval containing the origin) if and only if the following holds for all $v \in [0, 2\pi]$:

$$h''(v) + h(v) > 0 \quad .$$



Minkowski norms

Proposition 9. *Every strongly convex ovaloid which contains the origin $\mathcal{O} \in T_pM$, generates a unique Minkowski norm in that vector space.*

Minkowski norms

Definition 10. A Minkowski norm (generating the Finsler metric F) on a vector space V is a nonnegative function $F : V \to [\,0, \infty[$ with the following properties:

- F is C^∞ on $V - \{0\}$

- $F(\lambda \cdot y) = \lambda \cdot F(y)$ for all $\lambda > 0$ and for all $y \in V$

and:

Minkowski norms

- For every $y \in V - \{0\}$ the following symmetric bilinear form $g_y(u, v)$ is positive definite:

$$g_y(u,v) = u \cdot \left(\frac{1}{2}\,\mathrm{Hess}_y\, F^2(y)\right) \cdot v^\top$$
$$= \frac{1}{2}\frac{\partial^2}{\partial s \partial t}_{|s=0,t=0} F^2\left(y + s \cdot u + t \cdot v\right)$$

Minkowski norms

Proposition 11. *A Minkowski norm F in V is determined by its indicatrix $F^{-1}(1)$ – and vice versa:*

$$\mathcal{I} = F^{-1}(1) = \{w \in V \mid F(w) = 1\} \quad .$$

Minkowski norms

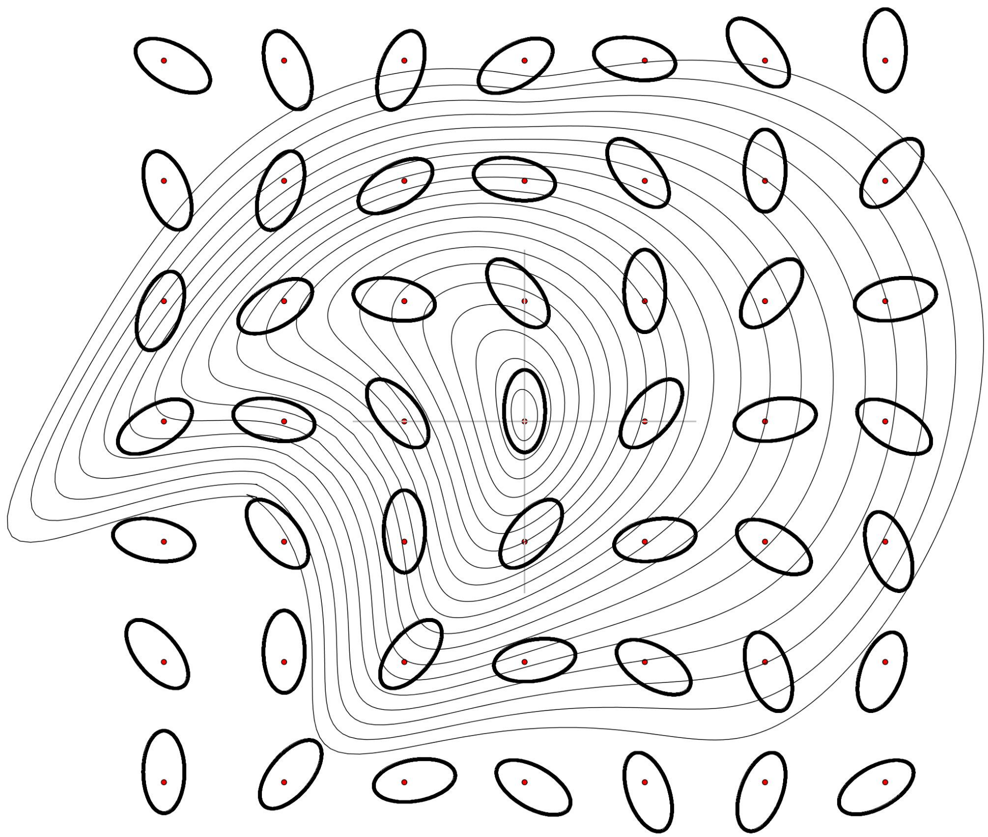
Proposition 12. *Every strongly convex ovaloid $\mathcal{O}$ in $V = T_pM$, which contains the origin $0 \in T_pM$, generates a unique Minkowski norm F in that vector space with indicatrix $\mathcal{O} = \mathcal{I}(p) = F^{-1}(1)$.*

Remark 13. Strong convexity (positive sectional curvature) of the ovaloid indicatrix is directly related to positive definiteness of the metric $g_y(u, v)$.

Final illustrations

A wildfire parametrization $\gamma(s, t)$ in a static field of Richards' type with relatively simple Zermelo data: a, b, c_1, c_2 are constants, only the rotation angle θ changes with position:

Final illustrations



Final illustrations

Finsler extremal rays from the center point of ignition:

Final illustrations

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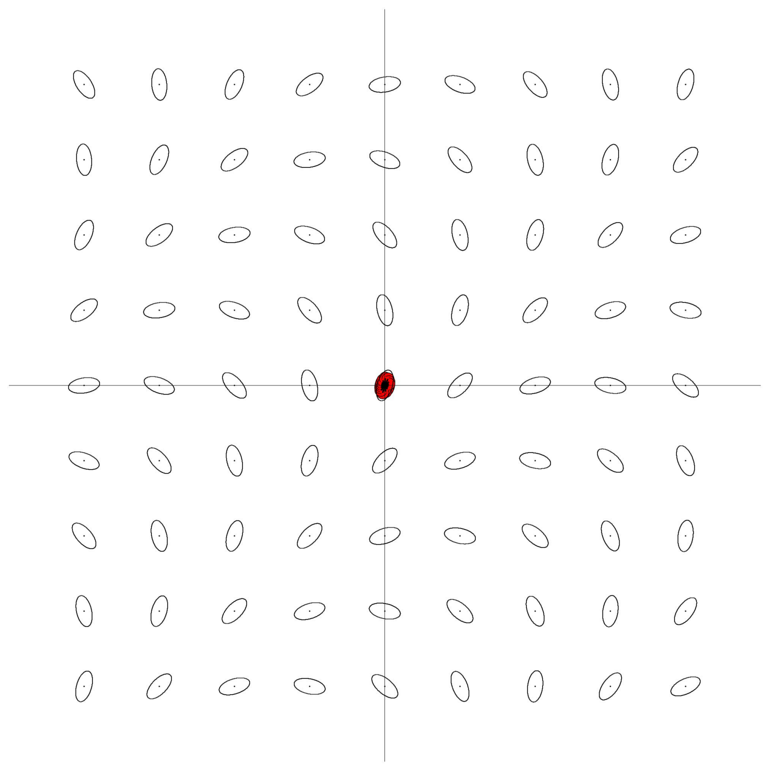
Final illustrations

A wildfire parametrization $\gamma(s,t)$ in a dynamic field of Riemannian type with simple Zermelo data: a, b, $c_1 = 0$, $c_2 = 0$ are constants, only the rotation angle θ changes with position and time:

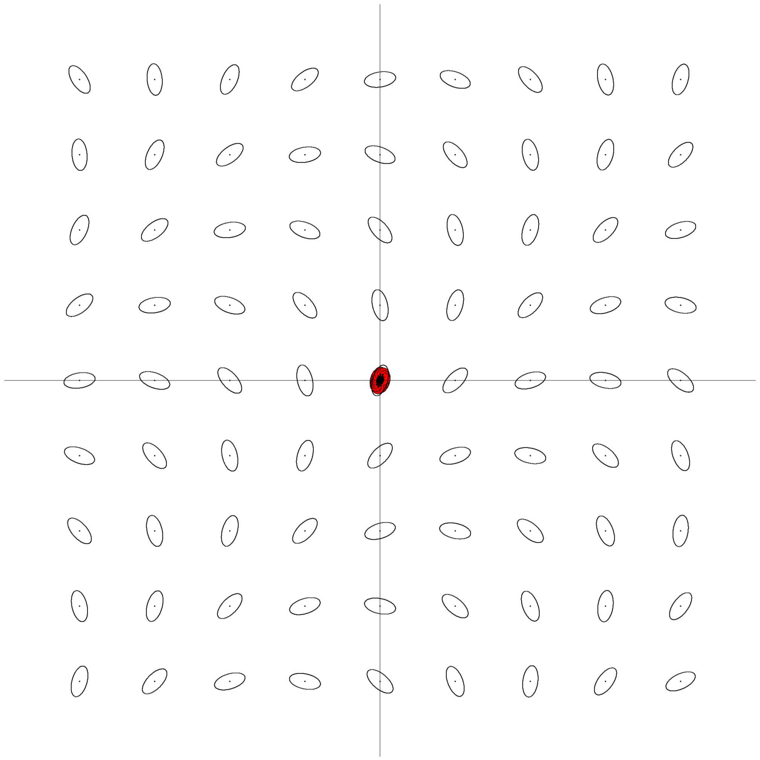
Final illustrations

Finsler extremal rays from the center point of ignition:



Final illustrations

Finsler extremal rays from the center point of ignition:



Thank you for your attention!

Comments, further questions?