



Geodætiske Kurver

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Technical University of Denmark

Georg Mohr vinderseminar, Sorø, 27.02.23

Korteste veje

*Fra Kastrup til Los Angeles
og så videre*



Korteste veje



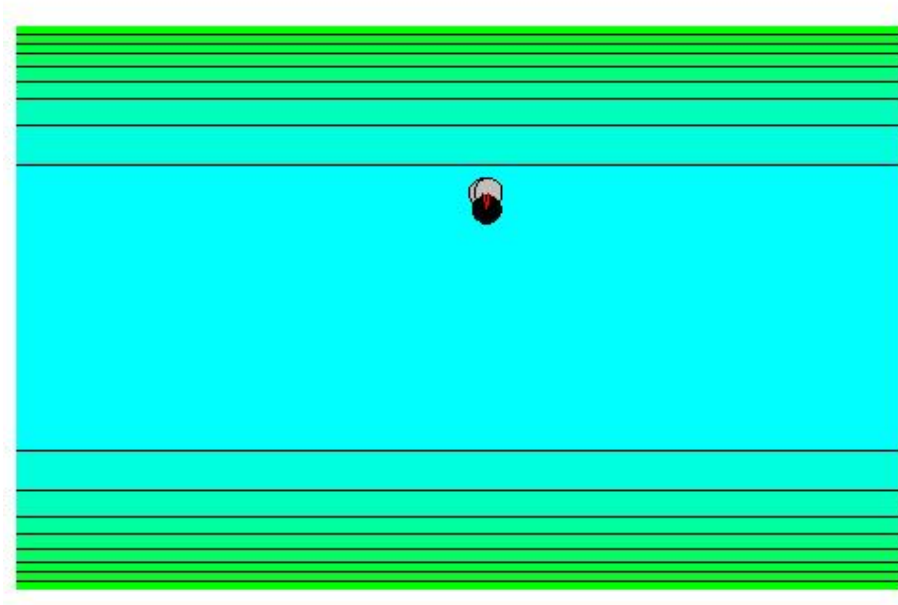
Mercator-projektionen

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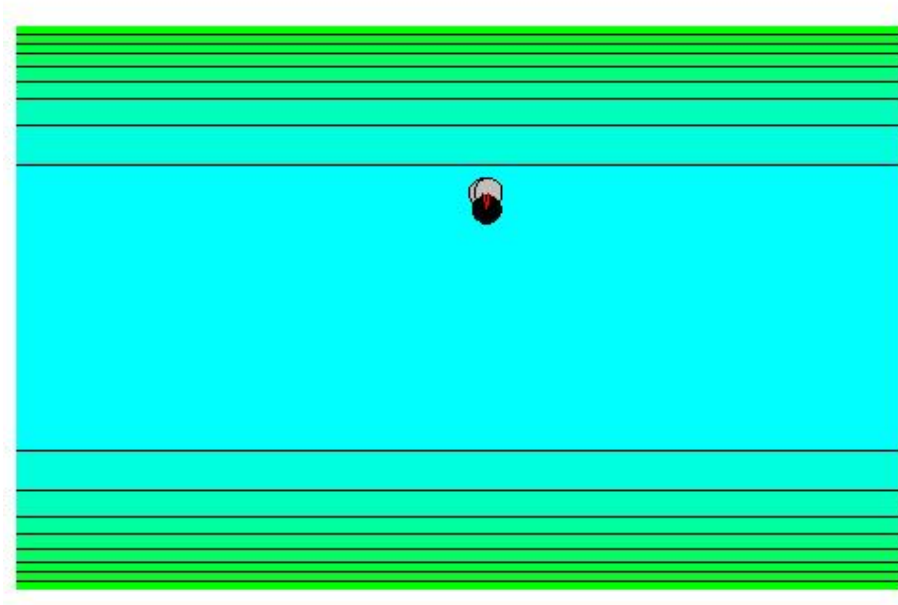


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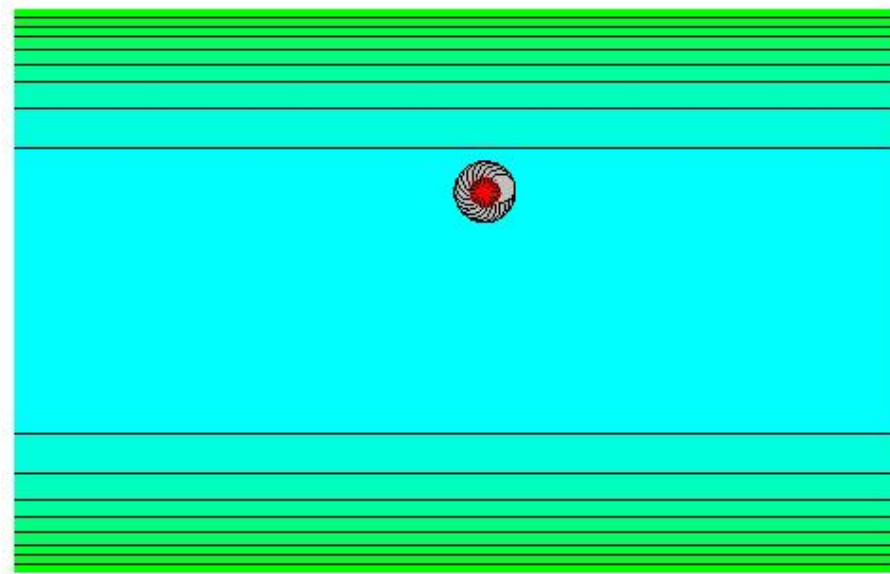
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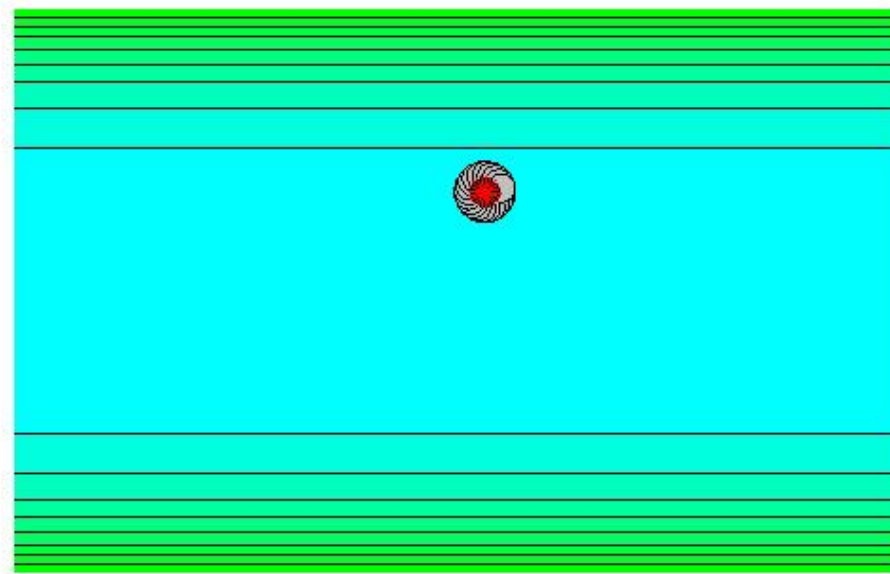
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Mercator-projektionen

Geometri og Topologi

Gauss-Bonnet's sætning

Gauss og Bonnet



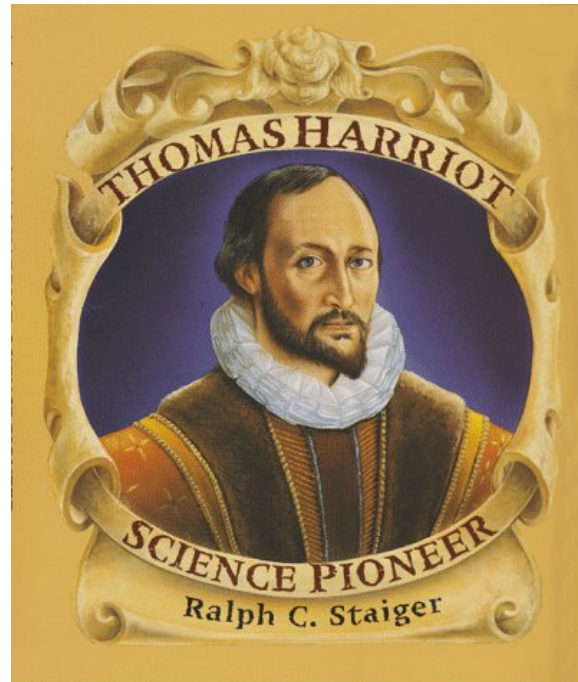
C. F. Gauss (1777 - 1855) og P. O. Bonnet (1819 - 1892)

Gauss-Bonnet's sætning

Sætning 1.

$$\int_S K \, d\mu = 2 \cdot \pi \cdot (F - E + V) \quad .$$

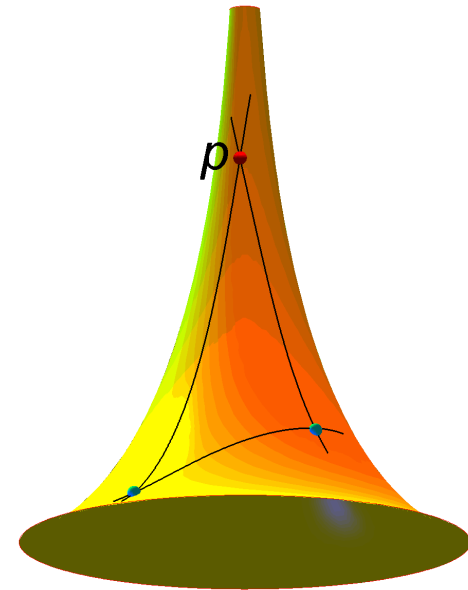
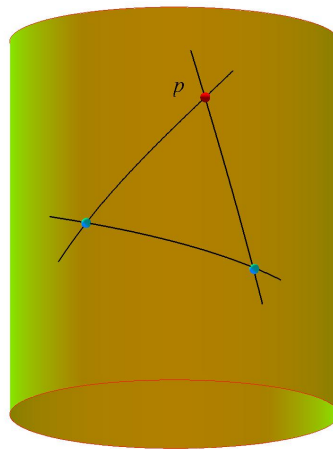
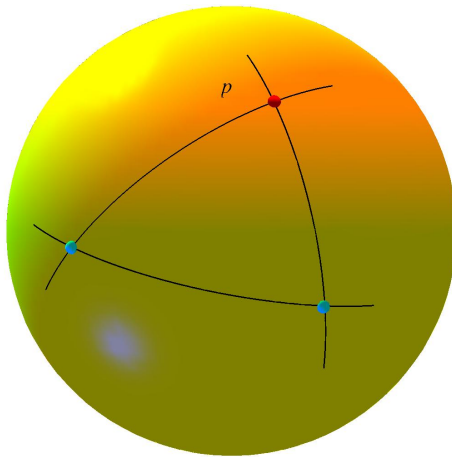
Thomas Harriot



T. Harriot (1560 - 1621)

Thomas Harriot

Fortællingen begynder med geodætiske trekanter på krumme flader. Fede, almindelige og tynde:



Thomas Harriot

Definition 2. Krumningen af en flade S i et punkt m defineres som en grænseværdi:

$$K(m) = \lim_{\Delta \rightarrow m} \left(\frac{\alpha_1 + \alpha_2 + \alpha_3 - \pi}{\text{Areal}(\Delta)} \right) ,$$

hvor α_i , $i = 1, 2, 3$, er vinklerne i Δ .

Thomas Harriot

Thomas Harriot viste, at krumningen af en kugleflade med radius 1 er $K = 1$ overalt.

Sætning 3. *Lad Δ betegne en vilkårlig geodætisk trekant på en kugleflade med radius 1. Dvs. de tre kanter i Δ er segmenter af storcirkler på kuglefladen. Så er*

$$\text{Areal}(\Delta) = \alpha_1 + \alpha_2 + \alpha_3 - \pi \quad ,$$

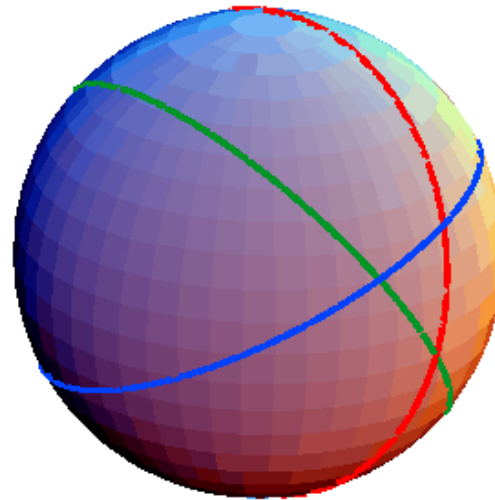
sådan at netop

$$\frac{\alpha_1 + \alpha_2 + \alpha_3 - \pi}{\text{Areal}(\Delta)} = 1$$

allerede før vi laver nogen grænseværdi for K .

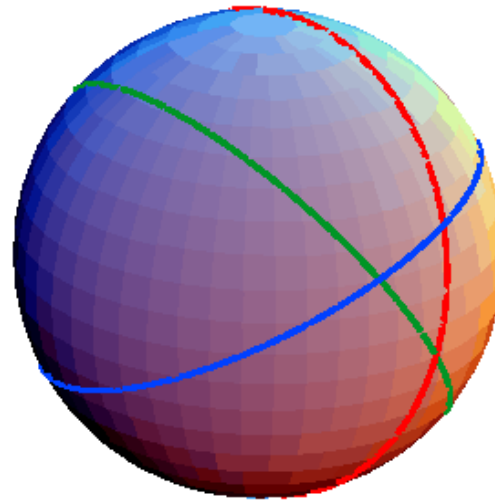
Thomas Harriot

Harriot's bevis er ret snedigt. Han deler først kuglefladen i to dele med samme areal $A = 2\pi$:



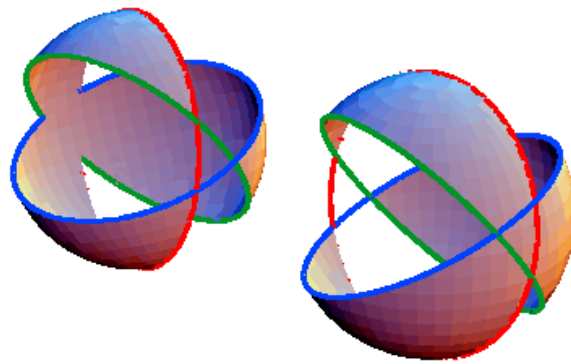
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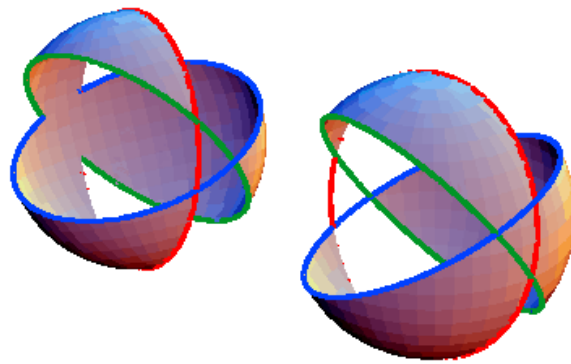
Thomas Harriot

Den forreste halvdel af kuglefladen deles i tre:



Thomas Harriot

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Thomas Harriot

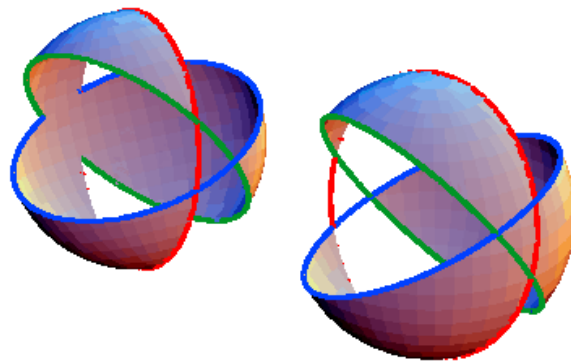
De tre 'sejl' D_i har hver et areal, der er proportional med deres respektive topvinkler:

$$\text{Areal}(D_i) = 2 \cdot \alpha_i \quad .$$

(Hvis $\alpha_i = 2 \cdot \pi$, så ville netop $\text{Areal}(D_i) = 4 \cdot \pi$.)

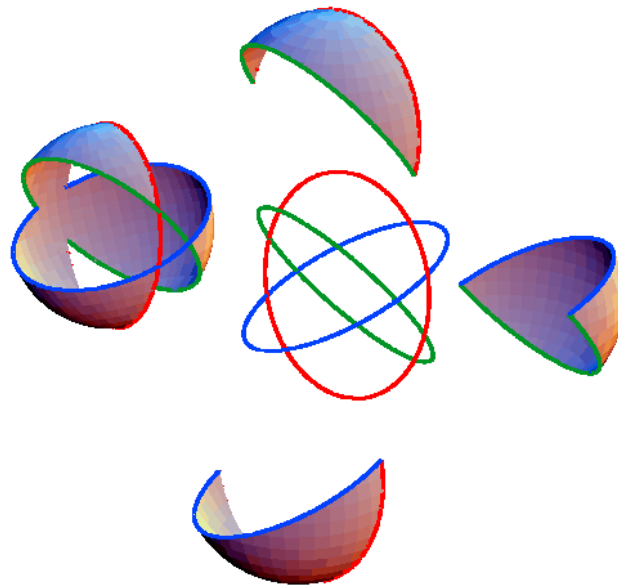
Thomas Harriot

Den forreste halvdel af kuglefladen delt i de tre sejl:



Thomas Harriot

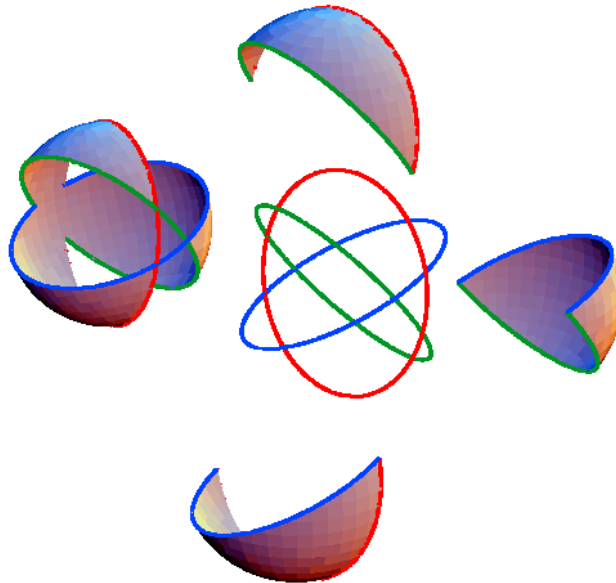
Harriot sætter sejlene sammen igen og korrigerer for det tredobbelte overlap af trekanten Δ :



$$A = 2 \cdot \pi = 2 \cdot (\alpha_1 + \alpha_2 + \alpha_3) - 2 \cdot \text{Areal}(\Delta) \quad .$$

Thomas Harriot

Harriot sætter sejlene sammen igen og korrigerer for det tredobbelte overlap af trekanten Δ :



$$A = 2 \cdot \pi = 2 \cdot (\alpha_1 + \alpha_2 + \alpha_3) - 2 \cdot \text{Areal}(\Delta) \quad ,$$

Thomas Harriot

Og heraf fås resultatet ved isolering af $\text{Areal}(\Delta)$:

$$2 \cdot \text{Areal}(\Delta) + 2 \cdot \pi = 2 \cdot (\alpha_1 + \alpha_2 + \alpha_3)$$

$$\text{Areal}(\Delta) + \pi = (\alpha_1 + \alpha_2 + \alpha_3)$$

$$\text{Areal}(\Delta) = \alpha_1 + \alpha_2 + \alpha_3 - \pi$$

$$\frac{\alpha_1 + \alpha_2 + \alpha_3 - \pi}{\text{Areal}(\Delta)} = 1$$

Thomas Harriot

Check Harriot's formel for en geodætisk trekant, som har toppunkt og vinkel $\pi/2$ i Nordpolen og som har de to andre hjørner på ækvator.

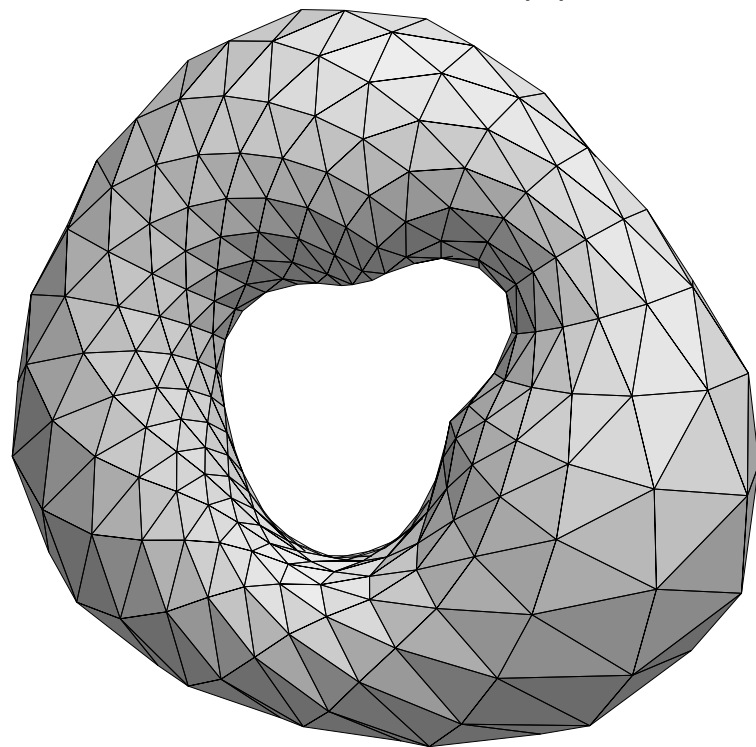
Korollar 4. *En kugleflade med radius 1 har konstant krumning 1.*

Korollar 5. *En kugleflade med radius R har konstant krumning $1/R^2$.*

Korollar 6. *Krumningen af en flade er invariant under enhver isometri af fladen. (Isometrier bevarer nemlig (pr. definition) både vinkler og arealer.)*

Korollar 7. *Krumningen af en plan er 0, så krumningen af en cylinder er derfor også 0.*

Triangulering af en flade



Her er F trekant-flader, E kanter, og V hjørnepunkter.

Triangulering af en flade

Vi antager, at alle trekanterne er geodætiske, altså med kanter, som er geodætiske kurver på fladen.

Så ved vi fra Harriot's sætning, at

$$\begin{aligned}\sum_{f=1}^{f=F} K \cdot \text{Areal}(\Delta_f) &= \sum_{f=1}^{f=F} (\alpha_1^f + \alpha_2^f + \alpha_3^f - \pi) \\ &= \sum_{f=1}^{f=F} (\alpha_1^f + \alpha_2^f + \alpha_3^f) - F \cdot \pi.\end{aligned}$$

Triangulering af en flade

Men i ethvert hjørnepunkt er summen af vinklerne omkring hjørnet $2 \cdot \pi$, så

$$\sum_{f=1}^{f=F} (\alpha_1^f + \alpha_2^f + \alpha_3^f) = 2\pi \cdot V \quad ,$$

$$\sum_{f=1}^{f=F} K \cdot \text{Areal}(\Delta_f) = 2\pi \cdot V - F \cdot \pi \quad .$$

Triangulering af en flade

Derudover har vi, at

$$2 \cdot E = 3 \cdot F$$

så

$$-2 \cdot E = -3 \cdot F$$

og

$$-F = 2 \cdot F - 3 \cdot F = 2 \cdot F - 2 \cdot E$$

hvoraf

$$\sum_{f=1}^{f=F} K \cdot \text{Areal}(\Delta_f) = 2\pi(F - E + V) \quad .$$

Gauss-Bonnet's sætning

Når trekkanterne underopdeles (nødvendigt for at K er veldefineret i grænsen) fås sætningen:

$$\int_S K \, dA = 2 \cdot \pi \cdot (F - E + V)$$

med *de samme værdier* for F , E og V som før, fordi underopdelingerne ikke ændrer højresiden!

Korollar 8. *Den vægtede alternerende sum $2 \cdot \pi \cdot (F - E + V)$ for en hvilken som helst triangulering af fladen S er en topologisk invariant, og det er totalkrumningen $\int_S K \, dA$ derfor også.*

Gauss-Bonnet's sætning

Vedrørende påstanden om underopdelingen: En standard forfining giver:

$$\begin{aligned} F^* &= 6 \cdot F \\ E^* &= 2 \cdot E + 6 \cdot F \\ V^* &= V + E + F \quad , \end{aligned} \tag{1}$$

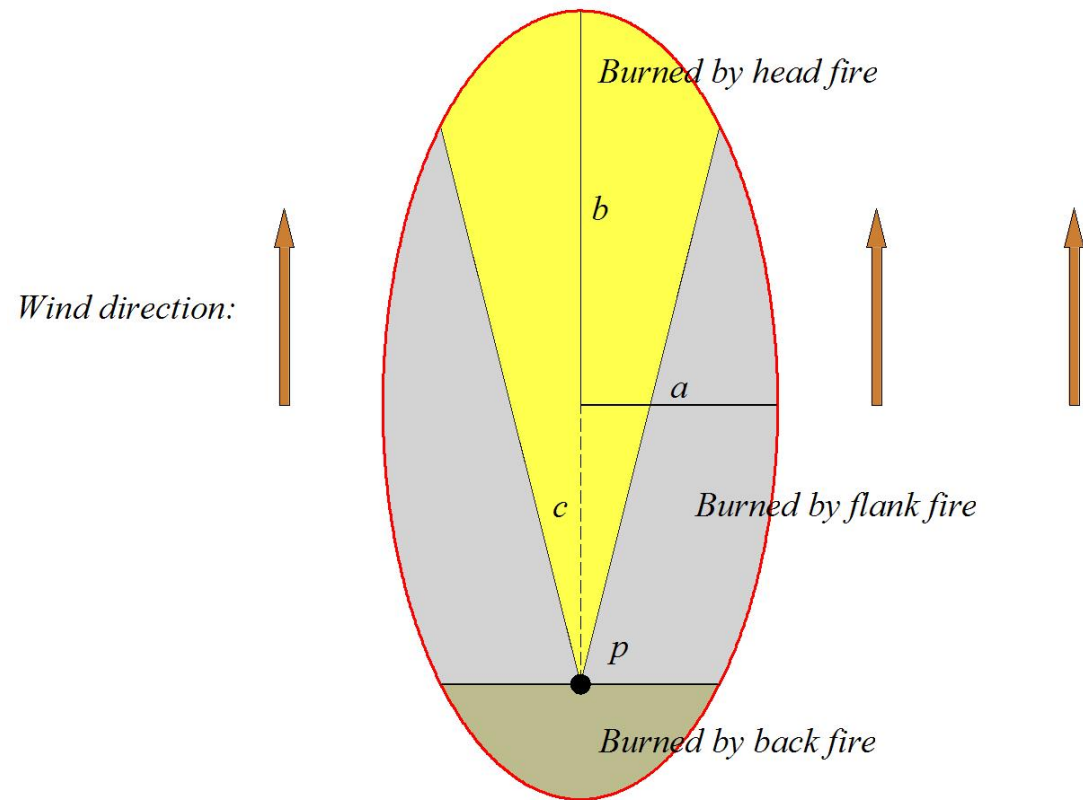
sådan at

$$\begin{aligned} F^* - E^* + V^* &= 6 \cdot F - 2 \cdot E - 6 \cdot F + V + E + F \\ &= F - E + V \quad . \end{aligned} \tag{2}$$

Real world analysis

Wildfire modelling

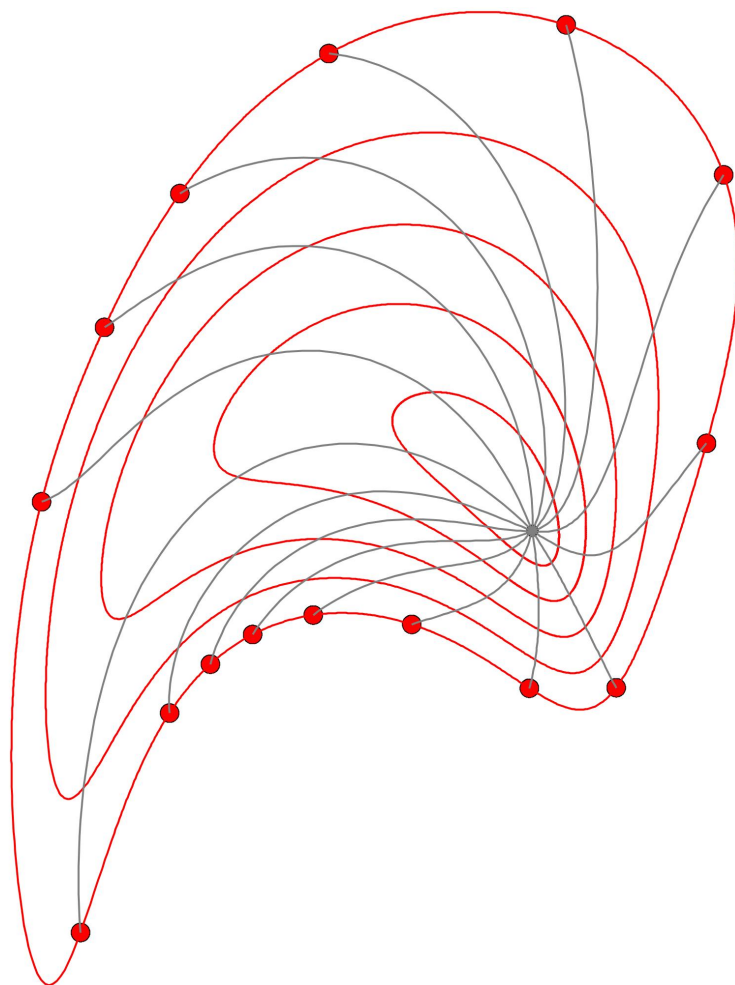
Firefighter's ellipse



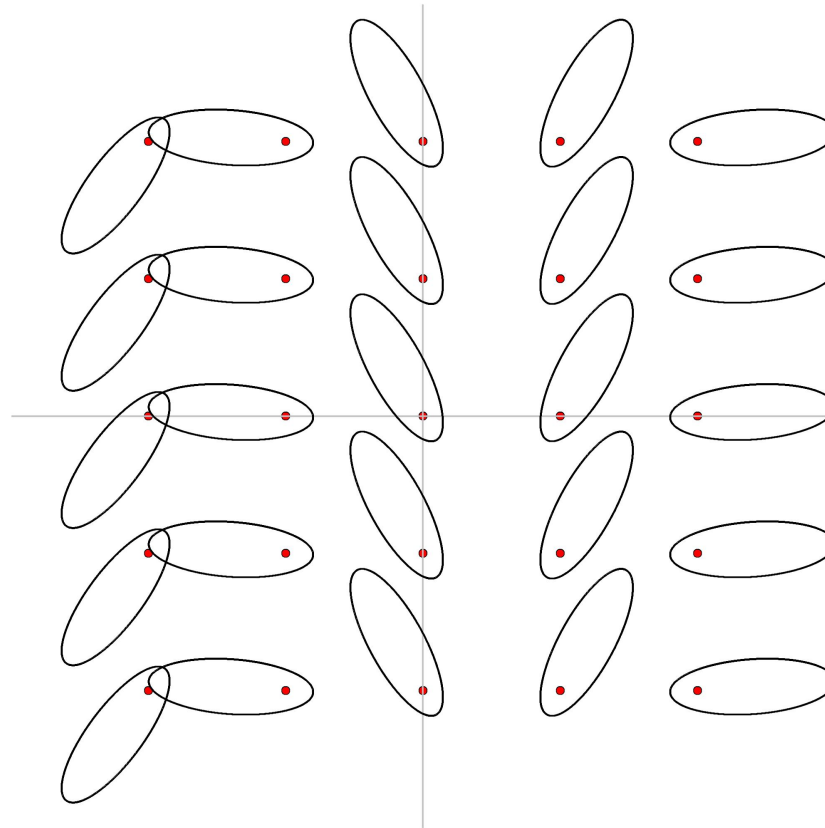
Motivation



Finsler geodesic circle frontals?



Yes, from an indicatrix field



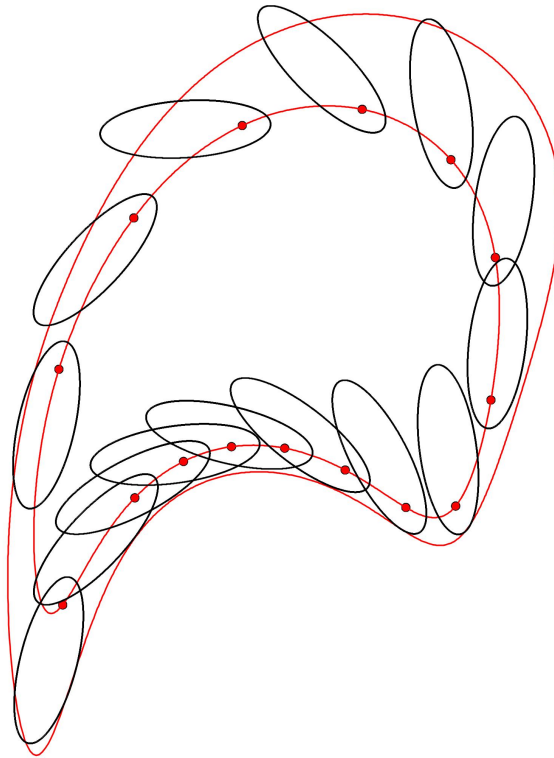
Huygens' principle

We ignite the fire at a given point and integrate the indicatrix data along each front to **envelope the next front via Huyghens' principle**:

A later front is obtained by **igniting a fire at each point** on the given front, let them **burn for a small time**, and then **take the outer envelope** of the ensuing fire domains.

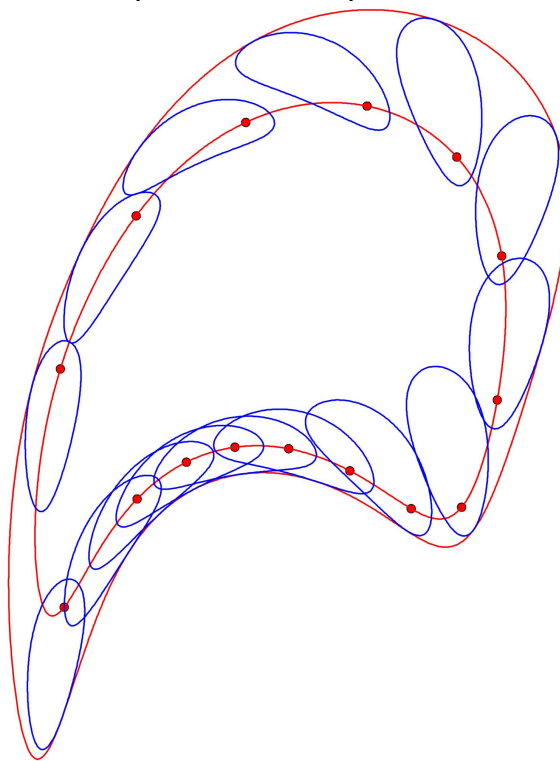
Huyghens' principle

The indicatrices cannot be used for this directly, since they are objects in the tangent bundle:

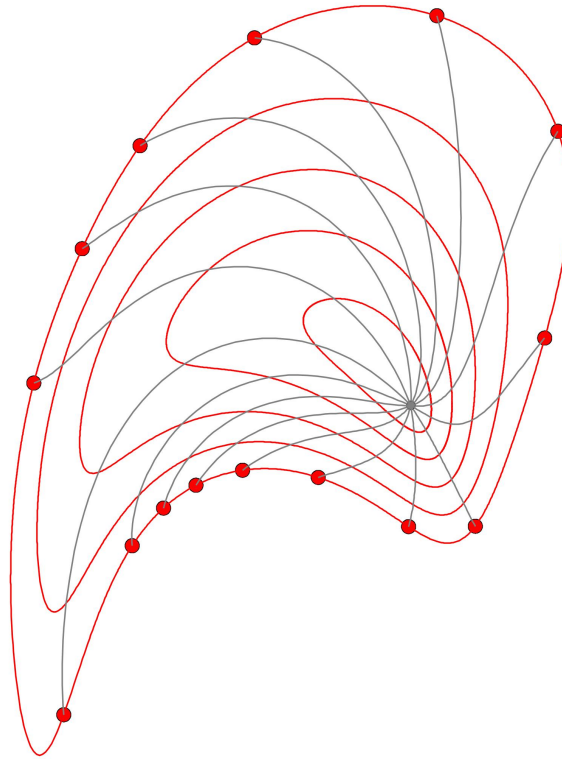


Huyghens' principle

The local indicatrix field must itself be geodesically integrated from the respective points of ignition:

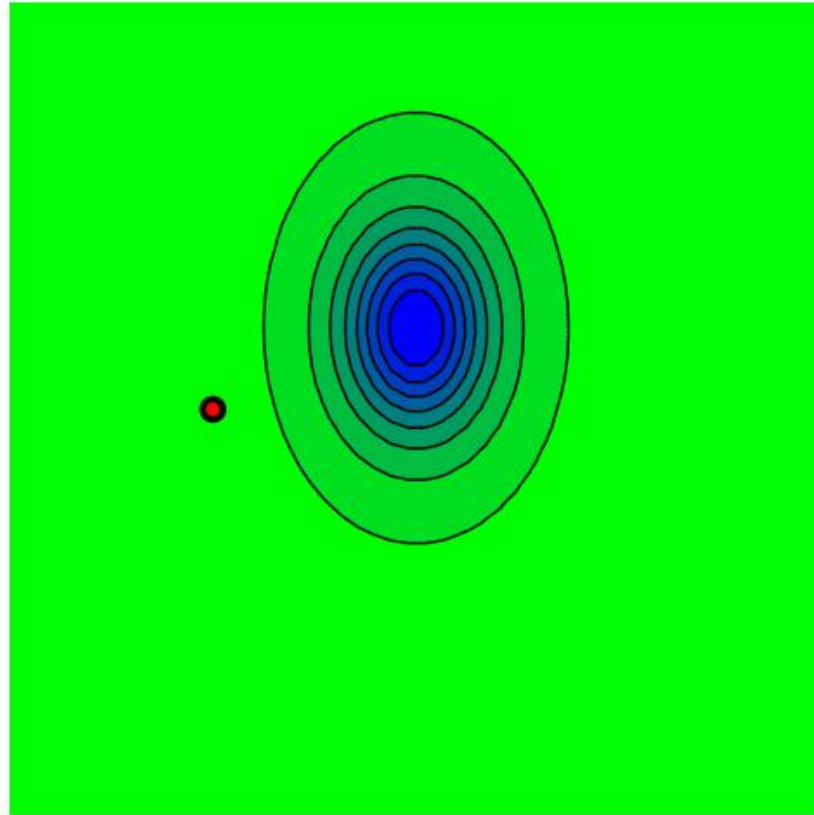


Finsler geodesic circle frontals!

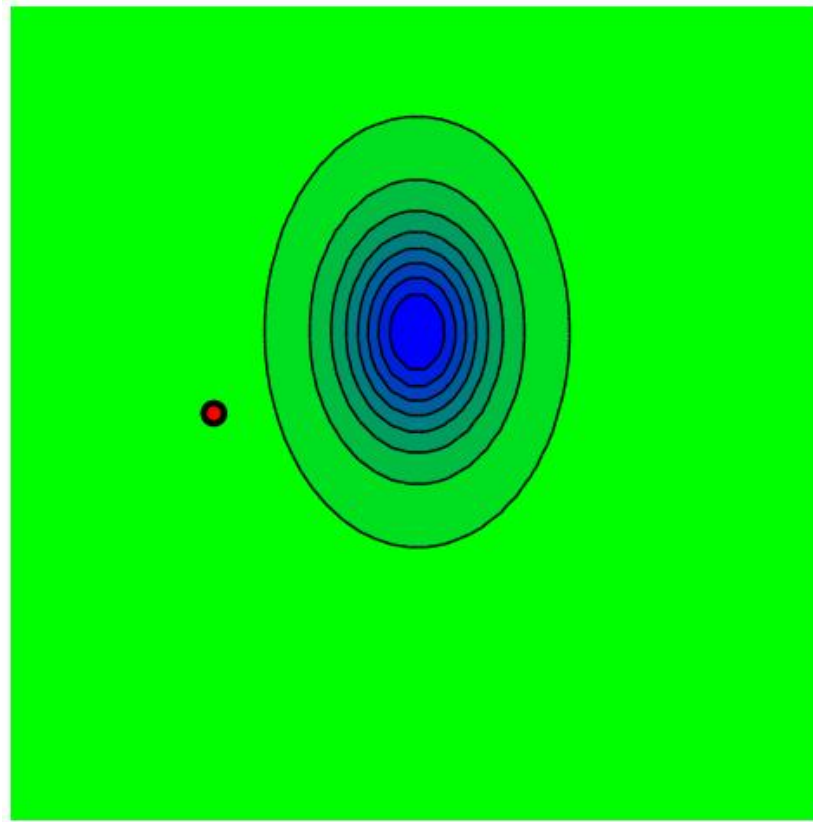


When the background Finsler metric is itself time-dependent:
See M. A. Javaloyes, E. Pendás-Recondo, and M. Sanchez:
A general model for wildfire propagation with wind and slope,
(2021)

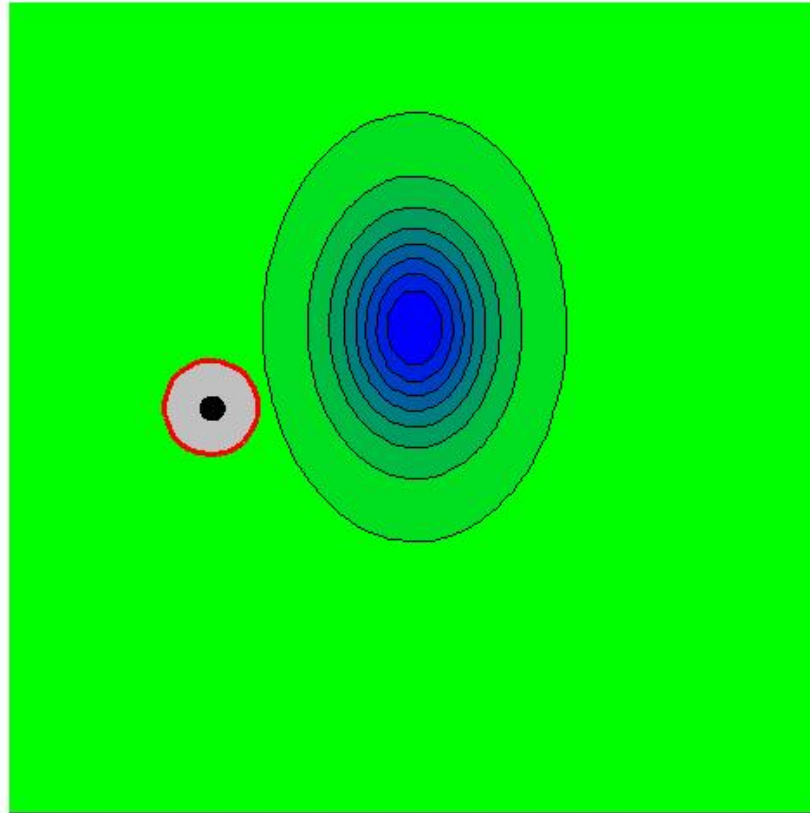
First experiments;
Indicating cut loci; No wind



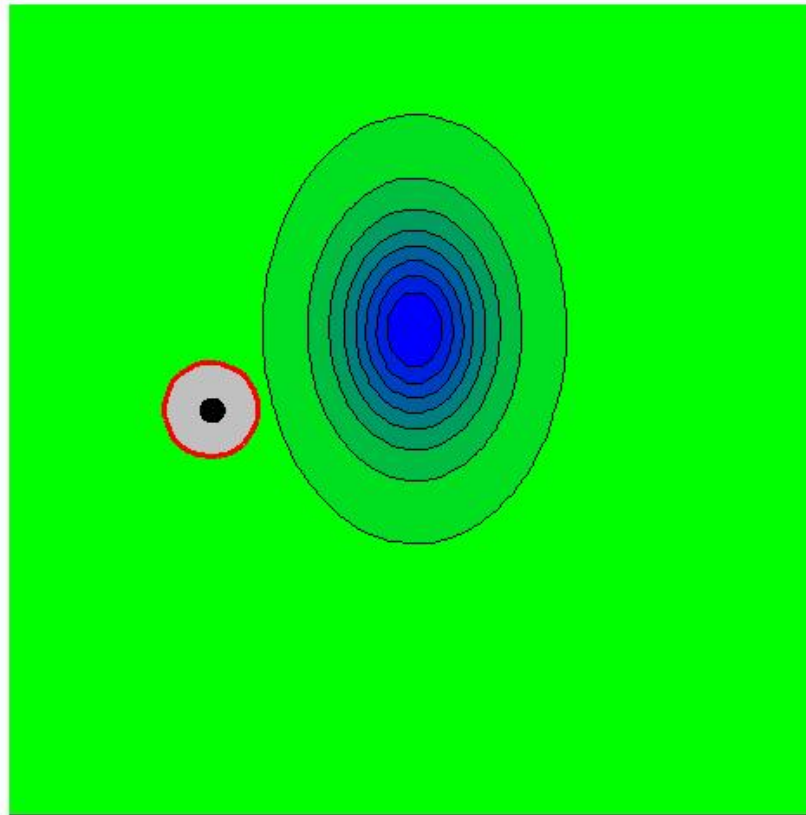
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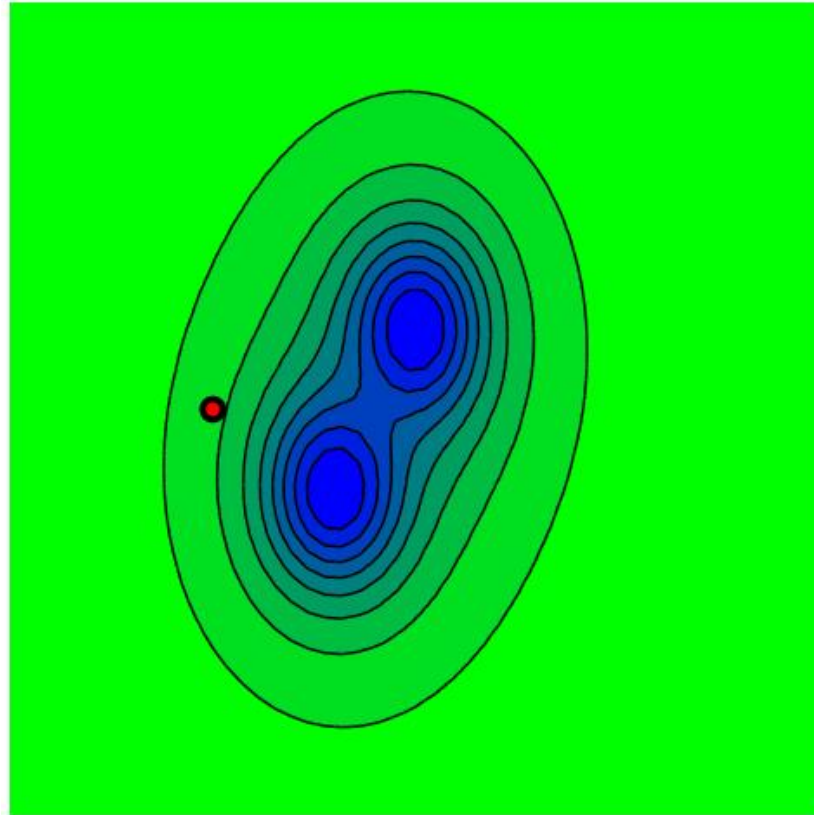
First experiments;
Showing cut loci; No wind



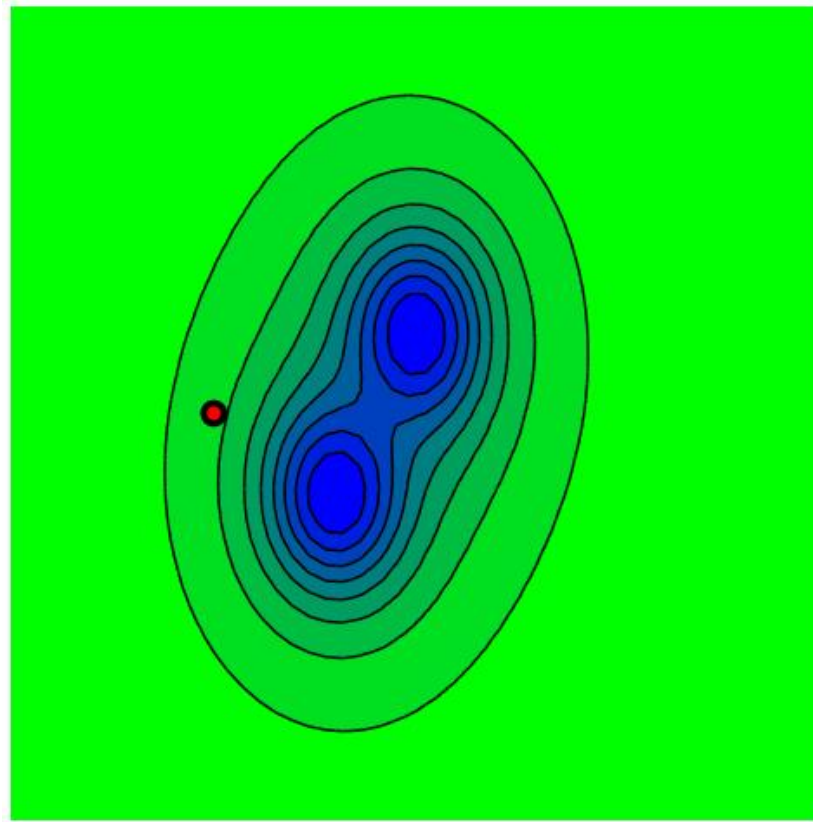
First experiments;
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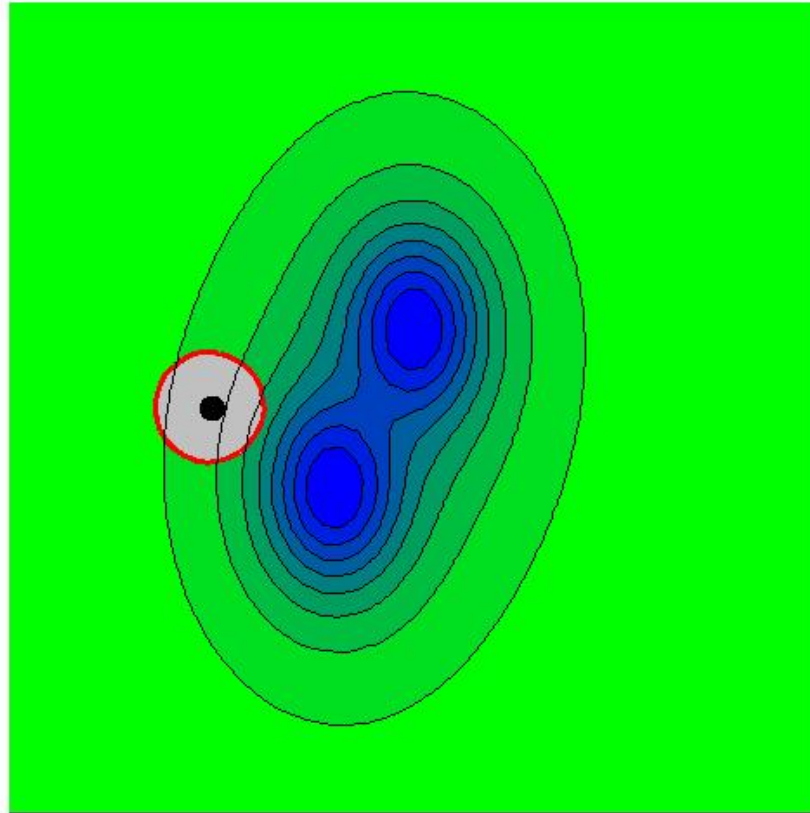
First experiments;
Indicating cut loci; No wind



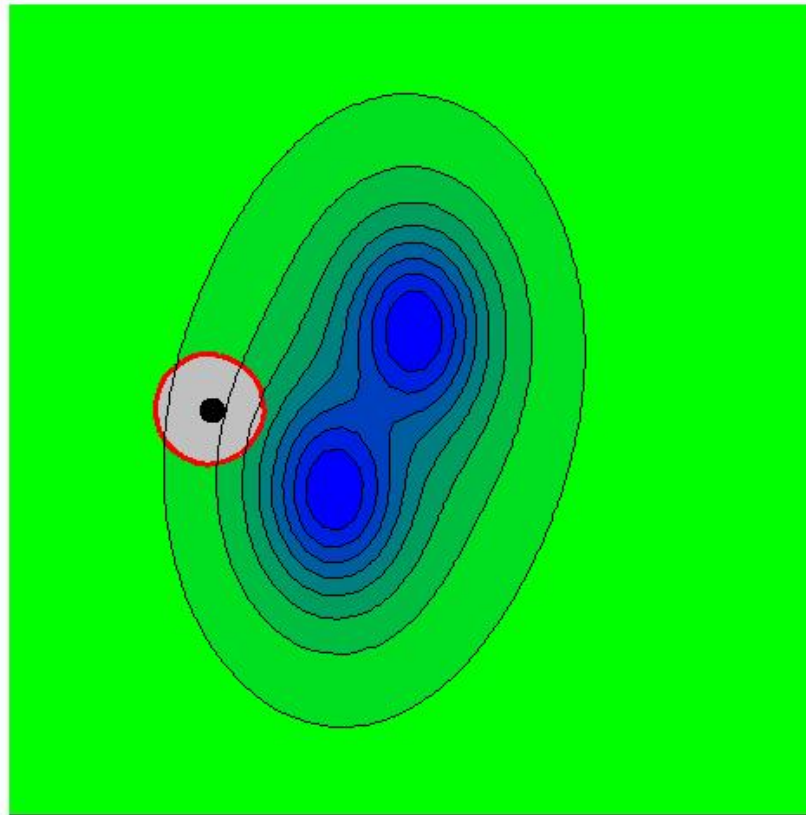
First experiments;
Indicating cut loci; No wind



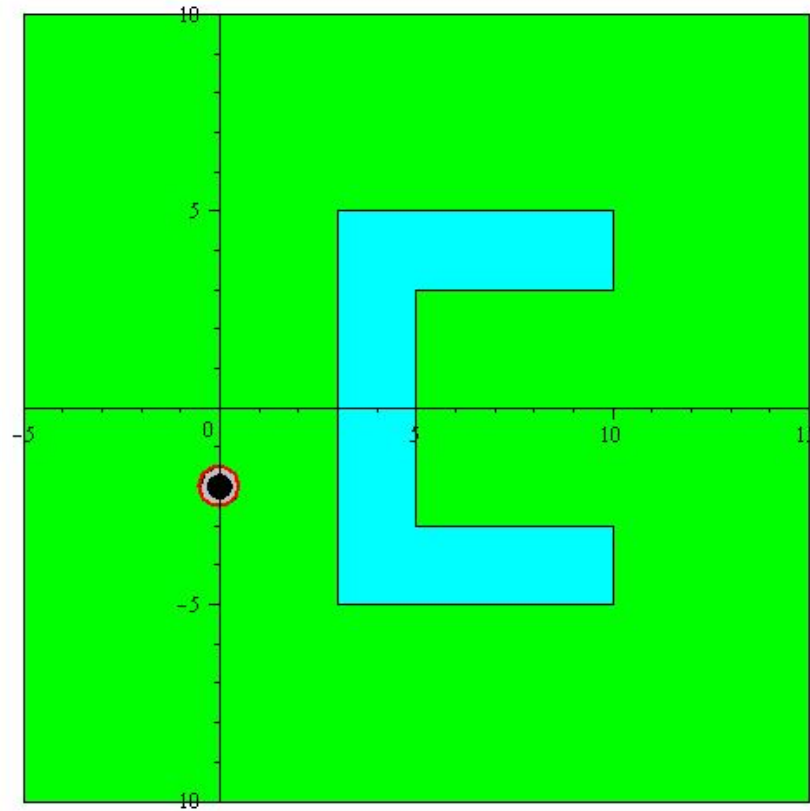
First experiments;
Showing cut loci; No wind



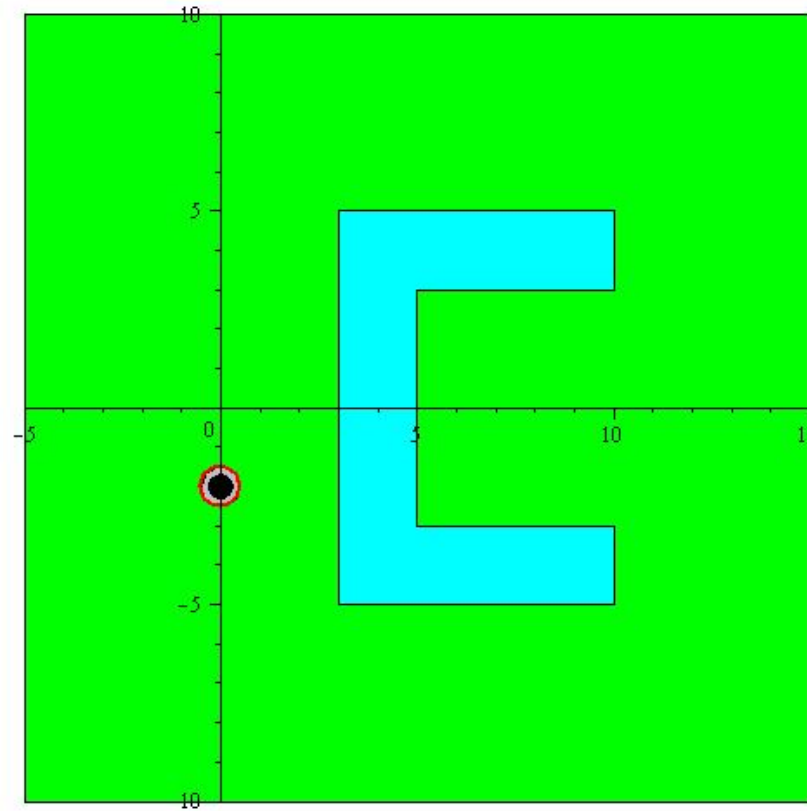
First experiments;
Showing cut loci; No wind



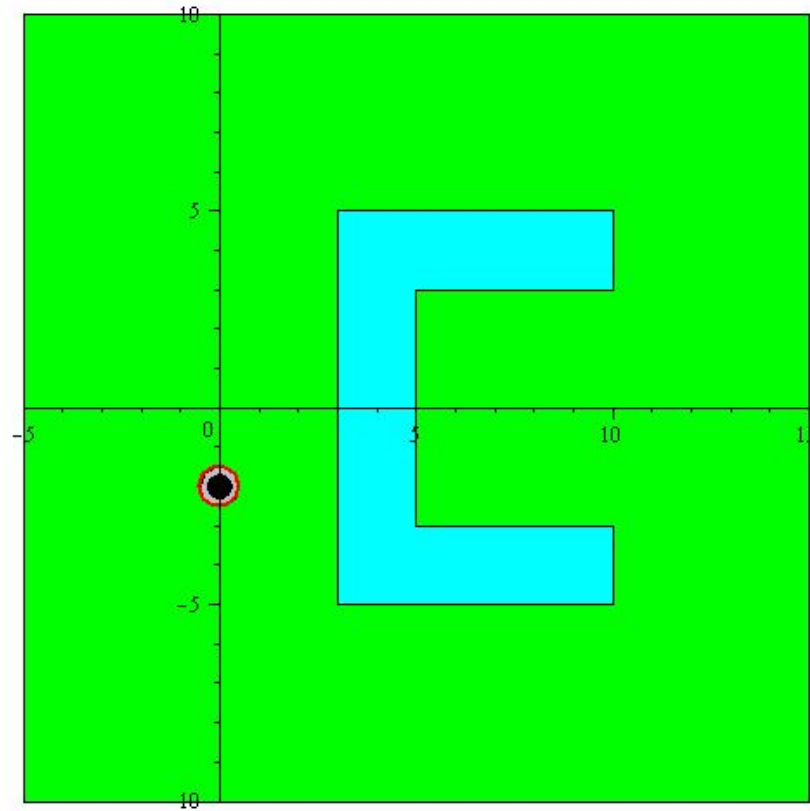
First experiments;
Showing cut locus; No wind



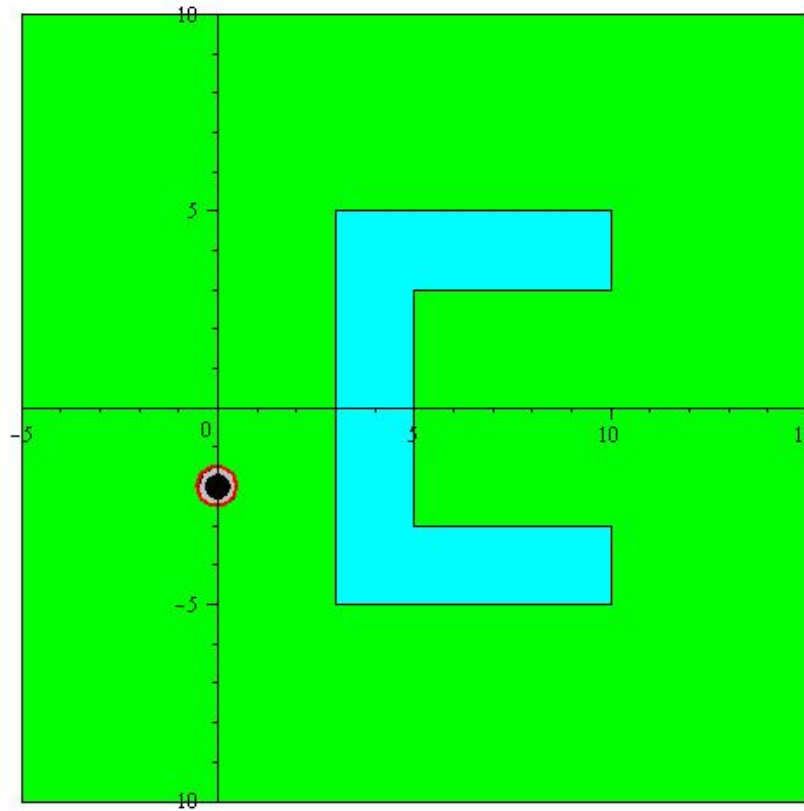
First experiments;
Showing cut locus; No wind



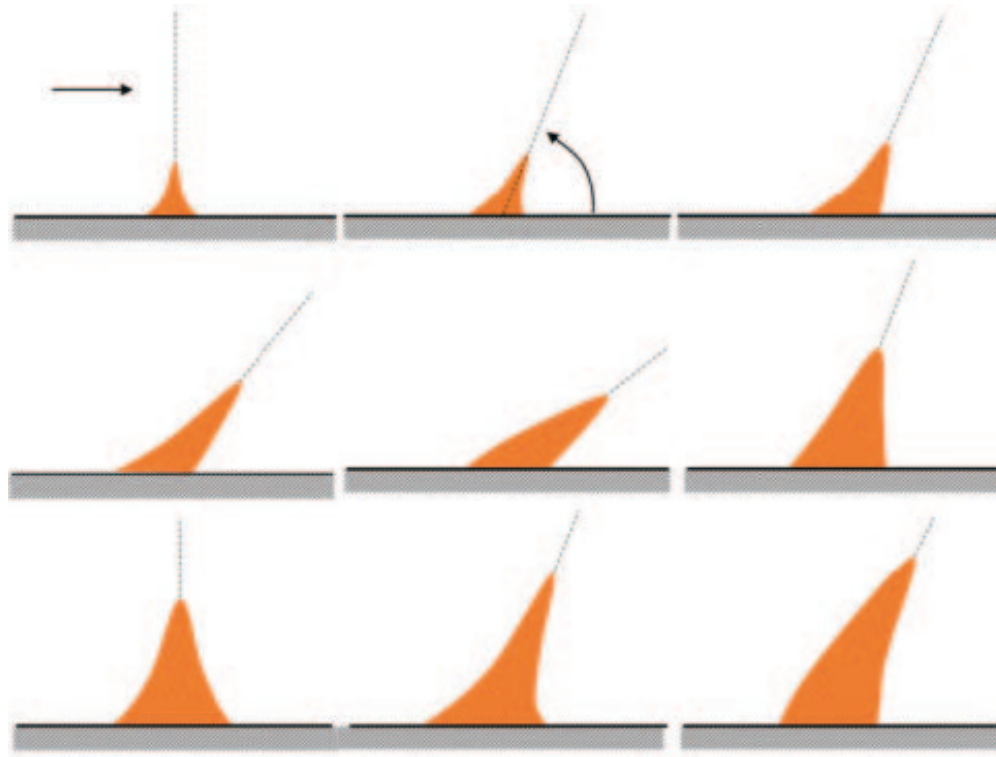
First experiments;
Showing cut locus; With wind



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Showing cut locus; With wind

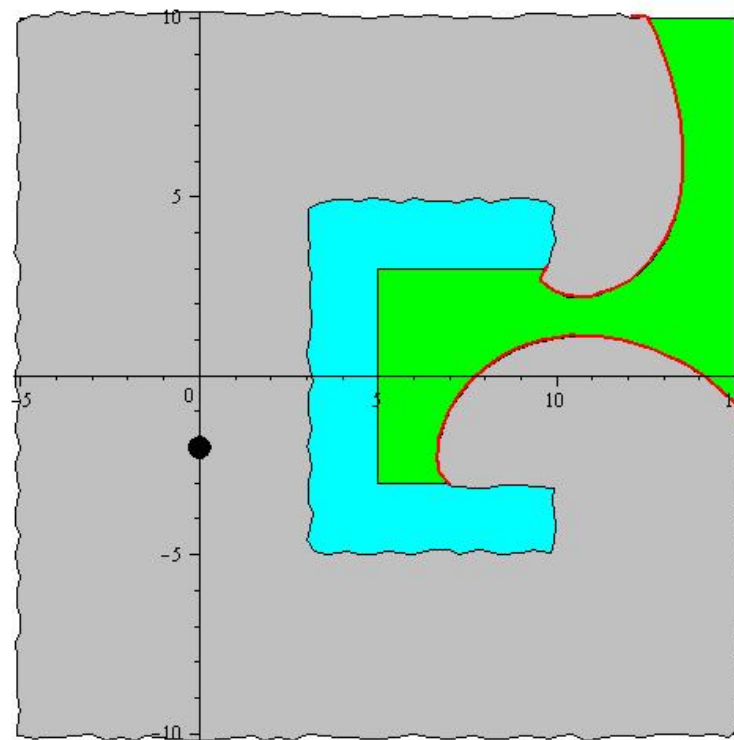


How to cope with oscillating frontals?



Domingos Xavier et al., 2021

How to cope with
physically interacting frontals
before the formation of cut loci?

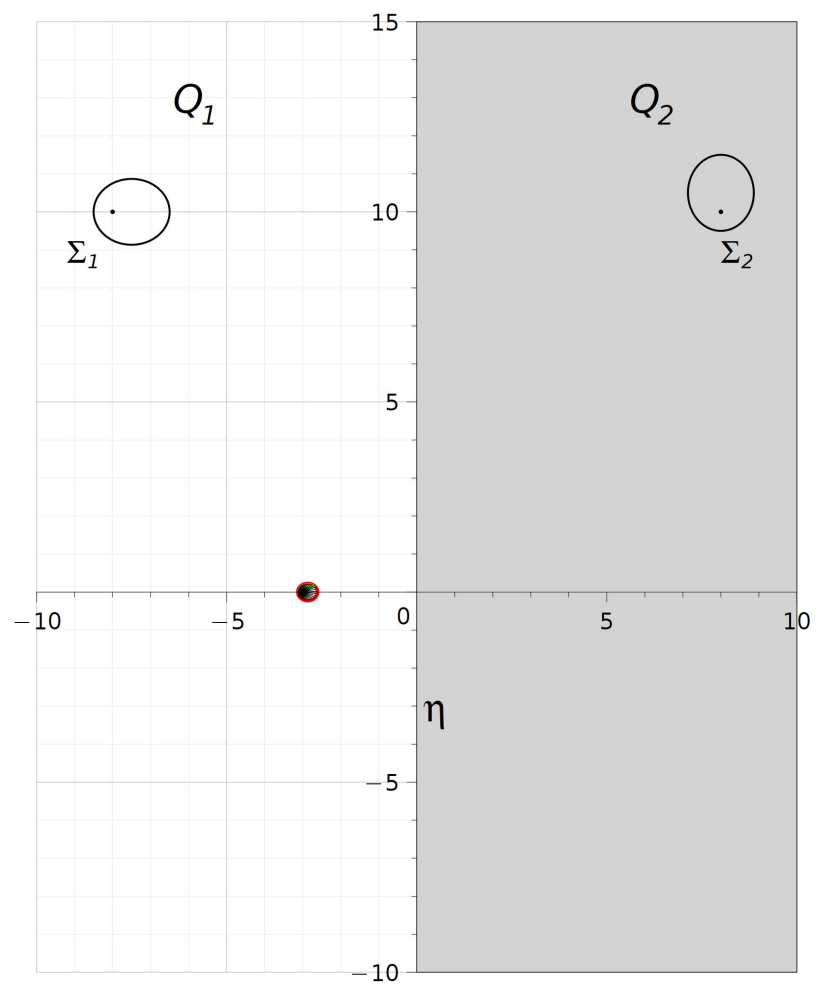


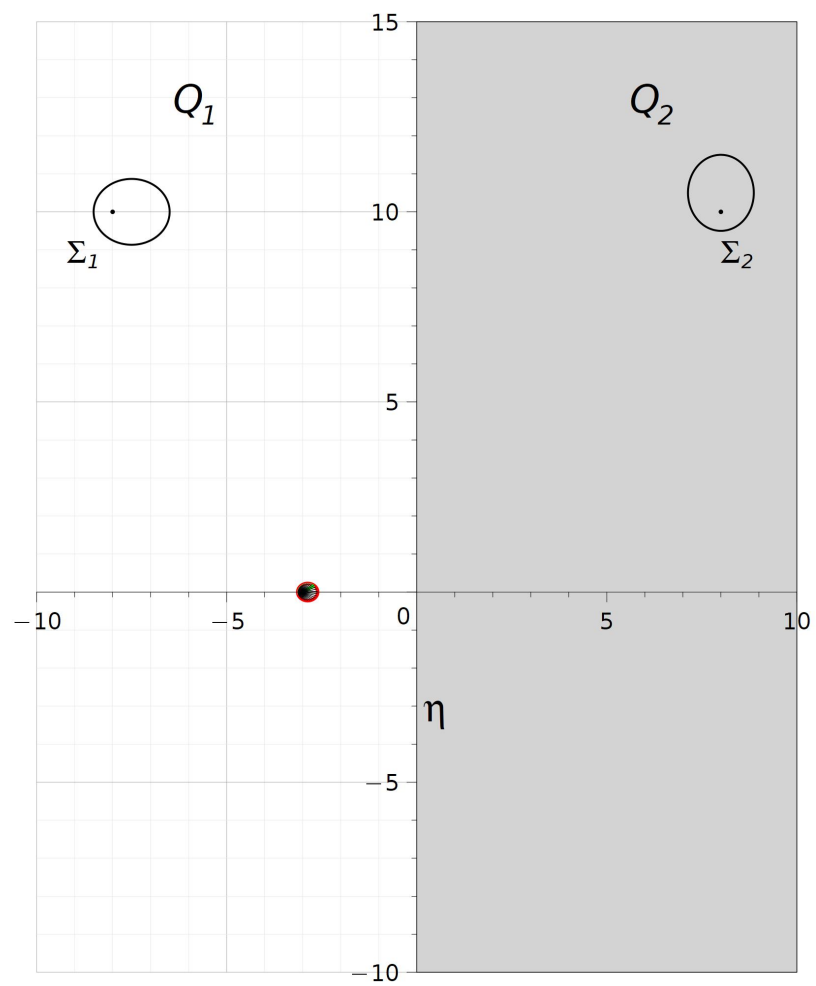
Finsler geometri

Joint with Enrique Pendás-Recondo:

Snell's law revisited and generalized via Finsler geometry,

International Journal of Geometric Methods in Modern Physics, 2023:







Tak for opmærksomheden!