DIFFERENTIAL GEOMETRY

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Contents

Chapter 0

Background Material

This chapter is a brief summary of some of the concepts from point set topology and geometry of curves and surfaces that are, to a certain degree, assumed knowledge in the course.

0.1 Point Set Topology

The idea of point set topology has its origins in the analysis of limits of real numbers, and especially continuity of functions. See [31] for a general introduction to topological spaces and – among them – metric spaces, which will be of most interest to us in the following chapters. The basic object is an *open set*. In $\mathbb{R}$, an open set is defined to be any union of open intervals (including the empty set). In $\mathbb{R}^n$ with its standard Euclidean distance function $d(x,y) = |x-y|$, open intervals are replaced by open balls: $B_\varepsilon(x_0) := \{x \in \mathbb{R}^n \mid |x-x_0| < \varepsilon\}$, is the open Euclidean ball of radius ε centered at x_0.
Recall that a set $U \subset \mathbb{R}$ is called open if, for any element $x_0 \in U$, there exists some number $\varepsilon > 0$ such that the open interval $B_\varepsilon(x_0)$ is also contained in U. Notice that, with this definition, an arbitrary *union* of open sets is obviously open. However an arbitrary *intersection* might not be: the single point set $\{0\}$ is a countably infinite intersection, $\cap_{n=1}^{\infty}(-1/n, 1/n)$, of open intervals, but it is not itself open.

Definition 0.1 A *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X, called open sets, such that

1. The empty set $\emptyset$ and X itself are both elements of $\mathcal{T}$.
2. Any union of elements of $\mathcal{T}$ is also an element of $\mathcal{T}$.
3. The intersection of any *finite* subcollection of $\mathcal{T}$ is also in $\mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X.

Basic examples are:

1. The *standard topology* on $\mathbb{R}$ or $\mathbb{R}^n$, defined with the ε-neighbourhood notion above.
2. The *trivial topology* on any set X, given by $\mathcal{T} = \{X, \emptyset\}$.
3. Or the other extreme, the *discrete topology*, where $\mathcal{T}$ consists of *all* subsets of X.

0.1.1 Basis for a topology

In general, it might be difficult to describe explicitly all elements of a topology. Instead, one talks about a *basis* for $\mathcal{T}$, that is, some sub-collection that generates $\mathcal{T}$ by taking arbitrary unions and finite intersections.

Definition 0.2 A *basis* for a topology on a set X is a collection $\mathcal{B}$ of subsets of X such that

1. For each $x \in X$ there is at least one set $B \in \mathcal{B}$ such that $x \in B$.
2. For any pair of sets B_1, B_2 in $\mathcal{B}$, if $x \in B_1 \cap B_2$ then there exists a third basis element B_3 such that $x \in B_3 \subset B_1 \cap B_2$.

The *toplogy $\mathcal{T}$ generated by a basis $\mathcal{B}$* is defined as follows: a set U belongs to $\mathcal{T}$ if, for each element $x \in U$, there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

EXERCISE 0.3

Find a basis for the standard topology on $\mathbb{R}$.

EXERCISE 0.4

Check that the "topology generated by a basis $\mathcal{B}$" is, in fact, a topology.

EXERCISE 0.5

Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Show that $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

0.1.2 Second Countability

In manifold theory, the proof of some useful tools requires a certain degree of "finiteness" in the topology, namely the second countability assumption. Recall that a set X is countable if its elements can be enumerated, $x_1, x_2, \ldots$, or, more precisely, there is a bijection between X and the natural numbers $\mathbb{N} := \{1, 2, 3, \ldots\}$.

Definition 0.6 A topological space X is said to be *second countable* if there exists a countable basis for its topology.

EXERCISE 0.7

Show that $\mathbb{R}$, with the standard topology, is second countable.

EXERCISE 0.8

Show that $\mathbb{R}$, with the *discrete* topology is not second countable.

Any subset A of a topological space X can always be given the *induced topology*, or *subspace topology* defined by declaring a set $V \subset A$ to be open if $V = U \cap A$, where U is some open subset of X. In symbols: $\mathcal{T}_A := \{U \cap A \mid U \in \mathcal{T}_X\}$. The subset A with this topology is called a *(topological) subspace* of X.

EXERCISE 0.9

Prove that any subspace of a second countable topological space is also second countable.

0.1.3 Hausdorff topological spaces

One other reasonable condition on a topological space is the following:

Definition 0.10 A topological space X satisfies the Hausdorff condition if for any two *distinct* points x and y in X there exist *disjoint* open sets U and V in X such that $x \in U$ and $y \in V$.

This property is already immediately satisfied by the standard topology in $\mathbb{R}^n$ and by the metric ball topology in any metric space. Since we will eventually model Riemannian manifolds via chart domains in $\mathbb{R}^n$, the Hausdorff condition on the manifold topology is but a natural condition to impose.

EXERCISE 0.11

Prove that the standard topology in $\mathbb{R}^n$ satisfies the Hausdorff condition.

The Hausdorff condition does not follow from second countability:

EXERCISE 0.12

Construct a set with a topology that is second countable but not Hausdorff. Hint: You may want to consider something like this set: $X = \mathbb{R} \cup \{*\}$, where $*$ is an element not in $\mathbb{R}$. And then declare a set U open if and only if $U \cap \mathbb{R}$ is open and, moreover, if $* \in U$, then $(U \cap \mathbb{R}) \cup \{0\}$ is a neighbourhood of 0 (i.e. an open set containing 0) in $\mathbb{R}$.

0.1.4 Continuity

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be continuous at a point x if, for any sequence x_n converging to x, the sequence $f(x_n)$ converges to $f(x)$. For a general topological space, the definition is given in terms of open sets:

Definition 0.13 A map $f : X \to Y$ between topological spaces is said to be *continuous* if for any open subset V of Y, the pre-image $f^{-1}(V)$ is an open subset of X.

Even for metric spaces like $\mathbb{R}$, this definition is often much easier to work with: consider the simple proof of the following fact:

Lemma 0.14 If $g : Y \to Z$ and $f : X \to Y$ are both continuous maps, then the composition $g \circ f : X \to Z$ is also continuous.

Proof. Let $V \subset Z$ be open. Then $g^{-1}(V)$ is open by continuity of g, and hence $f^{-1}(g^{-1}(V))$, which is the same as $(g \circ f)^{-1}(V)$, is open in X. □

EXERCISE 0.15

Using Definition 0.13, prove that the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is continuous. (Hint: one way to start is to observe that if $V \subset \mathbb{R}$ is an open set then V can be written as a union $\cup_\alpha (a_\alpha, b_\alpha)$ of open intervals.)

EXERCISE 0.16

Using again Definition 0.13, show that the map $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{for } x \leq 0 \\ x+1 & \text{for } x > 0 \end{cases} \tag{0.1}$$

is NOT a continuous map.

EXERCISE 0.17

In the vein of Definition 0.13, discuss the popular saying: "If you can draw the graph of a function $y = f(x)$ without lifting the pen from the paper, then f is continuous.

EXERCISE 0.18

Again in the vein of Definition 0.13, show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if the following holds true for all $x_0 \in \mathbb{R}$: There is an open interval I_{x_0} of $\mathbb{R}$ which contains x_0 and

$$f(x) = f(x_0) + \varepsilon(x - x_0) \quad \text{for all } x \in I_{x_0} \quad , \tag{0.2}$$

where ε is an epsilon function in the defining sense that $\varepsilon(t) \to 0$ for $t \to 0$.

Definition 0.19 A bijective map f from a topological space X into a topological space Y is called a homeomorphism if both f and f^{-1} are continuous maps.

EXERCISE 0.20

Show that the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is a homeomorphism.

0.2 Metric spaces

Some spaces admit a distance function:

Definition 0.21 Let X denote a non-empty set and suppose that the map $d : X \times X \mapsto \mathbb{R}_+ \cup \{0\}$ satisfies the following conditions:

1. for all x and y in X, $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$.
2. $d(x,y) = d(y,x)$ for all x and y in X.
3. $d(x,z) \leq d(x,y) + d(y,z)$ for all x, y, and z in X.

Then (X,d) is called a metric space with distance function d.

EXERCISE 0.22

Show that the following functions on $\mathbb{R}^2 \times \mathbb{R}^2$ are distance functions on $\mathbb{R}^2$, where we use coordinates $x = (x^1, x^2)$ and $y = (y^1, y^2)$ for points x and y:

1. $d_1(x,y) = |x^1 - y^1| + |x^2 - y^2|$
2. $d_2(x,y) = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2}$
3. $d_3(x,y) = \max\{|x^1 - y^1|, |x^2 - y^2|\}$.

A metric space becomes a topological space via the following definition of open sets.

Definition 0.23 Let (X,d) be a metric space and $U \subseteq X$. We define U to be open in X if for every $x \in U$ there exists $\varepsilon_x > 0$ so that the distance ball $B_{\varepsilon_x}(x)$ is contained in U. Here, the distance ball of radius $r > 0$ centered at $p \in X$ is defined by the distance function as follows:

$$B_r(p) = \{x \in X \,|\, d(x,p) < r\}. \tag{0.3}$$

The distance balls $B_r(p)$ in (X,d) are thence themselves open.

EXERCISE 0.24

Show that the distance balls of (X,d) constitute a basis for the topology defined by the open sets in definition 0.23.

0.3 Curves in Euclidean spaces of 2, 3, and n dimensions

We will use the curvature of surfaces in 3-space to develop an intuition for curvature in more general geometries. In turn, the curvature of surfaces can be defined and visualized using the notion of curvature of plane curves.

Definition 0.25 A *parameterized curve* in $\mathbb{R}^n$ is a differentiable map $\gamma : I \to \mathbb{R}^n$, from an interval $I \subset \mathbb{R}$, such that $\gamma'(t) \neq 0$ for all $t \in I$.

Note: Unless otherwise stated, the term *differentiable* will always mean differentiable *infinitely* many times. One can take the interval I to be either open or closed, as needed. If closed, then the derivatives are defined on the interior and should be continuously extendable to the end points.

Definition 0.26 Given a parameterized curve $\gamma : I_1 \to \mathbb{R}^n$, where I_1 is an open interval, a *change of coordinates* is a differentiable map: $\phi : I_1 \to I_2$ to another interval $I_2 \subset \mathbb{R}$, such that

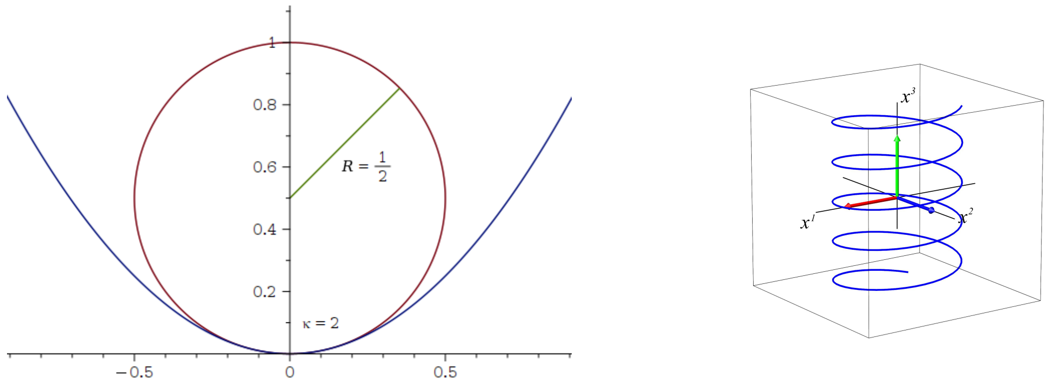


Figure 1: Left: two plane curves with the same curvature at the point $(0,0)$. Right: a curve in $\mathbb{R}^3$.

1. ϕ is *bijective* (i.e. injective and surjective),
2. $\phi'(t) \neq 0$ for all $t \in I_1$ (equivalently: ϕ^{-1} is also differentiable).

The map $\tilde{\gamma} : I_2 \to \mathbb{R}^n$ given by

$$\tilde{\gamma}(w) = \gamma(\phi^{-1}(w)),$$

is called a *re-parameterization* of γ with coordinate w.

Example 0.27

The plane curve $\gamma : (0,2\pi) \to \mathbb{R}^2$ given by $\gamma(t) = 3(\cos(t), \sin(t))$ can be reparameterized as:

$$\tilde{\gamma}(s) = 3(\cos(s/3), \sin(s/3)), \qquad s \in (0,6\pi).$$

Here the coordinate change is $\phi : (0,2\pi) \to (0,6\pi)$ given by

$$\phi(t) = 3t,$$

with inverse $\phi^{-1}(s) = s/3$. Let's check that ϕ satisfies conditions 1 and 2 of Definition 0.26:

1. $\phi : (0,2\pi) \to (0,6\pi)$ is *surjective*, since every element $s \in (0,6\pi)$ can be written $s = 3t$ where $t \in (0,2\pi)$, and ϕ is *injective* because whenever $\phi(t_1) = \phi(t_2)$ we have $t_1 = t_2$.
2. We have $\phi'(t) = 3$ and this is non-zero for all t. (Equivalently, the formula for the inverse, $\phi^{-1}(s) = s/3$ is differentiable).

Notice that the re-parameterized curve $\tilde{\gamma}$ in the example above has the property that the norm of the vector $\tilde{\gamma}'(s) = (\cos(s/3), \sin(s/3))$ is one: $||\tilde{\gamma}'(s)|| = \sqrt{\cos^2(s/3) + \sin^2(s/3)} = 1$.

Definition 0.28 Given a parameterized curve $\gamma : I \to \mathbb{R}^n$, the *tangent vector field* to γ is the map $\gamma' : I \to \mathbb{R}^n$, given by $\gamma'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$. The *speed* is the map: $||\gamma'|| : I \to \mathbb{R}$ given at each t by the norm of $\gamma'(t)$. The *length* of the curve is the integral of the speed over the whole interval:

$$L = \int_I ||\gamma'(t)||\mathrm{d}t.$$

Note that the *speed* of a curve depends on the parameterization. For example, for the curve in Example 0.27, we have

$$||\gamma'(t)|| = ||3(-\sin(t), \cos(t))|| = 3,$$

but

$$||\tilde{\gamma}'(s)|| = ||(-\sin(s/3), \cos(s/3))|| = 1,$$

The *length* however does not depend on the parameterization: for example with the curve above:

$$\int_I ||\gamma'(t)||\mathrm{d}t = \int_0^{2\pi} 3\mathrm{d}t = 6\pi,$$

and

$$\int_{I_2} ||\gamma'(s)||\mathrm{d}s = \int_0^{6\pi} 1\mathrm{d}s = 6\pi$$

A parameterization $\gamma(s)$ of a curve that has constant unit speed: $||\gamma'(s)|| = 1$ is called an *arc-length (or unit speed) parameterization*. The reason for the name "arc-length" is that, in that case:

$$s_1 - s_0 = \int_{s_0}^{s_1} \mathrm{d}s = \int_{s_0}^{s_1} ||\gamma'(s)||\mathrm{d}s,$$

is the length of the curve between s_0 and s_1. So lengths on the interval I are the same as corresponding arc-lengths on the curve.

EXERCISE 0.29

Prove that any regular curve $\gamma : [a,b] \to \mathbb{R}^n$ can be reparameterized by arc-length. (Hint: Set $\phi(t) = \int_a^t ||\gamma'(u)||\mathrm{d}u$).

Quantities like the speed that change if we reparameterize a curve are not considered geometric. This is because the parameter t is merely a way to *reference* the points on the curve in space. But the points in space are the real geometric objects. We are therefore interested in *geometric* quantities like the length L that do *not* change under reparameterization.

Another important geometric quantity of a plane curve is the *curvature function* $\kappa(t)$ which measures by how much the curve deviates from being a straight line at the point $\gamma(t)$.

Definition 0.30 Let $\gamma : I \to \mathbb{R}^2$ be a plane curve, parameterized by arc-length, i.e., with $||\gamma'(s)|| = 1$ for all $s \in I$. The *curvature function* $\kappa : I \to [0,\infty)$ is given by:

$$\kappa(s) = ||\gamma''(s)||.$$

For a curve $\gamma(t)$ that is not necessarily parameterized by arc-length we can compute the curvature function by the formula:

$$\kappa(t) = \frac{|\det(\gamma'(t), \gamma''(t)|}{||\gamma'(t)||^3}.$$

Differentiating $||\gamma'(s)||^2 = \gamma'(s) \cdot \gamma'(s) = 1$, we have:

$$0 = \gamma''(s) \cdot \gamma'(s) + \gamma'(s) \cdot \gamma''(s) = 2\gamma''(s) \cdot \gamma'(s),$$

so the vector $\gamma''(s)$ is *orthogonal* to the tangent vector $\gamma'(s)$ at each point. At points where $\kappa(s) \neq 0$ we define the *unit normal* n to the curve by:

$$n(s) = \frac{\gamma''(s)}{\kappa(s)}.$$

Note that the *unit tangent* $\mathbf{t}(s) = \gamma'(s)$ and unit normal $\mathbf{n}(s) = \gamma''(s)/\kappa(s)$ are well defined up to a change of sign. This is because there are two choices of arc-length parameterization, depending on which direction you run along the curve. Such a choice is called an *orientation* for the curve.

EXERCISE 0.31

Compute the curvature function κ for each of the following curves:

1. $\gamma_1(t) = R(\cos(t), \sin(t))$ where R is a positive constant,
2. $\gamma_2(t) = (t, t^2), \quad t \in \mathbb{R}$,
3. $\gamma_3(t) = (t, t^3), \quad t \in \mathbb{R}$.

0.4 The Frenet-Serret data for curves in $\mathbb{R}^3$

The local shape of a space curve is encoded into the following theorem/definition which – among other things – extends the previous notion of curvature to curves in one higher dimension. One of the purposes of these notes is to extend the relations below even further, namely to curves in non-Euclidean manifolds. This is the topic of Chapter 5.

Theorem 0.32 A regular space curve $p(t)$ with $p'(t) \neq 0$ for all t has curvature $\kappa(t)$, torsion $\tau(t)$ and an associated orthonormal basis (the so-called Frenet-Serret vectors) at each point $p(t)$ defined and found via the following expressions, where we have used the notation $v(t) = \|p'(t)\| > 0$:

$$\kappa(t) = \frac{\|p'(t) \times p''(t)\|}{v^3(t)}$$
$$\tau(t) = \frac{(p'(t) \times p''(t)) \cdot p'''(t)}{\|p'(t) \times p''(t)\|^2} \quad \text{when } \kappa(t) > 0 \tag{0.4}$$

$$e(t) = \frac{p'(t)}{v(t)}$$
$$g(t) = \frac{p'(t) \times p''(t)}{\|p'(t) \times p''(t)\|} \quad \text{when } \kappa(t) > 0 \tag{0.5}$$
$$f(t) = g(t) \times e(t) \quad \text{when } \kappa(t) > 0 \quad .$$

In these expressions, $e(t)$ is the unit tangent vector to the curve, $f(t)$ is that particular unit normal vector to the curve which points in the curvature direction, i.e. it corresponds to $n(t)$ for the planar curves. The third unit vector $f(t)$ is called the bi-normal vector – it is orthogonal to the so-called osculating plane for the curve, which spanned by $e(t)$ and $f(t)$.

Theorem 0.33 The Frenet-Serret vector functions satisfy the following ODE system:

$$\begin{aligned} e'(t) &= \quad v(t)\kappa(t)f(t) \\ f'(t) &= -v(t)\kappa(t)e(t) + v(t)\tau(t)g(t) \\ g'(t) &= \quad -v(t)\tau(t)f(t) \quad . \end{aligned} \tag{0.6}$$

The ODE system (5.27) gives rise to a natural *inverse problem*: How to reconstruct the curve (or at least just $e(t)$ from which an isometric version of the curve then follows by integration) from knowledge about $v(t)$, $\kappa(t)$, and $\tau(t)$. This is solved by the fundamental theorem for space curves – see the discussion and proof in [5]:

Theorem 0.34 Every regular curve in three-dimensional space, with non-zero curvature, has its shape completely determined by its curvature and torsion

0.5 Surfaces and their geometry

A surface in 3-space $\mathbb{R}^3$ is usually described either *implicitly* as the zero set of some real-valued function g, that is $S = \{(x,y,z) \mid g(x,y,z) = 0\}$, or explicitly as a *parameterized surface* with two parameters: $S = \{f(u,v) \mid (u,v) \in U\}$, where f is a map from $U \to \mathbb{R}^3$, and U is an open subset of $\mathbb{R}^2$. For example, we can represent the unit sphere $\mathbb{S}^2$ either as the set

$$\{x^2 + y^2 + z^2 - 1 = 0\},$$

or we can get most of the sphere as the image of the map $f : (0,2\pi) \times (-\pi/2, \pi/2) \to \mathbb{R}^3$ given by

$$f(u,v) = (\cos u \cos v, \sin u \cos v, \sin v).$$

EXERCISE 0.35

Use a computer to plot the parameterization $f(u,v)$ of the sphere $\mathbb{S}^2$ given above. Find the subset of $(0,2\pi) \times (-\pi/2, \pi/2)$ needed to plot just the upper hemisphere, that is $\{(x,y,z) \in S \mid z > 0\}$.

0.5.1 The tangent space to a surface

The *coordinate tangent vectors* for a parameterized surface $f(u,v)$ are given by $f_u := \frac{\partial f}{\partial u}$ and $f_v := \frac{\partial f}{\partial v}$. The condition that f is a valid parameterized surface (that is *regular*) is that f_u and f_v are linearly independent vectors at each point (u,v) in the domain. This is equivalent to the statement that the cross product $f_u \times f_v$ is non-zero. In this case, the vector subspace of $\mathbb{R}^3$ spanned by f_u and f_v is called the *tangent space* to S at (u,v).

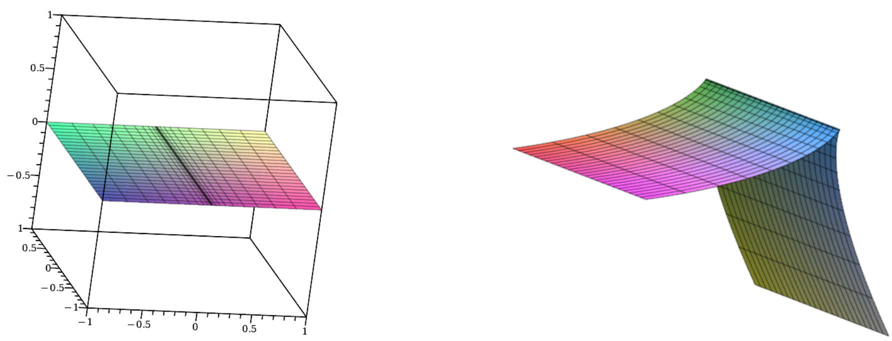


Figure 2: Left: the image of $f(u,v) = (u^3, v, 0)$. Right: the image of $f(u,v) = (u^2, u^3, v)$.

If the cross product is zero at some point, it might just mean that your parameterization is no good at that point - there may be another parameterization of the same image set which *is* regular.

EXERCISE 0.36

Show that the map $f : (-1,1) \times (-1,1) \to \mathbb{R}^3$, given by

$$f(u,v) = (u^3, v, 0),$$

is *not* a regular parameterized surface (Figure 2, left). Write down an alternative map $g : (-1,1) \times (-1,1) \to \mathbb{R}^3$ that has exactly the same image set in $\mathbb{R}^3$ and *is* regular.

Depending on the purpose, we often allow our parameterized surfaces to have *self-intersections* in the large but the regularity condition guarantees that if you restrict to a very small set in the uv-plane, then the image of this set will be a smooth surface with no self-intersections.

EXERCISE 0.37

Calculate the tangent vectors for the above parameterized sphere, and the cross-product $f_u \times f_v$.

EXERCISE 0.38

At which points on the sphere are our parameters invalid?

EXERCISE 0.39

Show analytically that $g : (-1,1) \times (-1,1) \to \mathbb{R}^3$ given by

$$g(u,v) = (u^2, u^3, v)$$

(see Figure 2) is not a regular parameterized surface along the line $\{u = 0\}$. Is there a regular surface with the same image as g?

EXERCISE 0.40

Plot the image set of the function $h : [-1,1] \times [-1,1] \to \mathbb{R}^3$, given by the formula $h(u,v) = (u, v^2, uv^3)$. For which points in $[-1,1] \times [-1,1]$ is this a regular parameterized surface? (Show this using using the formula). Does this fit with the image you plotted? (Note: we allow a parameterized surface to intersect itself).

EXERCISE 0.41

Using the set $[0,2\pi] \times \mathbb{R}$ as your parameter domain, give a parameterization of the infinite cylinder $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 4\}$. Plot this on an appropriate subdomain.

0.5.2 Surfaces of revolution:

A *curve* in the *xy*-plane is given by a function $\gamma\colon (a,b) \to \mathbb{R}^3$, from an open interval $(a,b) \subset \mathbb{R}$, of the form $\gamma(u) = (\alpha(u), \beta(u), 0)$, with the condition that $\gamma'(u) \neq 0$. The *surface of revolution* obtained by revolving this curve around the x-axis is parameterized as

$$f(u,v) = (\alpha(u), \beta(u)\cos(v), \beta(u)\sin(v)),$$

where $(u,v) \in (a,b) \times [0,2\pi]$. (The round cylinder of Exercise 0.41 is an example of revolving the line $(2,0,u)$, a curve in the xz-plane, about the z-axis.)

EXERCISE 0.42

Find a parameterization for the *catenoid* obtained by revolving the catenary $y = \cosh(x)$ about the x-axis. Plot this on an appropriate domain.

0.5.3 The unit normal

The cross-product of any two vectors is orthogonal to both of them. Therefore, if f is a regular surface, the vector $f_u \times f_v$ is orthogonal to the tangent plane of the surface. The *unit normal* is the unit length vector in this direction $\mathcal{N} = \frac{f_u \times f_v}{||f_u \times f_v||}$.

EXERCISE 0.43

Find the unit normal as a function of u and v for the parameterized sphere of Exercise 0.35. Is the unit normal pointing out of the sphere or into the center of the sphere?

EXERCISE 0.44

Find the unit normal to the cylinder of Exercise 0.41 at the points $(2,0,0)$ and $(0,2,0)$. Find the unit normal to the surface g of Exercise 0.39, as a function of u and v.

0.5.4 The first and second fundamental forms

Recall that lengths of vectors and angles between them are computed using the dot product:

$$||T|| = \sqrt{T \cdot T}, \qquad T \cdot U = \cos(\theta) \cdot ||T|| \cdot ||U||.$$

It is the dot product that allows us to measure things in $\mathbb{R}^n$. In integral calculus the above formulas are used to compute lengths of curves, areas of surfaces, and volumes of regions.

In order to measure things on a surface, we only need to know the value of the dot product restricted to the tangent space. This restriction is called the *first fundamental form*. For a parameterized surface, the tangent space is spanned by f_u and f_v, and so if we know the coefficients

$$E = f_u \cdot f_u, \qquad F = f_u \cdot f_v, \qquad G = f_v \cdot f_v,$$

we know the dot product of any tangent vectors. For an arbitrary pair of tangent vectors, T and U, written as linear combinations of f_u and f_v, that is: $T = af_u + bf_v$ and $U = cf_u + df_v$, the dot product is calculated by the matrix multiplication:

$$[a \;\; b] \cdot \begin{bmatrix} E & F \\ F & G \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = [a \;\; b] \cdot \mathcal{F}_I \cdot \begin{bmatrix} c \\ d \end{bmatrix} \quad ,$$

and $\mathcal{F}_I$ is the matrix of the first fundamental form matrix with respect to the given parameterization. Notions such as lengths, areas and angles on the surface are all defined in terms of the first fundamental form, and so the coefficients E, F and G come up in the formulae for these. Note that E, F and G will *change* if we change the parameterization of the surface.

To see how the surface is actually sitting inside $\mathbb{R}^3$, we use the *second fundamental form*, which is also a map which takes a pair of tangent vectors $T = af_u + bf_v$ and $U = cf_u + df_v$ and gives a number

$$[a \;\; b] \cdot \begin{bmatrix} L & M \\ M & N \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = [a \;\; b] \cdot \mathcal{F}_{II} \cdot \begin{bmatrix} c \\ d \end{bmatrix} \quad ,$$

where $\mathcal{F}_{II}$ denotes the second fundamental form matrix with respect to the given parametrization, i.e. if $\mathcal{N}$ is the unit normal, then

$$L = f_{uu} \cdot \mathcal{N}, \qquad M = f_{uv} \cdot \mathcal{N}, \qquad N = f_{vv} \cdot \mathcal{N}.$$

Since these coefficients take *second* derivatives of f and dot them with the normal, the second fundamental form $\mathcal{F}_{II}$ gives information about how the surface is bending away (in the direction of $\mathcal{N}$) from the tangent plane; so, for example, if the surface is a plane then $\mathcal{F}_{II}$ is the zero matrix. If a plane is bent into the shape of a cylinder, then $\mathcal{F}_{II}$ for the bent surface is not the zero matrix.

Another way to describe the geometric meaning of the second fundamental for is to observe that, differentiating $\mathcal{N} \cdot f_u = 0$ with respect to u we have: $\mathcal{N}_u \cdot f_u + \mathcal{N} \cdot f_{uu} = 0$, so $f_{uu} \cdot N = -\mathcal{N}_u \cdot f_u$ and so on, to get:

$$L = -\mathcal{N}_u \cdot f_u, \qquad M = -\mathcal{N}_u \cdot f_v = -\mathcal{N}_v \cdot f_u, \qquad N = -\mathcal{N}_v \cdot f_v,$$

so $\mathcal{F}_{II}$ is simply how the unit normal vector field along the surface is varying in space, since it contains all the information about the derivatives of $\mathcal{N}$.

0.5.5 The principal curvatures and directions:

If $T = a \cdot f_u + b \cdot f_v$ is a unit vector in the tangent space, i.e. $[a\ \ b] \cdot \mathcal{F}_I \cdot [a\ \ b]^* = 1$, then

$$[a\ \ b] \cdot \mathcal{F}_{II} \cdot [a\ \ b]^* \tag{0.7}$$

is a number which tells you how fast a curve in the surface which is tangent to T bends away from the surface. In other words, how *curved* the surface is in that direction. In general, (except at special points, called *umbilics*), there are two special directions in the surface where this bending is a maximum and a minimum respectively, at the point in question. These are called the *principal directions*. A curve in the surface which is always tangent to a principal direction is called a *principal curve*, or *line of curvature*.

These principal directions are the eigenvectors of the *Weingarten matrix*,

$$\mathcal{W} = \mathcal{F}_I^{-1} \cdot \mathcal{F}_{II}.$$

The eigenvalues are always real. At non-umbilic points, these eigenvalues are distinct, and are called the *principal curvatures*, which we denote by κ_1 and κ_2. To see why we have to use the eigenvectors of $\mathcal{W}$, not those of $\mathcal{F}_{II}$, note that, if T is a unit length eigenvector of $\mathcal{W}$, with $\mathcal{W} \cdot [a\ \ b]^* = \kappa \cdot [a\ \ b]^*$, then

$$[a\ \ b] \cdot \mathcal{F}_{II} \cdot [a\ \ b]^* = [a\ \ b] \cdot \mathcal{F}_I \left(\mathcal{F}_I^{-1} \cdot \mathcal{F}_{II} \cdot [a\ \ b]^* \right) = [a\ \ b] \cdot \mathcal{F}_I \cdot (\kappa \cdot [a\ \ b]^*) = \kappa,$$

so (comparing with equation (0.7)) the eigenvalue κ says how fast a curve in the direction of T bends away from the surface.

EXERCISE 0.45

The sphere, of Exercise 0.35 is a *totally umbilic* surface, that is, all points are umbilics, meaning that all directions at all points are principal directions. Show this by calculating the eigenvalues and eigenvectors of the Weingarten matrix.

EXERCISE 0.46

Calculate the principal directions and the principal curvatures for the cylinder (Exercise 0.41). Plot the cylinder again, this time displaying two principal curves through some particular point.

EXERCISE 0.47

Find the principal curvatures for the Catenoid (Exercise 0.42). What is special about the relationship between the two principal curvatures?

EXERCISE 0.48

Find the principal directions for the Catenoid, and plot some of the principal curves.

0.5.6 The Gauss and mean curvature:

The *Gauss curvature* of a surface is the product of the principal curvatures: $K = \kappa_1\kappa_2$. The most important thing about the Gauss curvature is its sign: if $K > 0$ at a point then both principal curves through that point are curving the same way, either into the direction of the normal, or away from the normal direction. If $K < 0$ then the two principal curves are bending in opposite directions. A special case is $K = 0$: if a surface has $K = 0$ everywhere, then it is called *flat* because it has the special property that it can be obtained from a "flat piece of paper" by bending, but not stretching. You can imagine this, because if $K = 0$ then one of the principal curvatures is zero, so in one direction the surface is not bending. This means you don't have to "stretch" the surface to bend it back to a flat piece of paper.
Note: Even if the eigenvalues of $\mathcal{W}$ are not distinct (at umbilics) we still define the Gauss curvature to be their product. For example, if there is one eigenvalue, $\kappa_1 = 3$, with multiplicity 2, then the Gauss curvature is $3 \times 3 = 9$.

EXERCISE 0.49

Find the Gauss curvature for the sphere, the plane $[u, v, 0]$, the cylinder and the catenoid.

EXERCISE 0.50

Plot the catenoid, colored by Gauss curvature. What is happening to the surface as u grows large?

The *mean curvature* is the average of the eigenvalues of the Weingarten matrix (even if they are not distinct), $H = \frac{1}{2}(\kappa_1 + \kappa_2)$. In contrast to the Gauss curvature, the sign of the mean curvature means nothing geometrically, because if you choose your unit normal to point in the opposite direction (for example by interchanging the parameters u and v), the mean curvature changes sign. (The *Gauss* curvature, on the other hand, does not depend on the choice of the unit normal direction).

EXERCISE 0.51

Calculate the mean curvature for the sphere, the cylinder, the plane and the catenoid.

EXERCISE 0.52

Calculate the Gauss and mean curvature for the surface g of Exercise 0.39. What happens to the mean curvature as u approaches 0?

0.6 Submanifolds of Euclidean space

0.6.1 Parameterized submanifolds

Recall that a map $f : \mathbb{R}^m \supset U \to \mathbb{R}^n$ is called *regular* if its derivative $\mathrm{d}f(p) : \mathbb{R}^m \to \mathbb{R}^n$ is regular at every point $p \in U$. In matrix form, the derivative is just the Jacobian matrix $J_f(p) = [\partial f^i / \partial x^j]|_p$, and an $m \times n$ matrix A is regular if it has the highest possible rank: $\mathrm{Rank}(A) = \min(m,n)$.

Parameterized submanifolds of arbitrary dimension are defined in the same way as regular curves in $\mathbb{R}^n$:

Definition 0.53 Let $n \geq m \geq 0$ be integers. An *m-dimensional* parameterized *submanifold of* $\mathbb{R}^n$ is given by an open subset $U \subset \mathbb{R}^m$ together with a regular smooth map $f : U \to \mathbb{R}^n$.

Note: we could add the condition that the map f be *injective* - however this is not very important for the moment.
The difference $(n-m)$ is called the *codimension* of the submanifold. We have already seen 1-dimensional examples (regular curves in $\mathbb{R}^n$) and also 2-dimensional parameterized submanifolds of $\mathbb{R}^3$.

EXERCISE 0.54

For each of f and h, check whether it defines a parameterized submanifold, and give the dimension and codimension.

$$f : \mathbb{R}^3 \to \mathbb{R}^4, \quad f(x,y,z) = \begin{pmatrix} x+y \\ x-y \\ x+z \\ x-z \end{pmatrix}^*, \qquad h : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^4 \quad h(u,v) = \begin{pmatrix} u \\ u^2 + \cos(v) \\ v^2 \\ uv \end{pmatrix}^*.$$

For a parameterized submanifold $f : U \to \mathbb{R}^n$, we can define the tangent space at a point $p \in U$, exactly as we did for parameterized surfaces: the fact that f is regular means that the columns:

$$f_{u_i} := \frac{\partial f}{\partial u_i}, \qquad i = 1, \ldots m,$$

of the Jacobian matrix, are all linearly independent. Therefore they define at each point p an m-dimensional vector subspace of $\mathbb{R}^n$, the *tangent space at p*, denoted $T_pU := \text{Span}\{f_{u_1}|_p, \ldots, f_{u_m}|_p\}$. The *first fundamental form* (or the *induced metric*) can be defined exactly as for surfaces, namely it is the inner product on T_pU obtained simply by restricting the dot product of $\mathbb{R}^n$:

$$\langle X, Y \rangle := X \cdot Y$$

where $X, Y \in T_pM \subset \mathbb{R}^n$. The matrix of the first fundamental form (the *metric matrix*) is computed in the same way as for surfaces:

$$G_{ij} = \langle f_{u_i}, f_{u_j} \rangle = f_{u_i} \cdot f_{u_j}.$$

As with surfaces the first fundamental form allows us to define lengths of curves and volumes of regions on a parameterized submanifold. As for curvature, this is a little more complicated to define if the codimension $n - m$ is greater than 1. We will, however, define curvature in a different way later.

0.6.2 General submanifolds of Euclidean space

In general, a submanifold of Euclidean space is a set that looks like a parameterized submanifold around every point. A classic example is the sphere defined by the position vectors $\mathbf{x}$ of unit length in $\mathbb{R}^n$: $\mathbb{S}^m := \{\mathbf{x} \in \mathbb{R}^{m+1} \mid \mathbf{x} \cdot \mathbf{x} = 1\}$. There is an open subset V of $\mathbb{S}^m$ around every point such that V is a parameterized submanifold of dimension m: for instance, the upper hemisphere, $\mathbb{S}^m_+ = \{\mathbf{x} \in \mathbb{S}^m \mid x_{n+1} > 0\}$ can be parameterized as $f : U \to \mathbb{R}^{m+1}$, where $U = \{\mathbf{u} \in \mathbb{R}^m \mid ||\mathbf{u}|| < 1\}$ and

$$f(\mathbf{u}) = (u_1, u_2, \ldots, u_m, \sqrt{1 - u_1^2 - \cdots - u_m^2}).$$

Similarly, for any $\mathbf{x} \in \mathbb{S}^m$, there is always a hemisphere, $\mathbb{S}^m_{j,\pm} = \{\mathbf{x} \in \mathbb{S}^m \mid \pm x_j > 0\}$ where an analogous parameterization can be used. It is necessary that parameterizations are *compatible* in the sense that there is a differentiable map from one to the other, and this will be discussed later.

We can give a precise definition of a submanifold of Euclidean space as follows, see [17, Chapter 2]:

Definition 0.55 A subset M of $\mathbb{R}^m$ is called a k-dimensional submanifold of $\mathbb{R}^m$ if: For each $p \in M$ there exists an open neighborhood W of p in $\mathbb{R}^n$, an open subset of U of 0 in $\mathbb{R}^k$, and a map $\phi : U \to \mathbb{R}^n$ with maximal rank of its Jacobian matrix in all of U, and such that $\phi : U \to \phi(U)$ is a homeomorphism with $\phi(U) = M \cap W$.

In particular, such a map ϕ stemming from this definition, is then a local parametrization of M, and $\phi(U)$ is itself a parametrized submanifold of $\mathbb{R}^n$.

EXERCISE 0.56

Show carefully that the unit sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ satisfies Definition 0.55.

Chapter 1

Introduction

In this chapter we first give a short survey of notation and some fundamental concepts and results, that we will use throughout these notes. The survey is mainly presented by simple examples which already in this chapter lead to the introduction of metric tensor fields for local Riemannian manifolds, i.e. the objects, that will play the key rôle throughout these notes.

> A key motivation for this introduction is found in the fact that a great many – very different – research works and disciplines study naturally appearing fields of ellipses (or ellipsoids) in the plane and in space, respectively, see for example the figures 1.1, 1.2, and 1.3 below. In cartography, for example, they appear as the so-called Tissot indicatrices, see Wiki: Tissot.
> In this chapter we will see – in a first glimpse – how such ellipse (ellipsoidal) fields give rise to metric tensor fields, and vice versa: A choice of metric tensor field in the plane gives us directly a field of ellipses in the plane, and a choice of metric tensor field in 3D space gives us a unique field of ellipsoids in space. We will refer to these geometric ellipse- (ellipsoidal-) fields as the indicatrix set or fingerprint of their corresponding metric tensor fields.
> The main purpose of these notes is to show how the notion of curvature can be well-defined from any given ellipsoidal field (in terms of its metric tensor field), and to eventually be able to understand the meaning of curvature via its influence on the shortest pathways through the ellipsoidal field.

1.1 Open sets

We begin by re-considering open sets $\mathcal{U}^2$ in $(\mathbb{R}^2, \cdot)$, where the dot is used to make clear, that initially we apply the usual dot-product in $\mathbb{R}^2$ and thence the usual Euclidean distance betweeen points p and q in $\mathbb{R}^2$: If p has coordinates $p = (p^1, p^2)$ and $q = (q^1, q^2)$ then

$$d_E(p,q) = \sqrt{(q^1 - p^1)^2 + (q^2 - p^2)^2} \quad . \tag{1.1}$$

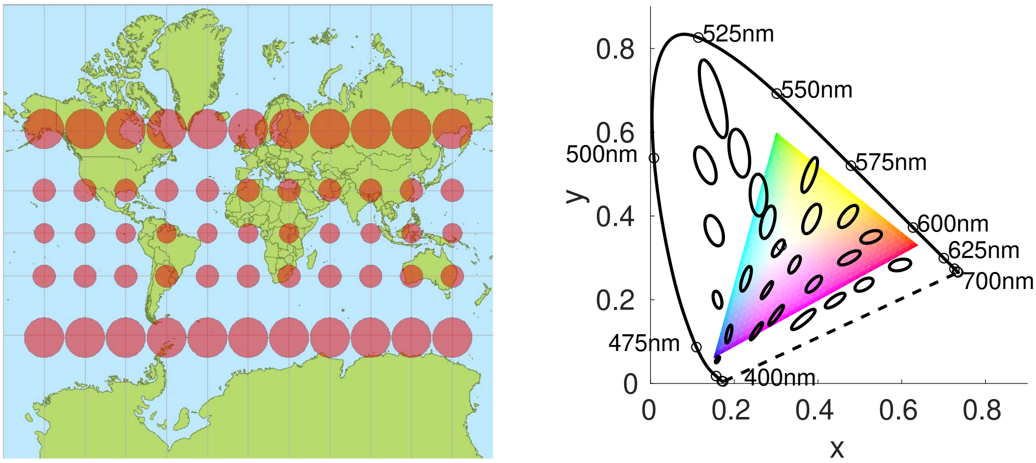


Figure 1.1: Left: Ellipses (in fact circles in this case) – the so-called Tissot indicatrices – from the Mercator projection of the globe. Right: Indicatrices from the metric of color space, see [10].

Note that for example $(p^2)^2$ is read out as follows: ["p two squared"]. At first sight it may seem clumsy to use upper indices for the coordinates in this way, but it will become efficient later when we introduce and apply Einstein's summation convention for tensor calculus and tensor analysis.

EXERCISE 1.1

Remember or look up in Chapter 0 what an open set in $(\mathbb{R}^2, \cdot)$ is. I.e. how is an *open set* defined in $(\mathbb{R}^2, \cdot)$?

EXERCISE 1.2

Let $\mathcal{U}_i^2$, $i = 1, 2, 3, 4$, denote the following subsets of $\mathbb{R}^2$ and find out (with an argument) which ones of these sets are open sets in $(\mathbb{R}^2, \cdot)$:

$$
\begin{aligned}
\mathcal{U}_1^2 &= \{(x^1, x^2) \in \mathbb{R}^2 \mid \quad (x^1)^2 + (x^2)^2 < 7 \quad \} \\
\mathcal{U}_2^2 &= \{(x^1, x^2) \in \mathbb{R}^2 \mid \quad (x^1)^2 + (x^2)^2 \leq 1 \quad \} \\
\mathcal{U}_3^2 &= \mathbb{R}^2 - \{(x^1, x^2) \in \mathbb{R}^2 \mid \quad x^2 = 0 \quad , \quad x^1 \leq 0 \quad \} \\
\mathcal{U}_4^2 &= \{(x^1, x^2) \in \mathbb{R}^2 \mid \quad x^2 > 0 \quad , \quad (x^1)^2 + (x^2)^2 \leq 7 \quad \} \quad .
\end{aligned}
\tag{1.2}
$$

EXERCISE 1.3

What is the corresponding definition of an *open set* $\mathcal{U}^n$ in $(\mathbb{R}^n, \cdot)$?

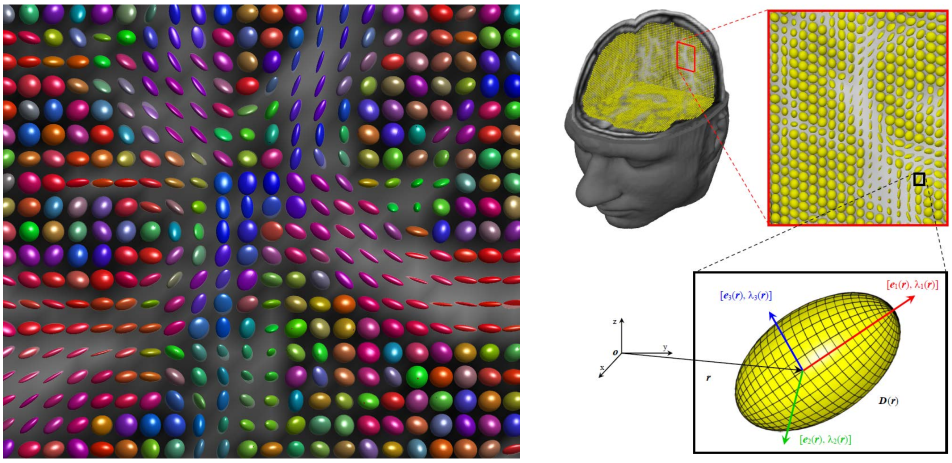


Figure 1.2: NMRI (Nuclear Magnetic Resonance Image) of a Corpus Callosum, see [13].

1.2 Coordinate expressions of maps $\mathbb{R}^n \mapsto \mathbb{R}^n$

Usually we will denote the coordinates in $\mathbb{R}^n$ by $(x^1, \cdots, x^n)$, and if we need another copy of $\mathbb{R}^n$ we can then denote the coordinates there by $(y^1, \cdots, y^n)$. This makes it possible to consider maps ϕ from an open subset $\mathcal{U}^n$ in one copy of $\mathbb{R}^n$ (with coordinates $(x^1, \cdots, x^n)$) onto its image $\phi(\mathcal{U}^n)$ in the other copy of $\mathbb{R}^n$ with coordinates $(y^1, \cdots, y^n)$. Correspondingly we denote such mappings as follows:

$$\begin{aligned} \phi \;:\; \mathcal{U} &\longmapsto \phi(\mathcal{U}) \\ \phi(x^1, \cdots, x^n) &= (\phi^1(x^1, \cdots, x^n), \cdots \phi^n(x^1, \cdots, x^n)) = (y^1, \cdots, y^n) \quad , \end{aligned} \tag{1.3}$$

where $\phi^j(x^1, \cdots, x^n)$ denotes the j'th coordinate y^j of the *image point* $\phi(x^1, \cdots, x^n)$.

Example 1.4

Let $\mathcal{U}$ denote all of $\mathbb{R}^2$ (which is itself an open set) and consider the map: $\phi \;:\; \mathbb{R}^2 \longmapsto \mathbb{R}^2$ defined by:

$$\phi(x^1, x^2) = (y^1, y^2) = (A \cdot [x^1 \; x^2]^*)^* \quad , \tag{1.4}$$

where $*$ denotes transposition and where A is a *regular* 2×2-matrix with elements a_{ij}, so that we can write:

$$\phi(x^1, x^2) = (y^1, y^2) = (a_{11} \cdot x^1 + a_{12} \cdot x^2, \; a_{21} \cdot x^1 + a_{22} \cdot x^2) \quad . \tag{1.5}$$

Such a map will be called a regular affine map. The image of $\mathbb{R}^2$ by the map ϕ is $\mathbb{R}^2$. The map is a bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$.

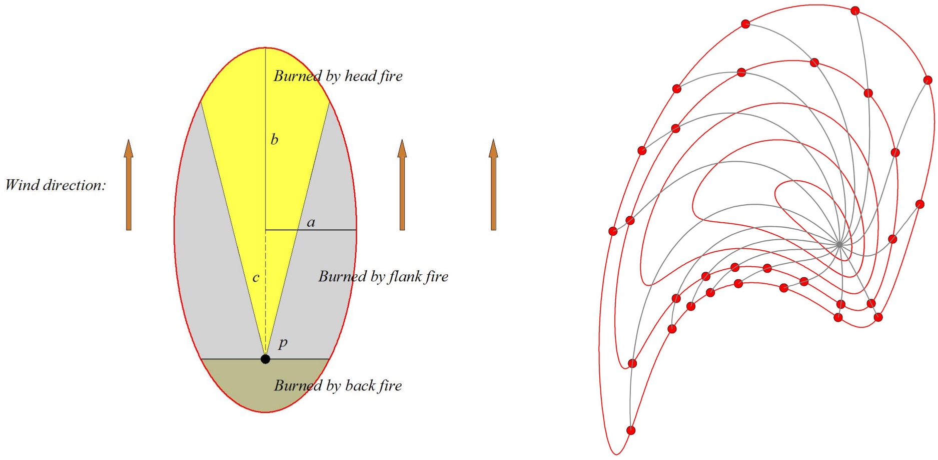


Figure 1.3: A canonical ellipse for short time ignition of wildfires, and a simulation of the spread of large wildfire frontals, obtained by 'integrating' the short time indicatrices, see [21].

EXERCISE 1.5

Why is the map ϕ a bijection?

In particular, the map ϕ has a well-defined inverse ϕ^{-1} – with the property: If $\phi(x^1,x^2) = (y^1,y^2)$ then $(x^1,x^2) = \phi^{-1}(y^1,y^2)$ and vice versa: If $\phi^{-1}(y^1,y^2) = (x^1,x^2)$ then $(y^1,y^2) = \phi(x^1,x^2)$.
The expression for the inverse map is in this example:

$$\phi^{-1}(y^1,y^2) = (x^1,x^2) = (A^{-1} \cdot [y^1\ y^2]^*)^* \quad , \tag{1.6}$$

where A^{-1} denotes the inverse of the regular matrix A.
The map ϕ is smooth – all partial derivatives of $\phi(x^1,x^2)$ with respect to x^1 and x^2 exist – in particular, the Jacobian matrix J_ϕ is well-defined and is, when evaluated at (x^1,x^2):

$$J_\phi(x^1,x^2) = A \quad , \quad \text{for all} \quad (x^1,x^2) \quad . \tag{1.7}$$

The inverse map ϕ^{-1} is also smooth – all partial derivatives of $\phi^{-1}(y^1,y^2)$ with respect to y^1 and y^2 exist – in particular, the Jacobian matrix $J_{\phi^{-1}}$ is again well-defined and is, when evaluated at (y^1,y^2):

$$J_{\phi^{-1}}(y^1,y^2) = A^{-1} \quad , \quad \text{for all} \quad (y^1,y^2) \quad . \tag{1.8}$$

We note, that the two Jacobian matrices are constants, they are regular, and one is the inverse of the other.

A more complicated (and more interesting and relevant) example is the following, which, however, should also be well-known:

Example 1.6

Let $\mathcal{U}$ denote the set $\mathcal{U}_3^2$ from exercise 1.2, i.e. $\mathbb{R}^2$ except the non-positive part of the x^1-axis:

$$\mathcal{U} = \mathcal{U}_3^2 = \mathbb{R}^2 - \{(x^1, x^2) \in \mathbb{R}^2 \,|\quad x^2 = 0 \quad , \quad x^1 \leq 0 \quad \} \quad , \tag{1.9}$$

and consider the map: $\phi \,:\, \mathcal{U} \longmapsto \phi(\mathcal{U})$ defined by:

$$\phi(x^1, x^2) = (y^1, y^2) = \left(\sqrt{(x^1)^2 + (x^2)^2}\,,\, \arg(x^1 + i \cdot x^2)\right) \quad , \tag{1.10}$$

where $\arg(z) \in]-\pi, \pi[$ denotes the argument (in that interval) of the complex number z. Correspondingly, $\sqrt{(x^1)^2 + (x^2)^2}$ is just the modulus of the complex number $x^1 + i \cdot x^2$. In other words, the map ϕ produces the polar coordinates of the point (x^1, x^2). We will therefore refer to this map as the polar map. When $\arg(x^1 + i \cdot x^2)) = \arccos\left(x^1 / \sqrt{(x^1)^2 + (x^2)^2}\right)$, (which is not always the case, see exercise 1.7), we can rewrite the expression for ϕ as follows:

$$\begin{aligned} \phi(x^1, x^2) &= (y^1, y^2) \\ &= (\phi^1(x^1, x^2)\,,\, \phi^2(x^1, x^2)) \\ &= \left(\sqrt{(x^1)^2 + (x^2)^2}\,,\, \arccos\left(\frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}}\right)\right) \quad , \end{aligned} \tag{1.11}$$

EXERCISE 1.7

For which points (x^1, x^2) in $\mathcal{U}$ is it true that $\arg(x^1 + i \cdot x^2) = \arccos\left(x^1 / \sqrt{(x^1)^2 + (x^2)^2}\right)$?

The image $\phi(\mathcal{U})$ is the following subset of $\mathbb{R}^2$ (with the coordinates (y^1, y^2)):

$$\phi(\mathcal{U}) = \mathcal{V} = \{(y^1, y^2) \in \mathbb{R}^2 \,|\, -\pi < y^2 < \pi \quad , \quad y^1 > 0\} \quad . \tag{1.12}$$

EXERCISE 1.8

Show that $\phi(\mathcal{U}) = \mathcal{V}$ is an open set in $\mathbb{R}^2$.

Claim: The polar map ϕ is a bijection of $\mathcal{U}$ onto $\mathcal{V}$.

EXERCISE 1.9

Prove that claim.

In particular, the polar map ϕ has a well-defined inverse map $\phi^{-1} \,:\, \mathcal{V} \mapsto \mathcal{U}$, i.e. the inverse polar map. The coordinate expression for the inverse polar map is:

$$\begin{aligned} \phi^{-1}(y^1, y^2) &= (x^1, x^2) \\ &= ((\phi^{-1})^1(y^1, y^2)\,,\, (\phi^{-1})^2(y^1, y^2)) \\ &= (y^1 \cdot \cos(y^2)\,,\, y^1 \cdot \sin(y^2)) \quad . \end{aligned} \tag{1.13}$$

Note that in this example ϕ^{-1} has a much simpler expression than ϕ.

Both the maps ϕ and ϕ^{-1} are smooth in their respective domains $\mathcal{U}$ and $\mathcal{V}$. In particular, they have well-defined Jacobian matrices. If we use the expression for ϕ in (1.11) we get:

$$J_\phi(x^1,x^2) = \begin{bmatrix} \frac{\partial\phi^1(x^1,x^2)}{\partial x^1} & \frac{\partial\phi^1(x^1,x^2)}{\partial x^2} \\ \frac{\partial\phi^2(x^1,x^2)}{\partial x^1} & \frac{\partial\phi^2(x^1,x^2)}{\partial x^2} \end{bmatrix} = \begin{bmatrix} \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} & \frac{x^2}{\sqrt{(x^1)^2+(x^2)^2}} \\ \frac{-x^2}{(x^1)^2+(x^2)^2} & \frac{x^1}{(x^1)^2+(x^2)^2} \end{bmatrix} \quad , \tag{1.14}$$

and similarly for the inverse map:

$$J_{\phi^{-1}}(y^1,y^2) = \begin{bmatrix} \frac{\partial(\phi^{-1})^1(y^1,y^2)}{\partial y^1} & \frac{\partial(\phi^{-1})^1(y^1,y^2)}{\partial y^2} \\ \frac{\partial(\phi^{-1})^2(y^1,y^2)}{\partial y^1} & \frac{\partial(\phi^{-1})^2(y^1,y^2)}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \cos(y^2) & -y^1\cdot\sin(y^2) \\ \sin(y^2) & y^1\cdot\cos(y^2) \end{bmatrix} \quad . \tag{1.15}$$

According to the chain rule, see proposition 1.11 and exercise 1.12 below, the two Jacobian matrices J_ϕ and $J_{\phi^{-1}}$ are the inverses of each other. In order to verify this in the present concrete example we need to apply the ϕ-correspondence between the two systems of coordinates, so that the matrices, that we want to compare, are expressed in the *same coordinates*. Specifically we have:

$$\begin{aligned} J_\phi^{-1}(x^1,x^2) &= \begin{bmatrix} \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} & \frac{x^2}{\sqrt{(x^1)^2+(x^2)^2}} \\ \frac{-x^2}{(x^1)^2+(x^2)^2} & \frac{x^1}{(x^1)^2+(x^2)^2} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} & -x^2 \\ \frac{x^2}{\sqrt{(x^1)^2+(x^2)^2}} & x^1 \end{bmatrix} \end{aligned} \tag{1.16}$$

In order to verify, that $J_\phi^{-1} = J_{\phi^{-1}}$ we insert $(x^1,x^2) = \phi^{-1}(y^1,y^2) = (y^1\cdot\cos(y^2), y^1\cdot\sin(y^2))$ in the above expression for $J_\phi^{-1}(x^1,x^2)$ and get

$$\begin{aligned} J_\phi^{-1}(y^1\cdot\cos(y^2), y^1\cdot\sin(y^2)) &= \begin{bmatrix} \cos(y^2) & -y^1\cdot\sin(y^2) \\ \sin(y^2) & y^1\cdot\cos(y^2) \end{bmatrix} \\ &= J_{\phi^{-1}}(y^1,y^2) \quad , \end{aligned} \tag{1.17}$$

which shows indeed that $J_\phi^{-1}(\phi^{-1}(y^1,y^2)) = J_{\phi^{-1}}(y^1,y^2)$ and, equivalently, $J_{\phi^{-1}}(\phi(x^1,x^2)) = J_\phi^{-1}(x^1,x^2)$.

Similarly, but redundantly, we could have substituted

$$(y^1,y^2) = \phi(x^1,x^2) = \left(\sqrt{(x^1)^2+(x^2)^2}\,,\ \arccos\left(\frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}}\right)\right) \tag{1.18}$$

into the expression for $J^{-1}_{\phi^{-1}}(y^1, y^2)$ and get:

$$
\begin{aligned}
J^{-1}_{\phi^{-1}}(y^1, y^2) &= \begin{bmatrix} \cos(y^2) & -y^1 \cdot \sin(y^2) \\ -\sin(y^2) & y^1 \cdot \cos(y^2) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \cos(y^2) & \sin(y^2) \\ \frac{-\sin(y^2)}{y^1} & \frac{\cos(y^2)}{y^1} \end{bmatrix} \\
&= \begin{bmatrix} \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} & \frac{x^2}{\sqrt{(x^1)^2+(x^2)^2}} \\ \frac{-x^2}{(x^1)^2+(x^2)^2} & \frac{x^1}{(x^1)^2+(x^2)^2} \end{bmatrix} \\
&= J_{\phi}(x^1, x^2) \quad ,
\end{aligned}
\tag{1.19}
$$

which shows that $J^{-1}_{\phi^{-1}}(\phi(x^1, x^2)) = J_{\phi}(x^1, x^2)$ or, equivalently, $J_{\phi}(\phi^{-1}(y^1,y^2) = J^{-1}_{\phi^{-1}}(y^1, y^2)$.

1.3 Diffeomorphisms

The polar map and the regular affine maps are examples of diffeomorphisms:

Definition 1.10 A map ϕ from an open subset $\mathcal{U}^n$ in $(\mathbb{R}^n, \cdot)$ onto its image $\phi(\mathcal{U}^n)$ in $\mathbb{R}^n$ is called a diffeomorphism if it is a smooth bijection of $\mathcal{U}^n$ onto an open subset $\phi(\mathcal{U}^n) = \mathcal{V}^n$ of $(\mathbb{R}^n, \cdot)$ with a smooth inverse map $\phi^{-1} : \mathcal{V}^n \longmapsto \mathcal{U}^n$.

Note that if ϕ is a diffeomorphism, then ϕ^{-1} is also a diffeomorphism.

For a general smooth map $\phi : \mathcal{U} \longmapsto \phi(\mathcal{V})$ – as considered in (1.3) – the Jacobian matrix is:

$$
J_{\phi}(x^1, \cdots, x^n) = \begin{bmatrix} \frac{\partial\phi^1(x^1,\cdots,x^n)}{\partial x^1} & \cdot & \frac{\partial\phi^1(x^1,\cdots,x^n)}{\partial x^n} \\ \cdot & \cdot & \cdot \\ \frac{\partial\phi^n(x^1,\cdots,x^n)}{\partial x^1} & \cdot & \frac{\partial\phi^n(x^1,\cdots,x^n)}{\partial x^n} \end{bmatrix}
\tag{1.20}
$$

The following general result, exemplified in the examples above, is a consequence of the chain rule.

Proposition 1.11 Let ϕ be a diffeomorphism from an open set $\mathcal{U}^n$ of $\mathbb{R}^n$ into $\mathbb{R}^n$. Then the Jacobian matrix J_ϕ evaluated at $(x^1, \cdots, x^n)$ is regular and it is the inverse of the Jacobian matrix of ϕ^{-1} evaluated at $(y^1, \cdots, y^n) = \phi(x^1, \cdots, x^n)$:

$$J_\phi(x^1, \cdots, x^n) = J_{\phi^{-1}}^{-1}(y^1, \cdots, y^n) = J_{\phi^{-1}}^{-1}(\phi(x^1, \cdots, x^n)) \quad . \tag{1.21}$$

EXERCISE 1.12

Remember, look up, or find out what the chain rule for sequences of mappings between copies of $\mathbb{R}^n$ is all about – and prove proposition 1.11. Hint: Use $\phi^{-1}(\phi(x^1, \cdots, x^n)) = (x^1, \cdots, x^n)$.

EXERCISE 1.13

Let ϕ denote the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by the expression:

$$\phi(x^1, x^2) = (y^1, y^2) = (e^{x^1} + x^2,\, x^1) \quad . \tag{1.22}$$

Show that ϕ is a diffeomorphism and find the Jacobian matrices $J_\phi(x^1, x^2)$ and $J_{\phi^{-1}}(y^1, y^2)$. Show that these Jacobian matrices are the inverses of each other.

EXERCISE 1.14

Let ϕ denote the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by the expression:

$$\phi(x^1, x^2) = (y^1, y^2) = (e^{x^1} + x^2,\, x^1 + e^{x^1}) \quad . \tag{1.23}$$

Show that ϕ is a diffeomorphism and find the Jacobian matrices $J_\phi(x^1, x^2)$ and $J_{\phi^{-1}}(y^1, y^2)$. Show that these Jacobian matrices are the inverses of each other.

EXERCISE 1.15

Let ϕ denote the map from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by the expression:

$$\phi(x^1, x^2, x^3) = (y^1, y^2, y^3) = (x^2, x^3, x^1 + x^2 + x^3) \quad . \tag{1.24}$$

Show that ϕ is a diffeomorphism and find the Jacobian matrices $J_\phi(x^1, x^2, x^3)$ and $J_{\phi^{-1}}(y^1, y^2, y^3)$. Show that these Jacobian matrices are the inverses of each other.

EXERCISE 1.16

Let ϕ denote the map from $\mathbb{R}^2$ into $\mathbb{R}^2$ given by:

$$\phi(x^1, x^2) = (y^1, y^2) = (x^1 + x^2,\ (x^1)^2 + (x^2)^2 + 2 \cdot x^1 \cdot x^2) \quad . \tag{1.25}$$

1. Show that ϕ is NOT a diffeomorphism of $\mathbb{R}^2$ onto its image in $\mathbb{R}^2$.

2. Show that ϕ is NOT a diffeomorphism of any open subset $\mathcal{U} \subset \mathbb{R}^2$ onto its image $\phi(\mathcal{U})$.

3. Find a function $f(x^1,x^2)$ and an open subset $\mathcal{U} \subseteq \mathbb{R}^2$ so that the following modified map IS a diffeomorphism of $\mathcal{U}$ onto the image $\psi(\mathcal{U})$:

$$\psi(x^1, x^2) = f(x^1,x^2) \cdot \phi(x^1,x^2) \quad , \quad (x^1,x^2) \in \mathcal{U} \quad . \tag{1.26}$$

Construct an explicit expression for the inverse $\psi^{-1}(y^1,y^2)$ and verify, that the corresponding Jacobian matrices for ψ and ψ^{-1} are also inverses of each other.

1.4 Parametrized surfaces in 3D space

In previous courses we have met an abundance of parametrized surfaces in 3D space. They are typically expressed by a smooth *regular map* r from a parameter domain $\mathcal{U}^2$ into $\mathbb{R}^3$ (see also Chapter 0):

$$\mathcal{S} : r(x^1,x^2) = (f(x^1, x^2), g(x^1, x^2), h(x^1, x^2)) \quad , \quad (x^1, x^2) \in \mathcal{U} \quad , \tag{1.27}$$

where f, g, and h are given smooth functions of two variables, see the example below.

The space $\mathbb{R}^3$ is used to coordinatize the points in Euclidean 3D space via a usual choice of a Cartesian coordinate system $\{O,x,y,z\}$, where O is the choice of origin of the coordinate system in space, and x, y, z mark the positive oriented choice of pairwise orthogonal coordinate axes.

We will pay special attention to the parameter domain $\mathcal{U}$ of a given parametrization of a given surface (or a given part of a surface). But, if ϕ is a diffeomorphism, $\phi : \mathcal{U} \longmapsto \mathcal{V}$, then we may as well use $\mathcal{V}$ as the parameter domain of the same (part of the) surface: To be specific, let $\phi(x^1,x^2) = (y^1,y^2)$ as before. Then the following map ρ from $\mathcal{V}$ into Euclidean 3D space covers precisely the same surface $\mathcal{S}$ as does the vector function r:

$$\rho(y^1,y^2) = r(\phi^{-1}(y^1,y^2)), \quad (y^1, y^2) \in \mathcal{V} \quad . \tag{1.28}$$

For both ρ and r we have the corresponding well-known (from calculus) Jacobian matrices $J_r(x^1,x^2)$ and $J_\rho(y^1,y^2)$

$$J_r(x^1,x^2) = \left[\left(\frac{\partial r}{\partial x^1}\right)^* \ \left(\frac{\partial r}{\partial x^2}\right)^* \right] = \begin{bmatrix} f'_{x^1} & f'_{x^2} \\ g'_{x^1} & g'_{x^2} \\ h'_{x^1} & h'_{x^2} \end{bmatrix} \quad . \tag{1.29}$$

$$J_{\rho}(y^1,y^2) = \left[\left(\frac{\partial \rho}{\partial y^1} \right)^* \left(\frac{\partial \rho}{\partial y^2} \right)^* \right] \quad . \tag{1.30}$$

They are not to be confused with our previous Jacobian matrices of the diffeomorphisms ϕ and ϕ^{-1} between the coordinate domains $\mathcal{U}$ and $\mathcal{V}$. In the present case (of parametrized surfaces) the coordinate domains are 2-dimensional, so $J_r(x^1,x^2)$ and $J_{\rho}(y^1,y^2)$ are (3×2)-matrices whereas $J_{\phi}(x^1,x^2)$ and $J_{\phi^{-1}}(y^1,y^2)$ are (2×2)-matrices. However, the Jacobians are, of course, related; from the chain rule we get:

$$J_{\rho}(y^1,y^2) = J_r(\phi^{-1}(y^1,y^2)) \cdot J_{\phi^{-1}}(y^1,y^2) \tag{1.31}$$

and similarly directly:

$$J_r(x^1,x^2) = J_{\rho}(\phi(x^1,x^2)) \cdot J_{\phi}(x^1,x^2) \quad . \tag{1.32}$$

It follows naturally, that the map ρ is regular if and only if r is regular – see exercise 1.18 below.

|||| Example 1.17

A (slit) paraboloid of revolution $\mathcal{P}$ is parametrized as follows via the domain $\mathcal{U} = \mathcal{U}_3^2$, i.e $\mathbb{R}^2$ minus the non-positive part of the x^1-axis: $r : \mathcal{U} \longmapsto \mathbb{R}^3$ given by the following expression:

$$r(x^1,x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2) \quad , \quad (x^1,x^2) \in \mathcal{U} \quad . \tag{1.33}$$

Now let $\phi : \mathcal{U} \longmapsto \mathcal{V} = \phi(\mathcal{U})$ denote the diffeomorphism that we analyzed in example 1.6, i.e.:

$$\phi(x^1,x^2) = (y^1,y^2) = \left(\sqrt{(x^1)^2 + (x^2)^2} \, , \, \arg(x^1 + i \cdot x^2) \right) \quad . \tag{1.34}$$

Then the combined map $r(\phi^{-1}) : \mathcal{V} \longmapsto \mathcal{U} \longmapsto \mathbb{R}^3$ with the concrete expression

$$\rho(y^1,y^2) = r(\phi^{-1}(y^1,y^2)) = (y^1 \cdot \cos(y^2), y^1 \cdot \sin(y^2), (y^1)^2) \quad , \quad (y^1,y^2) \in \mathcal{V} = \phi(\mathcal{U}) \quad . \tag{1.35}$$

is another regular parametrization of the very same (slit) paraboloid of revolution, see figure 1.4.

|||| EXERCISE 1.18

How can we be sure, that the parametrizations in (1.33) and (1.35) are indeed *regular*? Express the regularity-condition in terms of the Jacobian matrix J_r for the vector function r.

1.5 Curves in $\mathcal{U}$, in $\mathcal{V}$, and on a surface

We consider a regular parametrized smooth curve γ in $\mathcal{U}^2 \subseteq \mathbb{R}^2$ with a given expression

$$\gamma(t) = (\gamma^1(t), \gamma^2(t)) \quad , \quad t \in I \quad , \tag{1.36}$$

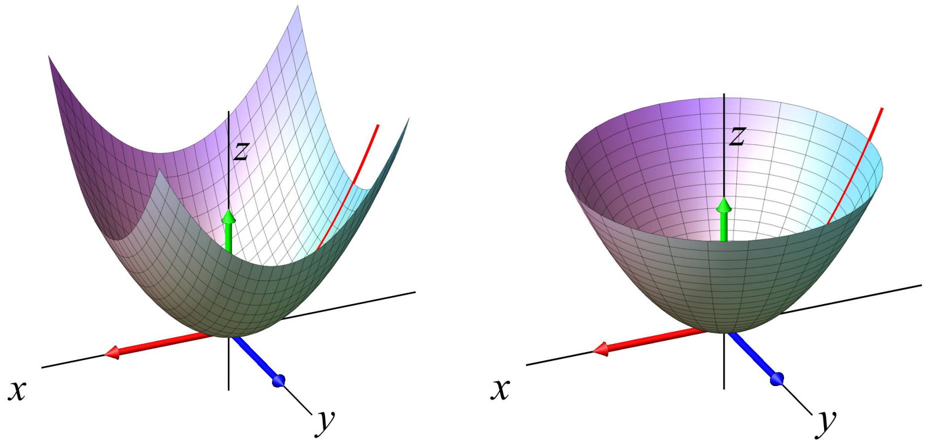


Figure 1.4: Portions of the same (slit) paraboloid of revolution parametrized in the two different ways defined in example 1.17. The left hand display is constructed from a slit square as $r(\Omega)$, where $\Omega = \mathcal{U} \cap (]-1,1[\times]-1,1[) \subset \mathcal{U}$, whereas the right hand display is constructed from a non-slit open rectangle in $\mathcal{V}$ as $\rho(\Lambda)$, where $\Lambda = \mathcal{V} \cap (]0,1[\times]-\pi,\pi[) \subset \mathcal{V}$. Note that the red curve on the surface, which projects to the non-positive part of the x-axis, $x \leq 0$, is *not* part of the surface.

where I denotes a specific t-interval, e.g. $I = [0,T]$ or $I = [a,b]$, depending on the context. Remember, that the curve γ is a regular curve if $\gamma'(t) \neq 0$ for all $t \in I$. The image of the curve γ by a diffeomorphism ϕ from $\mathcal{U}$ onto $\mathcal{V} = \phi(\mathcal{U})$ is then a regular curve η in $\mathcal{V}$. We have the following expression for η in terms of the parameter t:

$$\eta(t) = \phi(\gamma(t)) = \left(\phi^1(\gamma^1(t),\gamma^2(t)),\ \phi^2(\gamma^1(t),\gamma^2(t))\right) \quad , \quad t \in I \quad . \tag{1.37}$$

EXERCISE 1.19

How can we be sure, that $\eta = \phi(\gamma)$ is a *regular* curve?

Both of the two curves γ and η are mapped onto *the same curve* on the surface $\mathcal{S}$ in (1.27) by r and by $\rho = r(\phi^{-1})$, respectively, i.e. $r(\gamma) = r(\phi^{-1}(\eta)) = \rho(\eta)$.

Example 1.20

In continuation of example 1.17 we now consider a very simple specific curve γ in $\mathcal{U}$ and lift it (map it) to the slit paraboloid via r, see figure 1.5.

$$\gamma(t) = (\cos(t),\sin(t)) \quad , \quad t \in [-\pi/2,\pi/2] \quad . \tag{1.38}$$

Then

$$\eta(t) = (1,t) \quad , \quad t \in [-\pi/2,\pi/2] \quad , \tag{1.39}$$

and the ensuing curve on the surface $\mathcal{P}$ is parametrized as follows:

$$r(\gamma(t)) = r(\phi^{-1}(\eta(t))\rho(\eta(t)) = (\cos(t), \sin(t), 1) \quad , \quad t \in [-\pi/2, \pi/2] \quad . \tag{1.40}$$

Although we are considering one and the same curve *on the surface*, their representations γ and η in their respective parameter domains $\mathcal{U}$ and $\mathcal{V}$ are quite different, see also figure 1.6.

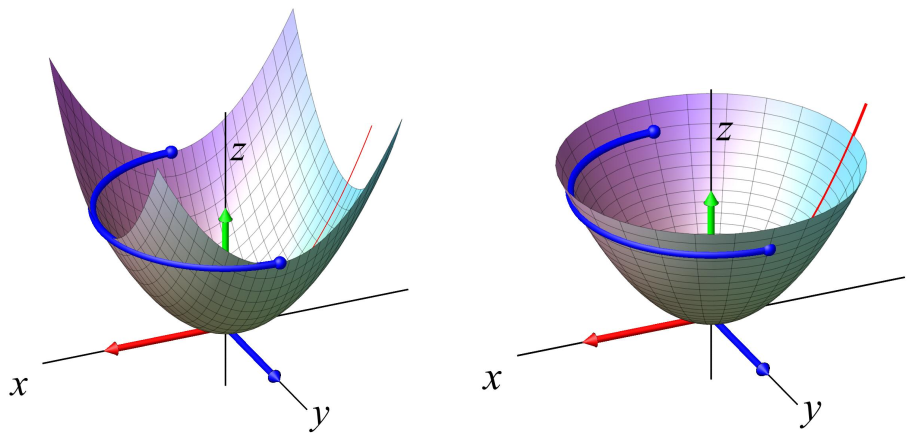


Figure 1.5: The two different parametrizations of the slit paraboloid – now with the (blue) curve $r(\gamma)$, which, by construction, is the same curve as $r(\phi^{-1}(\eta)) = \rho(\eta)$, see example 1.20.

1.6 Length of curves

We are now ready to study the length of regular curves on a regular parametrized surface $\mathcal{S}$ – from the viewpoint of the two different ϕ-related *parameter domains* $\mathcal{U}$ and $\mathcal{V}$ for the surface.

Let γ denote a smooth, regular, $t \in [a,b]$-parametrized curve in $\mathcal{U}$ with $\eta = \phi(\gamma)$ in $\mathcal{V}$. The image of γ on the regular surface $\mathcal{S}$ is then obtained via the regular surface map r, so that the length of the curve *on the surface* is given by the usual formula for lengths of curves in Euclidean 3D space:

$$L_E(r(\gamma)) = \int_a^b \|\frac{d}{dt} r(\gamma(t))\|_E \, dt \quad , \tag{1.41}$$

where $\|X\|_E$ denotes the Euclidean length of the vector X in $(\mathbb{R}^3, \cdot)$ with the usual dot-product.

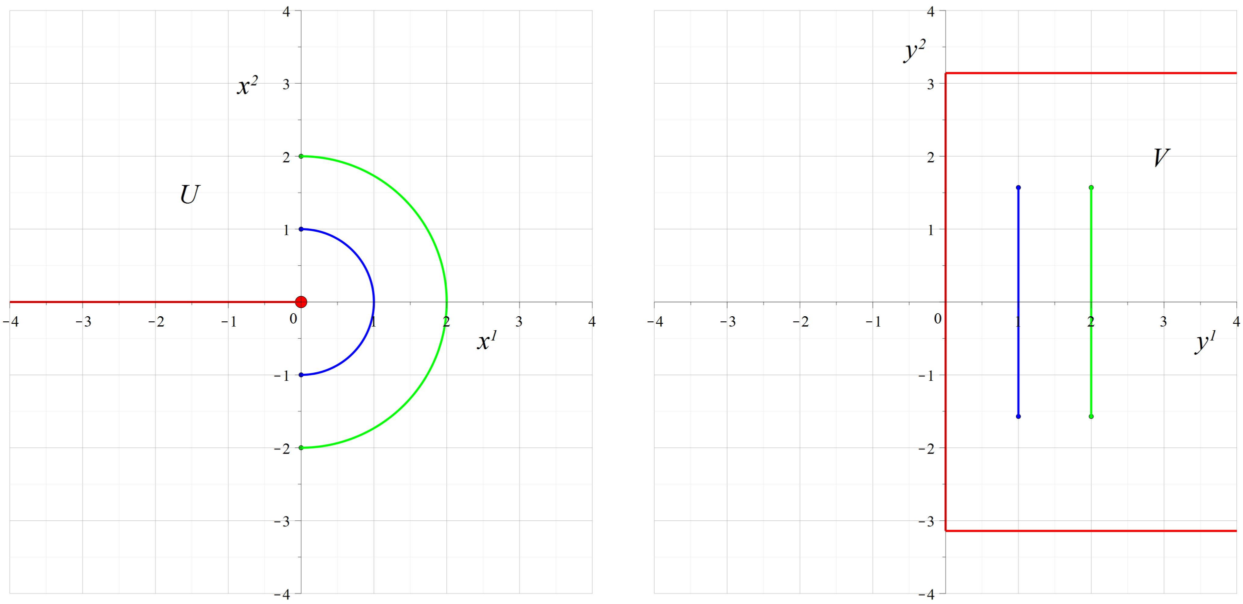


Figure 1.6: The two curves γ and η (in blue) in their respective parameter domains $\mathcal{U}$ and $\mathcal{V}$ indicated – see definitions in example 1.20. Also shown are two other curves (in green). Those green curves also have the same common image curve (not shown in figure 1.5) on $\mathcal{P}$.

For this we then calculate – using again the chain rule – and spelling everything out in detail:

$$\begin{aligned} \frac{d}{dt} r(\gamma(t)) &= \frac{d}{dt}(r(\gamma^1(t), \gamma^2(t))) \\ &= (\gamma^1)'(t) \cdot \frac{\partial r}{\partial x^1} + (\gamma^2)'(t) \cdot \frac{\partial r}{\partial x^2} \\ &= \left(J_r(\gamma(t)) \cdot \left[\begin{array}{c} (\gamma^1)'(t) \\ (\gamma^2)'(t) \end{array} \right] \right)^* \\ &= (J_r(\gamma(t)) \cdot (\gamma'(t))^*)^* \\ &= \gamma'(t) \cdot J_r^*(\gamma(t)) \quad . \end{aligned} \tag{1.42}$$

At this point we observe the (re-)appearance of the two tangent vectors in $\mathbb{R}^3$ to the coordinate curves of the parametrized surface : $\frac{\partial r}{\partial x^1}$ and $\frac{\partial r}{\partial x^2}$. When the parametrization is regular these two vectors span the 2-dimensional tangent plane of the surface in $\mathbb{R}^3$ at the point $r(x^1, x^2)$ – they constitute a basis of the vector space of all tangent vectors to the surface at that point.

Suppose we introduce the two standard basis vectors e_1 and e_2 in $\mathbb{R}^2$. Then γ' has the following coordinates with respect to that basis:

$$\gamma'(t) = (\gamma^1)'(t) \cdot e_1 + (\gamma^2)'(t) \cdot e_2 = \big((\gamma^1)'(t)\,,\,(\gamma^2)'(t)\big)_{\{e_1\,,\,e_2\}} \quad . \tag{1.43}$$

Then the equation (1.42) (in its second line) simply says that the coordinates of $\gamma'(t)$ with respect to the basis $\{e_1\,,\,e_2\}$ are exactly the same as the coordinates of $\frac{d}{dt}r(\gamma(t))$ with respect to the tangent plane basis $\{\frac{\partial r}{\partial x^1}\,,\,\frac{\partial r}{\partial x^2}\}$ for the surface at the point $r(\gamma(t))$. Later, in Chapter 8, we will study surfaces in $\mathbb{R}^3$ that are – in a sense to be made precise there – locally diffeomorphic to a parameter domain via suitable 'inverses' of their local parametrizations $r(x^1,x^2)$. Every regular smooth curve $\sigma(t)$ on the surface then has a local unique pre-image curve $\gamma(t)$ in the parameter domain. Moreover, to repeat, the tangent vector $\sigma'(t)$ of the surface curve at $\sigma(t)$ has the same coordinates with respect to $\{\frac{\partial r}{\partial x^1}\,,\,\frac{\partial r}{\partial x^2}\}$ as has the tangent vector $\gamma'(t)$ with respect to $\{e_1\,,\,e_2\}$. And, not to forget, the basis vectors $\{\frac{\partial r}{\partial x^1}\,,\,\frac{\partial r}{\partial x^2}\}$ are simply obtained by applying the Jacobian matrix $J_r(x^1,x^2)$ to the basis vectors in $\{e_1\,,\,e_2\}$, i.e. the basis vectors $\{\frac{\partial r}{\partial x^1}\,,\,\frac{\partial r}{\partial x^2}\}$ are the column vectors of the Jacobian matrix.

We get then:

$$\begin{aligned}
\|\frac{d}{dt}r(\gamma(t))\|_E^2 &= \left(\frac{d}{dt}r(\gamma(t))\right) \cdot \left(\frac{d}{dt}r(\gamma(t))\right) \\
&= \gamma'(t) \cdot J_r^*(\gamma(t)) \cdot (\gamma'(t) \cdot J_r^*(\gamma(t)))^* \\
&= \gamma'(t) \cdot (J_r^*(\gamma(t)) \cdot J_r(\gamma(t))) \cdot (\gamma'(t))^* \\
&= \gamma'(t) \cdot \begin{bmatrix} \frac{\partial r}{\partial x^1} \cdot \frac{\partial r}{\partial x^1} & \frac{\partial r}{\partial x^1} \cdot \frac{\partial r}{\partial x^2} \\ \frac{\partial r}{\partial x^2} \cdot \frac{\partial r}{\partial x^1} & \frac{\partial r}{\partial x^2} \cdot \frac{\partial r}{\partial x^2} \end{bmatrix} \cdot (\gamma'(t))^* \\
&= \gamma'(t) \cdot G_{\mathcal{U}}(x^1,x^2) \cdot (\gamma'(t))^* \quad .
\end{aligned} \tag{1.44}$$

Definition 1.21 The matrix appearing in this expression is what we will call the metric matrix function on $\mathcal{U}$ – in this case stemming from the surface S parametrized over $\mathcal{U}$. We will denote it $G_{\mathcal{U}}$, and when evaluated at a point (x^1,x^2) in $\mathcal{U}$ we write $G_{\mathcal{U}}(x^1,x^2)$:

$$G_{\mathcal{U}}(x^1,x^2) = \begin{bmatrix} \frac{\partial r}{\partial x^1} \cdot \frac{\partial r}{\partial x^1} & \frac{\partial r}{\partial x^1} \cdot \frac{\partial r}{\partial x^2} \\ \frac{\partial r}{\partial x^2} \cdot \frac{\partial r}{\partial x^1} & \frac{\partial r}{\partial x^2} \cdot \frac{\partial r}{\partial x^2} \end{bmatrix} = J_r^*(x^1,x^2) \cdot J_r(x^1,x^2) \quad . \tag{1.45}$$

Our paraboloid example will show an explicit expression for the metric matrix function $G_{\mathcal{U}}$ at (x^1,x^2) for this case:

Example 1.22

For the (slit) paraboloid $\mathcal{P}$, parametrized over $\mathcal{U}$, we have used the following parametrization:

$$r(x^1,x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2) \quad , \tag{1.46}$$

so that

$$G_{\mathcal{U}}(x^1,x^2) = \begin{bmatrix} 4(x^1)^2+1 & 4x^1 \cdot x^2 \\ 4x^1 \cdot x^2 & 4(x^2)^2+1 \end{bmatrix} \quad . \tag{1.47}$$

EXERCISE 1.23

Verify the expression for $G_{\mathcal{U}}(x^1,x^2)$ in (1.47).

In general, the length formula (1.41) – via (1.44) then reads in short form:

$$L_E(r(\gamma)) = \int_a^b \sqrt{\gamma'(t) \cdot G_{\mathcal{U}}(\gamma(t)) \cdot (\gamma'(t))^*}\, dt \quad . \tag{1.48}$$

EXERCISE 1.24

Show that the metric matrix $G_{\mathcal{U}}$ in (1.47) is not just symmetric but also positive definite.

Suppose that $G_{\mathcal{U}}(x^1,x^2)$ is given. What is then $G_{\mathcal{V}}(y^1,y^2)$ when $\phi(\mathcal{U}) = \mathcal{V}$, where ϕ is a diffeomorphism? The chain rule gives the answer:

$$\begin{aligned} G_{\mathcal{V}}(y^1,y^2) &= \begin{bmatrix} \frac{\partial r(\phi^{-1})}{\partial y^1} \cdot \frac{\partial r(\phi^{-1})}{\partial y^1} & \frac{\partial r(\phi^{-1})}{\partial y^1} \cdot \frac{\partial r(\phi^{-1})}{\partial y^2} \\ \frac{\partial r(\phi^{-1})}{\partial y^2} \cdot \frac{\partial r(\phi^{-1})}{\partial y^1} & \frac{\partial r(\phi^{-1})}{\partial y^2} \cdot \frac{\partial r(\phi^{-1})}{\partial y^2} \end{bmatrix} \\ &= J^*_{\phi^{-1}}(y^1,y^2) \cdot G_{\mathcal{U}}(x^1,x^2) \cdot J_{\phi^{-1}}(y^1,y^2) \\ &= J^*_{\phi^{-1}}(y^1,y^2) \cdot G_{\mathcal{U}}(\phi^{-1}(y^1,y^2)) \cdot J_{\phi^{-1}}(y^1,y^2) \quad . \end{aligned} \tag{1.49}$$

In short we may – and will – write this relation as:

$$G_{\mathcal{V}} = J^*_{\phi^{-1}} \cdot G_{\mathcal{U}}(\phi^{-1}) \cdot J_{\phi^{-1}} \quad . \tag{1.50}$$

EXERCISE 1.25

Verify the equations (1.49) – either in general, or just in the case of the slit paraboloid studied in the previous examples – over the open sets $\mathcal{U}$ and $\mathcal{V}$ – above.

Having shown (1.49) we also have the dual relation:

$$G_{\mathcal{U}}(x^1,x^2) = J^*_\phi(x^1,x^2) \cdot G_{\mathcal{V}}(\phi(x^1,x^2)) \cdot J_\phi(x^1,x^2) \quad . \tag{1.51}$$

In short we may – and will – also write this as:

$$G_{\mathcal{U}} = J^*_\phi \cdot G_{\mathcal{V}}(\phi) \cdot J_\phi \quad . \tag{1.52}$$

EXERCISE 1.26

Verify the equations (1.51) – either in general, directly from (1.49) (using proposition 1.11, or just in the case of the paraboloid studied in the previous examples.

Example 1.27

For the (slit) paraboloid $\mathcal{P}$ parametrized over $\mathcal{V}$ by

$$\rho(y^1,y^2) = (y^1 \cdot \cos(y^2),\, y^1 \cdot \sin(y^2)) \tag{1.53}$$

we get:

$$G_{\mathcal{V}}(y^1,y^2) = \begin{bmatrix} 4(y^1)^2+1 & 0 \\ 0 & (y^1)^2 \end{bmatrix} \quad . \tag{1.54}$$

EXERCISE 1.28

Verify this expression for $G_{\mathcal{V}}(y^1,y^2)$.

In the general setting, the $\mathcal{V}$-expression for the length of $\rho(\eta) = r(\phi^{-1}(\eta))$ is the following, which then, of course, must be the same as the length of $r(\gamma)$ since they are the same curve on the surface in question:

$$L_E(\rho(\eta)) = L_E(r(\phi^{-1}(\eta))) = \int_a^b \sqrt{\eta'(t) \cdot G_{\mathcal{V}}(\eta(t)) \cdot (\eta'(t))^*}\, dt \quad . \tag{1.55}$$

EXERCISE 1.29

Find the length of the blue curves in figure 1.5 by direct calculation of both $L_E(r(\gamma))$ and $L_E(r(\phi^{-1}(\eta)))$ from the respective formulas (1.48) and (1.55) – using r, γ, η, ϕ^{-1}, $G_{\mathcal{U}}$, and $G_{\mathcal{V}}$ as presented in the previous examples and exercises concerning the slit paraboloid $\mathcal{P}$.

EXERCISE 1.30

Show that the blue curve in figure 1.5 is not the shortest curve between its endpoints. Hint: You may want to use a circle, a parabola, or a broken straight line to connect the two endpoints in the $\mathcal{U}$-representation (or in the $\mathcal{V}$ representation) of the blue curve and then calculate the competing lengths via one of the formulas (1.48) or (1.55).

Observe, that the relations (1.50) and (1.52) do not refer to the actual surface – except via the construction of $G_{\mathcal{U}}$ in equation (1.51). In other words, given $G_{\mathcal{U}}$ and the diffeomorphism ϕ, we can find the matrix $G_{\mathcal{V}}$ directly from this formula without knowing more about the surface than what is encoded into $G_{\mathcal{U}}$.

Correspondingly there should be a more direct way to prove the relation (1.52) just from the key property that $G_{\mathcal{V}}$ and $G_{\mathcal{U}}$ must give the same length of ϕ-related curves γ and η:
We let $\eta(t) = \phi(\gamma(t))$ as above. Then

$$\eta'(t) = \left(J_\phi(\gamma(t)) \cdot (\gamma'(t))^*\right)^* = \gamma'(t) \cdot J_\phi^*(\gamma(t)) \quad . \tag{1.56}$$

The similarity between equation (1.56) and equation (1.42) is, of course, no coincidence. Think about it!

EXERCISE 1.31

Prove 1.56. Hint: The chain rule.

From the same-length-observation we have that the length integrands must be identical, otherwise the lengths themselves, the integrals, cannot in general be the same:

$$\gamma'(t) \cdot G_{\mathcal{U}} \cdot (\gamma'(t))^* = \eta'(t) \cdot G_{\mathcal{V}}(\phi) \cdot (\eta'(t))^* \quad . \tag{1.57}$$

We insert $\eta'(t)$ from (1.56) on the right hand side of the above equation and get:

$$\begin{aligned} \gamma'(t) \cdot G_{\mathcal{U}} \cdot (\gamma'(t))^* &= \left(J_\phi(\gamma(t)) \cdot (\gamma'(t))^*\right)^* \cdot G_{\mathcal{V}}(\phi) \cdot \left(\left(J_\phi(\gamma(t)) \cdot (\gamma'(t))^*\right)^*\right)^* \\ &= \left(J_\phi(\gamma(t)) \cdot (\gamma'(t))^*\right)^* \cdot G_{\mathcal{V}}(\phi) \cdot J_\phi(\gamma(t)) \cdot (\gamma'(t))^* \\ &= \gamma'(t) \cdot \left(J_\phi^* \cdot G_{\mathcal{V}}(\phi) \cdot J_\phi\right) \cdot (\gamma'(t))^* \quad , \end{aligned} \tag{1.58}$$

and, since this holds independent of the choice of curve γ, i.e. independent of the tangent vector $\gamma'(t)$, we get again:

$$G_{\mathcal{U}} = J_\phi^* \cdot G_{\mathcal{V}}(\phi) \cdot J_\phi \quad . \tag{1.59}$$

EXERCISE 1.32

We used the following claim: Two symmetric positive definite $n \times n$ matrices A and B are identical if they satisfy

$$V \cdot A \cdot V^* = V \cdot B \cdot V^* \quad \text{for all vectors} \quad V = (v^1, \cdots, v^n) \quad . \tag{1.60}$$

Give an argument for this claim. Is symmetry of A and B an important assumption for this claim to be true? Is positive definiteness of A and B an important assumption for the claim to be true?

1.7 Tangents to smooth curves in $\mathcal{U}$

We are now ready to formalize the findings and notations from the above sections.

In general we will *forget about the surfaces and their parametrizations* r and only work in the parameter domains $\mathcal{U}^n$ and with *given* positive definite symmetric matrix functions $G_{\mathcal{U}}$ on $\mathcal{U}^n$.

Some of the matrix functions that we will consider are naturally inherited from surfaces – as illustrated above with the slit paraboloid case – but not all. In this way we get the freedom to study many more and higher dimensional geometries than just the ones that come from surfaces in 3D space.

We first formalize precisely the tangent space of $\mathcal{U}^n$ at a point $p \in \mathcal{U}$ in the following way:

Definition 1.33 Since $\mathcal{U}^n$ is a subset of $\mathbb{R}^n$, the tangents of smooth curves in $\mathcal{U}$ through a given point p form a vector space of the same dimension n. The canonical basis vectors of $T_p\mathcal{U}$ are denoted by $e_1(p), \cdots, e_n(p)$. The tangent vector $e_i(p)$ is the tangent vector at p of the coordinate curve $(p^1, p^2, \cdots, p^{i-1}, p^i + t, p^{i+1}, \cdots, p^n)$ at p. Every tangent vector $V \in T_p\mathcal{U}$ has unique coordinates $(v^1, \cdots, v^n)$, so that

$$V = v^1 \cdot e_1(p) + \cdots + v^n \cdot e_n(p) \quad . \tag{1.61}$$

It is of importance to note, that each point in $\mathcal{U}$ in this way has its own tangent space and that any given tangent vector is associated to precisely one of the tangent spaces – see figure 1.7.

EXERCISE 1.34

Let $V = (3,2)$ in the tangent space $T_p\mathbb{R}^2$ where $p = (1,1)$. Construct an infinite family of *different* parametrized smooth curves γ_i, $i = 1, \cdots, \infty$ in $\mathbb{R}^2$, so that all the curves go through the point p and all the curves have the same tangent vector $V = (3,2)$ at p.

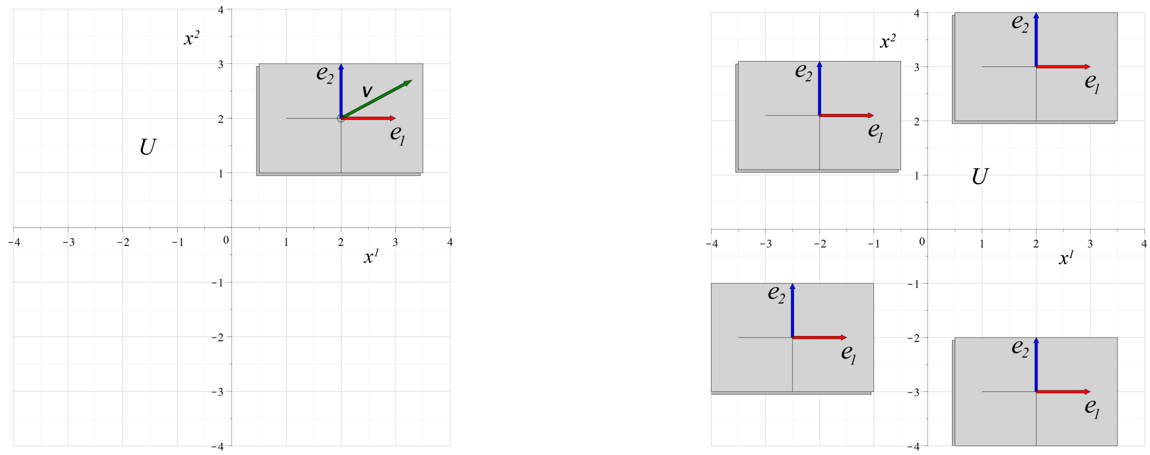


Figure 1.7: Left: One tangent plane $T_p\mathcal{U}^2$ with canonical basis vectors e_1 and e_2 and a tangent vector $V = v^1 \cdot e_1 + v^2 \cdot e_2$. Right: Tangent planes at various points in $\mathcal{U}$.

1.8 The metric tensor g from the matrix function $G_{\mathcal{U}}$

Let $G_{\mathcal{U}}$ be a positive definite symmetric $n \times n$-matrix function on an open subset $\mathcal{U}^n$ of $\mathbb{R}^n$. In each tangent space $T_p\mathcal{U}$ we define the metric tensor g_p as follows:

Definition 1.35 Let V and W denote arbitrary two vectors in $T_p\mathcal{U}$. Then the metric tensor g_p at p associated with the matrix function $G_{\mathcal{U}}$ at p is the following real valued function of the pair V and W:

$$g_p(V,W) = V \cdot G_{\mathcal{U}}(p) \cdot W^* \quad . \tag{1.62}$$

The defining property of a tensor is multilinearity:

EXERCISE 1.36

Show that the function g_p is linear in each of the two 'entries': For example, when W is kept fixed we have

$$g_p(V + k \cdot X, W) = g_p(V,W) + k \cdot g_p(X,W) \quad . \tag{1.63}$$

Notation 1.37 In the following we will often also use the following notation:

$$\langle V,W \rangle_p = g_p(V,W) \quad \text{for all } V \text{ and } W \text{ in } T_p\mathcal{U}. \tag{1.64}$$

Definition 1.38 Since the matrix function $G_{\mathcal{U}}(p) = G_{\mathcal{U}}(x^1, \cdots, x^n)$ is assumed to depend smoothly on the variables x^i in $\mathcal{U}$, the metric tensor $g_{(x^1,\cdots,x^n)}$ is also smooth. The metric tensor considered as a function of position is written as g or $g_{\mathcal{U}}$ and is called the metric tensor field on $\mathcal{U}^n$ associated with the matrix function $G_{\mathcal{U}}$.

At this point we can now have a second view towards the general notion of *indicatrices*, which was already illustrated and mentioned in figure 1.1, but now for any given metric tensor field $g_{\mathcal{U}}$ with its associated metric matrix function $G_{\mathcal{U}}$ on $\mathcal{U}^n$. They will help us to understand intuitively how the metric changes from point to point and eventually to understand some features of specific vector fields in $\mathcal{U}^n$ equipped with the metric $g_{\mathcal{U}}$. The formal definition of an indicatrix will be considered in definition 1.49 below. For now, the indicatrix of $g_{\mathcal{U}}$ at a given point p is the ellipse (or, in higher dimension, the ellipsoid) in the tangent space $T_p\mathcal{U}$ which is traced out by the g_{mU}-unit vectors, i.e. every position vector V in $T_p\mathcal{U}$ to a point on the indicatrix has $g_{\mathcal{U}}$-length $g_{\mathcal{U}}(V,V) = 1$, see figure 1.8. We will see briefly in section 1.12 below that if you know the indicatrix at p, then you also know the metric matrix function value $G_{\mathcal{U}}$ and thence the metric $g_{\mathcal{U}}$ at p.

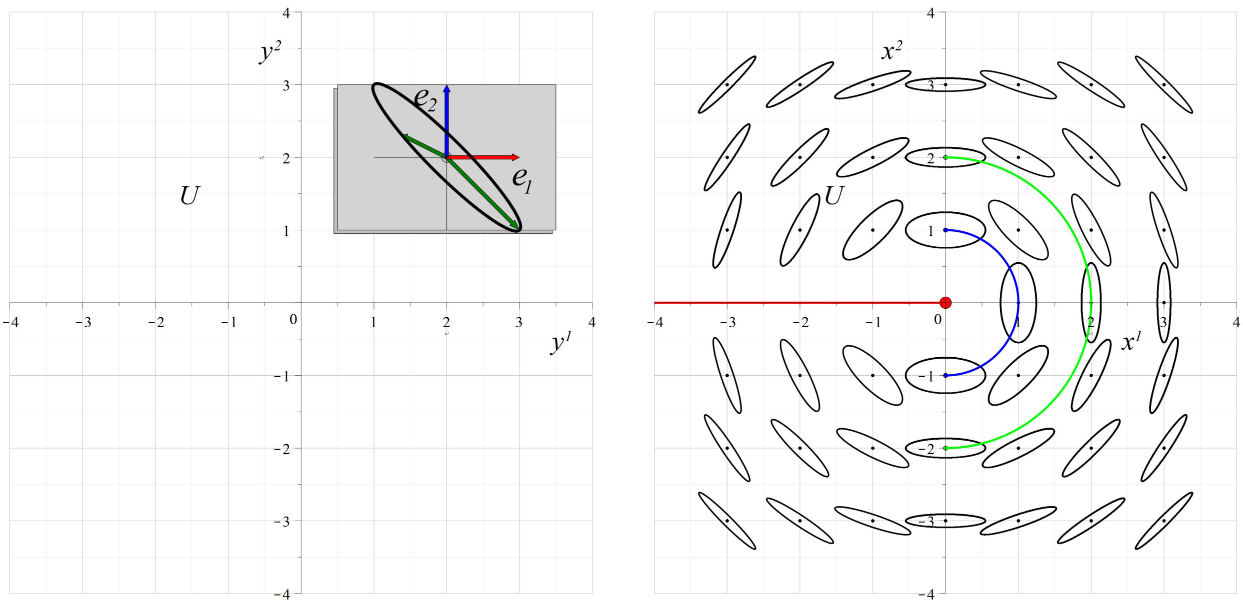


Figure 1.8: Left: One tangent plane with one indicatrix ellipse – the two green vectors have g-length 1 in the tangent plane – see definition 1.49. The position vectors to the ellipse all have g-length 1. Right: An indication of the full indicatrix field for the metric $g_{\mathcal{U}}$ in $\mathcal{U}$ – stemming from the slit paraboloid surface example $\mathcal{P}$ with its parametrization $r(x^1,x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2)$, see example 1.22.

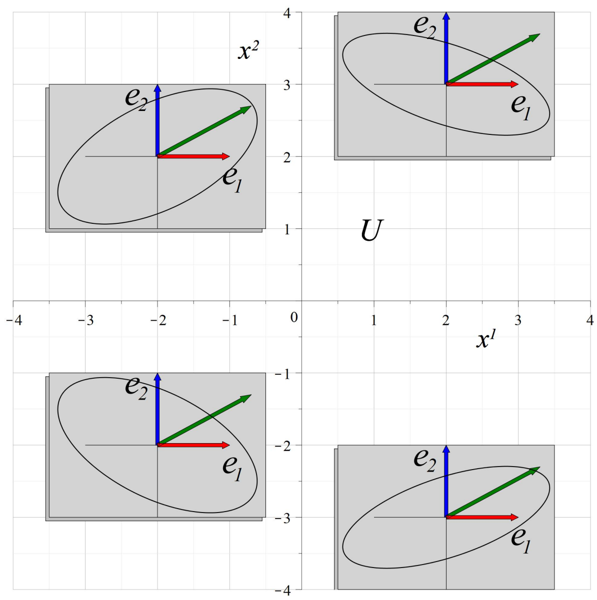


Figure 1.9: A first glimpse of a problem that we surely need to solve – and will solve in Chapter 3: Although the green vectors seem to be parallel transported versions of each other, they cannot be parallel because they clearly have different lengths with respect to the underlying metric that is represented by the respective ellipse indicatrices – and we want parallel transport of vectors to be length preserving.

The usual Euclidean dot product in $\mathbb{R}^n$ will now be denoted by g_E. For any two vectors V and U at $p \in \mathbb{R}^n$, i.e. V and U in $T_p\mathbb{R}^n$, with coordinates v^i and u^j with respect to the standard basis $\{e_1, \cdots, \mathrm{e}_n\}$, we have:

$$V = \sum_{i=1}^{i=n} v^i \cdot \mathrm{e}_i \quad , \quad U = \sum_{j=1}^{j=n} u^i \cdot \mathrm{e}_j$$

$$g_E(V,U) = V \cdot U = \sum_{k=1}^{k=n} v^k \cdot u^k = [v^1 \quad v^2 \quad \cdots \quad v^n] \cdot E \cdot \begin{bmatrix} u^1 \\ \cdot \\ \cdot \\ \cdot \\ u^n \end{bmatrix} \tag{1.65}$$

In other words, the constant metric matrix function G_E corresponding to g_E is the $(n \times n)$-identity matrix E. The indicatrix at a given point p is the unit sphere $\mathbb{S}_1^{n-1}$ with radius 1 centered at p – or, more precisely, centered at the origin of the tangent space $T_p\mathbb{R}^n$ at p.

1.9 Isometries and Local Riemannian Manifolds

Suppose now that we have two pairs $(\mathcal{U}^n, g_{\mathcal{U}})$ and $(\mathcal{V}^n, g_{\mathcal{V}})$ and a diffeomorphism ϕ such that the two matrix functions $G_{\mathcal{U}}$ and $G_{\mathcal{V}}$ associated with g_U and $g_{\mathcal{V}}$ are ϕ-related as in equations (1.50) and (1.52):

$$G_{\mathcal{V}} = J^*_{\phi^{-1}} \cdot G_{\mathcal{U}}(\phi^{-1}) \cdot J_{\phi^{-1}} \quad ,$$

and

$$G_{\mathcal{U}} = J^*_{\phi} \cdot G_{\mathcal{V}}(\phi) \cdot J_{\phi} \quad .$$

Then they both produce the same right hand side of the length integrals in equations (1.41) and (1.48):

$$\begin{aligned} L_{g_{\mathcal{U}}}(\gamma) &= \int_a^b \sqrt{g_{\mathcal{U}}(\gamma'(t), \gamma'(t))}\, dt \\ &= \int_a^b \sqrt{g_{\mathcal{V}}(\eta'(t), \eta'(t))}\, dt \\ &= L_{g_{\mathcal{V}}}(\eta) \quad . \end{aligned} \tag{1.66}$$

This common value we will from now on call the g-length of γ, or more precisely, the $g_{\mathcal{U}}$-length of γ. If we work in $\mathcal{V}$ we call it the $g_{\mathcal{V}}$-length of η.

Definition 1.39 Since the diffeomorphism ϕ in this way preserves g-lengths of curves if $g_{\mathcal{U}}$ and $g_{\mathcal{V}}$ are ϕ-related via (1.50) and (1.52), we will say that ϕ is an isometry between $(\mathcal{U}^n, g_{\mathcal{U}})$ and $(\mathcal{V}^n, g_{\mathcal{V}})$, in short: If such a diffeomorphism exists, then $(\mathcal{U}^n, g_{\mathcal{U}})$ and $(\mathcal{V}^n, g_{\mathcal{V}})$ are isometric.

To be isometric is an equivalence relation, and we can therefore define

Definition 1.40 A Local Riemannian Manifold, LRM^n, of dimension n is an equivalence class of isometric open subsets in $\mathbb{R}^n$ with metric tensors, $(\mathcal{U}, g_{\mathcal{U}})$.

EXERCISE 1.41

Look up, or find out, what *equivalence relations* and *equivalence classes* are all about, and show that the above *LRM*'s are well-defined in this way.

The main point of this abstraction is the following: If we want to calculate lengths of curves, areas or volumes of domains, and, as we shall see later, the curvatures of an *LRM*, then we can just do the calculations in any one of its representatives $(\mathcal{U}, g_{\mathcal{U}})$ using the corresponding particular metric tensor field.

Note that every Local Riemannian Manifold can be represented by just *one* pair $(\mathcal{U}^n, g_{\mathcal{U}})$. Any other (isometric) representation of the same *LRM* is obtained as $(\mathcal{V}, g_{\mathcal{V}})$, where ϕ is a diffeomorphism from the open set $\mathcal{U}$ onto $\mathcal{V}$, and where $g_{\mathcal{V}}$ is the metric tensor on $\mathcal{V}$ associated with the smooth metric matrix function $G_{\mathcal{V}}$ on $\mathcal{V}$ *constructed* via the equation (1.50) from the metric matrix function $G_{\mathcal{U}}$ associated with the given metric tensor $g_{\mathcal{U}}$. In other words, every choice of an open set $\mathcal{U}$ in $\mathbb{R}^n$ and a smooth symmetric positive definite matrix function $G_{\mathcal{U}}$ on $\mathcal{U}$ defines a unique Local Riemannian Manifold of dimension n which is represented by $(\mathcal{U}, g_{\mathcal{U}})$. And there are as many (isometric) representatives of the same *LRM* as there are diffeomorphisms on $\mathcal{U}$. The set of diffeomorphisms on an open set is huge – it is infinite dimensional, see Wiki: Diffeomorphism. If we are just given two matrix functions $G_{\mathcal{U}}(x^1, x^2)$ and $G_{\mathcal{V}}(y^1, y^2)$ it is usually difficult to decide if they are isometric via some diffeomorphism ϕ.

Throughout these notes we will use the monographs [4] by M. P. do Carmo, [19] by J. Lee, and [29] by Robbin and Salamon, respectively, as our main references. They are crisp and modern introductions to Riemannian geometry and they cover a great many geometric concepts, insights, and results that are beyond the scope of these notes.

1.10 Arc length (re-)parametrization

In this section we show in some detail how regular parametrized curves in a given Local Riemannian Manifold $(\mathcal{U}, g)$ can be re-parametrized so that they become arc-length parametrized.

We let $\gamma(t)$, $t \in I$, denote a given t-parametrized curve in $\mathcal{U}$, where I is an open connected interval in $\mathbb{R}$. We assume, that $\gamma(t)$ is a regular parametrization, so that $\|\gamma'(t)\|_g > 0$ for all $t \in I$. Now choose a point $\gamma(t_0)$ on the curve γ and let $S(t)$ denote the signed arc length of that segment of γ which is in between the two points $\gamma(t_0)$ and $\gamma(t)$ on γ:

$$S(t) = \int_{t_0}^{t} \|\gamma'(u)\|_g \, du \quad \text{for all } t \in I \quad . \tag{1.67}$$

Note that $S(t)$ is positive for $t > t_0$ and *negative* for $t < t_0$. The function is obviously increasing on the interval I, so there exists an inverse function $T(s)$ so that $T(s(t)) = t$ and $S(T(s)) = s$. The function $T(s)$ is also increasing on the corresponding S-interval, $S(I)$.
We now define the curve η as follows:

$$\eta(s) = \gamma(T(s)) \quad \text{for all } s \in S(I) \quad . \tag{1.68}$$

Obviously, the curve η then traces out the same path, the same curve, as γ and it is arc length parametrized, because of the following observations:

Since $T(s)$ and $S(t)$ are each others' inverse,

$$\begin{aligned} \frac{d}{dt}T(S(t)) &= 1 \\ S'(t)\cdot T'(s) &= 1 \quad , \end{aligned} \tag{1.69}$$

we get

$$\begin{aligned} \|\eta'(s)\|_g &= \|\frac{d}{ds}\eta(s)\|_g \\ &= \|\frac{d}{ds}\gamma(T(s))\|_g \\ &= |T'(s)|\cdot\|\frac{d}{dt}\gamma(t)_{|_{t=T(s)}}\|_g \\ &= |T'(s)|\cdot\|\gamma'(t)_{|_{t=T(s)}}\|_g \\ &= |T'(s)\cdot S'(t)| \\ &= 1 \quad , \end{aligned} \tag{1.70}$$

and therefore:

$$L_g(\eta([0,s])) = \int_0^s \|\eta'(u)\|_g\, du = s \quad . \tag{1.71}$$

To illustrate what is at stake we consider the following simplest possible curve and the steps that will reparametrize it to arc-length parametrization:

|||| Example 1.42

A straight line in the Euclidean plane $(\mathbb{R}^2, g_E)$ is analyzed:

$$\begin{aligned} \gamma(t) &= (3t-1,\, 3t-2,\, 0)\ ,\ t\in]-1,1[\\ \gamma'(t) &= (3,3,0) \\ \|\gamma'(t)\|_{g_E} &= 3\sqrt{2} \\ S(t) &= \int_{t_0=0}^{t} 3\sqrt{2}\, du \quad , \quad \text{from choosing } t_0 = 0 \\ S(t) &= 3\sqrt{2}\, t \\ T(s) &= \frac{s}{3\sqrt{2}} \quad , \quad \text{the inverse} \quad . \end{aligned} \tag{1.72}$$

Insert $T(s)$ for t in $\gamma(t)$. Then we get for all $s\in]S(-1),S(1)[=]-3\sqrt{2},\, 3\sqrt{2}[$:

$$\begin{aligned} \eta(s) &= \gamma(T(s)) = (3T(s)-1,\, 3T(s)-2,\, 0) \\ \eta(s) &= \left(\frac{s}{\sqrt{2}}-1,\, \frac{s}{\sqrt{2}}-2,\, 0\right) \quad . \end{aligned} \tag{1.73}$$

It is typically impossible to find one or both of the functions $S(t)$ and the inverse $T(s)$ expressed by known functions. Both functions can, however, be approximated to any desired precision e.g. by spline functions.

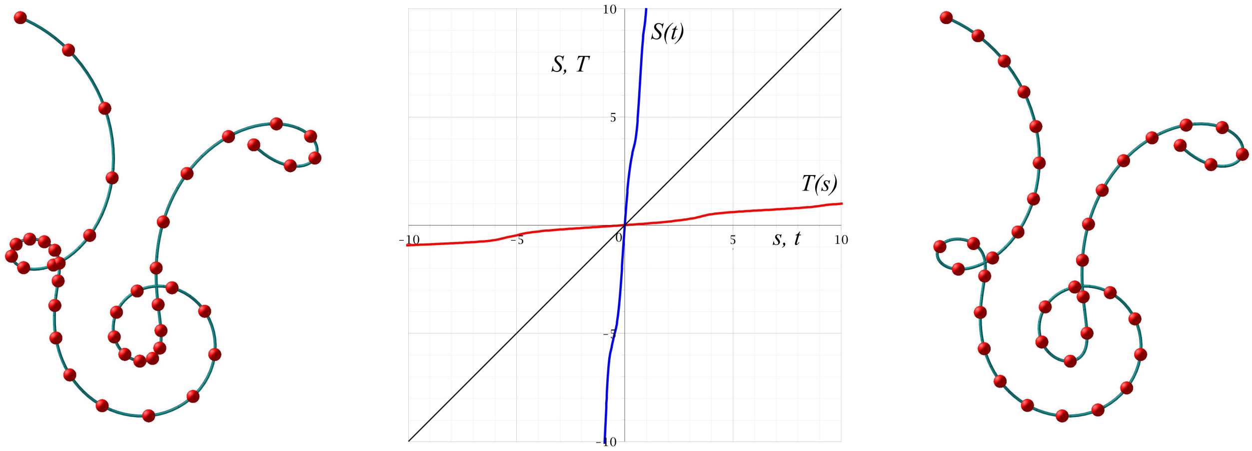


Figure 1.10: A planar curve in the Euclidean plane $(\mathbb{R}^2, g_E)$ is shown with at t-parametrization indicated to the left – with constant time difference between the marked points. The corresponding arc length parametrization for the same curve is indicated to the right – with constant arc length difference between the marked points. In the middle: The two functions $S(t)$ and $T(s)$ for s and t running through their respective intervals – both on the horizontal $'x'$-axis.

EXERCISE 1.43

Let $\gamma(t)$ denote the following curve in $(\mathbb{R}^2, g_E)$.

$$\gamma(t) = (t^3, 0, 0) \quad , \quad t \in \mathbb{R} \quad . \tag{1.74}$$

Show that this curve is *not* regularly parametrized by t. Show that it can, nevertheless, be reparametrized to become an arc length parametrized curve $\eta(s)$.

1.11 Angles, areas, and g-orthonormal bases

A metric tensor field $g = g_{\mathcal{U}}$ gives us much more than just lengths of vectors in tangent spaces $T_p\mathcal{U}$ (and thence of curves in $\mathcal{U}$), for example:

Definition 1.44 Let V and W be two vectors in the *same* tangent space $T_p\mathcal{U}$. The unique g-angle $\measuredangle_g(V,W) \in [0,\pi]$ between the two vectors is then given by the relation:

$$\cos\left(\measuredangle_g(V,W)\right) = \frac{g(V,W)}{\|V\|_g \cdot \|W\|_g} \quad , \tag{1.75}$$

where $\|V\|_g$ denotes $\sqrt{g(V,V)}$.

Definition 1.45 Let V and W be two vectors in the *same* tangent space $T_p\mathcal{U}$. The g-area $A_g(V,W)$ of the parallellogram spanned by the two vectors in $T_p\mathcal{U}$ is:

$$A_g(V,W) = \sqrt{\|V\|_g^2 \cdot \|W\|_g^2 - g(V,W)^2} \quad . \tag{1.76}$$

On the other hand, this powerful generality comes with a price. By now it should be clear, that we cannot directly compare vectors with each other *unless they belong to the same tangent space*.

Example 1.46

Consider, the metric induced from the paraboloid in example 1.22:

$$G_{\mathcal{U}}(x^1,x^2) = \begin{bmatrix} 4(x^1)^2+1 & 4x^1\cdot x^2 \\ 4x^1\cdot x^2 & 4(x^2)^2+1 \end{bmatrix} \quad . \tag{1.77}$$

Then we have for lengths of the basis vectors $e_1(x^1,x^2)$ and $e_2(x^1,x^2)$ in the tangent space $T_{(x^1,x^2)}\mathcal{U}$ at the point (x^1,x^2):

$$\begin{aligned} \|e_1(x^1,x^2)\|_g &= \sqrt{4(x^1)^2+1} \\ \|e_2(x^1,x^2)\|_g &= \sqrt{4(x^2)^2+1} \quad , \end{aligned} \tag{1.78}$$

so they are clearly not (in general) g-unit vectors.

EXERCISE 1.47

Find in this example the g-angle between e_1 and e_2 and the g-area of the parallellogram spanned by the two vectors. Express the angle and the area as functions of (x^1,x^2).

Of course there exist infinitely many pairs of vectors in each tangent space that *are* g-orthonormal and they are obviously quite useful for many purposes.

EXERCISE 1.48

Following the example 1.46: Construct for every point (x^1,x^2) in $\mathcal{U}$ two g-orthonormal vectors $E_1(x^1,x^2)$ and $E_2(x^1,x^2)$ in the tangent space $T_{(x^1,x^2)}\mathcal{U}$, meaning that (with $g = g_{\mathcal{U}}$):

$$\begin{aligned} &\|E_1(x^1,x^2)\|_g = \|E_2(x^1,x^2)\|_g = 1 \quad , \quad \text{and} \\ &g(E_1(x^1,x^2)\,,\,E_2(x^1,x^2)) = 0 \quad . \end{aligned} \tag{1.79}$$

Express $E_1(x^1,x^2)$ and $E_2(x^1,x^2)$ as a linear combinations of e_1 and e_2,so that e.g.:

$$E_1(x^1,x^2) = k^1(x^1,x^2)\cdot e_1 + k^2(x^1,x^2)\cdot e_2 \quad . \tag{1.80}$$

There are many solutions to this problem. Solve it so that the coefficient functions are smooth functions of (x^1, x^2).

1.12 From metric to indicatrix – and back

In an open set $\mathcal{U}^2$ in $\mathbb{R}^2$ we consider a given metric tensor g_p at the point $p = (x^1, x^2) \in \mathcal{U}$. The corresponding ellipse, ellipsoid, or indicatrix, *in the tangent space* $T_p\mathcal{U}$ is then defined as follows:

Definition 1.49 Let g_p denote the metric tensor corresponding to $G_{\mathcal{U}}(p)$. Then the indicatrix of g at p is:

$$I(p) = \{V \in T_p\mathcal{U} \,|\, g_p(V,V) = V \cdot G_{\mathcal{U}} \cdot V^* = 1\} \quad . \tag{1.81}$$

Since $G_{\mathcal{U}}(p)$ is positive definite, i.e. $g_p(V,V) > 0$ for all non-zero V in $T_p\mathcal{U}$, the indicatrix $I(p)$ is an ellipse.

EXERCISE 1.50

Prove that claim. Hint:

$$\begin{aligned} g_p(V,V) = V \cdot G_{\mathcal{U}}(p) \cdot V^* &= [v^1 \quad v^2] \cdot \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\ &= g_{11} \cdot (v^1)^2 + 2g_{12} \cdot v^1 \cdot v^2 + g_{22} \cdot (v^2)^2 \quad , \end{aligned} \tag{1.82}$$

so that $g_p(V,V) = 1$ is a second order polynomial equation in the variables v^1 and v^2. Since $G_{\mathcal{U}}$ is positive definite, the solutions $V = v^1 \cdot e_1 + v^2 \cdot e_2$ form an ellipse in the tangent space spanned by e_1 and e_2. More hints: See Math 1: See eNote 22: Quadratic equations.

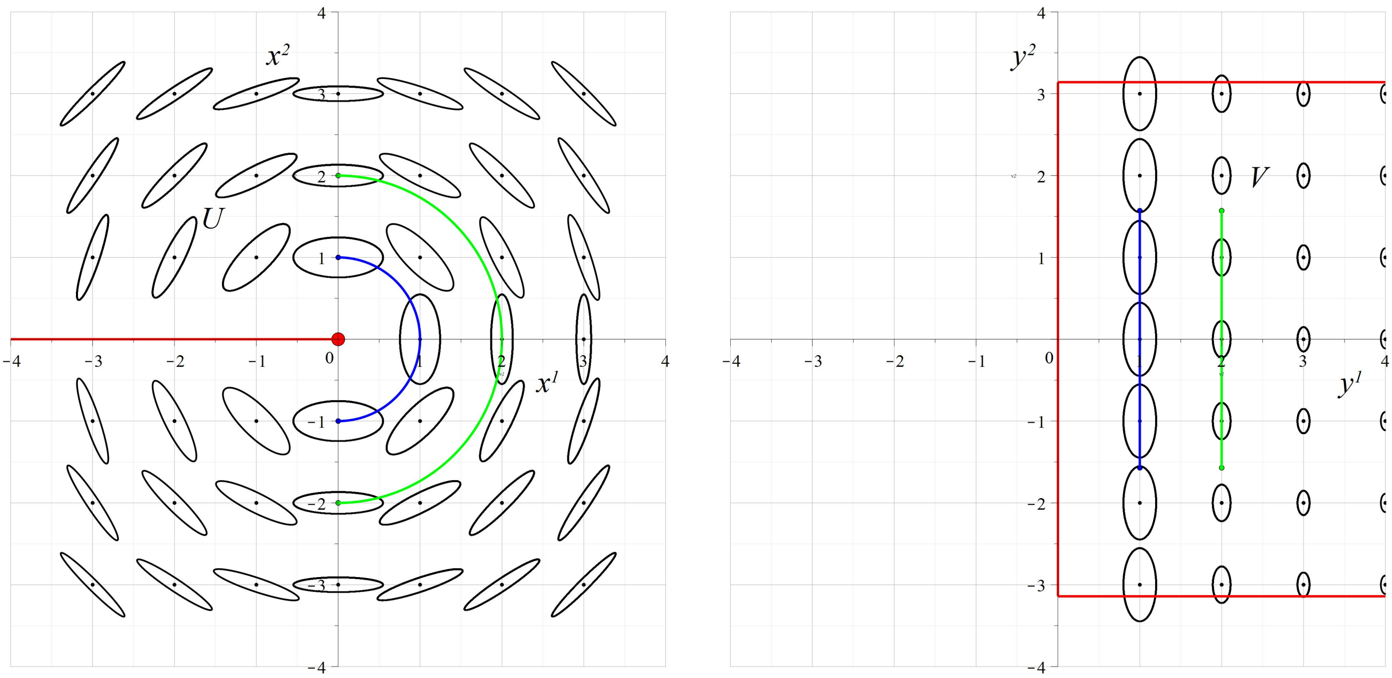


Figure 1.11: A comparison of the metric matrix fingerprints from $\mathcal{P}$ in $\mathcal{U}$ and $\mathcal{V}$, respectively. In both cases the green curve is clearly longer than the blue curve. The two green curves have the same length and the two blue curves have the same length. $(\mathcal{U}, g_{\mathcal{U}})$ and $(\mathcal{V}, g_{\mathcal{V}})$ are isometric and represent the same Local 2-dimensional Riemannian Manifold.

EXERCISE 1.51

Let $G_{\mathcal{U}}$ denote the following metric matrix (at some given point p):

$$G_{\mathcal{U}} = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix} \quad . \tag{1.83}$$

Find an equation for the corresponding indicatrix $I(p)$ (using tangent space variables v^1 and v^2). Find a parametrization of the indicatrix ellipse (using t as parameter) in the form $V(t) = f(t) \cdot e_1 + h(t) \cdot e_2$, where e_1 and e_2 are the canonical tangent space basis vectors at p and f and h are suitable functions of t.

EXERCISE 1.52

In a tangent space $T_p\mathcal{U}^2$ with canonical basis vectors e_1 and e_2 a parametrized indicatrix is given as follows:

$$V(t) = 3 \cdot \cos(t) \cdot e_1 + 7 \cdot \sin(t) \cdot e_2 \quad , \quad t \in [0, 2\pi] \quad . \tag{1.84}$$

Find the corresponding metric matrix $G_{\mathcal{U}}$ at p.

EXERCISE 1.53

In a tangent space $T_p\mathcal{U}^2$ with canonical basis vectors e_1 and e_2 a parametrized indicatrix is given as follows:

$$V(t) = \left(3\cdot\cos(t) - 7\cdot\sqrt{3}\cdot\sin(t)\right)\cdot e_1 + \left(3\cdot\sqrt{3}\cdot\cos(t) + 7\cdot\sin(t)\right)\cdot e_2 \quad , \quad t\in[0,2\pi] \quad . \tag{1.85}$$

Find the corresponding metric matrix $G_{\mathcal{U}}$ at p.

EXERCISE 1.54

In a tangent space $T_p\mathcal{U}^2$ with canonical basis vectors e_1 and e_2, and corresponding vector coordinates v^1 and v^2, an indicatrix is given by the following equation:

$$7\cdot(v^1)^2 + 6\cdot v^1\cdot v^2 + (v^2)^2 \quad . \tag{1.86}$$

Find the corresponding metric matrix $G_{\mathcal{U}}$ at p.

1.13 The Poincaré disk and half-plane

We consider two important geometric models, the Poincaré disk- and half plane models:
The disk model is $(\mathcal{U}, g_{\mathcal{U}})$, where:

$$\mathcal{U} = \{(x^1,x^2)\in\mathbb{R}^2 \,|\, ((x^1)^2 + (x^2)^2 < 1\,\}$$

$$G_{\mathcal{U}}(x^1,x^2) = \frac{4}{(1-(x^1)^2-(x^2)^2)^2}\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad . \tag{1.87}$$

The half plane model is $(\mathcal{V}, g_{\mathcal{V}})$, where:

$$\mathcal{V} = \{(y^1,y^2)\in\mathbb{R}^2 \,|\, y^2 > 0\,\}$$

$$G_{\mathcal{V}}(y^1,y^2) = \frac{1}{(y^2)^2}\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad . \tag{1.88}$$

The corresponding fingerprints, the respective indicatrix fields, are shown in figure 1.12.

EXERCISE 1.55

Let γ_1 and γ_2 denote the following two curves in *the disk model*:

$$\begin{aligned}\gamma_1(t) &= (0,\, 2\cdot t) \quad , \quad t\in[-1/4,\, 1/4]\\ \gamma_2(t) &= \left(\frac{1}{2},\, 2\cdot t\right) \quad , \quad t\in[-1/4,\, 1/4] \quad .\end{aligned} \tag{1.89}$$

Find the $g_{\mathcal{U}}$ length of these two curves. Construct another curve that connects the two endpoints of γ_2 but is shorter than γ_2.

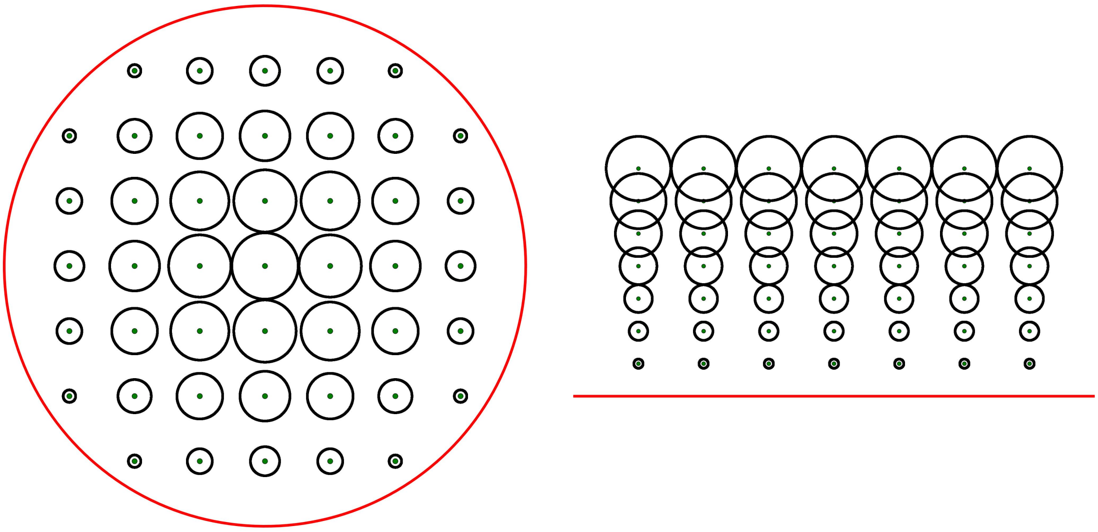


Figure 1.12: The unique (scaled) fingerprint indicatrix fields for the Poincaré metric in a disk and in a half plane, respectively. In these displays the two sets of unit fingerprints have been scaled down by factors of 4 and 7, repectively.

The two models are isometric, i.e. they are representatives for the same Local Riemannian Manifold (in fact it is a Global Riemannian manifold, because, as we shall see later, these models are complete – they cannot be extended). The actual isometry ϕ from the half plane model to the disk model is known as the Cayley transformation, see Wiki: Cayley Transform. It was shown by David Hilbert, that the Poincaré metrics cannot be realized from any surface in 3D, see Wiki: Hilbert's Theorem.

The Cayley mapping is most elegantly expressed in terms of complex variables: Suppose $z = y^1 + i \cdot y^2$, then the image by the Cayley map of z and thence of (y^1, y^2) in the halfplane model is (x^1, x^2) in the disk model, which is likewise represented by the complex number $w = x^1 + i \cdot x^2$ defined as follows:

$$w = \mathrm{Cay}(z) = \frac{z - i}{z + i} \quad . \tag{1.90}$$

The inverse mapping is then as simple:

$$z = \mathrm{Cay}^{-1}(w) = i \cdot \frac{1 + w}{1 - w} \quad . \tag{1.91}$$

In the real coordinates (y^1, y^2) and (x^1, x^2) the expressions for Cay and Cay^{-1} are thence the

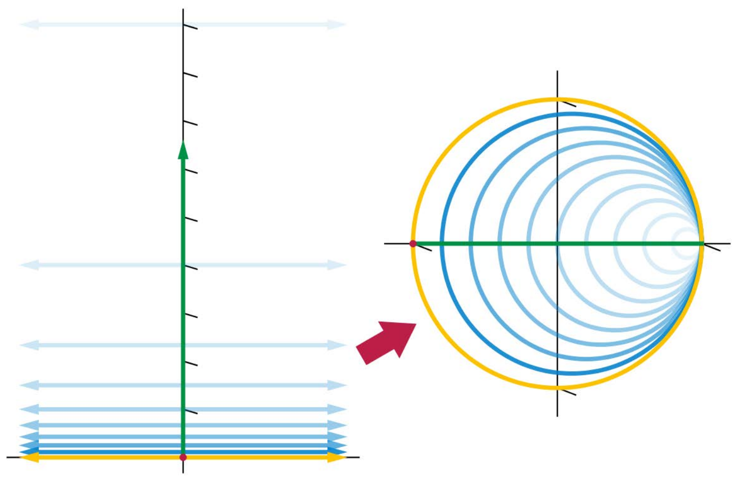


Figure 1.13: An indication of the Cayley transform ϕ^{-1} from the Poincaré half plane to the Poincaré disk. The Cayley map is an isometry between the two models, see exercise 1.56.

following:

$$\begin{aligned} \mathrm{Cay}(y^1,y^2) &= \left(\frac{1}{(y^1)^2+(1+y^2)^2}\right)\cdot\left((y^1)^2+(y^2)^2-1\,,\ -2(y^1)\right) \\ \mathrm{Cay}^{-1}(x^1,x^2) &= \left(\frac{1}{1+(x^1-x^2)^2}\right)\cdot\left(-2(x^2)\,,\ 1-(x^1)^2-(x^2)^2\right) \quad . \end{aligned} \tag{1.92}$$

In either case it is a fairly simple matter to show that Cay is indeed an isometry via the appropriate Jacobian:

$$G_{\mathcal{V}}(y^1,y^2) = J^*_{\mathrm{Cay}}(y^1,y^2)\cdot G_{\mathcal{U}}(\mathrm{Cay}(y^1,y^2))\cdot J_{\mathrm{Cay}}(y^1,y^2) \quad . \tag{1.93}$$

EXERCISE 1.56

Use one of the expressions above for the Cayley transfom to reconstruct the figure 1.13 – using your favorite graphics tool.

In the next example we show, that the so-called pseudosphere surface is isometric to subsets of the Poincaré models:

Example 1.57

The pseudosphere is the surface in 3D space obtained by revolving the following generator curve around the z-axis:

$$\eta(t) = (h(t)\,,\,0\,,\,f(t)) \quad t \geq 0 \quad , \tag{1.94}$$

where

$$\begin{aligned} h(t) &= \frac{1}{\cosh(t)} \\ f(t) &= t - \tanh(t) \quad . \end{aligned} \tag{1.95}$$

The simplest parametrization of the pseudosphere is then

$$r(t,v) = (h(t)\cdot\cos(v)\,,\,h(t)\cdot\sin(v)\,,\,f(t)) \quad , \quad t \in\,]0,\infty[\quad , \quad v \in [-\pi,\pi] \quad . \tag{1.96}$$

Note that

$$\|\eta'(t)\| = \tanh(t) \quad , \tag{1.97}$$

so that the above parametrization of η is not regular at $t = 0$. Correspondingly, the pseudosphere cannot be extended regularly to include negative values of the parameter t.

En passant we also note, that metric matrix function induced by the parametrization (1.96) is found to be:

$$G_r(t,v) = \begin{bmatrix} \tanh^2(t) & 0 \\ 0 & 1/\cosh^2(t) \end{bmatrix} \quad . \tag{1.98}$$

A more interesting – and a bit more complicated – parametrization of the same surface is the following:

$$\rho(y^1,y^2) = \left(h(\operatorname{arcosh}(y^2))\cdot\cos(x^1)\,,\,h(\operatorname{arcosh}(y^2))\cdot\sin(y^1)\,,\,f(\operatorname{arcosh}(y^2))\right) \quad , \tag{1.99}$$

where now $y^2 \in\,]1,\infty[$ and $y^1 \in [-\pi,\pi]$. Note that this parametrization clearly covers the pseudosphere bijectively; here we have $\operatorname{arcosh}(1) = 0$, so y^2 is necessarily constrained to $]1,\infty[$. With this parametrization ρ of the pseudosphere we obtain the following familiar metric matrix for the pseudosphere:

$$G_\rho(y^1,y^2) = \begin{bmatrix} 1/(y^2)^2 & 0 \\ 0 & 1/(y^2)^2 \end{bmatrix} \quad . \tag{1.100}$$

Therefore the pseudosphere is isometric to the following (lifted) half strip in the Poincaré half plane:

$$\mathcal{V}_P = \{(y^1,y^2) \in \mathcal{V} = \mathbb{R}\times\mathbb{R}_+ \,|\, y^2 > 1\} \quad , \tag{1.101}$$

and thence it is isometric to the 'wedge' $\mathcal{U}_P = \phi^{-1}(\mathcal{V}_P)$ in the Poincaré disk, where ϕ^{-1} denotes the isometric Cayley transform from $\mathcal{V}$ onto $\mathcal{U}$, see figure 1.14.

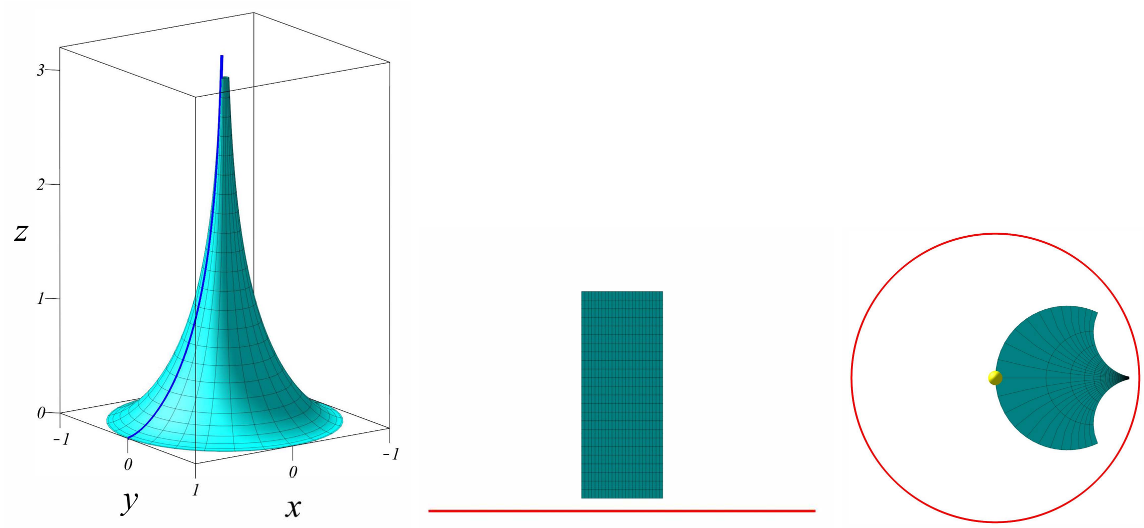


Figure 1.14: Left: The first of the two considered parametrizations of the pseudosphere – truncated from above at $z = 3$. Middle to right: The pseudosphere is isometrically represented in the Poincaré halfplane and in the Poincaré disk, respectively.

1.14 Isometric surfaces in 3D space

We have seen – at length – above in the main example 1.6 concerning the slit Paraboloid how a surface in 3D in a natural way gives rise to a Local Riemannian Manifold of dimension 2. And also indicated (via Hilbert's theorem) that not every 2-dimensional Local Riemannian Manifold can be so obtained from a surface in 3D space.

To complete the picture we now illustrate by example, how different surfaces in 3D can give rise to the *same* Local Riemannian Manifold – so that these surfaces in this precise sense are locally isometric surfaces.

Example 1.58

We let $\mathcal{W}$ denote the following open set in $\mathbb{R}^2$

$$\mathcal{W} = \{ \left(x^1, x^2\right) \in \mathbb{R}^2 \,|\, -\pi < x^2 < \pi \} \quad , \tag{1.102}$$

and consider the following family of surfaces $\mathcal{H}_t$ parametrized by $(x^1,x^2) \in \mathcal{W}$ and $t \in]-\pi, \pi[$ as follows:

$$\mathcal{H}_t \quad : \quad r_t(x^1,x^2) = (x_t(x^1,x^2), y_t(x^1,x^2), z_t(x^1,x^2)) \quad , \tag{1.103}$$

where the 3 coordinate functions for the vector function r_t are respectively:

$$\begin{aligned} x_t(x^1,x^2) &= \cos(t)\cdot\sinh(x^1)\cdot\sin(x^2) + \sin(t)\cdot\cosh(x^1)\cdot\cos(x^2)\,, \\ y_t(x^1,x^2) &= -\cos(t)\cdot\sinh(x^1)\cdot\cos(x^2) + \sin(t)\cdot\cosh(x^1)\cdot\sin(x^2)\,, \\ z_t(x^1,x^2) &= x^2\cdot\cos(t) + x^1\cdot\sin(t) \quad . \end{aligned} \tag{1.104}$$

Two of the surfaces, $\mathcal{H}_0$ (the helicoid), $\mathcal{H}_{\pi/2}$ (the catenoid) and the intermediate surface $\mathcal{H}_{\pi/4}$ are displayed in figure 1.15. The boundaries that are not part of the surfaces are indicated in red – corresponding to the two boundary components $x^2 = \pm\pi$ in the parameter domain $\mathcal{W}$. An animation of the family can be found in Wiki: Catenoid. All members of the family are examples of so-called minimal surfaces, see Wiki: Minimal Surfaces.

As already alluded to, all the surfaces $\mathcal{H}_t$ are pairwise locally isometric – they are surface representations of one and the same Local Riemannian Manifold $(\mathcal{W}, g_{\mathcal{W}})$ with the following metric matrix function for all $t \in]-\pi, \pi[$:

$$G_{\mathcal{W}}(x^1,x^2) = \cosh^2(x^1) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{1.105}$$

EXERCISE 1.59

Show that the metric matrix function for all members of the surface family in example 1.58 is independent of t and is given by (1.105).

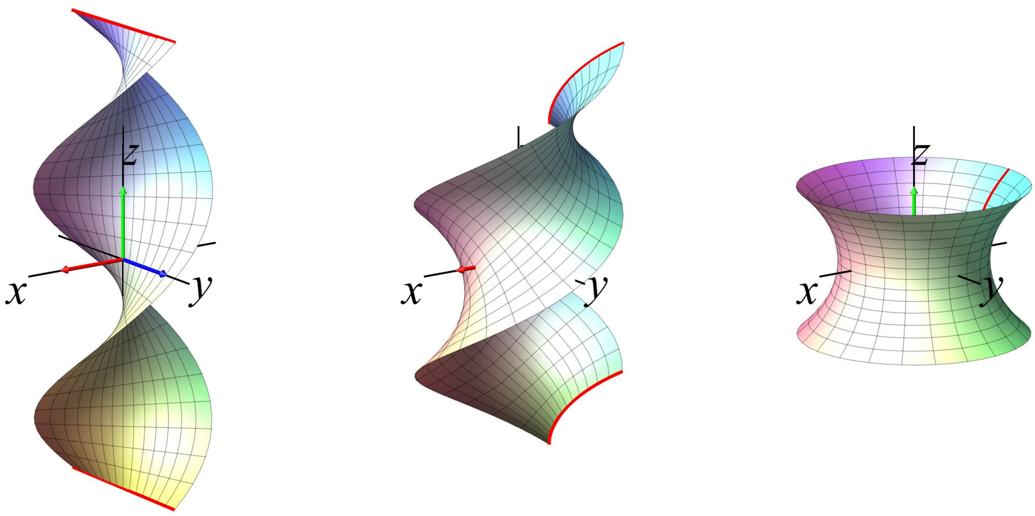


Figure 1.15: Three members of the family of pairwise isometric minimal surfaces $\mathcal{H}_t$, $t \in]-\pi, \pi[$.

EXERCISE 1.60

We consider again the same parameter domain as in example 1.58:

$$\mathcal{W} = \{(x^1, x^2) \in \mathbb{R}^2 \,|\, -\pi < x^2 < \pi\} \quad , \tag{1.106}$$

and consider the following simple family of (ribbon) surfaces Q_t parametrized by $(x^1,x^2) \in \mathcal{W}$ and

$t \in]1, \infty[$ as follows:

$$Q_t \quad : \quad r_t(x^1, x^2) = \left(t \cdot \cos(x^2/t) - t\,,\, t \cdot \sin(x^2/t)\,,\, x^1\right) \quad . \tag{1.107}$$

Show that all the surfaces in the family Q_t are isometric to each other and give a geometric description of the family.

Further experiments and results concerning ribbon surfaces are reported in [3] and [28].

1.15 Outlook: Statistics on a Riemannian manifold

Suppose a number of points are given on the paraboloid $\mathcal{P}$ as illustrated in figure 1.16. The points are clearly distributed and elongated in the vertical direction along a meridian curve on the paraboloid. This curve obviously carries great significance for the statistical analysis of the point distribution.

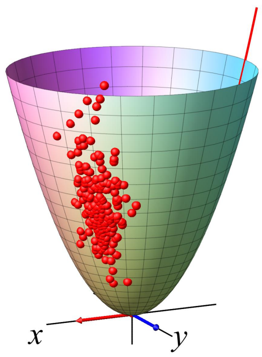


Figure 1.16: Point set distribution on $\mathcal{P}$.

The elongated shape of the point set on $\mathcal{P}$ is, however, not immediate from its representations neither in $\mathcal{U}$ nor in $\mathcal{V}$, see figure 1.17. I.e. the usual *Euclidean linear regression* of the images in the parameter domains does not work well. We need to take the respective metrics $g_{\mathcal{U}}$ and $g_{\mathcal{V}}$ into account in order to set up a proper notion of "linear" regression in the parameter domains.

If we include the indicatrix field fingerprints of the metrics $g_{\mathcal{U}}$ and $g_{\mathcal{V}}$ in $\mathcal{U}$ and $\mathcal{V}$ we get a first visual indication of the elongation of the point set in the correct directions, see figure 1.18.

The ellipse indicatrices close to the point set are themselves elongated in the direction where the Euclidean length is longer than the g-length of the vectors in the corresponding tangent spaces. In

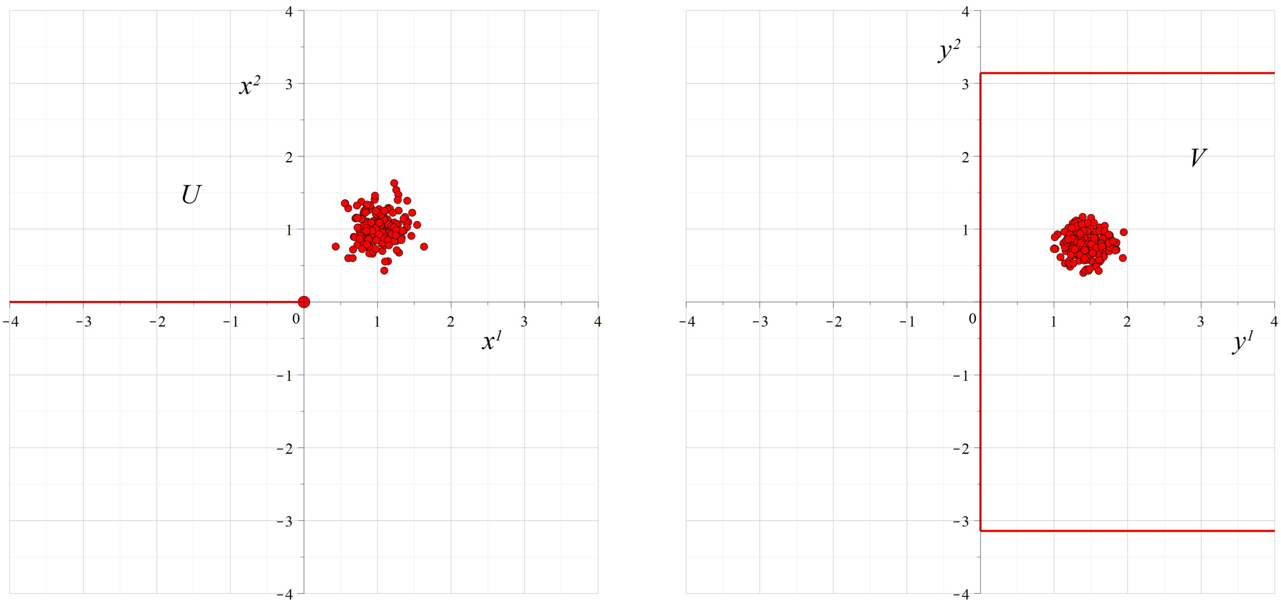


Figure 1.17: The $\mathcal{U}$- and $\mathcal{V}$-representations of the point set from figure 1.16.

the g-orthogonal direction (which in this particular setting corresponds to the Euclidean orthogonal direction) the Euclidean length of tangent vectors are shorter than the g-length of the vectors.

If the point set is more or less circular distributed (in the Euclidean sense) – as is the case in both $\mathcal{U}$ and $\mathcal{V}$ – it means, that the point set is actually elongated (in the g sense) in the direction which is g-orthogonal to the direction of elongation of the g-indicatrices. This observation is then in perfect accordance with the visual inspection of the point set on the paraboloid.

In the following chapters we will eventually be able to perform a correct "linear" regression on a data set in a Local Riemannian Manifold. For example, the notion of *center of mass* of the set will be generalized to the so-called *Grove-Karcher mean* or *Frechet mean* of the set, and, moreover, the well-known Euclidean notion of *Principal Component Analysis* is generalized to *Principal Geodesic Analysis*in Riemannian manifolds.

One seminal example from real life showing the application of invariant analysis on Riemannian manifolds for statistical purposes is in the paper by T. Fletcher et al.: [7].

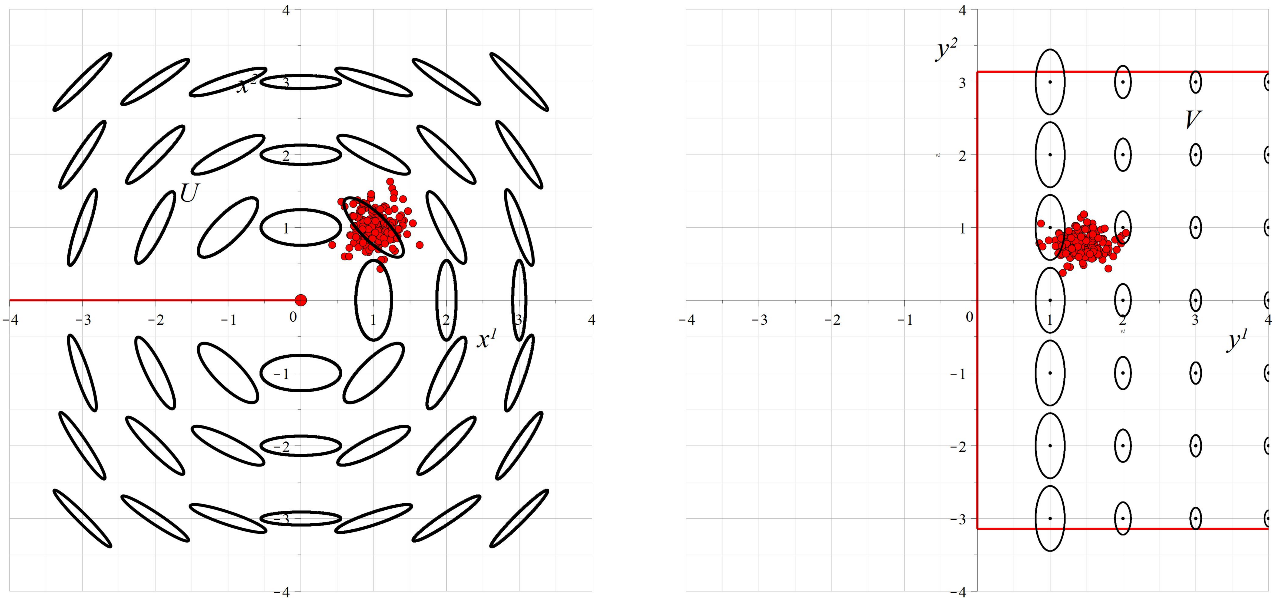


Figure 1.18: The respective indicatrix fields show clearly in both coordinate domains $\mathcal{U}$ and $\mathcal{V}$, that the displayed point sets are metrically elongated in the radial direction.

Chapter 2

Vector fields and the Lie derivative

2.1 Vector fields and their coordinate representations

We consider a Local Riemannian Manifold $(\mathcal{U}^n, g)$ represented by an open set $\mathcal{U}^n$ in $\mathbb{R}^n$ and a given metric tensor field $g = g_{\mathcal{U}}$ on $\mathcal{U}^n$. Let $V(p)$ denote a vector in the tangent space $T_p\mathcal{U}$ at $p \in \mathcal{U}$. Then $V(p)$ is a linear combination of the canonical vectors $e_1(p), \cdots, e_n(p)$ that span $T_p\mathcal{U}$, and the coefficients are called $v^i(p)$:

$$V(p) = \sum_{i=1}^{i=n} v^i(p) \cdot e_i(p) = (v^1(p), \cdots, v^n(p))_{|\{e_1, \cdots, e_n\}} \quad . \tag{2.1}$$

Suppose that we choose a vector $V(x^1, \cdots, x^n)$ in each tangent space $T_{(x^1, \cdots, x^n)}\mathcal{U}$ for every point $(x^1, \cdots, x^n) \in \mathcal{U}$. Then

$$V(x^1, \cdots, x^n) = \sum_{i=1}^{i=n} v^i(x^1, \cdots, x^n) \cdot e_i(x^1, \cdots, x^n) \quad . \tag{2.2}$$

Definition 2.1 We will say that V is a smooth vector field in $\mathcal{U}$ if the coordinate functions $v^k(x^1, \cdots, x^n)$ are smooth functions of $(x^1, \cdots, x^n)$ for every $k = 1, \cdots, n$.

Notation 2.2 The set of smooth vector fields on $(\mathcal{U}, g)$ is from now on denoted by $\mathfrak{X}(\mathcal{U})$, and the set of smooth functions on $(\mathcal{U}, g)$ is denoted by $\mathfrak{F}(\mathcal{U})$.

Proposition 2.3 Let ϕ denote an isometry between two representations $(\mathcal{U}, g_{\mathcal{U}})$ and $(\mathcal{V}, g_{\mathcal{V}})$ of the same *LRM*. If $V \in \mathfrak{X}(\mathcal{U})$ then $(J_\phi \cdot V^*)^* \in \mathfrak{X}(\mathcal{V})$. And vice versa, if $W \in \mathfrak{X}(\mathcal{V})$ then $(J_{\phi^{-1}} \cdot W^*)^* \in \mathfrak{X}(\mathcal{U})$. In other words, the smoothness property of a vector field is an isometry invariant property (in fact a diffeomorphism invariant property).

Proof. Since ϕ and ϕ^{-1} are diffeomorphisms, the Jacobian matrices J_ϕ and $J_{\phi^{-1}}$ are smooth matrix functions on $\mathcal{U}$ and $\mathcal{V}$ respectively. Therefore the product matrix functions $(J_\phi \cdot V^*)^*$ and $(J_{\phi^{-1}} \cdot W^*)^*$ are also smooth. □

Notation 2.4 Note that the $*$ notation for transposition of vector coordinates into a column (respectively a row) matrix is in fact obsolete in the sense that when we calculate e.g. the vector $(J_\phi \cdot V^*)^*$, then it is clear that the matrix multiplication has to be done in precisely this way in order to have the types of the matrices agree. Therefore, from now on we will in general – and without lack of concensus – drop the explicit use of transpositions in the *notation* of such evaluations. For example, in the case just considered we will write shorthand $J_\phi \cdot V$ and even (when no confusion is immediate) $J_\phi(V)$ (for $(J_\phi \cdot V^*)^*$. To avoid the mentioned confusion, note that $J_\phi(V)$ is then the image of V by the Jacobian map between tangent spaces, whereas the notation $J_\phi(p)$ is the Jacobian evaluated at the point p.

Example 2.5

We use the Local Riemannian Manifold induced by the paraboloid $\mathcal{P}$ in 3D space $\mathbb{R}^3$ – and represented by $(\mathcal{U}^2, g_{\mathcal{U}})$ – in the main example 1.6 from chapter 1, and consider the following vector field in $\mathcal{U}$ given by its coordinate functions with respect to the canonical basis at each point:

$$V = (v^1(x^1,x^2),\ v^2(x^1,x^2)) = \left(\sqrt{(x^1)^2+(x^2)^2}\,,\, x^2 \cdot \sin(x^1-x^2)\right)_{|\{e_1,e_2\}} \quad . \tag{2.3}$$

EXERCISE 2.6

Show that this vector field V is smooth in the open subset $\mathcal{U}$. Hint: Note that $\mathcal{U}$ does not contain $(0,0)$.

The corresponding vector field W in $\mathcal{V}$ is determined via ϕ by

$$W(y^1,y^2) = (J_\phi(\phi^{-1}(y^1,y^2)) \cdot V^*(\phi^{-1}(y^1,y^2)))^* \quad . \tag{2.4}$$

Note that we need to apply ϕ^{-1} in order to express the coordinate functions of W in (y^1,y^2)-coordinates. The two vector fields are displayed in figure 2.1.

Both of the vector fields V in $\mathcal{U}$ and W in $\mathcal{V}$ are representatives of one and the same vector field $\widehat{V}$ on the paraboloid $\mathcal{P}$, as illustrated in figure 2.2 and obtained via the $\mathcal{P}$-defining vector function r in the usual way:

$$\begin{aligned} \widehat{V}(x^1,x^2) &= v^1(x^1,x^2)\cdot\left(\frac{\partial r}{\partial x^1}\right) + v^2(x^1,x^2)\cdot\left(\frac{\partial r}{\partial x^2}\right) \\ &= \widehat{W}(y^1,y^2) \qquad (2.5) \\ &= w^1(y^1,y^2)\cdot\left(\frac{\partial r(\phi^{-1})}{\partial y^1}\right) + w^2(y^1,y^2)\cdot\left(\frac{\partial r(\phi^{-1})}{\partial y^2}\right) \quad . \end{aligned}$$

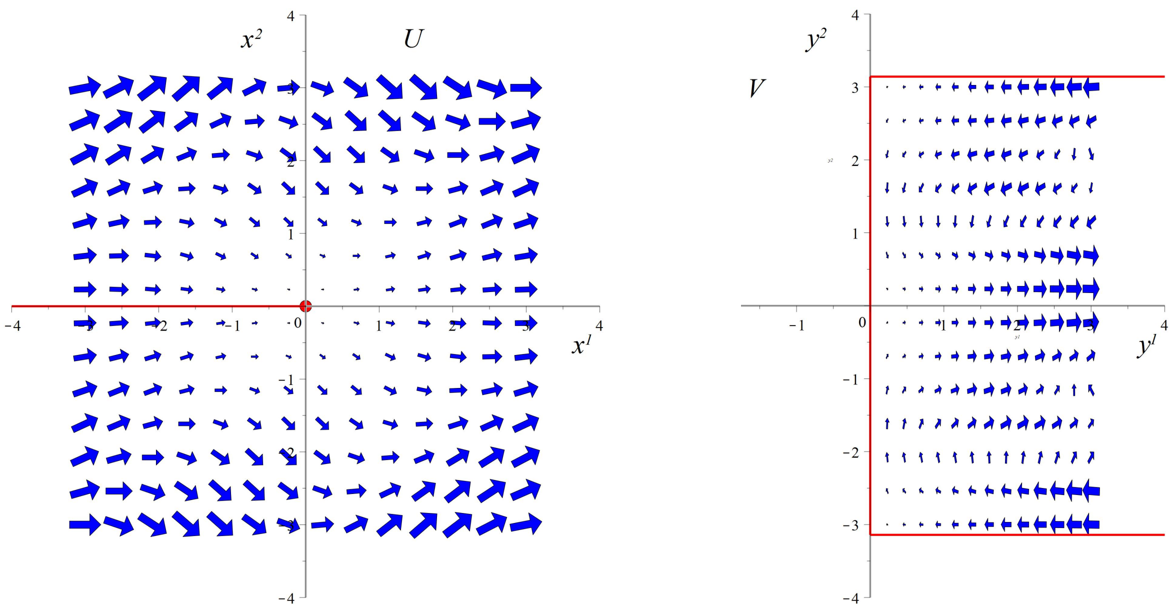


Figure 2.1: The vector fields V (left) and W (right) from example 2.5 by (2.3) and (2.4) represented in $\mathcal{U}$ and in $\mathcal{V}$.

Although the two vector fields V and W are ϕ-related they *look* very different from the Euclidean viewpoint. For example, it is evident, that the usual Euclidean divergence of the two vector fields are quite different from each other. In order to define a consistent divergence of vector fields on (Local) Riemannian Manifolds we therefore need to come up with a definition that is coordinate invariant in the sense that it will give *the same result in every isometric representation* of the *LRM*. This will be done – using the invariant metric tensor g – in a later chapter.

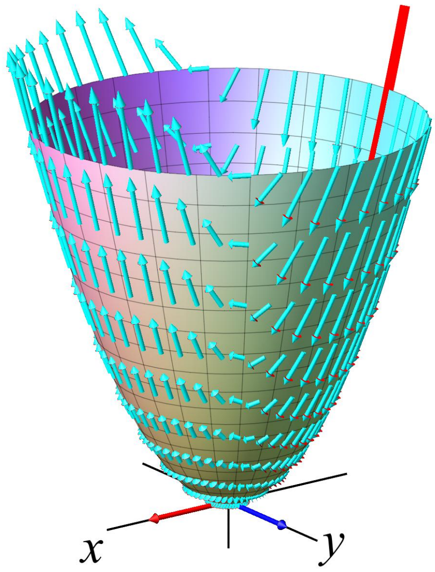


Figure 2.2: The lift to the paraboloid of the vector fields displayed in figure 2.1 from example 2.5.

2.2 Vector fields as derivations of functions

The directional derivative $X(f)$ of a smooth function f in the direction of a vector X at a point $p \in \mathcal{U}$ is well known:

Definition 2.7 Let $X = \sum_{i=1}^{i=n} v^i \cdot e_i$ be a vector field in $\mathfrak{X}(\mathcal{U})$. Then the X-directional derivative of f at each point p is:

$$X_p(f) = \sum_{i=1}^{i=n} v^i(p) \cdot \frac{\partial f}{\partial x^i}_{|p} \quad . \tag{2.6}$$

In particular we have for each of the canonical basis vectors e_i at each point:

$$e_i(f) = \frac{\partial f}{\partial x^i} \quad . \tag{2.7}$$

In this sense, every vector is associated with a derivation of smooth functions at each point, where the notion of derivation is defined as follows:

Definition 2.8 A derivation w at a point p is by definition a linear map $w : \mathfrak{F}(\mathcal{U}) \longmapsto \mathbb{R}$ which satisfies the product-differentiation-rule:

$$w(f \cdot h)_{|p} = f(p) \cdot w(h)_{|p} + h(p) \cdot w(g)_{|p} \tag{2.8}$$

EXERCISE 2.9

Show that the directional derivative defined by a vector X as in (2.6) is a derivation in the sense of definition 2.8.

There are no more derivations than there are vectors and vice versa:

Proposition 2.10 Every derivation at p is a directional derivative by a unique vector at p and vise versa, to every vector at p corresponds precisely one unique derivation.

Proof. We refer to [18, p. 53] for a fairly straightforward proof of this fact. The proof applies a version of Taylor's theorem which may be of independent interest. □

Whenever convenient we will therefore write and use vectors as derivations in this sense – this is a strong alternative to the more geometric conceptualization of a vector as a velocity vector (with length and direction) for a motion along a time-parametrized curve through the point in question. In short we write correspondingly:

Notation 2.11 Let $X = \sum_i v^i \cdot e_i \in T_p\mathcal{U}$. Then we write – see (2.6) and (2.7):

$$X_p = \sum_{i=1}^{i=n} v^i(p) \cdot \frac{\partial}{\partial x^i}_{|p} \quad . \tag{2.9}$$

The following observation is immediate:

Proposition 2.12 Suppose that $X \in \mathfrak{X}(\mathcal{U})$ and $f \in \mathfrak{F}(\mathcal{U})$. Then $X(f)$ is itself a smooth function on $\mathcal{U}$, i.e. $X(f) \in \mathfrak{F}(\mathcal{U})$.

EXERCISE 2.13

Let $\mathcal{U}^2 = \mathbb{R}^2$ and let $X \in \mathfrak{X}(\mathcal{U}^2)$ and $Y \in \mathfrak{X}(\mathcal{U}^2)$ denote the following two vector fields in terms of the canonical basis vectors e_i, $i = 1,2$, in each tangent space $T_{(x^1,x^2)}$ of $\mathcal{U}^2$:

$$\begin{aligned} X(x^1,x^2) &= e^{x^2} \cdot e_1 + (x^1)^3 \cdot e_2 = \left(e^{x^2},\, (x^1)^3\right) \\ Y(x^1,x^2) &= -x^2 \cdot e_1 + x^1 \cdot e_2 = \left(-x^2,\, x^1\right) \quad . \end{aligned} \tag{2.10}$$

Let $f \in \mathfrak{F}(\mathcal{U})$ denote the following smooth function on $\mathbb{R}^2$:

$$f(x^1, x^2) = x^1 \cdot x^2 \quad . \tag{2.11}$$

Calculate the following six functions on $\mathbb{R}^2$ expressed in terms of the coordinates x^1 and x^2: $X(f)$, $Y(f)$, $X(X(f))$, $X(Y(f))$, $Y(X(f))$, and $Y(Y(f))$.

2.3 Flows that preserve indicatrices

Some vector fields are constructed in such a way that – or have the property that – they 'respect' the metric tensor field g of the *LRM* in the sense that if we push the g-indicatrix fingerprint in the direction of the vector field and with the speed determined everywhere by the vector field, then the indicatrix field, the full fingerprint field, is mapped onto itself. One of the purposes of the present chapter is to make this special type of relation between the metric tensor and the vector field precise. When such a relation holds we will naturally say that the vector field generates a local isometry of $(\mathcal{U}, g)$ into itself – see section 2.7 below. Such a vector field is called a Killing vector field after Wilhelm Killing.

Example 2.14

A first – and simple – example of a (to be shown below in exercise 2.39) Killing vector field on the representations $(\mathcal{U}, g_{\mathcal{U}})$ and $(\mathcal{V}, g_{\mathcal{V}})$ for the paraboloid example in chapter 1 is displayed in figure 2.3. The vector fields are induced from the rotation of the paraboloid around its axis as illustrated in figure 2.5. The representation of the vector field is given in $\mathcal{U}$ as follows (using the canonical basis in all tangent spaces $T_p\mathcal{U}$):

$$V(x^1, x^2) = \left(-x^2, x^1\right) \quad , \tag{2.12}$$

so that the corresponding expression in $\mathcal{V}$ is (using the notational convention of deleting transpositions and using the canonical basis in all tangent spaces $T_q\mathcal{V}$):

$$W(y^1, y^2) = J_\phi(\phi^{-1}(y^1, y^2)) \cdot V(\phi^{-1}(y^1, y^2)) = (0, 1) \quad . \tag{2.13}$$

EXERCISE 2.15

Show that the V-corresponding vector field W in $\mathcal{V}$ is indeed given by the simple expression in (2.13).

In order to make precise how a given vector field actually moves the points around in $\mathcal{U}$ we need to study its integral curves and its corresponding flow map.

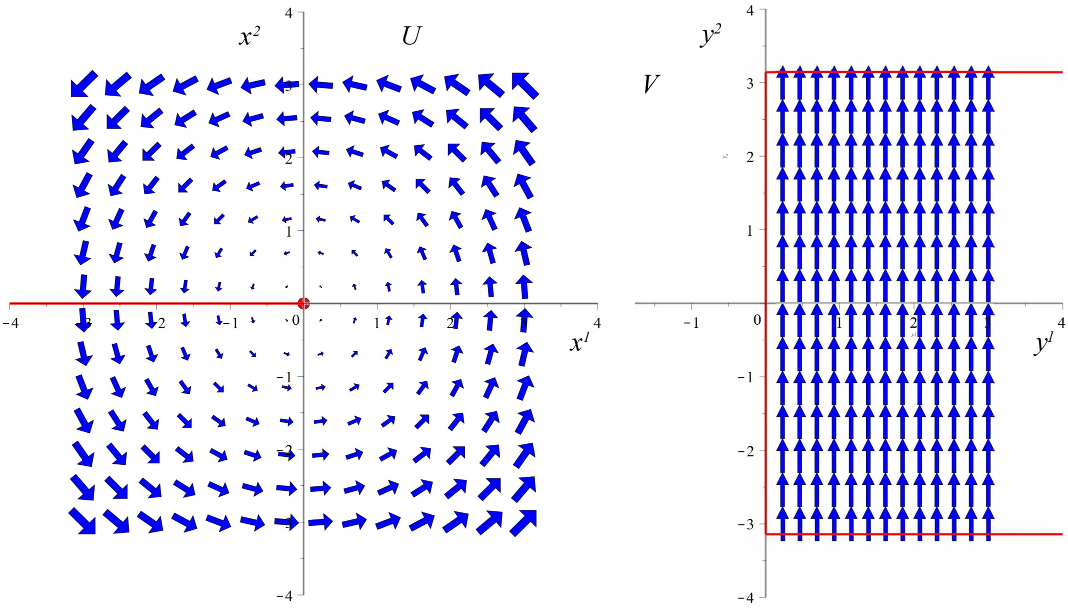


Figure 2.3: The vector fields in $\mathcal{U}$ and $\mathcal{V}$ generated by a rotation of the paraboloid $\mathcal{P}$ around its axis of symmetry, see figure 2.5.

2.4 Integral curves and flow maps

Definition 2.16 Let $V \in \mathfrak{X}(\mathcal{U})$. A t-parametrized curve γ, $t \in I$, in $\mathcal{U}$, is called an integral curve of V if

$$\gamma'(t) = V(\gamma(t)) \quad \text{for all } t \in I \quad . \tag{2.14}$$

In 2D and in $\mathcal{U}$-coordinates it means explicitly that the coordinate functions of $\gamma(t) = (\gamma^1(t), \gamma^2(t))$ satisfy the first order ordinary differential equation system determined by the vector field $V(x^1, x^2) = (v^1(x^1, x^2), v^2(x^1, x^2))$:

$$\begin{aligned} (\gamma^1)'(t) &= v^1(\gamma^1(t), \gamma^2(t)) \quad \text{and} \\ (\gamma^2)'(t) &= v^2(\gamma^1(t), \gamma^2(t)) \quad \text{for all } t \in I \quad . \end{aligned} \tag{2.15}$$

Proposition 2.17 As usual we let ϕ denote a diffeomorphism. If γ is an integral curve of V in $\mathcal{U}$, then $\phi(\gamma)$ is an integral curve of the corresponding vector field $W = J_\phi \cdot V$ in $\phi(\mathcal{U}) = \mathcal{V}$ – and vice versa.

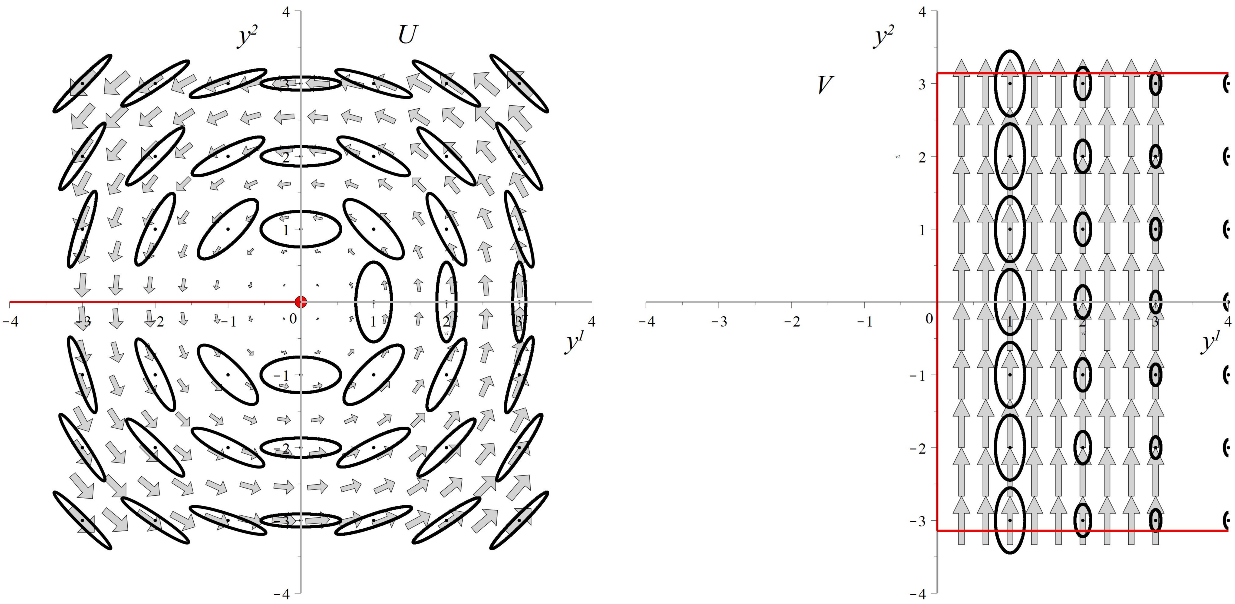


Figure 2.4: The rotational vector field on $\mathcal{P}$ has these representations in $\mathcal{U}$ and in $\mathcal{V}$. They are studied in example 2.14 – here they are displayed together with the fingerprint indicatrix fields of the metric tensors $g_{\mathcal{U}}$ (left) and $g_{\mathcal{V}}$ (right), respectively.

EXERCISE 2.18

For all positive values of k we let $\gamma_k(t) = (k \cdot \cos(t), k \cdot \sin(t))$, $t \in]-\pi, \pi[$, $k \in \mathbb{R}_+$. Show that all the curves γ_k, $k > 0$, are integral curves of the vector field $V = (-x^2, x^1)$ in example 2.14 and that they all satisfy that $\gamma_k(0)$ has zero second coordinate. Show that there are no other curves that are integral curves $\eta(t)$ of V and whose second coordinate function $\eta^2(t)$ is zero for $t = 0$. Why are we assuming that $k > 0$? Could we have considered $k < 0$?

Definition 2.19 The local flow map θ_t generated by a smooth vector field $X \in \mathfrak{X}(\mathcal{U})$ on $\mathcal{U}$ is the mapping determined locally by the integral curves $\alpha(t, p)$ of X, i.e. the integral curves defined by $\alpha(0, p) = p$ for each p. The time domains of the integral curves are thus always assumed to contain $t = 0$.

$$\theta_t(p) = \alpha(t, p) \quad \text{for all } p \in \mathcal{U} \text{ and for all } -\varepsilon < t < \varepsilon \quad , \tag{2.16}$$

where ε is a sufficiently small value.

The mapping defined in definition 2.19 is the one that will be used to push open domains forward along the vector field in $\mathcal{U}$. If p is held constant, then the mapping $t \longmapsto \theta_t(p)$ is just the integral curve $\alpha(t, p)$. On the other hand, if t is held constant then the mapping $q \longmapsto \theta_t(q)$ defines a map which lets every point q of any given open subset of $\mathcal{U}$ flow for exactly time t, if only t is

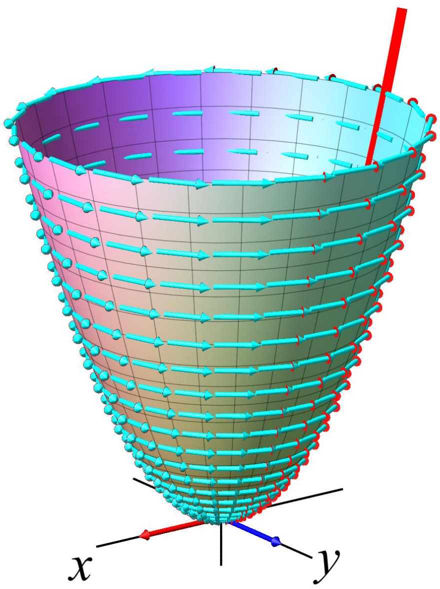


Figure 2.5: The rotational vector field on $\mathcal{P}$ whose representations in $\mathcal{U}$ and in $\mathcal{V}$ are studied in example 2.14

sufficiently small in absolute value. If this latter condition is not satisfied we may not be able to flow the subset within $\mathcal{U}$ for all the time t.

The most fundamental observations about local flows θ_t, that are generated by smooth vector fields in this way, are contained in the following proposition. We refer to [18, pp. 212–214] where the proof is developed in all necessary details.

Proposition 2.20

1. θ_0 does not move anything – it is the identity map on $\mathcal{U}$.
2. $\theta_s(\theta_t) = \theta_t(\theta_s) = \theta_{s+t}$ whenever s, t, and $s+t$ are sufficiently small in absolute values. In other words, you can flow for time $s+t$ by first flowing for time t and then afterwards for time s.
3. At each point $p \in \mathcal{U}$, for sufficiently small t, the map θ_t is a local diffeomorphism on a sufficiently small open set Ω containing p with inverse map $\theta_t^{-1} = \theta_{-t}$ on $\theta_t(\Omega)$. This is the most important property. It is based on the fact that integral curves never intersect each other (why is that?). And it means in particular, that the flow map θ_t for every allowed t-value has well-defined Jacobians J_{θ_t} and $J_{\theta_t}^{-1} = J_{\theta_{-t}}$, respectively.

EXERCISE 2.21

Suppose V is the vector field in $\mathcal{U} = \mathbb{R}^2 \setminus \{(x^1, x^2) \in \mathbb{R}^2 \mid x^2 = 0,\ x^1 \leq 0\}$ from example 2.14:

$$V(x^1, x^2) = \left(-x^2, x^1\right) \quad . \tag{2.17}$$

Find the expression of the integral curve $\alpha(t, (x^1, x^2))$ of V through the point $(x^1, x^2) \in \mathcal{U}$. Show that Property 3 of Proposition 2.20 is satisfied.

EXERCISE 2.22

Let V be the following vector field in $\mathbb{R}^2$:

$$V(x^1, x^2) = \left(a \cdot x^1 + b \cdot x^2,\ c \cdot x^1 + d \cdot x^2\right) \quad , \tag{2.18}$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is a constant regular matrix} \quad . \tag{2.19}$$

Find the expression of the integral curve $\alpha(t, (x^1, x^2))$ of V through the point $(x^1, x^2) \in \mathbb{R}^2$ – either in general for any choice of regular coefficient matrix A, or just for some example.

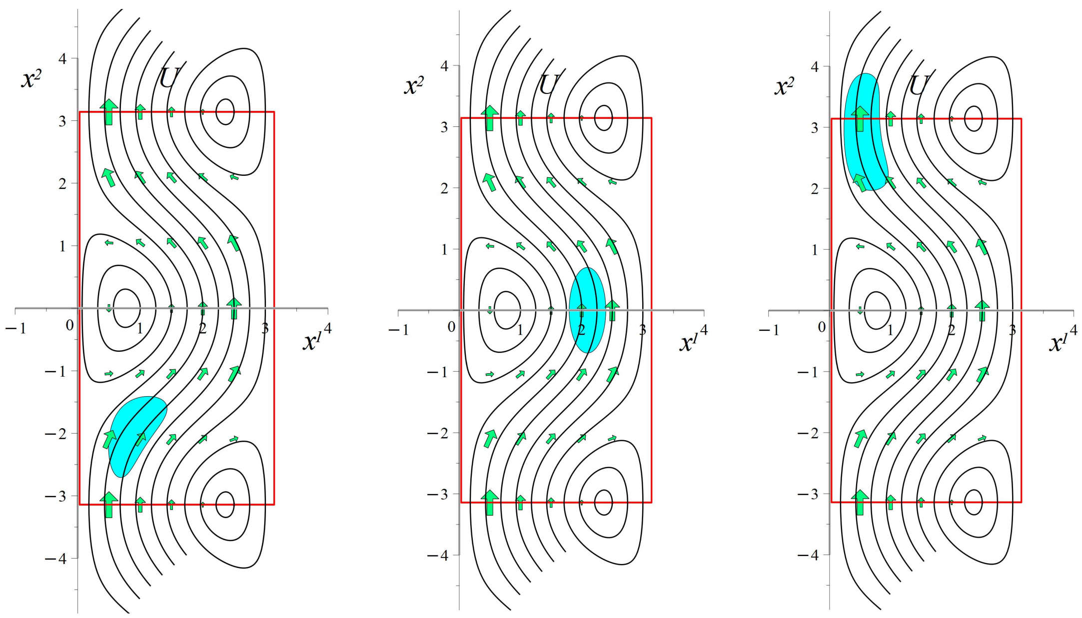


Figure 2.6: A blob (open set) is floating along the integral curves of a vector field. At each instance of time t the image of the blob is diffeomorphic to the original blob at time $t = 0$. The diffeomorphism is obtained directly by the flow map θ_t which is generated by the vector field, see definition 2.19.

2.5 Pulling back vectors along integral curves

We will use the flow map θ_t for a vector field $X \in \mathfrak{X}(\mathcal{U})$ to move – and then compare – vectors along integral curves for X.

We consider one integral curve $\theta_t(p)$, $t \in I$, and use the Jacobian $J_{\theta_{-t}}$ of the inverse flow map diffeomorphism θ_{-t} at the point $q = \theta_t(p)$ to give us a vector $J_{\theta_{-t}}(Y_q)$ in $T_p\mathcal{U}$ for every $Y_q \in T_q\mathcal{U}$. Observe that we are using short hand notation $J_{\theta_{-t}}(Y_q)$ for the image of the vector Y_q by the Jacobian $J_{\theta_{-t}}$ at q – as forewarned in the notation-box 2.4 above.

Definition 2.23 The vector $\widehat{Y}_t = J_{\theta_{-t}}(Y_q) \in T_p\mathcal{U}$ is called the pull back (at p) of the vector $Y_q \in T_q\mathcal{U}$ (at q). So the vector Y_q is pulled back from q to p by the backwards flow map θ_{-t} associated with the forward flow map θ_t for the vector field X.

The pulled-back vector $\widehat{Y}_t$ can now (and only now) be *compared* with the value Y_p of the vector field Y at p. This motivates V. I. Arnol'd's metaphorical name, the fisherman's derivative, see [1], for the following derivative, which is actually named after Sophus Lie:

Definition 2.24 We let θ_t denote the local flow map around $p \in \mathcal{U}$ for a vector field $X \in \mathfrak{X}(\mathcal{U})$ and let $Y \in \mathfrak{X}(\mathcal{U})$ denote another smooth vector field on $\mathcal{U}$. Then we define the Lie derivative of Y along X at p as follows:

$$L_X Y_{|_p} = \lim_{t \mapsto 0} \left(\frac{\widehat{Y}_t - Y_p}{t} \right) = \lim_{t \mapsto 0} \left(\frac{J_{\theta_{-t}}(Y_q) - Y_p}{t} \right) \quad . \tag{2.20}$$

The Lie derivative can be interpreted as a vector field, i.e. as a derivation on functions, namely as follows (this identification is expressed explicitly in proposition 2.28 below):

Definition 2.25 The Lie bracket of two vector fields X and Y in $\mathfrak{X}(\mathcal{U})$ at p is denoted $[X\,,\,Y]_{|_p}$ and defined as follows:

$$[X\,,\,Y](f) = X(Y(f)) - Y(X(f)) \quad \text{for all functions } f \in \mathfrak{F}(\mathcal{U}) \tag{2.21}$$

The Lie bracket is itself a derivation, i.e. a vector field, so that $[X, Y] \in \mathfrak{X}(\mathcal{U})$ whenever X and Y are in $\mathfrak{X}(\mathcal{U})$:

EXERCISE 2.26

Prove this claim, i.e. show – or give examples which illustrate – that whenever X and Y are two vector fields in $\mathfrak{X}(\mathcal{U})$ then there exists a vector field $V \in \mathfrak{X}(\mathcal{U})$ such that

$$[X,Y](f) = V(f) \quad \text{for all } f \in \mathfrak{F}(\mathcal{U}). \tag{2.22}$$

Hint: See equation (2.27) or exercise 2.29 below.

The result of this exercise 2.26 is a bit surprising, because we should expect second derivatives of f to appear on the right hand side $X(Y(f)) - Y(X(f))$. But they cancel each other – because with our general smoothness assumptions on f we have $f''_{x^i x^j} = f''_{x^j x^i}$ for all i and j – and we are left with first order derivatives of f and thence a derivation.

The Lie bracket satisfies the so-called Jacobi identity that we shall need later:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad . \tag{2.23}$$

EXERCISE 2.27

Show the Jacobi identity (2.23). Hint: Substitute the definition of the bracket and see that the resulting 12 terms cancel in pairs!

The Lie bracket is identical to the Lie derivative – you may want to look up the proof of this fact in [18, p. 229]; it is not difficult:

Proposition 2.28

$$L_X Y = [X\,,\,Y] \quad . \tag{2.24}$$

In other words, $L_X Y$ is itself a vector field in $\mathfrak{X}(\mathcal{U})$, and

$$L_X Y = [X\,,\,Y] = -[Y\,,\,X] = -L_Y X \quad . \tag{2.25}$$

In coordinates we therefore obtain the following expression for $L_X Y$. We let X and Y denote the two vector fields in $\mathfrak{X}(\mathcal{U})$ with coordinate functions v^i, $i = 1, \cdots, n$, and w^j, $j = 1, \cdots, n$:

$$\begin{aligned} X &= \sum_{i=1}^{i=n} v^i \cdot \frac{\partial}{\partial x^i} = \sum_{i=1}^{i=n} v^i \cdot e_i \\ Y &= \sum_{i=1}^{i=n} w^i \cdot \frac{\partial}{\partial x^i} = \sum_{i=1}^{i=n} w^i \cdot e_i \quad , \end{aligned} \tag{2.26}$$

$$
\begin{aligned}
L_X Y &= [X\,,\,Y] \\
&= \sum_{i=1}^{i=n} \left(X(w^i) - Y(v^i) \right) \cdot \frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \left(v^j \cdot \frac{\partial w^i}{\partial x^j} - w^j \cdot \frac{\partial v^i}{\partial x^j} \right) \cdot \frac{\partial}{\partial x^i} \quad ,
\end{aligned}
\tag{2.27}
$$

so that the Lie bracket is indeed a (vector field) derivation with vector coordinates given by the parenthesis above.

EXERCISE 2.29

Returning to exercise 2.13: Find the vector-coordinates for the derivation $[X,Y]$ determined by the two vector fields given in that exercise:

$$
\begin{aligned}
X(x^1, x^2) &= e^{x^2} \cdot e_1 + (x^1)^3 \cdot e_2 = \left(e^{x^2}\,,\,(x^1)^3 \right) \\
Y(x^1, x^2) &= -x^2 \cdot e_1 + x^1 \cdot e_2 = \left(-x^2\,,\,x^1 \right) \quad .
\end{aligned}
\tag{2.28}
$$

Show by direct calculation that $[X,Y](f) = X(Y(f)) - Y(X(f))$ for the function f given in that exercise.

EXERCISE 2.30

Let X and Y be two canonical vector fields in $\mathcal{U}$: $X = e_1$ and $Y = e_2$. Show that $[X,Y] = 0$.

EXERCISE 2.31

Let X and Y be the following vector fields in $\mathcal{U} = \mathbb{R}^2 - \{(0,0)\}$ in terms of the canonical basis fields e_1 and e_2:

$$
\begin{aligned}
X(x^1, x^2) &= \left(x^1, x^2 \right) \\
Y(x^1, x^2) &= \left(-x^2, x^1 \right) \quad .
\end{aligned}
\tag{2.29}
$$

Show that $[X,Y] = 0$.

EXERCISE 2.32

We consider a diffeomorphism ψ of an open subset Ω of $\mathbb{R}^2$ onto $\psi(\Omega) \subset \mathbb{R}^2$ with a given expression $\psi(x^1, x^2)$ and inverse map $\psi^{-1}(y^1, y^2)$. Let V and W denote the vector fields in $\psi(\Omega)$:

$$
\begin{aligned}
V(y^1, y^2) &= \frac{\partial \psi}{\partial x^1}(\psi^{-1}(y^1, y^2)) \\
W(y^1, y^2) &= \frac{\partial \psi}{\partial x^2}(\psi^{-1}(y^1, y^2)) \quad .
\end{aligned}
\tag{2.30}
$$

Show that $[V, W] = 0$. Hint: Try out the claim for the well known diffeomorphism from Chapter 1:

$$\phi(x^1, x^2) = (\sqrt{(x^1)^2 + (x^2)^2}, \arg(x^1 + i \cdot x^2)) \quad , \tag{2.31}$$

where $(x^1, x^2) \in \Omega$ consisting of all points in $\mathbb{R}^2$ except the non-positive part of the x^1-axis.

EXERCISE 2.33

Let X and Y be the following unit vector fields in Euclidean $\mathbb{R}^2 - \{(0,0)\}$:

$$\begin{aligned} X(x^1, x^2) &= \left(\frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}}, \frac{x^2}{\sqrt{(x^1)^2 + (x^2)^2}} \right) \\ Y(x^1, x^2) &= \left(\frac{-x^2}{\sqrt{(x^1)^2 + (x^1)^2}}, \frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}} \right) \quad . \end{aligned} \tag{2.32}$$

Show that in this case the Lie bracket is the following *non-vanishing* vector field:

$$[X, Y] = \left(\frac{x^2}{(x^1)^2 + (x^1)^2}, \frac{-x^1}{(x^1)^2 + (x^2)^2} \right) \quad . \tag{2.33}$$

2.6 Pulling back the metric tensor along integral curves

In the same way as above we can use the Jacobian of the flow map to define what we will call the pull back tensor of the metric tensor. Again we 'pull back' the metric tensor from the point $q = \theta_t(p)$ to the point p. But note, that we now do it by *pushing forward* given vectors V_p and W_p from the tangent space at p to the tangent space at q via the *forward flow* θ_t and evaluate the metric g_q on the two (pushed-forward-) vectors at q. This evaluation gives a non-negative real number, of course, which can then be compared with the evaluation of g_p on V_p and W_p. Formally, it is all contained in the definition:

Definition 2.34 Again we let θ_t denote the local flow map around $p \in \mathcal{U}$ for a vector field $X \in \mathfrak{X}(\mathcal{U})$ which maps p to q, and let g denote the given metric tensor field on $\mathcal{U}$. The (pulled-back-) metric tensor $\widehat{g}_t$ at p is defined by its values on vectors V_p and W_p in the tangent space $T_p\mathcal{U}$ at p as follows:

$$\widehat{g}_t(V_p, W_p) = g_q(J_{\theta_t}(V_p), J_{\theta_t}(W_p)) \quad \text{for all } V_p \text{ and } W_p \text{ in } T_p\mathcal{U} \quad . \tag{2.34}$$

Note, that we use again shorthand notation $J_{\theta_t}(V_p)$ for the vector obtained by using the Jacobian map J_{θ_t} at p on the vector V_p at p, cf. the notation box 2.4.

The real values of $\widehat{g}_t(V_p, W_p)$ can freely be compared with the values $g(V_p, W_p)$, and the following limit is well defined and a clear measure of the change of the metric tensor field (and thence of the indicatrix field) during the flow along the vector field X:

Definition 2.35 We define the Lie derivative of g along X at p in much the same way as we defined $L_X Y$ in definition 2.24:

$$L_X(g)_{|_p} = \lim_{t \mapsto 0} \left(\frac{\widehat{g}_t - g_p}{t} \right) = \frac{d}{dt} \widehat{g}_t \quad . \tag{2.35}$$

In coordinates we get the following expression for the Lie derivative of the metric tensor, see again [18]:

Proposition 2.36 Let $(\mathcal{U}, g)$ be a Local Riemannian Manifold and let $g_{ij}(x^1, \cdots, x^n)$ denote the components of the metric matrix function $G_{\mathcal{U}}$, i.e. $g_{ij} = g(e_i, e_j)$ with respect to the canonical basis vectors e_i, $i = 1, \cdots, n$, in each tangent space $T_p\mathcal{U}$. Let $V \in \mathfrak{X}(\mathcal{U})$ be a vector field in $\mathcal{U}$ with coordinates $v^k(x^1, \cdots, x^n)$, $k = 1, \cdots, n$. Then the Lie derivative of g has the following coordinate expression for each choice of i and j:

$$L_V(g)(e_i, e_j) = \sum_{k=1}^{k=n} \left(v^k \cdot \frac{\partial g_{ij}}{\partial x^k} + g_{jk} \cdot \frac{\partial v^k}{\partial x^i} + g_{ik} \cdot \frac{\partial v^k}{\partial x^j} \right) \quad . \tag{2.36}$$

2.7 Local isometries: Killing vector fields

We are now ready to define the notion of a Killing vector field as already alluded to in section 2.3:

Definition 2.37 Let $(\mathcal{U}, g)$ denote an *LRM* and let $V \in \mathfrak{X}(\mathcal{U})$ denote a vector field on $\mathcal{U}$. Then V is called a Killing vector field on $(\mathcal{U}, g)$ if

$$L_V(g)_{|_p} = 0 \quad \text{for all } p \in \mathcal{U} \quad . \tag{2.37}$$

From the construction above it now makes perfect sense to say that the flow map of a Killing vector field is an infinitesimal isometry, because we get directly from equation (2.35) that $\frac{d}{dt}\widehat{g}_t = 0$ at all points $p \in \mathcal{U}$.

Since the flow map θ_t of a Killing vector field preserves the metric g in the sense of equation (2.37), it's Jacobian also preserves the indicatrix field of g in $\mathcal{U}$, i.e. the indicatrix at p is mapped onto the indicatrix at $q = \theta_t(p)$ by J_{θ_t}. Moreover, since the Jacobian is just the linearized version of the flow map itself, it means that if we consider a small blob $\Omega_p \subset \mathcal{U}$, i.e. a small open subset of $\mathcal{U}$ around p which resembles/approximates the indicatrix of g at p, then the images $\theta_t(\Omega_p)$ of the blob by the flow map will correspondingly approximate the indicatrices of g along the flow path $\theta_t(p)$ – see the figures 2.7 and 2.10 below.

In coordinates we have, in consequence of the expression for the Lie derivative of the metric in proposition 2.36:

Proposition 2.38 Let $(\mathcal{U}, g)$ be a Local Riemannian Manifold and let $g_{ij}(x^1, \cdots, x^n)$ denote the components of the metric matrix function $G_{\mathcal{U}}$. Let $V \in (\mathcal{U})$ be a vector field in $\mathcal{U}$ with coordinates $v^k(x^1, \cdots, x^n)$, $k = 1, \cdots, n$. Then V is a Killing vector field if and only if its component functions v^k satisfy the following equation for all $i = 1, \cdots, n$ and $j = 1, \cdots, n$:

$$\sum_{k=1}^{k=n} \left(v^k \cdot \frac{\partial g_{ij}}{\partial x^k} + g_{jk} \cdot \frac{\partial v^k}{\partial x^i} + g_{ik} \cdot \frac{\partial v^k}{\partial x^j} \right) = 0 \quad . \tag{2.38}$$

Below we will apply these equations to check if a given vector field is Killing or not in a given Local Riemannian Manifold.

EXERCISE 2.39

Show that the vector field in example 2.14 is indeed a Killing field on the paraboloid $\mathcal{U}$ with the metric tensor $g_{\mathcal{U}}$ determined by the usual metric matrix function:

$$G_{\mathcal{U}}(x^1, x^2) = \begin{bmatrix} 4(x^1)^2 + 1 & 4x^1 \cdot x^2 \\ 4x^1 \cdot x^2 & 4(x^2)^2 + 1 \end{bmatrix} \quad . \tag{2.39}$$

Hint: Use that g_{ij} is precisely the (i, j)'th element in the metric matrix function and then apply (2.38).

EXERCISE 2.40

If you add two Killing vector fields, do you then get a Killing vector field? If your answer is [yes], then prove it; if your answer is [no], then give a counterexample.

EXERCISE 2.41

If you multiply a Killing vector field by a constant real factor do you then get a Killing vector field? If your answer is [yes], then prove it; if your answer is [no], then give a counterexample.

EXERCISE 2.42

Prove that the space of Killing vector fields on a given LRM is a real vector space.

A coordinate invariant version of the property of being Killing is also supplied by [24, p. 251]:

Proposition 2.43 A vector field $X \in \mathfrak{X}(\mathcal{U})$ on $(\mathcal{U}, g)$ is Killing if and only if

$$X(\langle Y, Z\rangle) = \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle \quad \text{for all } Y, \text{ and } Z \text{ in } \mathfrak{X}(\mathcal{U}) \quad , \tag{2.40}$$

where we have used the notation $\langle Y, Z\rangle$ for $g(Y,Z)$ etc.

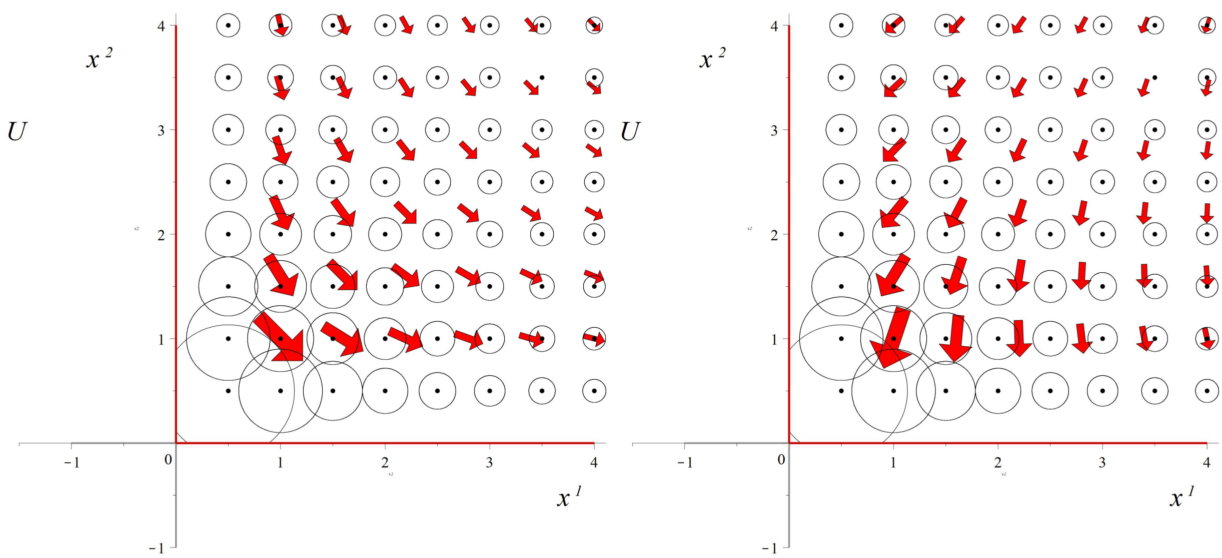


Figure 2.7: Two Killing vector fields in a Local Riemannian Manifold $(\mathcal{U}, g)$, where $\mathcal{U}$ is the open first quadrant of $\mathbb{R}^2$, and the metric tensor is represented by its indicatrix field fingerprint, see example 2.44.

Killing vector fields obviously play a key role for the analysis of isometries of Riemannian manifolds, but they are also important in General Relativity, e.g. for the proper definition of static spacetimes, see e.g. [24] and [12].

We consider further examples of Killing vector fields in the following example/exercise – see figure 2.7.

EXERCISE 2.44

We let $\mathcal{U}$ denote the open first quadrant of $\mathbb{R}^2$ and let $g_{\mathcal{U}}$ be the metric tensor on $\mathcal{U}$ with the following metric matrix function – see figure 2.7

$$G_{\mathcal{U}} = \left((x^1)^2 + (x^2)^2\right) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad x^1 > 0 \, , \, x^2 > 0 \quad . \tag{2.41}$$

The two vector fields X and Y on display in 2.7 are given by their coordinate functions as follows:

$$\begin{aligned} X &= \left(\frac{x^1}{(x^1)^2 + (x^2)^2} \, , \, \frac{-x^2}{(x^1)^2 + (x^2)^2} \right) \\ Y &= \left(\frac{x^1 - 2x^2}{(x^1)^2 + (x^2)^2} \, , \, \frac{-2x^1 - x^2}{(x^1)^2 + (x^2)^2} \right) \quad . \end{aligned} \tag{2.42}$$

Show that X and Y are Killing vector fields on $(\mathcal{U}, g_{\mathcal{U}})$.

In general, let α, β, and k denote any set of three real constants. Show that all of the following vector fields are Killing vector fields in $(\mathcal{U}, g_{\mathcal{U}})$:

$$\begin{aligned} X_{\alpha,\beta} &= \left(\frac{\alpha \cdot x^1 + \beta \cdot x^2}{(x^1)^2 + (x^2)^2} \, , \, \frac{\beta \cdot x^1 - \alpha \cdot x^2}{(x^1)^2 + (x^2)^2} \right) \\ Z_k &= (-k \cdot x^2 \, , \, k \cdot x^1) \quad . \end{aligned} \tag{2.43}$$

2.8 All Killing vector fields in the Euclidean plane

We want to find all the Killing fields in the Euclidean plane $(\mathbb{R}^2, g_E)$, where g_E of course denotes the Euclidean metric tensor which is the one associated with the simplest of all metric matrix functions:

$$G_U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{2.44}$$

Suppose that we consider an unknown vector field

$$V(x^1, x^2) = (f(x^1, x^2) \, , \, h(x^1, x^2) \quad , \tag{2.45}$$

where f and h are two unknown functions of the two variables x^1 and x^2. We then insert the vector field $V = (f, h)$ into the Killing equation (2.38) and get three coupled partial differential equations for f and h:

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x^1} \\ 0 &= \frac{\partial f}{\partial x^2} + \frac{\partial h}{\partial x^1} \\ 0 &= \frac{\partial h}{\partial x^2} \quad . \end{aligned} \tag{2.46}$$

EXERCISE 2.45

Show that we get these three equations by inserting V into the Killing equation. Why are there only three equations?

We conclude from the first and the third equation in (2.46) that f does not depend on x^1, i.e. f only depends on x^2, and that h does not depend on x^2, i.e. h only depends on x^1. The second equation in (2.46) then shows that both of the derivatives $\frac{\partial f}{\partial x^2}$ and $\frac{\partial h}{\partial x^1}$ are constant and that the sum of these constants is 0.

EXERCISE 2.46

Why are these derivatives $\frac{\partial f}{\partial x^2}$ and $\frac{\partial h}{\partial x^1}$ necessarily constants as claimed in the above reasoning?

From this we then have:

$$\begin{aligned} \frac{\partial f}{\partial x^2} &= C \\ \frac{\partial h}{\partial x^1} &= -C \quad . \end{aligned} \tag{2.47}$$

We treat the situation in two cases:

Case 1: If $C = 0$ we get the vector solutions

$$V = V(x^1, x^2) = (f, h) = (\alpha, \beta) \quad , \quad \text{where } \alpha \text{ and } \beta \text{ are constants.} \tag{2.48}$$

In short, any constant vector V is a Killing vector field in the Euclidean plane. This is no surprise, since pure translations in the Euclidean plane are well-known isometries. The integral curves are parallel straight lines that are parametrized so that they all have the same tangent vector (α, β).

Case 2: If $C \neq 0$ we get the solutions

$$V = V(x^1, x^2) = (C \cdot (x^2 - b), \, -C \cdot (x^1 - a)) \quad , \quad \text{where } a \text{ and } b \text{ are constants.} \tag{2.49}$$

These vector fields generate rotations around the fixed point (a, b) – the integral curve through $p = (x_0^1, x_0^2)$ is the following, where $t \in I = [-\pi, \pi[$:

$$\alpha(t, p) = (a, b) + (x_0^1 - a, \, x_0^2 - b) \cdot \cos(C \cdot t) - (b - x_0^2, \, x_0^1 - a) \cdot \sin(C \cdot t)) \quad . \tag{2.50}$$

In short, all Killing vector fields in the Euclidean plane stem from either parallel translations or rotations around a fixed point.

EXERCISE 2.47

Verify that equation (2.50) gives the correct expression for all the integral curves of the Killing field V in equation (2.49).

2.9 Killing vector fields on the 2-sphere in 3D space

As for the Euclidean plane we can also find all the Killing fields on the sphere (of, say, radius 1) in $(\mathbb{R}^3, g_E)$ (where now g_E denotes the Euclidean metric tensor in 3D space).

The unit sphere $\mathcal{S}$ has the following parametrization:

$$\mathcal{S} \quad : \quad r(x^1, x^2) = \left(\sin(x^1)\cdot\cos(x^2),\, \sin(x^1)\cdot\sin(x^2),\, \cos(x^1)\right) \quad , \tag{2.51}$$

where (x^1, x^2) is restricted to the following open set in $\mathbb{R}^2$:

$$\mathcal{U} = \{(x^1, x^2) \in \mathbb{R}^2 \,|\, 0 < x^1 < \pi \quad \text{and} \quad -\pi < x^2 < \pi\,\} \tag{2.52}$$

The metric tensor induced in $\mathcal{U}$ from the unit sphere parametrization is then given by the following metric matrix function:

$$G_{\mathcal{U}}(x^1, x^2) = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(x^1) \end{bmatrix} \quad . \tag{2.53}$$

EXERCISE 2.48

Show that the metric matrix function for the unit sphere has the expression given in equation (2.53). Check that it is positive definite for all (x^1, x^2) in $\mathcal{U}$.

Every Killing vector field V on the sphere in 3D is induced by a rotation of the sphere around an axis through its center. If the axis has direction vector $r(x_0^1, x_0^2)$ then the corresponding Killing vector field is simply

$$V(x^1, x^2) = r(x_0^1, x_0^2) \times r(x^1, x^2) \quad , \quad (x^1, x^2) \in \mathcal{U} \quad , \tag{2.54}$$

where $\times$ denotes the usual cross product in $\mathbb{R}^3$. See examples in figure 2.8.

EXERCISE 2.49

Show that a rotation (with angular velocity 1) in 3D space around the axis through $(0,0,0)$ defined by the direction $r(x_0^1, x_0^2)$ gives rise to the velocity vector field $V(x^1, x^2)$ in equation (2.54) at the points $r(x^1, x^2)$ on the unit sphere $\mathcal{S}$.

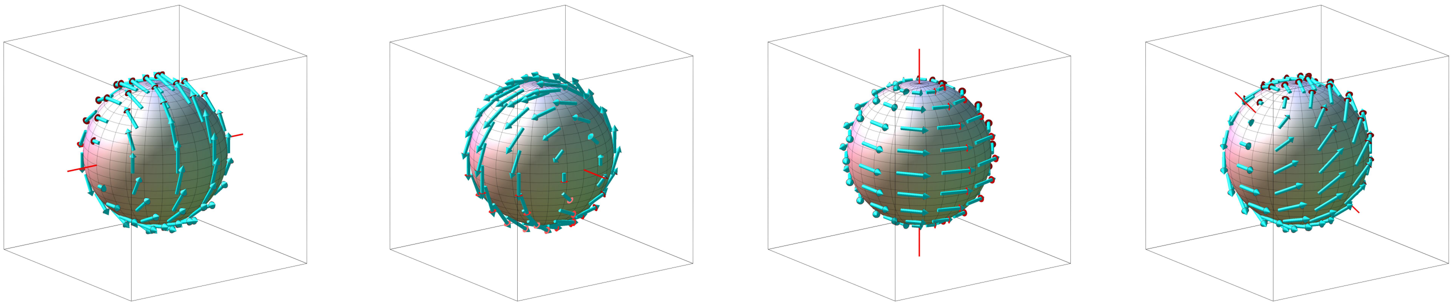


Figure 2.8: Killing vector fields on the sphere with various choices of (red) axes. The first three from the left are generated by rotations around the x-axis, the y-axis, and the z-axis, respectively. The rightmost field is generated by a rotation around the axis with direction vector $(\sqrt{2}, 0, \sqrt{2})/2$.

We now construct the corresponding Killing vector fields in $\mathcal{U}$ (for the first three leftmost vector fields in figure 2.8 corresponding to the axis rotations in 3D around the x-axis, the y-axis, and the z-axis, respectively – when the sphere is parametrized by the standard parametrization $r(x^1, x^2)$ as in (2.51).

The axis rotation Killing fields have the following representations in $\mathcal{U}$ with respect to the canonical basis at each tangent plane in $\mathcal{U}$:

$$\begin{aligned} V_x(x^1, x^2) &= (-\sin(x^2), -\cos(x^2) \cdot \cot(x^1)) \\ V_y(x^1, x^2) &= (\cos(x^2), -\sin(x^2) \cdot \cot(x^1)) \\ V_z(x^1, x^2) &= (0, 1) \quad . \end{aligned} \tag{2.55}$$

EXERCISE 2.50

Show that V_x, V_y, and V_z are the representations in $\mathcal{U}$ of the following three axis rotation Killing fields

$$\begin{aligned} V_1(x^1, x^2) &= (1, 0, 0) \times r(x^1, x^2) \\ V_2(x^1, x^2) &= (0, 1, 0) \times r(x^1, x^2) \\ V_3(x^1, x^2) &= (0, 0, 1) \times r(x^1, x^2) \quad . \end{aligned} \tag{2.56}$$

Hint: Use the x^1- and x^2-derivatives of the vector function $r(x^1, x^2)$ to construct the vector fields on the sphere in 3D – in the same way as in example 2.5.

The three vector fields V_x, V_y, and V_z are displayed in figure 2.9.

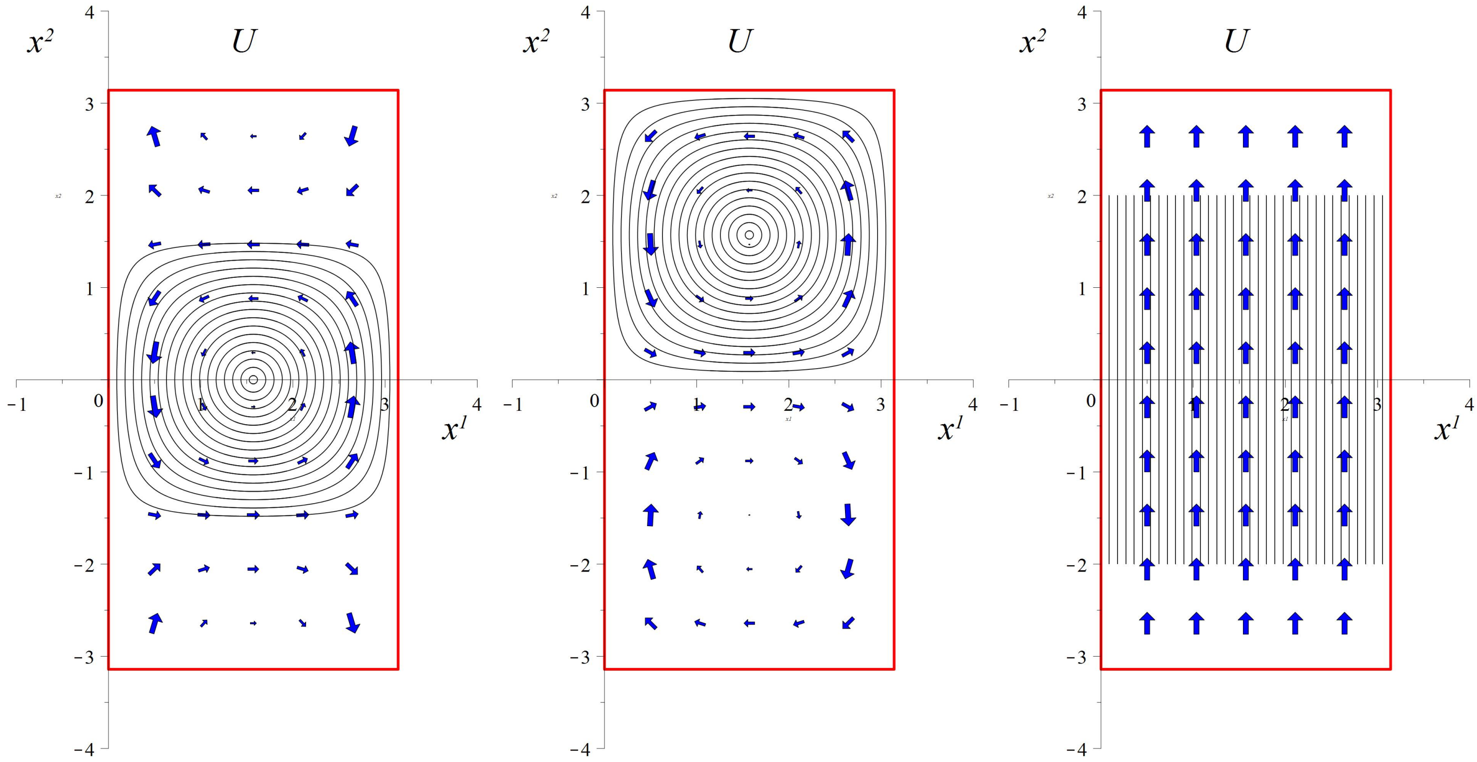


Figure 2.9: Representations in $\mathcal{U}$ of axis-Killing vector fields (plus a few of their integral curves) from the corresponding rotations of the unit sphere as displayed in figure 2.8.

EXERCISE 2.51

Verify that the three vector fields in equation (2.55) all satisfy the Killing equations in (2.38).

The axis rotation vector fields for the sphere considered above in their $\mathcal{U}$ representations are fairly simple examples of Killing vector fields stemming from the corresponding isometric rotations of the sphere. The rotation about a non-coordinate-axis line through the center of the sphere is shown in the rightmost display of figure 2.8 and gives rise to a somewhat more complicated Killing field in $\mathcal{U}$ – it is actually already on display in figure 2.6 above, where we show a blob following the flowlines of the Killing vector field. The flow map is an isometry, so the blob has constant area for all times along the flow.

Here in figure 2.12 we then insert the (finger print) indicatrix field obtained from the unit sphere into figure 2.6 and observe, that the vector field flow map indeed does deform the blob in accordance with the deformation of the indicatrices, so that the blob in particular is inspected to keep its area invariant during the flow.

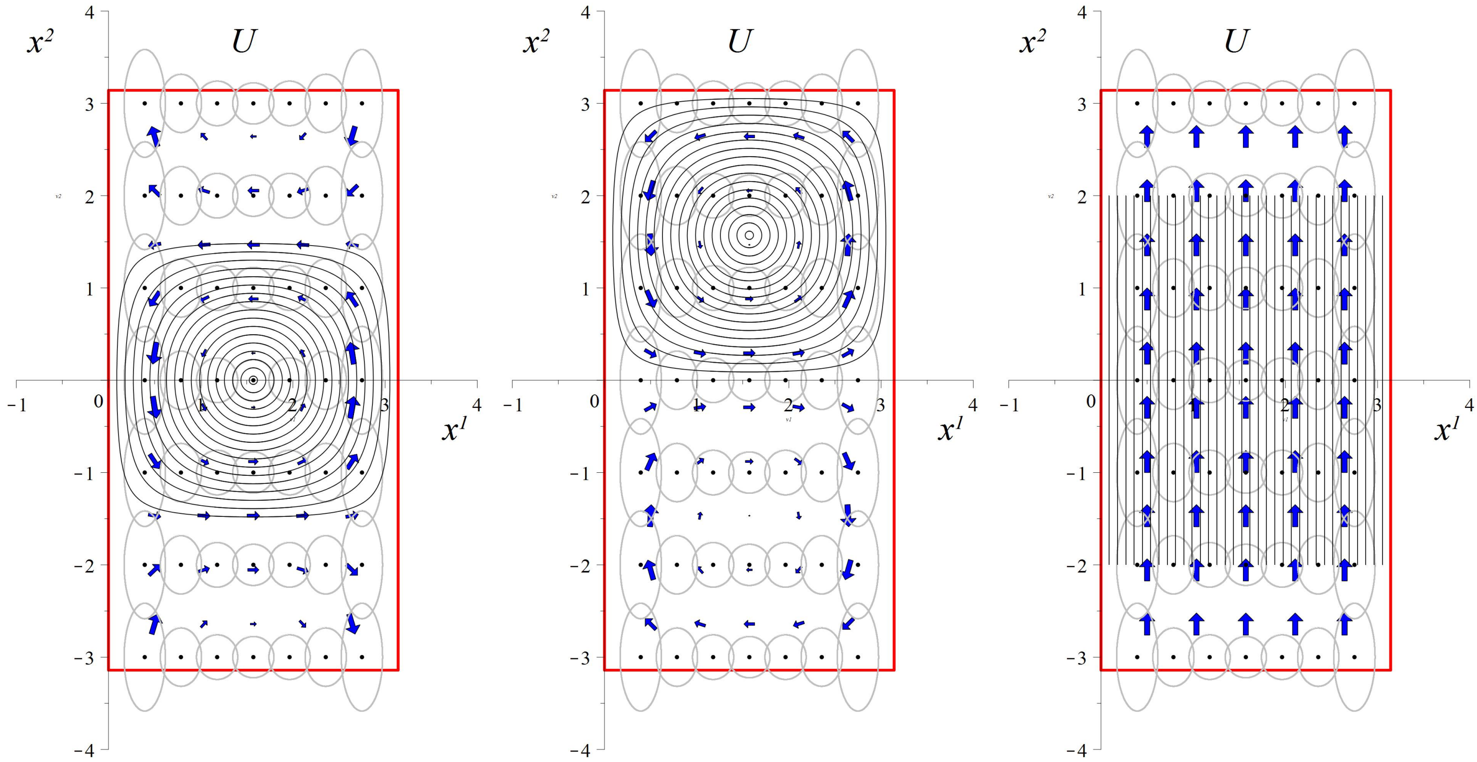


Figure 2.10: The metric tensor fingerprint from the sphere in $\mathcal{U}$ together with the axis rotations Killing fields in figure 2.9. The fingerprint is clearly mapped into itself when the individual indicatrices are deformed along the flow maps of the vector fields. I.e. the flow maps are local (in fact in this case global) isometries.

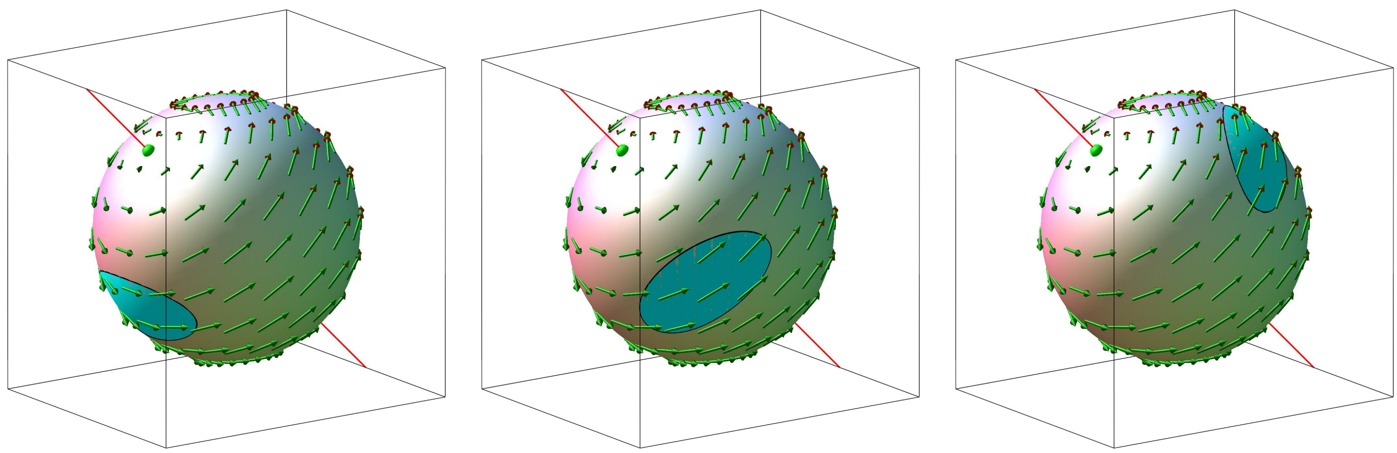


Figure 2.11: The rotation of a blob around a skew axis, i.e. not a coordinate axis.

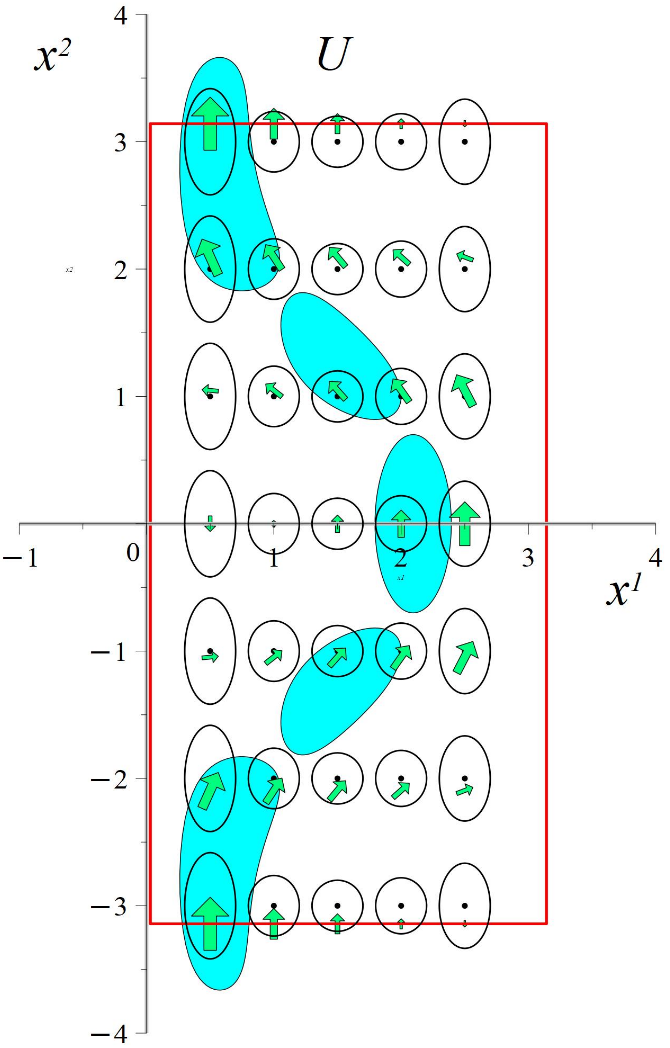


Figure 2.12: The representation (in $(\mathcal{U}, g_{\mathcal{U}})$) of the fixed blob on a rotating unit sphere.

Chapter 3

The Levi-Civita connection

3.1 The acceleration problem

We consider a regular smooth curve γ in the Euclidean plane $(\mathcal{U} = \mathbb{R}^2, g_E)$ with its Euclidean metric tensor g_E. Suppose γ is unit speed, i.e. it is parametrized by arc length s measured with sign from the point $\gamma(0) = p = (p^1, p^2)$. Then we have:

$$\begin{aligned} \gamma(s) &= (\gamma^1(s), \gamma^2(s)) \quad , \quad s \in \mathbb{R} \quad , \\ \gamma'(s) &= ((\gamma^1)'(s), (\gamma^2)'(s)) \quad , \\ \|\gamma'(s)\|_{g_E} &= 1 \quad , \quad \text{so that } s \text{ is indeed the signed arc length from } p\text{:} \\ s &= \int_0^s \|\gamma'(t)\|_{g_E}\, dt \quad . \end{aligned} \tag{3.1}$$

EXERCISE 3.1

A parametrized circle with center $(0,0)$ and radius R in $(\mathbb{R}^2, g_E)$ is given as follows:

$$\gamma(s) = (R \cdot \cos(s/R), R \cdot \sin(s/R)) \quad , \quad s \in]-R\cdot\pi, R\cdot\pi[\quad . \tag{3.2}$$

Show that the expression in equation (3.2) gives a unit speed parametrization of the circle.

In the Euclidean space $(\mathbb{R}^n, g_E)$ we *can* – and very often do – compare two vectors in *different* tangent spaces via the usual Euclidean parallel transport. For example, we can – in that particular representation of the Local Riemannian Manifold – compare the tangent vector $\gamma'(s_0 + t)$ in the tangent space $T_{\gamma(s_0+t)}$ with $\gamma'(s_0)$ in the tangent space $T_{\gamma(s_0)}$. In fact, we can directly consider the difference between the two vectors $\gamma'(s_0 + t)$ and $\gamma'(s_0)$, divide that difference by t and let t go to 0. Then we get the well known Euclidean acceleration of the parametrized curve γ at $\gamma(s_0)$, and from this construction it is natural to name the result of this operation by $\gamma''(s_0)$. In the given

Euclidean setting we have:

$$\begin{aligned} \mathrm{acc}_\gamma(s_0) &= \lim_{t \mapsto 0} \left(\frac{\gamma'(s_0+t) - \gamma'(s_0)}{t} \right) \\ &= \gamma''(s_0) = \left((\gamma^1)''(s_0) \, , \, (\gamma^2)''(s_0) \right) \quad . \end{aligned} \tag{3.3}$$

EXERCISE 3.2

Let γ be any regular arclength parametrized curve in $(\mathbb{R}^2, g_E)$. Show that $\mathrm{acc}_\gamma(s_0)$ is always orthogonal to the tangent (velocity) vector $\gamma'(s_0)$ for all s_0 in the parameter interval I for γ:

$$g_E\left(\mathrm{acc}_\gamma(s) \, , \, \gamma'(s)\right) = 0 \quad , \quad \text{for all } s \in I. \tag{3.4}$$

Hint: Note that $g_E(V \, , \, W) = V \cdot W$ is just the usual dot-product in $(\mathbb{R}^2, g_E)$ and consider the s-derivative of both sides of the equation $\|\gamma'(s)\|^2_{g_E} = 1$.

The acceleration problem is now the following: From chapters 1 and 2 it is evident, that we cannot just subtract (the coordinates of) two vectors in different tangent spaces and hope to get an isometry invariant and well defined acceleration from the second derivative of the coordinate functions of the curve as done in equation (3.3). In other words, we need to define the important notion of acceleration in a more refined way than equation (3.3), so that it becomes well defined in every representation $(\mathcal{U}, g_U)$ for any given Local Riemannian Manifold.

The notion of acceleration of a motion along a given curve is of paramount importance for an abundance of applications – just think of Newton's second law. So we need to solve the 'acceleration problem', so that the calculation of that important entity can be done consistently in any representation of a given Local Riemannian Manifold.

As a dramatic illustration of this 'acceleration problem' we consider again the polar map ϕ which was introduced in the previous chapters:

Example 3.3

We consider the Euclidean plane and represent that LRM in two different – but isometric ways – $(\mathcal{U}, g_{\mathcal{U}})$ and $(\mathcal{V}, g_{\mathcal{V}})$, respectively, corresponding to using ordinary Cartesian coordinates in the plane and ordinary polar coordinates in the plane.

So, we let ϕ denote the polar map defined on the open set $\mathcal{U} = (\mathbb{R}^2 \text{ minus the non-positive } x\text{-axis})$, which produces polar coordinates (y^1, y^2) as follows:

$$\phi(x^1, x^2) = \left(\sqrt{(x^1)^2 + (x^2)^2} \, , \, \arg(x^1 + i \cdot x^2) \right) = (y^1, y^2) \quad . \tag{3.5}$$

To repeat: In $\mathcal{U}$ we consider the Euclidean metric g_E so that $(\mathcal{U}, g_E)$ is then surely a direct representative of the LRM, the Euclidean plane, that we are considering. (Note that this *LRM* is quite different from the one constructed from the paraboloid $\mathcal{P}$ (with inherited metric from the ambient Euclidean 3D space) in chapter 1. The present $(\mathcal{U}, g_E)$ is in the similar sense constructed from the Euclidean plane $r(x^1, x^2) = (x^1, x^2, 0)$ in 3D.)

The metric matrix function in $\mathcal{U}$ is then simply:

$$G_E(x^1, x^2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{3.6}$$

Now let $\mathcal{V} = \phi(\mathcal{U})$, the open half strip in $\mathbb{R}^2$ considered before:

$$\mathcal{V} = \{(y^1, y^2) \,|\, y^1 > 0 \quad \text{and} \quad -\pi < y^2 < \pi\} \quad . \tag{3.7}$$

Then, in order for $(\mathcal{V}, g_V)$ to become a representative for the same *LRM* as $(\mathcal{U}, g_E)$ we must define $g_{\mathcal{V}}$ so that its corresponding metric matrix function is the following – see equation (1.15) in chapter 1:

$$\begin{aligned} G_{\mathcal{V}}(y^1, y^2) &= J^*_{\phi^{-1}} \cdot G_E(\phi^{-1}) \cdot J_{\phi^{-1}} \\ &= J^*_{\phi^{-1}} \cdot J_{\phi^{-1}} \\ &= \begin{bmatrix} \cos(y^2) & -y^1 \cdot \sin(y^2)) \\ \sin(y^2)) & y^1 \cdot \cos(y^2)) \end{bmatrix}^* \cdot \begin{bmatrix} \cos(y^2) & -y^1 \cdot \sin(y^2)) \\ \sin(y^2)) & y^1 \cdot \cos(y^2)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & (y^1)^2 \end{bmatrix} \quad . \end{aligned} \tag{3.8}$$

The circle in exercise 3.1 is mapped by ϕ into a curve η with the ϕ-induced parametrization:

$$\begin{aligned} \eta(s) &= \phi(\gamma(s)) = \phi\left(R \cdot \cos\left(s/R\right),\, R \cdot \sin\left(s/R\right)\right) \\ &= (R,\, s/R) \quad , \quad s \in \,]-R \cdot \pi,\, R \cdot \pi[\quad , \end{aligned} \tag{3.9}$$

so that the tangent (velocity) vector along η is:

$$\eta'(s) = (0, 1/R) \quad . \tag{3.10}$$

Note that by construction via ϕ – which is an isometry – the curve η is automatically parametrized by arc length.

EXERCISE 3.4

Check that

$$\|\eta'(s)\|_{g_{\mathcal{V}}} = 1 \quad \text{for all} \quad s \in \,]-R \cdot \pi,\, R \cdot \pi[\quad . \tag{3.11}$$

Observe, that if we now just blindly calculate the second derivatives of the coordinate functions of the parametrization $\eta(s)$ in $\mathcal{V}$ we get

$$\eta''(s) = (0,0) \quad . \tag{3.12}$$

whereas, if we calculate the (correct) acceleration of γ in $\mathcal{U}$ we get

$$\gamma''(s) = \left(\frac{-\cos(s/R)}{R} , \frac{-\sin(s/R)}{R} \right) \quad . \tag{3.13}$$

In other words, the two calculations are not in any way compatible – and we know the reason: The brute force derivation of $\eta'(s)$ resulting in $\eta''(s)$ presumes the (forbidden) subtraction of tangent vectors from different tangent spaces.

At this point you may think, that in the previous chapter we actually *did* compare vectors in different tangent spaces, namely via the (Jacobian of the) flow map defined by a vector field in $\mathcal{U}$. However, in the present situation we only have one curve with its tangent vectors. It is not obvious how to extend these vectors to a vector field around the curve in a consistent way, so that the corresponding flow map can be used for a proper definition of the acceleration of the motion along the curve – in a way similar to the construction of the Lie derivative.

The solution to the acceleration problem – which will eventually give the *same acceleration vector* in every representation of a given *LRM* – is obtained via a clever modification of the (forbidden, coordinate dependent) brute force differentiation of the coordinate functions of the tangent vectors η' along the curves η in question. The needed modification involves derivatives of the metric matrix function. The key concept to be established for this to work is called covariant differentiation of γ' along γ:

3.2 Covariant differentiation of vector fields along curves

The tangent vectors $\gamma'(t)$, $t \in I$, to a given curve γ – not necessarily arc length parametrized – is but one example of a smooth vector field *along a curve*.

Notation 3.5 Let γ denote a regular smooth curve in $\mathcal{U}^n$. At each point $\gamma(t)$, $t \in I$, we let $V(t)$ denote a vector in the tangent space $T_{\gamma(t)}\mathcal{U}$ and denote its coordinates with respect to the canonical basis in $T_{\gamma(t)}(\mathcal{U})$ by $v^i(t)$, $i = 1, \cdots, n$, so that

$$V(t) = \sum_{i=1}^{i=n} v^i(t) \cdot e_i \quad , \quad t \in I \quad . \tag{3.14}$$

If all the coordinate functions $v^i(t)$ are smooth functions of t in the interval I, then we will say that V is a smooth vector field along γ. The set of smooth vector fields along γ are denoted by $\mathfrak{X}(\gamma)$.

The tangent vectors γ' along a smooth regular curve γ clearly form a smooth vector field along γ, so that $\gamma' \in \mathfrak{X}(\gamma)$.

Note that if X is a vector field in $\mathcal{U}$, i.e. $X \in \mathfrak{X}(\mathcal{U})$, then the restriction X_γ of X to the smooth regular curve γ is a vector field along γ, so that $X_\gamma \in \mathfrak{X}(\gamma)$. Conversely, if $V \in \mathfrak{X}(\gamma)$, then there are several ways to extend the vector field along γ to a vector field $X \in \mathfrak{X}(\mathcal{U})$.

EXERCISE 3.6

Let γ denote a simple smooth closed curve in $\mathcal{U} = \mathbb{R}^2$. Simple means: without self-intersections. How would you construct a smooth extension of a given $V \in \mathfrak{X}(\gamma)$ along γ to a vector field $X \in \mathfrak{X}(\mathcal{U})$ in all of $\mathcal{U}$?

As motivated above we want to define a vector derivative (with respect to t) of any smooth vector field V along $\gamma(t)$, $t \in I$, so that the vector derivative is itself a smooth vector field along γ and so that the derivative gives the same result (modulo the isometry-diffeomorphisms) in any isometric representation of the *LRM*, that we are considering. When we have done that, we have also solved the 'acceleration problem' alluded to above.

Definition 3.7 Let $(\mathcal{U}, g)$ represent a Local Riemannian Manifold. A covariant derivative of a vector field V along a given curve γ in $(\mathcal{U}, g)$ parametrized by $t \in I$ is a mapping:

$$\frac{D}{dt} \quad : \quad \mathfrak{X}(\gamma) \longmapsto \mathfrak{X}(\gamma) \tag{3.15}$$

which satisfies the following natural derivation-conditions for all V, W in $\mathfrak{X}(\gamma)$ and for all smooth functions f on I:

$$\begin{aligned} \frac{D}{dt}(V(t) + W(t)) &= \frac{D}{dt}V(t) + \frac{D}{dt}W(t) \\ \frac{D}{dt}(f(t)\cdot V(t)) &= f'(t)\cdot V(t) + f(t)\cdot \frac{D}{dt}V(t) \quad . \end{aligned} \tag{3.16}$$

Moreover – and this is the key condition that we have been looking for – the covariant derivative must be compatible with the metric in the following sense:

$$\frac{d}{dt}g(V(t), W(t)) = g\left(\frac{D}{dt}V(t)\,,\,W(t)\right) + g\left(V(t)\,,\,\frac{D}{dt}W(t)\right) \tag{3.17}$$

EXERCISE 3.8

Show by example that the usual time-derivative $\frac{d}{dt}$ *does not* satisfy all the three conditions in equations (3.16) and (3.17) . Hint: You may want to use ingredients from example 3.3.

EXERCISE 3.9

Show that if we consider the rare case that the metric tensor is Euclidean, i.e. $g = g_E$, then the usual time derivative *does* satisfy the three conditions in equations (3.16) and (3.17).

Therefore, in consequence of exercise 3.9, if we can show that covariant derivatives exist and are unique – we shall introduce one more condition below so that this indeed will be the case – then it must reduce to the usual time derivative when the metric in the *LRM* representation $(\mathcal{U}, g)$ is the Euclidean $g = g_E$.

3.3 Defining properties of the Levi-Civita connection

In order to nail down a unique covariant derivative (of vector fields along curves) we first define a much more powerful and useful operator, the Levi-Civita connnection. It is the single most important object in these notes and from here onwards we will refer to it again and again.

The Levi-Civita connection is a derivation of vector fields along *vector fields*. It satisfies conditions which are similar to the requirements for covariant derivatives of vector fields along curves.

Definition 3.10 Let again $(\mathcal{U}, g)$ represent a Local Riemannian Manifold. The Levi-Civita connection ∇ on $(\mathcal{U}, g)$ is the following mapping:

$$\begin{aligned} \nabla \quad : \quad \mathfrak{X}(\mathcal{U}) \times \mathfrak{X}(\mathcal{U}) &\longmapsto \mathfrak{X}(\mathcal{U}) \\ (X \quad , \quad Y) &\longmapsto \nabla_X Y \quad , \end{aligned} \tag{3.18}$$

which satisfies the following conditions for all vector fields X, Y, and Z in $\mathfrak{X}(\mathcal{U})$ and for all smooth functions f and h in $\mathfrak{F}(\mathcal{U})$:

$$\nabla_{(f \cdot X + h \cdot Y)} Z = f \cdot \nabla_X Z + h \cdot \nabla_Y Z \tag{3.19}$$

$$\nabla_X (Y + Z) = \nabla_X (Y) + \nabla_X (Z) \tag{3.20}$$

$$\nabla_X (f \cdot Y) = f \cdot \nabla_X (Y) + X(f) \cdot Y \tag{3.21}$$

$$X(g(Y, Z)) = g(\nabla_X (Y), Z) + g(Y, \nabla_X Z) \tag{3.22}$$

$$\nabla_X (Y) - \nabla_Y (X) - [X, Y] = 0 \quad . \tag{3.23}$$

Although this seems to be quite a massive set of non-transparent properties to meet, there are several miraculous payoffs: As already alluded to in the formulation of the definition above, the Levi-Civita connection mapping ∇ always exists and it is unique! See the proof below. Moreover, it will give us a direct solution to the acceleration problem mentioned above. In the same vein it will give us a well defined notion of parallel transport, of geodesics, and of curvature tensors. But first, the most important theorem about the Levi-Civita connection:

Theorem 3.11 Given a Local Riemannian Manifold represented by $(\mathcal{U}, g)$. Then there exists a unique mapping ∇ which satisfies all the conditions in definition (3.10).

Proof. A slick and very nice coordinate-free version of the proof is given in [4, p. 55]. It is highly appropriate to repeat it here – and we do that almost verbatim:

We assume initially, that such a connection ∇ exists. We must then show that it is unique. From the assumption on metric compatibility expressed in equation (3.22) we have:

$$X(g(Y,Z)) = g(\nabla_X(Y), Z) + g(Y, \nabla_X Z) \tag{3.24}$$

$$Y(g(Z,X)) = g(\nabla_Y(Z), X) + g(Z, \nabla_Y X) \tag{3.25}$$

$$Z(g(X,Y)) = g(\nabla_Z(X), Y) + g(X, \nabla_Z Y) \quad . \tag{3.26}$$

Now add the two first equations (3.24) and (3.25) together and subtract the third equation (3.26). Then use the symmetry expressed in equation (3.23) and obtain:

$$\begin{aligned} &X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &= g([X, Z], Y) + g([Y, Z], X) + g([X, Y], Z) + 2 \cdot g(Z, \nabla_Y X) \quad . \end{aligned} \tag{3.27}$$

In consequence we therefore have:

$$\begin{aligned} g(Z, \nabla_Y X) = &\frac{1}{2} \cdot (X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))) \\ -&\frac{1}{2} \cdot (g([X, Z], Y) + g([Y, Z], X) + g([X, Y], Z)) . \end{aligned} \tag{3.28}$$

The expression in equation (3.28) shows that the connection ∇ is uniquely determined by the metric g. Hence, if it exists, then it is also unique. The existence follows from the same equation (3.28) because we can *use it as a definition* of $\nabla_Y X$ and then show, that this specific definition actually gives rise to a connection, which satisfies all the conditions in definition 3.10. □

EXERCISE 3.12

Think about the last lines in the proof above: Why does (3.28) show that the connection ∇ is uniquely determined by the metric g? What about the Lie brackets appearing in the expression?

EXERCISE 3.13

Show that (3.28) actually can be used to *construct a candidate* for a connection, which satisfies all the conditions in definition 3.10.

Notation 3.14 Although somewhat redundant (because ∇ follows from g), we will from now on write the representatives for a Local Riemannian Manifold as follows: $(\mathcal{U}, g, \nabla)$ – in particular when we make explicit use of the associated Levi-Civita connection ∇.

Equation (3.28) is obviously of instrumental importance for establishing the Levi-Civita connection. But it also gives us directly the coordinates of the connection map. We first define these coordinates as follows:

Definition 3.15 We have at each point p in $\mathcal{U}$:

$$(\nabla_{e_i} e_j)_p = \sum_{k=1}^{k=n} \Gamma_{ij}^k(p) \cdot e_k \quad . \tag{3.29}$$

The coordinate functions $\Gamma_{ij}^k(p)$ appearing in this linear combination of the basis vectors at p are called the Christoffel symbols for ∇ on $(\mathcal{U}, g, \nabla)$.

Then, by insertion of $X = e_i$, $Y = e_j$ and $Z = e_\ell$ into (3.28) and observing that all the Lie brackets $[e_i\,,\,e_j]$ vanish (according to exercise 2.30 in chapter 2) we get, on the left hand side of equation (3.28):

$$\begin{aligned} g\left(Z,\,\nabla_Y X\right) &= g\left(e_\ell\,,\,\nabla_{e_j} e_i\right) \\ &= g\left(e_\ell\,,\,\sum_{k=1}^{k=n} \Gamma_{ji}^k \cdot e_k\right) \\ &= \sum_{k=1}^{k=n} \Gamma_{ji}^k \cdot g(e_\ell\,,\,e_k) \quad \text{for all indices } i,\ j, \text{ and } \ell \quad . \end{aligned} \tag{3.30}$$

Equating this with the right hand side of equation 3.28 we have:

$$\sum_{k=1}^{k=n} \Gamma_{ji}^k \cdot g(e_\ell, e_k) = \frac{1}{2} \cdot \left(e_i(g(e_j, e_\ell)) + e_j(g(e_\ell, e_i)) - e_\ell(g(e_i, e_j))\right) \quad . \tag{3.31}$$

Using shorthand index notation for the elements g_{ij} in the metric matrix function G associated with g in $(\mathcal{U}, g, \nabla)$ we obtain:

$$\sum_{k=1}^{k=n} \Gamma_{ji}^{k} \cdot g_{\ell k} = \frac{1}{2} \cdot \left(e_i(g_{j\ell}) + e_j(g_{\ell i}) - e_\ell(g_{ij}) \right) \quad , \tag{3.32}$$

and finally, using the derivation interpretation of $e_i = \frac{\partial}{\partial x^i}$ we have equivalently:

$$\sum_{k=1}^{k=n} \Gamma_{ji}^{k} \cdot g_{\ell k} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x^i} g_{j\ell} + \frac{\partial}{\partial x^j} g_{\ell i} - \frac{\partial}{\partial x^\ell} g_{ij} \right) \quad , \tag{3.33}$$

In order to extract a pure formula (in terms of g and the derivatives of g) for the Christoffel symbols we introduce the following

Notation 3.16 Since the metric matrix function G is positive definite it has an inverse G^{-1} whose elements we call g^{ij}, so that

$$\sum_{k=1}^{k=n} g_{ik} \cdot g^{kj} = \delta_i^j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{3.34}$$

We can then multiply both sides of the equation (3.33) by G^{-1} which amounts to the following:

$$\begin{aligned} \sum_{\ell=1}^{\ell=n} \left(g^{\ell m} \cdot \sum_{k=1}^{k=n} \Gamma_{ji}^{k} \cdot g_{\ell k} \right) &= \Gamma_{ji}^{m} \\ &= \frac{1}{2} \cdot \sum_{\ell=1}^{\ell=n} \left(\frac{\partial}{\partial x^i} g_{j\ell} + \frac{\partial}{\partial x^j} g_{\ell i} - \frac{\partial}{\partial x^\ell} g_{ij} \right) \cdot g^{\ell m} \quad . \end{aligned} \tag{3.35}$$

In this way we have now isolated the calculation of the Christoffel symbols. One observation from equation (3.35) is that since G and G^{-1} are symmetric matrices, the Christoffel symbols are also symmetric in the lower indices:

Proposition 3.17

$$\Gamma_{ij}^{m} = \Gamma_{ji}^{m} = \frac{1}{2} \cdot \sum_{\ell=1}^{\ell=n} \left(\frac{\partial}{\partial x^i} g_{j\ell} + \frac{\partial}{\partial x^j} g_{\ell i} - \frac{\partial}{\partial x^\ell} g_{ij} \right) \cdot g^{\ell m} \quad . \tag{3.36}$$

Note that in any Euclidean Local Riemannian Manifold $(\mathcal{U}, g_E, \nabla)$ all the Christoffel symbols vanish because all the entries of the metric matrix function are (the simplest possible) constants!

Before showing that the Levi-Civita connection produces a unique covariant differentiation along curves we first note the following:

Proposition 3.18 Let V and W be two vector fields in $(\mathcal{U}, g, \nabla)$. Then, in order to calculate the vector $\nabla_V W$ at some given point p in $\mathcal{U}$, we do not need to know all the values of V – only the value of V at p. This follows readily from the expression below, where we use:

$$\begin{aligned} V &= \sum_{i=1}^{i=n} v^i \cdot e_i \\ W &= \sum_{i=1}^{i=n} w^i \cdot e_i \quad . \end{aligned} \tag{3.37}$$

With these V and W we get:

$$(\nabla_V(W))_p = \sum_{k=1}^{k=n} \left(\sum_{ij} v^i(p) \cdot w^j(p) \cdot \Gamma^k_{ij}(p) + V_p(w^k) \right) \cdot e_k \tag{3.38}$$

Note that equation (3.38) shows in particular, that $(\nabla_V W)_p$ only depends on the value of $V \in T_p\mathcal{U}$ *at the point* p and on the values of W *in the direction* of W. In other words, we do not need a full vector field $V \in \mathfrak{X}(\mathcal{U})$ in order to evaluate the ∇-derivative of the vector field W with respect to V at p, but we do need the vector field W to be given in the direction of V_p in order to find and use the directional derivatives $V_p(w^k)$, $k = 1, \cdots, n$.

Proof.

$$\begin{aligned} \nabla_V W &= \sum_i v^i \cdot \nabla_{e_i} \left(\sum_j w^j \cdot e_j \right) \\ &= \sum_{ij} v^i \cdot w^j \cdot \nabla_{e_i} e_j + \sum_{ij} v^i \cdot e_i(w^j) \cdot e_j \\ &= \sum_{ij} \sum_{k=1}^{k=n} \left(v^i \cdot w^j \cdot \Gamma^k_{ij} + V(w^k) \right) \cdot e_k \quad , \end{aligned} \tag{3.39}$$

which shows that $(\nabla_V W)_p$ depends on $v^i(p)$, $w^k(p)$, and the derivatives $V_p(w^k)$ of the coordinate functions w^k by V at p. □

In passing we note that Killing vector fields are characterized by the following property, which is expressed in terms of the Levi-Civita connection:

Proposition 3.19 A vector field $X \in \mathfrak{X}(\mathcal{U})$ is Killing if and only if

$$g(\nabla_V X, W) + g(\nabla_W X, V) = 0 \quad \text{for all vector fields } V \text{ and } W \text{ in } \mathfrak{X}(\mathcal{U}) \quad . \tag{3.40}$$

Proof. This follows from the previous characterization of Killing vector fields in chapter 2:

$$X(g(V, W)) = g([X,V], W) + g(V, [X, W]) \quad , \tag{3.41}$$

because this equation is equivalent to both of the following equations (using equation (3.23)):

$$\begin{aligned} 0 &= g(\nabla_X V, W) - g([X,V], W) + g(\nabla_X W, V) - g([X,W], V) \\ 0 &= g(\nabla_V X, W) + g(\nabla_W X, V) \quad . \end{aligned} \tag{3.42}$$

□

3.4 Back to the covariant differentiation

The following result now shows that the Levi-Civita connection ∇ determines a unique covariant differentiation of vector fields along curves. This is not surprising in view of the massive derivation-type conditions satisfied already by the connection ∇:

Proposition 3.20 Let ∇ denote the Levi-Civita connection map of an *LRM* represented by $(\mathcal{U}, g, \nabla)$. Let γ denote a regular smooth curve in $\mathcal{U}$ parametrized by $t \in I$, and let $V \in \mathfrak{X}(\gamma)$. Suppose that V is the restriction of a vector field $W \in \mathfrak{X}(\mathcal{U})$ to the curve γ so that $V(t) = W(\gamma(t))$. Then there is a unique covariant derivative $\frac{D}{dt}$ along γ satisfying Definition 3.7, and it has the property:

$$\frac{D}{dt}V(t) = \nabla_{\gamma'} W \quad \text{along } \gamma \quad , \tag{3.43}$$

where the left hand side makes sense at each point along γ in view of proposition 3.18. In standard coordinates in $\mathcal{U}$ with the induced standard basis fields $\{e_1, \cdots, e_n\}$ we get, for any vector field $V \in \mathfrak{X}(\gamma)$ with $V(t) = \sum_{i=1}^{i=n} v^i(t) \cdot e_i$ along γ:

$$\frac{D}{dt}V(t) = \sum_k \left(\frac{dv^k}{dt} + \sum_{ij} v^j(t) \cdot (\gamma^i)'(t) \cdot \Gamma^k_{ij}(\gamma(t)) \right) \cdot e_k \quad . \tag{3.44}$$

Proof. Suppose first that we have found a covariant derivative $\frac{D}{dt}$ that satisfies the conditions in definition 3.7. Then we must show that it is unique. With $V(t) = \sum_{i=1}^{i=n} v^i(t) \cdot e_i$ we get from equations (3.16):

$$\frac{D}{dt}V(t) = \sum_{j=1}^{j=n} \frac{dv^j}{dt} \cdot e_j + \sum_{j=1}^{j=n} v^j(t) \cdot \frac{D}{dt} e_j \quad , \tag{3.45}$$

where – via equations (3.17) and (3.19) – for all index values j:

$$\begin{aligned}\frac{D}{dt}e_j &= \nabla_{\gamma'}e_j = \nabla_{\sum_i(\gamma^i)'\cdot e_i}e_j \\ &= \sum_i(\gamma^i)'\cdot\nabla_{e_i}e_j \\ &= \sum_i(\gamma^i)'\cdot\sum_k\Gamma^k_{ij}\cdot e_k \quad .\end{aligned} \tag{3.46}$$

Inserting this into equation (3.45) we get:

$$\frac{D}{dt}V(t) = \sum_k\left(\frac{dv^k}{dt} + \sum_{ij}v^j(t)\cdot(\gamma^i)'(t)\cdot\Gamma^k_{ij}(\gamma(t))\right)\cdot e_k \tag{3.47}$$

This relation shows the uniqueness of the covariant derivative – stemming from the wanted properties and from the existence and uniqueness of the connection ∇ and its Christoffel symbols. The existence is also guaranteed by the equation (3.47). In fact, we can again just *define* a covariant derivative by this equation and then check that it satisfies the needed conditions. □

EXERCISE 3.21

Show that if we define a candidate for a covariant derivative by equation (3.47), then it satisfies all the conditions of definition 3.7.

In other words, the uniqueness and existence of the Levi-Civita connection induces the unique existence of a covariant derivative along any smooth curve.

3.5 The solution to the acceleration problem

At this point we have no other choice but to define the notion of acceleration of a motion along a parametrized curve as follows:

Definition 3.22 Let $(\mathcal{U}, g, \nabla)$ represent a given *LRM*, and let γ be a t-parametrized curve in $\mathcal{U}$, $t \in I$. Then the acceleration vector for the corresponding motion along γ is simply:

$$\mathrm{acc}_\gamma(t) = \frac{D}{dt}\gamma'(t) \quad \text{for all } t \in I \tag{3.48}$$

In terms of coordinates this definition is then equivalent to the following, using equation (3.47) with $v^k(t) = (\gamma^k)'(t)$:

$$\mathrm{acc}_\gamma(t) = \frac{D}{dt}\gamma'(t) = \sum_k\left(\frac{d^2}{dt^2}\gamma^k(t) + \sum_{ij}(\gamma^i)'(t)\cdot(\gamma^j)'(t)\cdot\Gamma^k_{ij}(\gamma(t))\right)\cdot e_k \quad . \tag{3.49}$$

And it works! Let us have a look again at example 3.3:

Example 3.23

The Christoffel symbols of the Levi-Civita connection of $(\mathcal{V}, g_{\mathcal{V}}, \nabla)$ are the following 8 functions on $\mathcal{V}$ – calculated via the formula in proposition 3.17:

$$\begin{aligned}
\Gamma^1_{1\,1}(y^1,y^2) &= 0 \\
\Gamma^2_{1\,1}(y^1,y^2) &= 0 \\
\Gamma^1_{1\,2}(y^1,y^2) &= \Gamma^1_{2\,1}(y^1,y^2) = 0 \\
\Gamma^2_{1\,2}(y^1,y^2) &= \Gamma^2_{2\,1}(y^1,y^2) = \frac{1}{y^1} \\
\Gamma^1_{2\,2}(y^1,y^2) &= -y^1 \\
\Gamma^2_{2\,2}(y^1,y^2) &= 0 \quad .
\end{aligned} \tag{3.50}$$

EXERCISE 3.24

Check these identities for the respective Christoffel symbol functions.

We insert the Christoffel symbols into equation (3.49) and obtain the acceleration vector for the curve η in $\mathcal{V}$ – using, of course, that only two of the Christoffel symbols are non-zero:

$$\mathrm{acc}_{\eta}(s) = \left(\frac{-y^1}{R^2}\right) \cdot e_1 = \left(\frac{-\eta^1(s)}{R^2}\right) \cdot e_1 = -\left(\frac{1}{R}\right) \cdot e_1 = \left(-\frac{1}{R}, 0\right) \quad . \tag{3.51}$$

Using the metric $g_{\mathcal{V}}$ we see, that the acceleration vector has the correct length $1/R$ and, moreover, it is isometrically ϕ-related to the corresponding acceleration vector in $(\mathcal{U}, g_{\mathcal{U}})$ that we previously calculated in example 3.3:

$$\begin{aligned}
J_{\phi^{-1}}(\mathrm{acc}_{\eta}(s)) &= \begin{bmatrix} \cos(s) & -R\cdot\sin(s) \\ \sin(s) & R\cdot\cos(s)) \end{bmatrix} \cdot \begin{bmatrix} -1/R \\ 0 \end{bmatrix} \\
&= \left(\frac{-\cos(s/R)}{R}, \frac{-\sin(s/R)}{R}\right) \\
&= \mathrm{acc}_{\gamma}(s) \quad , \quad s \in]-R\cdot\pi, R\cdot\pi[
\end{aligned} \tag{3.52}$$

– precisly as expected and as needed: The acceleration vector field along a parametrized curve is invariant under isometric diffeomorphisms – we get the same acceleration in every representation of any given Local Riemannian Manifold.

In prolongation of exercise 3.2 we now have in all generality, that if a curve is parametrized by constant speed, then the acceleration vector is orthogonal to the curve:

EXERCISE 3.25

Let γ be any regular arclength parametrized curve in any representation of a given *LRM*, $(\mathcal{U}, g_{\mathcal{U}})$. Show that $\mathrm{acc}_\gamma(s)$ is always orthogonal to the tangent (velocity) vector $\gamma'(s)$ for all s in the parameter interval I for γ:

$$g_{\mathcal{U}}\left(\mathrm{acc}_\gamma(s)\,,\,\gamma'(s)\right) = g_{\mathcal{U}}\left(\frac{D}{dt}\gamma'(s)\,,\,\gamma'(s)\right) = 0 \quad , \quad \text{for all } s \in I. \tag{3.53}$$

Hint: Note that

$$\frac{d}{ds}g_{\mathcal{U}}\left(\gamma'(s)\,,\,\gamma'(s)\right) = 0 \quad . \tag{3.54}$$

3.6 Gravity-induced accelerated motion on the paraboloid

The notion of acceleration plays a well-known and instrumental rôle in the study of analytical mechanics and dynamical systems. We briefly illustrate the relevance of the covariant derivative in this setting via a couple of examples:

Example 3.26

Suppose we consider again the paraboloid introduced in chapter 1:

$$\mathcal{P} \quad : \quad r(x^1, x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2) \quad . \tag{3.55}$$

The tangent plane in $\mathbb{R}^3$ to $\mathcal{P}$ at the point $r(x^1\,,\,x^2)$ is spanned by the two tangent vectors:

$$\begin{aligned} \frac{\partial}{\partial x^1} r(x^1, x^2) &= \left(1, 0, 2\cdot x^1\right) \\ \frac{\partial}{\partial x^2} r(x^1, x^2) &= \left(0, 1, 2\cdot x^2\right) \quad . \end{aligned} \tag{3.56}$$

A particle of mass 1 moving *on the surface* of $\mathcal{P}$ under the sole influence of gravity $G = (0, 0, -1)$ then solves Newton's second law: The acceleration of the particle on $\mathcal{P}$ is equal to the projection of G into the surface (tangent plane) at each point along its track. This latter projection is obtained in as follows: First we find the unit normal vector N to the surface at each point $r(x^1, x^2)$:

$$N(x^1, x^2) = \frac{r'_{x^1} \times r'_{x^2}}{\|r'_{x^1} \times r'_{x^2}\|_E} \quad , \tag{3.57}$$

where $\| * \|_E$ denotes the usual Euclidean norm in $\mathbb{R}^3$. The result is

$$N(x^1, x^2) = (-2\cdot x^1\,,\,-2\cdot x^2, 1)\,/\,\sqrt{1 + 4(x^1)^2 + 4(x^2)^2} \quad . \tag{3.58}$$

The projection of G into the tangent plane of the surface at $r(x^1, x^2)$ is then (using fat $\bullet$ for the usual dot-product in Euclidean $\mathbb{R}^3$):

$$\begin{aligned}\mathrm{proj}(G) &= G - (G \bullet N)\cdot N \\ &= \frac{1}{1+4\cdot(x^1)^2+4\cdot(x^2)^2}\cdot\left(-2\cdot x^1\,,\; -2\cdot x^2\,,\; -4\cdot(x^1)^2-4\cdot(x^2)^2\right) \quad .\end{aligned} \tag{3.59}$$

The projection $\mathrm{proj}(G)$ is then a linear combination of r'_{x^1} and r'_{x^2}:

$$\mathrm{proj}(G) = \alpha\cdot r'_{x^1} + \beta\cdot r'_{x^2} \quad . \tag{3.60}$$

Solving this equation for α and β gives the coordinates of the projection with respect to the induced basis vectors in the tangent plane:

$$\begin{aligned}\mathrm{proj}(G) &= (\alpha,\beta)_{\{\frac{\partial r}{\partial x^1},\frac{\partial r}{\partial x^2}\}} \\ &= \frac{1}{1+4\cdot(x^1)^2+4\cdot(x^2)^2}\cdot\left(-2\cdot x^1\,,\; -2\cdot x^2\right)_{\{\frac{\partial r}{\partial x^1},\frac{\partial r}{\partial x^2}\}} \quad .\end{aligned} \tag{3.61}$$

These coordinate functions can also be found (in a much simpler and more direct way via the metric) as the coordinates of the negative *g-gradient* of the potential (height) function $z(x^1,x^2) = (x^1)^2 + (x^2)^2$ on the surface determined by the gravity vector $(0,0,-1)$. The g-gradient will be introduced below in section 3.10.1 – see exercise 3.57.

These coordinates are thence also the *coordinates* of the representation of the force vector in the standard basis $\{e_1, e_2\}$ of the Local Riemannian Manifold model $(\mathcal{U}, g, \nabla)$ of the paraboloid – this identity between coordinates of tangent vectors in $T_p\mathcal{U}$ and tangent vectors to the surface at $r(p)$ was discussed in section 1.4 in chapter 1.

The motion, the parametrized curve $\gamma(t) = (\gamma^1(t)\,,\,\gamma^2(t))$ must therefore satisfy:

$$\mathrm{acc}_\gamma(t) = \left(\frac{-2\cdot x^1}{1+4\cdot(x^1)^2+4\cdot(x^2)^2}\,,\;\frac{-2\cdot x^2}{1+4\cdot(x^1)^2+4\cdot(x^2)^2}\right)_{|x^1=\gamma^1(t),x^2=\gamma^2(t)} \quad . \tag{3.62}$$

The equations of motion to be solved are then contained in the following second order ordinary differential equation system:

$$\frac{D}{dt}\gamma'(t) = \left(\frac{-2\cdot x^1}{1+4\cdot(x^1)^2+4\cdot(x^2)^2}\,,\;\frac{-2\cdot x^2}{1+4\cdot(x^1)^2+4\cdot(x^2)^2}\right)_{|x^1=\gamma^1(t),x^2=\gamma^2(t)} \tag{3.63}$$

with suitable initial conditions – to be chosen: $\gamma(0) = p$ and $\gamma'(0) = V_0$.

These equations are a bit complicated – not least because the left hand side also contains the Christoffel symbols when we write out the covariant derivative of γ' along γ.

We display the non-zero Christoffel symbol functions here – they are also useful for exercise 3.71 below.

Only four of the Christoffel symbols do not vanish, and they appear in pairs as follows:

$$\Gamma^1_{11}(x^1,x^2) = \Gamma^1_{22}(x^1,x^2) = \frac{4\cdot x^1}{1+4\cdot(x^1)^2+4\cdot(x^2)^2}$$

$$\Gamma^2_{11}(x^1,x^2) = \Gamma^2_{22}(x^1,x^2) = \frac{4\cdot x^2}{1+4\cdot(x^1)^2+4\cdot(x^2)^2} \quad . \tag{3.64}$$

The solution curve $(\gamma^1(t),\gamma^2(t))$ with initial conditions $\gamma(0)=(1,0)$ and $\gamma'(0)=(1,1)$ can be obtained by numerical solution and is displayed in the parameter domain in figure 3.1. The solution is then lifted into the paraboloid (for various choices of duration of time) in figure 3.2. The boundedness of the solution curve is a natural consequence of the preservation of total mechanical energy, i.e. potential plus kinetic energy. This is but one example of the intimate relationship between differential geometry and the theory of dynamical systems. A much simpler example is given in exercise 3.27.

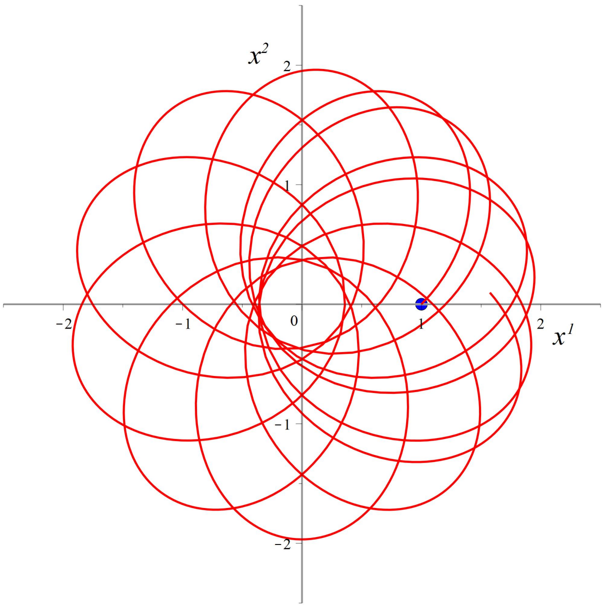


Figure 3.1: See example 3.26. Gravity induces acceleration and makes the particle move around the center point, which corresponds to the bottom point of the paraboloid – se figure 3.2.

EXERCISE 3.27

Repeat the steps in the above example 3.26, but now for an inclined plane instead of the paraboloid – with the same gravity $G=(0,0,-1)$ in the ambient 3D space. I.e.:

$$P \quad : \quad r(x^1,x^2) = \left(x^1,x^2,\,\alpha\cdot x^1\right) \tag{3.65}$$

for some α. This situation is considerably simpler than the setting in the paraboloid example above because all the Christoffel symbols vanish! Find the exact solution to the motion problem with

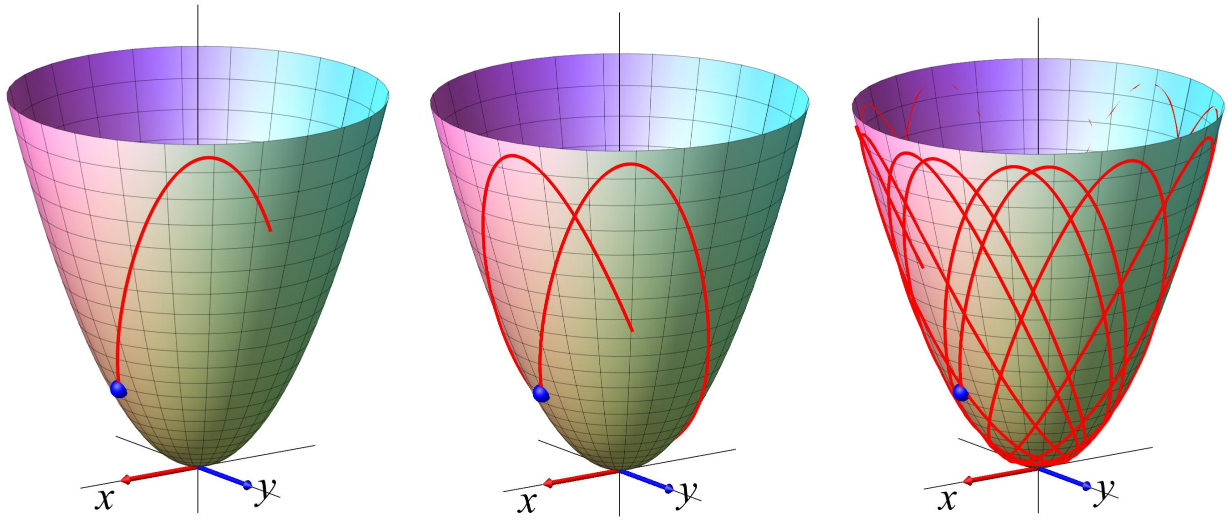


Figure 3.2: The solution curve from figure 3.1 is – for various durations of time – lifted into the paraboloid considered in example 3.26.

given/chosen initial conditions for the case of any inclined plane, corresponding to any fixed value of α. What happens if $\alpha = 0$? Find the exact solution to the motion problem with given/chosen initial conditions for the case of a *vertical plane* (which needs a slight modification of the expression for its parametrization).

3.7 Parallel transport of vectors along curves

The most general g-compatible notion of parallel transport of a vector V from a tangent space $T_p\mathcal{U}$ at p to a vector in another tangent space $T_q\mathcal{U}$ at q is defined by:

Definition 3.28 Let $(\mathcal{U}, g, \nabla)$ denote a Local Riemannian Manifold. Let γ denote a regular smooth curve in $\mathcal{U}$ from a point p to a point q, parametrized by $t \in I = [a,b]$, so that $\gamma(a) = p$ and $\gamma(b) = q$. Let $V(t)$ denote a vector field along γ, $V \in \mathfrak{X}(\gamma)$. Then V is called a parallel vector field along the curve γ if and only if

$$\frac{D}{dt}V(t) = 0 \quad \text{for all} \quad t \in I = [a,b] \quad . \tag{3.66}$$

In coordinates this is equivalent to the following condition via equation (3.47), where we use $V(t) = \sum_k v^k(t) \cdot e_k$ and $\gamma(t) = (\gamma^1(t), \cdots, \gamma^n(t))$:

$$0 = \sum_k \left(\frac{dv^k}{dt} + \sum_{ij} v^j(t) \cdot (\gamma^i)' \cdot \Gamma^k_{ij}(\gamma(t)) \right) \cdot e_k \quad \text{for all } t \in I = [a,b]. \tag{3.67}$$

Moreover, we naturally say that the vector $V(a) \in T_p\mathcal{U}$ has been parallel transported to the vector $V(b) \in T_q\mathcal{U}$ along γ.

EXERCISE 3.29

Show that in a Euclidean Local Riemannian Manifold $(\mathcal{U}, g_E, \nabla)$ parallel transport of a vector consists of the usual process of keeping the coordinate functions for the vector field constant – completely independent of the curve along which the vector is transported. Hint: All the Christoffel symbols vanish.

EXERCISE 3.30

Let $(\mathcal{V}, g_{\mathcal{V}}, \nabla)$ denote Poincaré's half plane model that we encountered in chapter 1. I.e. we have $y^2 > 0$ and

$$G_{\mathcal{V}}(y^1, y^2) = \frac{1}{(y^2)^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{3.68}$$

Let η denote the horizontal straight line: $\eta(t) = (t\,,\,1)$ in $\mathcal{V}$ and let $V_0 \in T_{\gamma(0)}\mathcal{V}$ denote the vector $V_0 = (0\,,\,1)$ in the tangent space at $\eta(0) = (0,1)$. The vector V_0 is parallel transported along η and produces the vector $V(t)$ at $\eta(t)$ so that

$$\frac{D}{dt}V(t) = 0 \quad \text{and} \quad V(0) = V_0 = (0,1) \quad . \tag{3.69}$$

Find the coordinate functions $v^i(t)$, $i = 1,2$, for $V(t) = v^1(t)\cdot e1 + v^2(t)\cdot e_2$.

In general, parallel transport depends on the curve along which we solve the transport problem:

Example 3.31

We let C denote a half-circle of radius R and center $(0,0)$ in the half plane model $(\mathcal{V}, g_{\mathcal{V}}, \nabla)$ as considered in the exercise 3.30 above. The transport curve is then:

$$C \quad : \quad \gamma(t) = (R\cdot\sin(t)\,,\,R\cdot\cos(t)) \quad , \quad t \in I =]-\pi/2\,,\,\pi/2[\quad , \tag{3.70}$$

and if we write $V(t) = (v^1(t)\ v^2(t))$, then the first order differential equation system for V which is equivalent to parallel transport of $V_0 = (a,b)$ along the half circle from point $\gamma(0) = (0,R)$ is the following:

$$\begin{aligned} 0 &= \frac{d}{dt}v^1(t) - v^2(t) + \tan(t)\cdot v^1(t) \\ 0 &= \frac{d}{dt}v^2(t) + v^1(t) + \tan(t)\cdot v^2(t) \quad . \end{aligned} \tag{3.71}$$

with the said initial condition: $V(0) = V_0 = (a,b)$.

EXERCISE 3.32

Show that (3.71) are the equations for parallel transport in this setting.

The solution is fairly simple:

$$\begin{aligned} v^1(t) &= \frac{1}{2}\cdot(a + a\cdot\cos(2t) + b\cdot\sin(2t)) \\ v^2(t) &= \frac{1}{2}\cdot(b + b\cdot\cos(2t) - a\cdot\sin(2t)) \quad , \quad t \in I =]-\pi/2, \pi/2[\quad . \end{aligned} \tag{3.72}$$

EXERCISE 3.33

Verify, that the coordinate functions in (3.72) do solve the parallel transport problem under consideration.

In figure 3.3 we display the parallel transport along a half circle for a specific choice of initial vector $V(0) = (a,b)$ so that the transport can be directly compared with another parallel transport along the horizontal line between two given points on the half circle – as studied in exercise 3.30. It is apparent from the figure, that parallel transport depend – in this case quite significantly – on the curve along which it is constructed. The values of two vector fields are identical at the rightmost common endpoint of the curves, but quite different at the leftmost common endpoint of the curves.

Proposition 3.34 Parallel transport preserves lengths of vectors and angles between vectors. Precisely, we have the following: Let $V \in \mathfrak{X}(\gamma)$ and $W \in \mathfrak{X}(\gamma)$ be two parallel vector fields along γ parametrized by $t \in I = [a,b]$. Then

$$g(V(t),W(t)) = g(V(a),W(a)) = g(V(b),W(b)) \quad \text{for all } t \in I. \tag{3.73}$$

Proof. This is one of the fine consequences of the metric compatibility of the connection ∇ and thence of the covariant derivative. In fact, we can just calculate the relevant t-derivative from property (3.17):

$$\frac{d}{dt}g(V(t),W(t)) = g\left(\frac{D}{dt}V(t), W(t)\right) + g\left(V(t), \frac{D}{dt}W(t)\right) = 0 \quad , \tag{3.74}$$

because $\frac{D}{dt}V(t) = 0$ and $\frac{D}{dt}W(t) = 0$, so that $g(V(t),W(t))$ is constant along γ. □

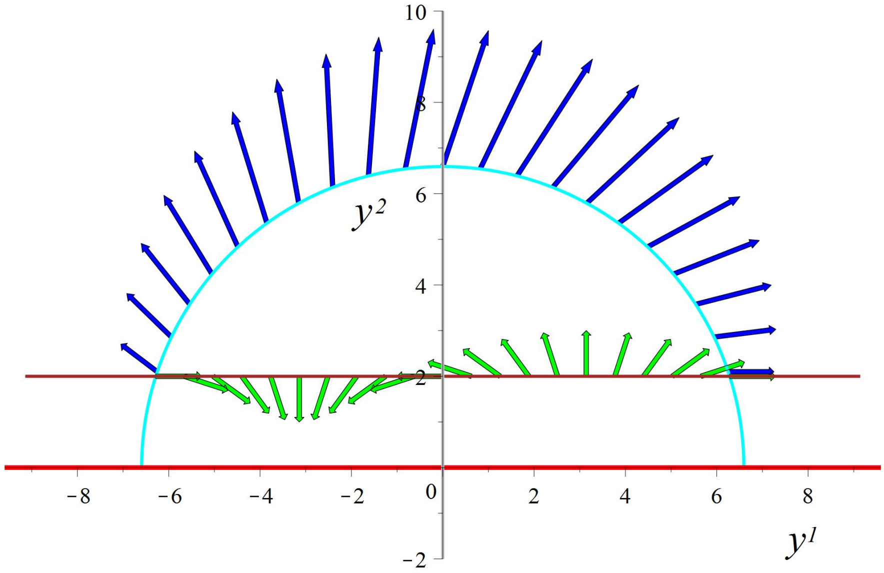


Figure 3.3: Two parallel vector fields along two curves in the Poincaré halfplane. The green vector field is parallel along the horizontal straight line – see exercise 3.30. The blue vector field is parallel along the half circle – see example 3.31. Note that the blue and the green vector fields agree at the right hand intersection point between the two curves but not at the left hand intersection point.

EXERCISE 3.35

Why does this proposition show, that lengths of vectors and angles between vectors are preserved under parallel transport?

Each canonical basis vector field e_k, $k \in \{1, \cdots, n\}$, is always a member of the set $\mathfrak{X}(\mathcal{U})$. If we restrict e_k to a regular smooth curve γ parametrized by $t \in I$ in $\mathcal{U}$ we obtain a vector field $e_k(t)$ along γ, ie. an element in $\mathfrak{X}(\mathcal{U})$. In general, none of these vector fields $e_k(t)$ is parallel along γ. Moreover, as we have discussed at length, the vectors e_k do not in general form a g-orthonormal basis in any given tangent space of $(\mathcal{U}, g, \nabla)$.

Using parallel transport along γ we now construct a very useful replacement of – or alternative to – the canonical basis along γ:

Definition 3.36 Let γ be a regular smooth curve in $(\mathcal{U}, g, \nabla)$ parametrized by $t \in I = [a, b]$, $a < 0 < b$. Let $(E_i)_{\gamma(0)}$, $i = 1, \cdots, n$, denote a g-orthonormal basis of the tangent space $T_{\gamma(0)}\mathcal{U}$. Then construct parallel transports $E_i(t)$ of all the vectors $(E_i)_{\gamma(0)}$ along γ, i.e.

$$\frac{D}{dt}E_i(t) = 0 \quad \text{for all } t \in I, \text{ and} \quad E_i(0) = (E_i)_{\gamma(0)} \quad . \tag{3.75}$$

From proposition 3.34 we infer, that the vector fields $E_i \in \mathfrak{X}(\gamma)$, $i = 1, \cdots n$, form a g-orthonormal basis in each tangent space $T_{\gamma(t)}\mathcal{U}$ along γ, $t \in I$. This n-tuple of vector fields is called a parallel frame along γ.

Note that each choice of a g-orthonormal basis in the tangent space of $\mathcal{U}$ at $\gamma(0)$ gives rise to a unique parallel frame along γ.

Proposition 3.37 Let $V \in \mathfrak{X}(\gamma)$ and let E_i, $i = 1, \cdots, n$, denote a parallel frame along γ parametrized by $t \in I$. Then for each t we obtain unique coordinate functions $V^i(t)$ for V with respect to the frame $E_i(t)$, $i = 1, \cdots, n$, along γ:

$$V(t) = \sum_i V^i(t) \cdot E_i(t) \quad , \quad t \in I \quad . \tag{3.76}$$

If V is itself parallel along γ, then every coordinate function $V^i(t)$ is constant.

Proof. The last claim follows directly from:

$$\begin{aligned} 0 = \frac{D}{dt}V(t) &= \frac{D}{dt}\left(\sum_i V^i(t) \cdot E_i(t)\right) \\ &= \sum_i (V^i)'(t) \cdot E_i(t) + \sum_i V^i(t) \cdot \frac{D}{dt}E_i(t) \\ &= \sum_i (V^i)'(t) \cdot E_i(t) \quad , \end{aligned} \tag{3.77}$$

which implies that $(V^i)'(t) = 0$ for all t and all i – because the frame vectors $E_i(t)$ are, in particular, linearly independent. We conclude, that $V^i(t)$ is a constant for each i. □

EXERCISE 3.38

Recall the construction of g-orthonormal bases from a previous exercise in chapter 1, or find out or look up the procedure known as Gram-Schmidt orthonormalization.

Notation 3.39 Since parallel transport along curves in $(\mathcal{U}, g, \nabla)$ will play a significant and instrumental rôle also in the following, we will introduce a special notation to keep track of the setting and what goes on. Suppose $V(t)$ is obtained by parallel transport of $V_0 \in T_{\gamma(t_0)}\mathcal{U}$ along γ in $\mathcal{U}$. Then we may consider $V(t)$ as the result of the parallel transport operation on V_0 and write as follows:

$$\Pi_{\gamma}^{t_0,t} V_0 = V(t) \quad \text{for all } t \in I. \tag{3.78}$$

The mapping $\Pi_{\gamma}^{t_0,t}$ from $T_{\gamma(t_0)}\mathcal{U}$ to $T_{\gamma(t)}\mathcal{U}$ preserves the g-lengths of vectors according to proposition 3.34 – in other words, it is a tangent space isometry.

3.8 Pulling back vectors along curves via parallel transport

The covariant derivative $\frac{D}{dt}$ and the Levi-Civita connection map ∇ are both 'contained' so much in the concept of parallel transport along curves, that they can be reconstructed from it:

Proposition 3.40 Let $(\mathcal{U}, g, \nabla)$ denote a Local Riemannian Manifold. Let X and Y be vector fields in $\mathfrak{X}(\mathcal{U})$ and let γ be a t-parametrized integral curve for X through $p = \gamma(t_0)$ in $\mathcal{U}$, $t \in I$. Then

$$\begin{aligned}
(\nabla_X Y)_p &= (\nabla_{\gamma'} Y)_p \\
&= \left(\frac{D}{dt} Y(t)\right)_{t=t_0} \\
&= \left(\frac{d}{dt} \Pi_{\gamma}^{t,t_0}\left(Y_{\gamma(t)}\right)\right)_{t=t_0} \\
&= \lim_{t \longmapsto t_0} \left(\frac{\Pi_{\gamma}^{t,t_0}\left(Y_{\gamma(t)}\right) - Y_p}{t - t_0}\right) \quad ,
\end{aligned} \tag{3.79}$$

where Π_{γ}^{t,t_0} denotes parallel transport along γ *from* $\gamma(t)$ *to* $\gamma(t_0) = p$. In this sense, then, this particular parallel transport *pulls back* the vectors from $T_{\gamma(t)}\mathcal{U}$ to $T_p\mathcal{U}$ along γ.

Proof. We just have to observe that the limit gives the covariant derivative of Y at p. For this we choose a parallel frame E_i along γ – any such frame will do, i.e. a parallel frame determined by any choice of a g-orthonormal basis $E_i(0)$ in $T_{\gamma(t_0)}\mathcal{U}$. Then

$$Y(t) = \sum_i Y^i(t) \cdot E_i(t) \quad \text{for all } t \in I, \tag{3.80}$$

and thence

$$\left(\frac{D}{dt}Y(t)\right)_{t=t_0} = \sum_i \left(Y^i\right)'(t_0)\cdot E_i(t_0) \tag{3.81}$$

On the other hand we also have

$$\Pi_\gamma^{t,t_0}\left(Y_{\gamma(t)}\right) = \sum_i Y^i(t)\cdot E_i(t_0) \quad , \tag{3.82}$$

and

$$Y_p = \sum_i Y^i(t_0)\cdot E_i(t_0) \tag{3.83}$$

so that

$$\Pi_\gamma^{t,t_0}\left(Y_{\gamma(t)}\right) - Y_p = \sum_i \left(Y^i(t) - Y^i(t_0)\right)\cdot E_i(t_0) \tag{3.84}$$

and therefore

$$\lim_{t\longmapsto t_0}\left(\frac{\Pi_\gamma^{t,t_0}\left(Y_{\gamma(t)}\right) - Y_p}{t-t_0}\right) = \sum_i \left(Y^i\right)'(t_0)\cdot E_i(t_0) = \left(\frac{D}{dt}Y(t)\right)_{t=t_0} \quad , \tag{3.85}$$

which is what we wanted to show. □

3.9 A first defining glimpse of geodesics

Geodesics are very important curves that will be discussed further in the next chapter. They are special so-called autoparallel curves.

Definition 3.41 A given regular smooth curve γ in $(\mathcal{U}, g, \nabla)$ parametrized by $t \in I$ is called an autoparallel curve if its tangent vector field is parallel along the curve - in other words if its acceleration vector field is 0:

$$\mathrm{acc}_\gamma(t) = \frac{D}{dt}\gamma'(t) = 0 \quad \text{for all } t\in I \quad . \tag{3.86}$$

Proposition 3.42 If a curve is autoparallel, then it has constant speed.

Proof. Again, the metric compatibility of covariant differentiation gives:

$$\frac{d}{dt}g\left(\gamma'(t), \gamma'(t)\right) = 2\cdot g\left(\frac{D}{dt}\gamma'(t), \gamma'(t)\right) = 0 \quad . \tag{3.87}$$

□

Definition 3.43 A geodesic in $(\mathcal{U}, g, \nabla)$ is an autoparallel curve whose constant speed is 1.

Note: it is common in the literature to define a geodesic to simply be any autoparallel curve.

Definition 3.44 The geodesic curvature of a unit speed curve γ in $(\mathcal{U}, g, \nabla)$ is then simply the g-length of the corresponding acceleration vector, i.e. of the covariant derivative of the unit tangent vector field along γ. We call it κ_γ^g:

$$\kappa_\gamma^g(s) = \| \operatorname{acc}_\gamma(s) \|_g = \| \frac{D}{dt}\gamma'(s) \|_g \quad . \tag{3.88}$$

The following observation is immediate:

Proposition 3.45 Every geodesic has zero geodesic curvature.

Definition 3.46 A pre-geodesic is a regular smooth curve γ in $(\mathcal{U}, g, \nabla)$ that satisfies the following condition:

$$\frac{D}{dt}\gamma'(t) = \rho(t) \cdot \gamma'(t) \quad \text{for all } t \in I, \tag{3.89}$$

where ρ is a smooth function of t.

Proposition 3.47 A regular smooth curve γ in $(\mathcal{U}, g, \nabla)$ is a pre-geodesic if and only if it can be reparameterized to a geodesic curve.

Proof. Suppose first that γ is a pre-geodesic. Set:

$$s(t) = \int_{t_0}^{t} ||\gamma'(u)||_g \, \mathrm{d}u.$$

Then, $s'(t) > 0$, so s is an increasing function and $s : I \to J$ is a diffeomorphsim onto some interval J, and we can consider the reparameterization $\alpha : J \to \mathcal{U}$, $\alpha(s) := \gamma(t(s))$, or we may write:

$$\gamma(t) = \alpha(s(t)),$$

so

$$\gamma'(t) = \alpha'(s)\frac{\mathrm{d}s}{\mathrm{d}t} = \alpha'(s(t))||\gamma'(t)||_g.$$

Since γ is a pre-geodesic, we have:

$$\begin{aligned}
\rho(t)\gamma'(t) &= \frac{D}{\mathrm{d}t}\gamma'(t) \\
&= \nabla_{\gamma'(t)}\gamma'(t) \\
&= ||\gamma(t)||_s \nabla_{\alpha'(s(t))}(\alpha'(s(t))||\gamma'(t)||_g) \\
&= ||\gamma(t)||_s \left(\left(\frac{\mathrm{d}}{\mathrm{d}s}||\gamma'(t)||_g \right) \alpha'(s) + ||\gamma'(t)||\nabla_{\alpha'(s)}\alpha'(s) \right)
\end{aligned}$$

The last expression on the right hand side above is a decomposition into components that are respectively proportional to $\gamma'(t)$, and orthogonal to $\gamma'(t)$, since $\alpha'(s(t))$ is proportional, and $\nabla_{\alpha'(s)}\alpha'(s)$ is orthogonal to $\gamma'(t)$. Comparing this with the left hand side, namely $\rho(t)\gamma'(t)$, it follows that:

$$\nabla_{\alpha'(s)}\alpha'(s) = 0,$$

in other words $\alpha(s)$ is autoparallel. Since we have already parameterized it by arc-length, α is a geodesic.

Conversely, suppose a regular curve η can be reparameterized as a geodesic. Then we have a geodesic α with arc length parameter $s \in J$, and a smooth increasing function h of u with $h(u) = s$, such that $\eta(u) = \alpha(h(u))$. Then, since $\frac{D}{ds}\alpha'(s) = 0$, and $\|\alpha'(s)\|_g = 1$ for all s, we get:

$$\begin{aligned}
\frac{D}{du}\eta'(u) &= h''(u) \cdot \alpha'(h(u)) \\
&= \left(\frac{h''(u)}{\|\eta'(u)\|} \right) \cdot \eta'(u)) \quad \text{for all } u \in J.
\end{aligned}$$

In other words, η is a pre-geodesic, with $\rho(u) = h''(u)/||\eta''(u)||$.

□

EXERCISE 3.48

Explain why $\nabla_{\alpha'(s)}\alpha'(s)$ is orthogonal to $\gamma'(t)$ in the first half of the proof of Proposition 3.47.

EXERCISE 3.49

Why does $\psi(s) = \eta(s/c)$ give speed 1 if the speed of $\eta(u)$ is c?

In figure 3.4 we illustrate that every circle in the Poincaré halfplane which interects the y^1-axis orthogonally is a pregeodesic: Tangent vectors are mapped into tangent vectors by parallel transport. In other words, such half circles can be reparametrized so that they become (arclength parametrized) geodesics.

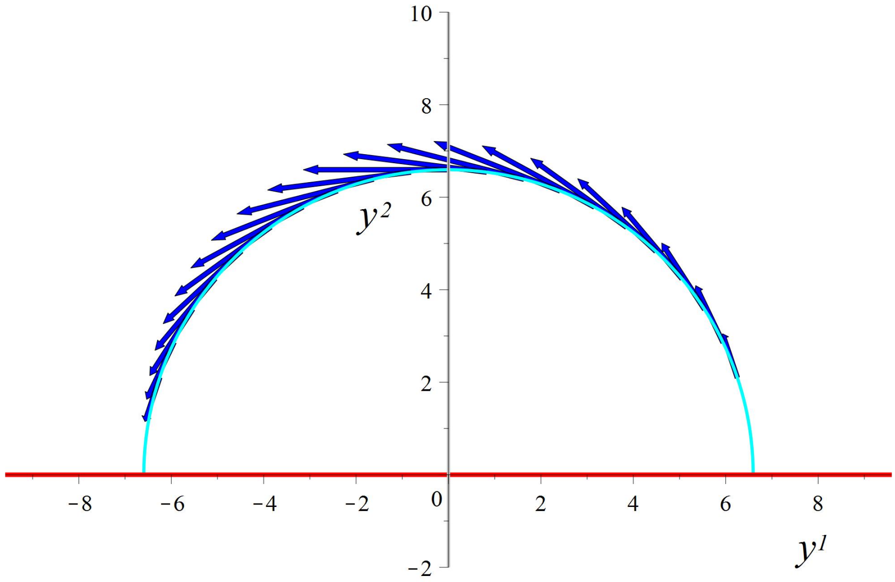


Figure 3.4: Parallel transport of a vector along the special half circle considered also in example 3.31. The vector field is everywhere tangent to the circle, so the circle can be reparametrized to become autoparallel and hence geodesic.

3.10 Useful operators on Local Riemannian Manifolds

Before finishing this chapter on the – somewhat abstract – definition of the Levi-Civita connection, we will consider some very useful operators on functions and vector fields, that are more or less direct spin-offs from this connection. They are of instrumental importance for studying heat kernels, spectral geometry, and a number of deep theorems in global geometric analysis.

3.10.1 The Gradient

We let $(\mathcal{U}, g, \nabla)$ denote a Local Riemannian Manifold. Then we define:

Definition 3.50 Let $f \in \mathfrak{F}(\mathcal{U})$. Then the gradient vector field of f is defined as the unique vector field $\mathrm{grad}(f) \in \mathfrak{X}(\mathcal{U})$ which satisfies

$$g(\mathrm{grad}(f), X) = X(f) \quad \text{for all } X \in \mathfrak{X}(\mathcal{U}). \tag{3.90}$$

Recall that $X(f) = (d/dt)f(\eta(t))_{|t=0}$ for any curve $\eta(t)$, $t \in]-\varepsilon, \varepsilon[$, with $\eta'(0) = X$.

EXERCISE 3.51

Show that in any Euclidean setting $(\mathcal{U}, g_E, \nabla)$ the above definition gives the usual Euclidean coordinates for the gradient of a given function f on $\mathcal{U}$.

EXERCISE 3.52

Show that in the general setting of an LRM $(\mathcal{U}, g, \nabla)$ we have the following coordinate expression of the gradient of a function $f \in \mathfrak{F}(\mathcal{U})$ at the point $p = (x^1, \cdots, x^n)$:

$$\operatorname{grad}(f)(p) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} g^{k\ell} \cdot \frac{\partial f}{\partial x^\ell} \cdot e_k \quad . \tag{3.91}$$

Definition 3.53 We consider a function $f \in \mathfrak{F}(\mathcal{U})$. A point $p \in \mathcal{U}$ is a stationary point for f in $(\mathcal{U}, g, \nabla)$ if $\operatorname{grad}(f)(p) = 0$.

EXERCISE 3.54

We consider a 2-dimensional LRM, $(\mathbb{R}^2, g, \nabla)$ with metric g whose metric matrix function is

$$G_{\mathcal{U}}(x^1, x^2) = \begin{bmatrix} 1 & 1 \\ 1 & 2 + (x^1)^2 \end{bmatrix} \quad . \tag{3.92}$$

Let $f \in \mathfrak{F}(\mathcal{U})$ be defined by

$$f(x^1, x^2) = (x^1)^2 + (x^2)^2 \quad . \tag{3.93}$$

Find all the stationary points for f in $(\mathcal{U}, g, \nabla)$.

EXERCISE 3.55

Consider $(\mathcal{U}, g, \nabla)$ and $f \in \mathfrak{F}(\mathcal{U})$. Show that p is a stationary point for f if and only if

$$\frac{\partial f}{\partial x^i} = 0 \quad \text{for all} \quad i = 1, \cdots, n \quad . \tag{3.94}$$

Note that exercise 3.55 literally says that the notion of *stationary point* is completely independent of the metric!

EXERCISE 3.56

Consider $(\mathcal{U}, g, \nabla)$ and $f \in \mathcal{U}$. Let $\mathcal{K}_c(f)$ denote the following level set in $\mathcal{U}$:

$$\mathcal{K}_c(f) = \{q \in \mathcal{U} \,|\, f(q) = c\} \quad . \tag{3.95}$$

Let $V \in T_p\mathcal{U}$ denote a tangent vector to $\mathcal{K}_c(f)$ at some point p (assuming that a tangent vector exists). Show that

$$g(V, \operatorname{grad}(f)(p)) = 0 \quad . \tag{3.96}$$

Hint: The function f is constant along $\mathcal{K}_c(f)$ so $V(f) = 0$.

EXERCISE 3.57

Suppose $f \in \mathfrak{F}(\mathcal{U})$ is the potential function for some (conservative) force vector field $F \in \mathfrak{X}(\mathcal{U})$, then by definition (of what it means to be a potential function):

$$F = -\operatorname{grad}(f) \quad . \tag{3.97}$$

In example 3.26 the force vector field for the motion on the paraboloid is therefore $-\operatorname{grad}(h)(x^1, x^2)$, where $h(x^1, x^2) = (x^1)^2 + (x^2)^2$ is the height function potential restricted to the surface, i.e. it is the third (z-)coordinate of the vector function that parametrizes the surface. Show that this claim is true, i.e. show the second equation of the following line concerning example 3.26

$$F = -\operatorname{grad}(h)(x^1, x^2) = \left(\frac{-2 \cdot x^1}{1 + 4 \cdot (x^1)^2 + 4 \cdot (x^2)^2} \, , \, \frac{-2 \cdot x^2}{1 + 4 \cdot (x^1)^2 + 4 \cdot (x^2)^2} \right) \quad . \tag{3.98}$$

Below in subsection 3.11.1 we shall use this much simpler method to construct the induced force field (from the ambient gravitational field) on a torus and solve the acceleration problem on that surface.

3.10.2 The Hessian

Definition 3.58 Let $f \in \mathfrak{F}(\mathcal{U})$. Then the Hessian of f is defined as the following operator on two vector fields X and Y in $\mathfrak{X}(\mathcal{U})$:

$$\operatorname{Hess}(f)(X, Y) = X(Y(f)) - (\nabla_X Y)(f) \quad \text{for all } X \text{ and } Y \text{ in } \mathfrak{X}(\mathcal{U}) \tag{3.99}$$

EXERCISE 3.59

Show that

$$\mathrm{Hess}(f)(X,Y) = g\left(\nabla_X \mathrm{grad}(f)\,,\, Y\right) \quad . \tag{3.100}$$

Hint: Use the metric compatibility of ∇ for this:

$$X\left(g\left(\mathrm{grad}(f)\,,\, Y\right)\right) = g\left(\nabla_X \mathrm{grad}(f)\,,\, Y\right) + g\left(\mathrm{grad}(f)\,,\, \nabla_X Y\right) \quad , \tag{3.101}$$

and then use the definition of $\mathrm{grad}(f)$.

EXERCISE 3.60

Show that if two vector fields V and W in $\mathfrak{X}(\mathcal{U})$ are expressed as follows in terms of the basis vector fields $\{e_1, \cdots, e_n\}$,

$$\begin{aligned} V &= \sum_i v^i \cdot e_i \\ W &= \sum_j w^j \cdot e_j \quad , \end{aligned} \tag{3.102}$$

then we get:

$$\mathrm{Hess}(f)(V,W) = \sum_i \sum_j v^i \cdot w^j \cdot \mathrm{Hess}(f)(e_i, e_j) \quad . \tag{3.103}$$

EXERCISE 3.61

Show that $\mathrm{Hess}(f)(X,Y) = \mathrm{Hess}(f)(Y,X)$. Thus $\mathrm{Hess}(f)$ is a symmetric quadratic form (like the metric g). Show by example, however, that $\mathrm{Hess}(f)$ is *not necessarily* positive definit.

EXERCISE 3.62

Show that the coordinate expression for $\mathrm{Hess}(f)(V,W)$ in $(\mathcal{U}, g, \nabla)$ is the following, where we use again $V = \sum_i v^i \cdot e_i$, $W = \sum_j w^j \cdot e_j$, and $p = (x^1, \cdots, x^n)$:

$$\mathrm{Hess}(f)(V,W)(p) = \sum_i \sum_j v^i \cdot w^j \cdot \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{i\,j} \cdot \frac{\partial f}{\partial x^k} \right) \quad . \tag{3.104}$$

EXERCISE 3.63

Suppose that $p \in \mathcal{U}$ is a stationary point for $f \in \mathfrak{F}(\mathcal{U})$. Show that then we have:

$$\mathrm{Hess}(f)(X,Y)(p) = X(Y(f))(p) \quad . \tag{3.105}$$

Definition 3.64 A point $p \in \mathcal{U}$ is called a strict local minimum point for $f \in \mathfrak{F}(\mathcal{U})$ if p is a stationary point for f and $\operatorname{Hess}(f)(X,Y)(p)$ is positive definite, meaning that $\operatorname{Hess}(f)(X,X)(p) > 0$ for all non-zero $X \in T_p\mathcal{U}$. Similarly, p is a strict local maximum point for f if p is a stationary point for f and $\operatorname{Hess}(f)(X,Y)(p)$ is negative definite.

EXERCISE 3.65

Show that in any Euclidean setting $(\mathcal{U}, g_E, \nabla)$ the definition of $\operatorname{Hess}(f)(X,Y)(p)$ gives the usual Euclidean matrix function for the Hessian of a given function f on $\mathcal{U}$.

3.10.3 The Divergence

It follows from the definition 3.10 of the Levi-Civita connection ∇ that for any given fixed vector field $V \in \mathfrak{X}(\mathcal{U})$ the mapping

$$\nabla.V \quad : \quad X \longmapsto \nabla_X V \tag{3.106}$$

maps $X \in T_p\mathcal{U}$ to $\nabla_X V \in T_p\mathcal{U}$ at $p \in \mathcal{U}$ *linearly*.

Remember, that equation 3.38 shows that $\nabla_X V$ only depends on the value of X at the point p, i.e. we do not need a full vector field X in order to evaluate the ∇-derivative of the vector field V with respect to X at p.

Note that the Hessian of a function f is clearly related to this mapping. Indeed, if we choose $V = \operatorname{grad} f$ we get

$$\nabla.\operatorname{grad}(f) \quad : \quad X \longmapsto \nabla_X \operatorname{grad}(f) \quad . \tag{3.107}$$

If we denote this particular linear map by $H(f) = \nabla.\operatorname{grad}(f)$ we get:

$$\operatorname{Hess}(f)(X,Y) = g\,(H(f)(X)\,,\,Y) \tag{3.108}$$

and thus via equation (3.100) in exercise 3.59:

$$\operatorname{Hess}(f)(e_i, e_j) = g\,(H(f)(e_i)\,,\,e_j) \quad . \tag{3.109}$$

In general, whenever we are given a vector field $V \in \mathfrak{X}(\mathcal{U})$ the linear map (of the vector space $T_p\mathcal{U}$ into itself):

$$\nabla.V \quad : \quad T_p\mathcal{U} \longmapsto T_p\mathcal{U} \tag{3.110}$$

has a matrix representation with respect to the basis $\{e_1, \cdots, e_n\}$, i.e. there exists a unique matrix A with elements a_i^j such that

$$\nabla_{e_i} V = \sum_j a_i^j \cdot e_j \quad . \tag{3.111}$$

From this we get:

$$g(\nabla_{e_i} V, e_k) = g\left(\sum_j a_i^j \cdot e_j, e_k\right) = \sum_j a_i^j \cdot g_{jk} \tag{3.112}$$

$$\sum_k g(\nabla_{e_i} V, e_k) \cdot g^{k\ell} = \sum_{jk} a_i^j \cdot g_{jk} \cdot g^{k\ell} = \sum_\ell a_i^\ell \quad , \tag{3.113}$$

and thence

$$\begin{aligned} \sum_k \sum_\ell g(\nabla_{e_\ell} V, e_k) \cdot g^{k\ell} &= \sum_\ell a_\ell^\ell \\ &= \mathrm{trace}(A) \\ &= \mathrm{trace}(\nabla.V) \quad . \end{aligned} \tag{3.114}$$

EXERCISE 3.66

As indicated by the last equality in equation (3.114), the trace of the linear mapping $\nabla.V$ is independent of the *chosen* basis $\{e_1, \cdots, e_n\}$ in $T_p\mathcal{U}$. Let $\{b_1, \cdots, b_n\}$ denote *another basis* in $T_p\mathcal{U}$, i.e. there is a regular matrix D with elements d_i^j so that

$$e_i = d_i^j \cdot b_j \quad . \tag{3.115}$$

Suppose that B is the matrix of $\nabla.Y$ with respect to the basis $\{b_1, \cdots, b_n\}$. Show that $\mathrm{trace}(B) = \mathrm{trace}(A)$, so that, indeed, $\mathrm{trace}(\nabla.V)$ is independent of chosen basis. Hint: Remember, look up, find out, or prove that the matrices A and $D^{-1} \cdot A \cdot D$ have the same trace.

Definition 3.67 The divergence $\mathrm{div}(V)$ of a vector field $V \in \mathfrak{X}(\mathcal{U})$ is the smooth function in $\mathcal{U}$ defined by the invariant trace found above:

$$\begin{aligned} \mathrm{div}(V) &= \mathrm{trace}(\nabla.V) \\ &= \sum_k \sum_\ell g(\nabla_{e_\ell} V, e_k) \cdot g^{k\ell} \end{aligned} \tag{3.116}$$

Proposition 3.68 If we use a g-orthonormal basis $\{E_1, \cdots, E_n\}$ at each tangent space $T_p\mathcal{U}$, so that $g_{ij} = \delta_i^j$, then the expression for $\mathrm{div}(V)$ is naturally simplified as follows:

$$\mathrm{div}(V) = \sum_\ell g(\nabla_{E_\ell} V, E_\ell) \quad . \tag{3.117}$$

We are now ready to introduce the Christoffel symbols into the divergence formula and get two explicit coordinate expressions for $\mathrm{div}(V)$, one of which, however, does not contain the Christoffel symbols directly:

Proposition 3.69 In the coordinate-generated basis $\{e_1, \cdots, e_n\}$ in $\mathcal{U}$, suppose $V \in \mathfrak{X}(\mathcal{U})$ has coordinate functions v^i, $i = 1, \cdots, n$, i.e. $V = \sum_i v^i \cdot e_i$. Then

$$\mathrm{div}(V) = \sum_i \frac{\partial}{\partial x^i} v^i + \sum_i \sum_j \Gamma^i_{ij} \cdot v^j \quad . \tag{3.118}$$

In terms of the (determinant of) the metric matrix function G for g we also have the following short expression for the divergence:

$$\mathrm{div}(V) = \frac{1}{\sqrt{\mathrm{Det}(G)}} \cdot \sum_i \frac{\partial}{\partial x^i} \left(v^i \cdot \sqrt{\mathrm{Det}(G)} \right) \quad . \tag{3.119}$$

Proof. We apply the definition 3.67 and equation (3.38):

$$\begin{aligned} \nabla_{e_\ell} V &= \sum_i \sum_j \sum_m \left(\delta^i_\ell \cdot v^j \cdot \Gamma^m_{ij} + e_\ell(v^m) \right) \cdot e_m \\ &= \sum_j \sum_m \left(v^j \cdot \Gamma^m_{\ell j} + e_\ell(v^m) \right) \cdot e_m \quad . \end{aligned} \tag{3.120}$$

We insert into (3.116) and get:

$$\begin{aligned} \mathrm{div}(V) &= \sum_k \sum_\ell g \left(\sum_j \sum_m \left(v^j \cdot \Gamma^m_{\ell j} + e_\ell(v^m) \right) \cdot e_m \, , \, e_k \right) \cdot g^{k\ell} \\ &= \sum_k \sum_\ell \sum_j \sum_m \left(v^j \cdot \Gamma^m_{\ell j} + e_\ell(v^m) \right) \cdot g_{mk} \cdot g^{k\ell} \\ &= \sum_\ell \sum_j \sum_m \left(v^j \cdot \Gamma^m_{\ell j} + e_\ell(v^m) \right) \cdot \delta^\ell_m \\ &= \sum_\ell \sum_j \left(v^j \cdot \Gamma^\ell_{\ell j} + \frac{\partial}{\partial x^\ell}(v^\ell) \right) \\ &= \sum_\ell \frac{\partial}{\partial x^\ell}(v^\ell) + \sum_\ell \sum_j v^j \cdot \Gamma^\ell_{\ell j} \quad , \end{aligned} \tag{3.121}$$

which is equivalent to equation (3.118) – modulo a suitable renaming of the indices. When performing the differentiations of $\sqrt{\mathrm{Det}(G)}$ that are needed for the expression (3.119), then the Christoffel symbols will re-appear from there and show equivalence with (3.118). □

EXERCISE 3.70

We construct two isometric Riemannian manifolds $(\mathcal{U}^2, g, \nabla_g)$ and $(\mathcal{V}^2, h, \nabla_h)$ in the following way: Let G denote the following given metric matrix function for the metric g in $\mathcal{U}$:

$$G(x^1, x^2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 + (x^1)^2 \end{bmatrix} \tag{3.122}$$

and let ϕ denote the very simple diffeomorphism from $\mathcal{U}$ to $\mathcal{V}$:

$$\phi(x^1, x^2) = (-x^2, x^1) \tag{3.123}$$

with inverse diffeomorphism:

$$\phi^{-1}(y^1, y^2) = (y^2, -y^1) \quad . \tag{3.124}$$

The corresponding Jacobians are then:

$$\begin{aligned} J_\phi(x^1, x^2) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ J_{\phi^{-1}}(y^1, y^2) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad . \end{aligned} \tag{3.125}$$

The metric h and its corresponding metric matrix function $H(y^1, y^2)$ are now *constructed* so that ϕ *becomes an isometry*:

$$H(y^1, y^2) = J^*_{\phi^{-1}}(y^1, y^2) \cdot G(\phi^{-1}(y^1, y^2) \cdot J_{\phi^{-1}}(y^1, y^2) = \begin{bmatrix} 1 + (y^2)^2 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{3.126}$$

We consider a vector field $V \in \mathfrak{X}(\mathcal{U})$:

$$V(x^1, x^2) = (x^1 \,,\, 3 \cdot x^2 - x^1) \quad , \tag{3.127}$$

which is then ϕ-related to the following vector field $W \in \mathfrak{X}(\mathcal{V})$:

$$W(y^1, y^2) = J_\phi(\phi^{-1}(y^1, y^2)) \cdot V(\phi^{-1}(y^1, y^2)) = (3 \cdot y^1 + y^2 \,,\, y^2) \quad . \tag{3.128}$$

In order to calculate and compare the g-divergence of V and the h-divergence of W we finally need the Christoffel symbols for g and h, respectively.

Show that the non-zero Christoffel symbols for g are:

$$(\Gamma_g)^2_{21} = (\Gamma_g)^2_{12} = \frac{x^1}{1 + (x^1)^2} \quad \text{and} \quad (\Gamma_g)^1_{22} = -x^1 \quad . \tag{3.129}$$

Show that the non-zero Christoffel symbols for h are:

$$(\Gamma_h)^1_{21} = (\Gamma_h)^1_{12} = \frac{y^2}{1 + (y^2)^2} \quad \text{and} \quad (\Gamma_h)^2_{11} = -y^2 \quad . \tag{3.130}$$

Use these ingredients to show that

$$\mathrm{div}_g(V)(x^1,x^2) = \frac{5\cdot(x^1)^2+4}{1+(x^1)^2} \tag{3.131}$$

and

$$\mathrm{div}_h(W)(y^1,y^2) = \frac{5\cdot(y^2)^2+4}{1+(y^2)^2} \quad , \tag{3.132}$$

and finally, that the two divergences therefore agree at corresponding points:

$$\mathrm{div}_g(V)(x^1,x^2) = \mathrm{div}_h(W)(\phi(x^1,x^2)) \quad . \tag{3.133}$$

A computationally more complicated exercise relates to a previous wish (from chapter 2) along the same lines:

EXERCISE 3.71

In chapter 2, example 2.5, figure 2.1, and figure 2.2 we considered two ϕ-related vector fields $V(x^1,x^2)$ and $W(y^1,y^2)$ in their respective ϕ-isometric representations of the Local Riemannian Manifold defined by the standard paraboloid of revolution. Show that the divergences of the two vector fields agree at corresponding ϕ-related points. Hint: You can use the Christoffel symbol functions in $(\mathcal{U},g_{\mathcal{U}},\nabla)$ that are on display in equation (3.64). The other useful set of non-zero Christoffel symbol functions, i.e. the ones in $(\mathcal{V},g_{\mathcal{V}},\nabla)$, are the following:

$$\begin{aligned}
\Gamma^1_{11}(y^1,y^2) &= \frac{4\cdot y^1}{1+4\cdot(y^1)^2} \\
\Gamma^2_{12}(y^1,y^2) = \Gamma^2_{21}(y^1,y^2) &= \frac{1}{y^1} \\
\Gamma^1_{22}(y^1,y^2) &= \frac{-y^1}{1+4\cdot(y^1)^2} \quad .
\end{aligned} \tag{3.134}$$

EXERCISE 3.72

Let $f \in \mathfrak{F}\mathcal{U})$ and $V \in \mathfrak{X}(\mathcal{U})$. Show that

$$\operatorname{div}(f \cdot V) = f \cdot \operatorname{div}(V) + g(\operatorname{grad}(f), V) = f \cdot \operatorname{div}(V) + V(f) \quad . \tag{3.135}$$

3.10.4 The Laplacian

Definition 3.73 The Laplacian $\Delta(f) \in \mathfrak{F}(\mathcal{U})$ of a function $f \in \mathfrak{F}(\mathcal{U})$ is defined by

$$\Delta(f) = \operatorname{div}(\operatorname{grad}(f)) \quad . \tag{3.136}$$

Using standard coordinates in $(\mathcal{U}, g, \nabla)$ with metric matrix function G, this definition is equivalent to the following expressions (via Proposition 3.69 and exercise 3.74 below):

$$\begin{aligned} \Delta(f) &= \sum_i \sum_j g^{ij} \cdot \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma^k_{ij} \cdot \frac{\partial f}{\partial x^k} \right) \\ &= \frac{1}{\sqrt{\operatorname{Det}(G)}} \cdot \sum_i \frac{\partial}{\partial x^i} \left(\sum_j g^{ij} \cdot \sqrt{\operatorname{Det}(G)} \cdot \frac{\partial f}{\partial x^j} \right) \quad . \end{aligned} \tag{3.137}$$

EXERCISE 3.74

Show that in coordinates in $\mathcal{U}$ we have the following expressions for the Laplacian:

$$\begin{aligned} \Delta(f) &= \operatorname{trace}(\nabla_{.} \operatorname{grad}(f)) \\ &= \sum_i \sum_j g\left(\nabla_{e_i} \operatorname{grad}(f),\, e_j\right) \cdot g^{ij} \\ &= \sum_i \sum_j g\left(H(f)(e_i),\, e_j\right) \cdot g^{ij} \\ &= \sum_i \sum_j \operatorname{Hess}(f)(e_i, e_j) \cdot g^{ij} \\ &= \sum_i \sum_j g^{ij} \cdot \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma^k_{ij} \cdot \frac{\partial f}{\partial x^k} \right) \quad . \end{aligned} \tag{3.138}$$

EXERCISE 3.75

Show the following general identities for any functions h and f in $\mathfrak{F}(\mathcal{U})$ and for any vector field X in $\mathfrak{X}(\mathcal{U})$:

$$\begin{aligned} \mathrm{grad}(f\cdot h) &= f\cdot\mathrm{grad}(h) + h\cdot\mathrm{grad}(f) \\ \mathrm{div}(f\cdot X) &= X(f) + f\cdot\mathrm{div}(X) \\ \mathrm{div}(h\cdot\mathrm{grad}(f)) &= h\cdot\Delta(f) + g(\mathrm{grad}(f)\,,\,\mathrm{grad}(h)) \\ \Delta(f\cdot h) &= h\cdot\Delta(f) + f\cdot\Delta(h) + 2\cdot g(\mathrm{grad}(f)\,,\,\mathrm{grad}(h)) \quad . \end{aligned} \tag{3.139}$$

3.11 Example: The standard torus in 3D

A torus $\mathcal{T}$ in 3D Euclidean space can be parametrized as follows:

$$\mathcal{T} \quad : \quad r(x^1,x^2) = \left((2+\cos(x^1)\cdot\cos(x^2)\,,\,(2+\cos(x^1)\cdot\sin(x^2)\,,\,\sin(x^1)\right) \quad , \tag{3.140}$$

where $(x^1,x^2)\in\mathcal{U} =]-\pi,\pi[\,\times\,]-\pi,\pi[$. The induced metric matrix function is then:

$$G_{\mathcal{U}} = \begin{bmatrix} 1 & 0 \\ 0 & (2+\cos(x^1))^2 \end{bmatrix} \quad . \tag{3.141}$$

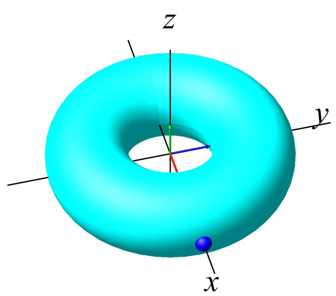


Figure 3.5: The torus $\mathcal{T}$ defined in equation (3.140).

The *non-zero* Christoffel symbol functions corresponding to the above choice of parametrization of the torus are:

$$\begin{cases} \Gamma^2_{12}(x^1,x^2) &= \Gamma^2_{21}(x^1,x^2) = -\frac{\sin(x^1)}{2+\cos(x^1)} \\ \Gamma^1_{22}(x^1,x^2) &= \sin(x^1)\cdot(2+\cos(x^1)) \end{cases} \tag{3.142}$$

3.11.1 Gravity induced motion on the torus

The induced height function potential on the torus is

$$h(x^1,x^2) = \sin(x^1) \quad . \tag{3.143}$$

The gradient of the potential function is then

$$\mathrm{grad}(f)(x^1,x^2) = \big(\cos(x^1)\,,\,0)\big) \quad . \tag{3.144}$$

We apply the general observation from exercise 3.57, that the induced gravitational force in the surface is

$$F(x^1,x^2) = -\,\mathrm{grad}(f)(x^1,x^2) \quad . \tag{3.145}$$

Newton's second law then gives the differential equations for the motion on the torus:

$$\mathrm{acc}_\gamma(t) = \frac{D}{dt}\gamma'(t) = \nabla_{\gamma'(t)}\gamma'(t) = -\,\mathrm{grad}(f)(\gamma(t)) \quad , \quad t \in \mathbb{R} \quad , \tag{3.146}$$

that is – according to equation (3.49):

$$\nabla_{\gamma'(t)}\gamma'(t) = \big(-\cos(\gamma^1(t))\,,\,0)\big) \quad , \tag{3.147}$$

so that the two coordinate equations to be solved are

$$\left\{ \begin{array}{ll} (\gamma^1)''(t) + \sum_{ij}(\gamma^i)'(t)\cdot(\gamma^j)'(t)\cdot\Gamma^1_{ij}(\gamma(t)) & = -\cos(\gamma^1(t)) \\ (\gamma^2)''(t) + \sum_{ij}(\gamma^i)'(t)\cdot(\gamma^j)'(t)\cdot\Gamma^2_{ij}(\gamma(t)) & = 0 \quad , \end{array} \right. \tag{3.148}$$

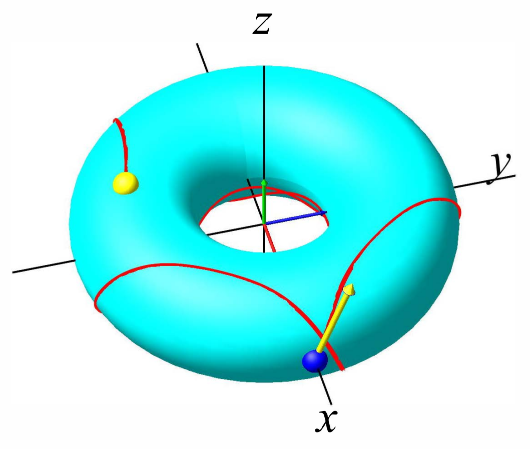
which reduces to the following, when we apply the non-zero Christoffel symbols:

$$\left\{ \begin{array}{ll} (\gamma^1)''(t) + (\gamma^2)'(t)\cdot(\gamma^2)'(t)\cdot\Gamma^1_{22}(\gamma(t)) & = -\cos(\gamma^1(t)) \\ (\gamma^2)''(t) + 2\cdot(\gamma^1)'(t)\cdot(\gamma^2)'(t)\cdot\Gamma^2_{12}(\gamma(t)) & = 0 \quad , \end{array} \right. \tag{3.149}$$

so finally we get:

$$\left\{ \begin{array}{ll} (\gamma^1)''(t) + (\gamma^2)'(t)\cdot(\gamma^2)'(t)\cdot(2+\cos(\gamma^1(t)))\cdot\sin(\gamma^1(t)) & = -\cos(\gamma^1(t)) \\ (\gamma^2)''(t) - 2\cdot(\gamma^1)'(t)\cdot(\gamma^2)'(t)\cdot\frac{\sin(\gamma^1(t))}{2+\cos(\gamma^1(t))} & = 0 \quad . \end{array} \right. \tag{3.150}$$

Two solution curves (with two different initial conditions) are presented in figure 3.6. It is a remarkable fact, that the solution curves to such geometric Newtonian systems can also be obtained as geodesics of a suitable metric on the configuration space (the torus in the present example) – see [26].



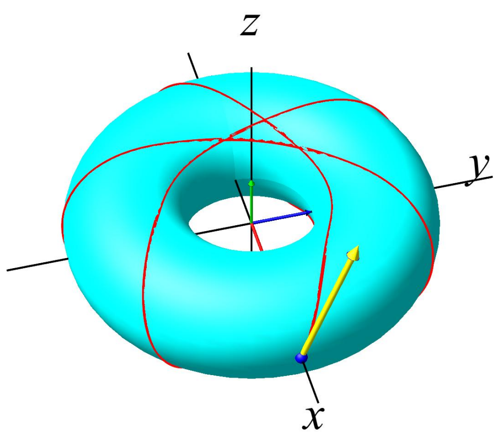


Figure 3.6: The torus $\mathcal{T}$ and two tracks of a particle moving (for the same duration of time) on the torus under the influence of gravity – with two different initial conditions given by the yellow initial velocity vectors at $(1,0,0)$.

3.12 Example: Laplace's equation on surfaces of revolution

We consider a general surface of revolution in 3D Euclidean space with smooth and regular generator curve

$$\gamma(s) = (f(s), 0, h(s)) \quad , \quad s \in \mathbb{R} \quad , \tag{3.151}$$

and assume without lack of generality that $(f'(s))^2 + (h'(s))^2 = 1$, so that γ is arc length parametrized by s.

The corresponding surface of revolution is then itself parametrized as follows:

$$r(x^1, x^2) = \left(f(x^1) \cdot \cos(x^2),\, f(x^1) \cdot \sin(x^2),\, h(x^1)\right) \quad , \tag{3.152}$$

where $(x^1, x^2) \in \mathcal{U} = \mathbb{R} \times]-\pi, \pi[$.

With this parametrization the surface has the following metric matrix function in $\mathcal{U}$:

$$g = \begin{bmatrix} 1 & 0 \\ 0 & h(x^1)^2 \end{bmatrix} \quad , \tag{3.153}$$

and therefore the following non-zero Christoffel symbol functions – compare with the previous examples above concerning the torus and the paraboloid:

$$\begin{aligned} \Gamma^2_{12}(x^1, x^2) &= \Gamma^2_{21}(x^1, x^2) = \frac{h'(x^1)}{h(x^1)} \\ \Gamma^1_{22}(x^1, x^2) &= -h(x^1) \cdot h'(x^1) \quad . \end{aligned} \tag{3.154}$$

The Laplacian of a function $f \in \mathcal{U}$ is then the following:

$$\Delta(f)(x^1,x^2) = \frac{\partial^2 f}{\partial(x^1)^2} + \frac{1}{h^2(x^1)} \cdot \frac{\partial^2 f}{\partial(x^2)^2} + \frac{1}{h(x^1)} \cdot \frac{\partial h}{\partial x^1} \cdot \frac{\partial f}{\partial x^1} \quad . \tag{3.155}$$

EXERCISE 3.76

Show the expression for the Laplacian of f presented in equation (3.155).

We now consider but one application of this specific expression for the Laplacian on general 2D manifolds with a metric as given in equation (3.153).

3.12.1 Capacity and effective resistance

Laplace's equation $\Delta(f) = 0$ plays a fundamental role in an abundance of contexts – in applications and in mathematics. One such context is that of electric potential theory on surfaces and more generally on Local Riemannian manifolds $(\mathcal{U}, g, \nabla)$ equipped with (or made of) a homogeneous conducting material. For example, for the surface of revolution considered above we let Ω denote the (annulus) domain given by

$$\Omega(R) = \{(x^1,x^2) \,|\, \rho \leq x^1 \leq R\} \quad . \tag{3.156}$$

Then $\Omega = \Omega(R)$ has two (circular) boundary components $\partial\Omega_\rho$ and $\partial\Omega_R$ corresponding to $x^1 = \rho$ and $x^1 = R$, respectively.

Suppose now that we engage an electric potential of value 0 on all of $\partial\Omega_\rho$ and of value 1 on all of $\partial\Omega_R$. This will then generate a potential function u in all of $\Omega \subset \mathcal{U}$ and a corresponding current vector field $\mathrm{grad}(u)$ with zero divergence, i.e.

$$\mathrm{div}(\mathrm{grad}(u)) = \Delta(u) = 0 \quad . \tag{3.157}$$

Obviously, we would then like to find a potential function $u(x^1,x^2)$ so that:

$$\begin{cases} \Delta(u)(x^1,x^2) & = 0 \quad \text{for} \quad (x^1,x^2) \in \Omega \\ u(x^1,x^2) & = 0 \quad \text{at the boundary} \quad \partial\Omega_\rho \\ u(x^1,x^2) & = 1 \quad \text{at the boundary} \quad \partial\Omega_R \quad . \end{cases} \tag{3.158}$$

For surfaces of revolution there is a nice (and unique) solution u to this boundary value problem, that only depends on x^1, namely the following, where h is the function that defines the metric g in equation (3.153):

$$u(x^1,x^2) = \frac{\int_\rho^{x^1} \frac{1}{h(w)}\, dw}{\int_\rho^{R} \frac{1}{h(w)}\, dw} \quad . \tag{3.159}$$

EXERCISE 3.77

Show that the function in (3.159) in fact does solve the PDE boundary value problem in (3.158).

EXERCISE 3.78

Show the claim above stating that the function u defined in equation (3.159) is the only solution to the boundary value problem in (3.158). Hint: Assume that v is another solution different from u. Then $u-v$ is zero on *both* boundary components $\partial\Omega_\rho$ and $\partial\Omega_R$ and $\Delta(u-v)=0$. Moreover, $u-v$ must have a maximum point (not necessarily a strict maximum point) or a minimum point (not necessarily a strict minimum point) in the interior of Ω. This contradicts the famous maximum principle (for regular elliptic operators), see e.g. [14, 9], so v cannot be a solution different from u.

The current I entering into Ω through the boundary component $\partial\Omega_\rho$ is then the length, $2\pi\cdot h(\rho)$, of that boundary times the constant component of the gradient of u in the direction of the inwards pointing unit normal $e_1=(1,0)$ to the boundary, i.e.

$$\begin{aligned}
I &= 2\pi\cdot h(\rho)\cdot g(\text{grad}(u),e_1)_{|x^1=\rho} \\
&= 2\pi\cdot h(\rho)\cdot e_1(u)_{|x^1=\rho} \\
&= 2\pi\cdot h(\rho)\cdot \frac{\partial}{\partial x^1}u(x^1,x^2)_{|x^1=\rho} \\
&= 2\pi\cdot h(\rho)\cdot \frac{d}{dx^1}\left(\frac{\int_\rho^{x^1}\frac{1}{h(w)}\,dw}{\int_\rho^{R}\frac{1}{h(w)}\,dw}\right)_{|x^1=\rho} \\
&= \frac{2\pi}{\int_\rho^{R}\frac{1}{h(w)}\,dw} \quad .
\end{aligned} \tag{3.160}$$

Since this current is obtained by a potential difference of 1 between the boundaries, the value of I is called the capacity $\text{Cap}(\Omega)$ of Ω, and $\text{R}_{\text{eff}}(\Omega)=1/I$ is called the effective resistance of the domain Ω:

$$\begin{aligned}
\text{Cap}(\Omega(R)) &= \frac{2\pi}{\int_\rho^{R}\frac{1}{h(w)}\,dw} \\
\text{R}_{\text{eff}}(\Omega(R)) &= \frac{1}{2\pi}\cdot\int_\rho^R \frac{1}{h(w)}\,dw
\end{aligned} \tag{3.161}$$

EXERCISE 3.79

Show that for any surface of revolution in 3D we have for any fixed value of the in-radius ρ: $\text{Cap}(\Omega(R=\infty))=0$, i.e. the effective resistance of the portion $\Omega(R)$ of a surface of revolution goes to infinity when R goes to infinity. Hint: Show first that $h(x^1)\leq h(\rho)+x^1$.

Definition 3.80 Suppose we allow any positive function h in the fundamental metric matrix (3.153) (so that the metric does not necessarily stem from a surface of revolution in 3D Euclidean space). Such a manifold is called a warped product manifold of dimension 2 with warping function h.

EXERCISE 3.81

Find examples of functions h so that the corresponding capacity values $\mathrm{Cap}(\Omega(\infty))$ are positive for any fixed value of the in-radius ρ, i.e. so that the effective resistance to infinity is finite for such annuli in warped product 2-manifolds.

EXERCISE 3.82

Show that the Poincaré disk – and thence also the Poincaré half plane – is isometric to the 2D warped product which has warping function $h(x^1) = \sinh(x^1)$, $x^1 \geq 0$. Note that there is a slight problem with this setting at $x^1 = 0$ since there, at this pole point, the warped product metric degenerates. We can so far neglect this, because here we are only interested in annuli with in-radii $\rho > 0$: Show that the annulus $\Omega = \Omega_{\rho,R}$ has $\mathrm{Cap}(\Omega(R = \infty)) > 0$ for every positive value of the in-radius ρ.

Chapter 4

The Exponential map

Geodesic curves are of instrumental importance for the local and global geometric analysis of Riemannian manifolds and of what goes on inside them. We first repeat the definition of a geodesic:

4.1 Recap on geodesics

Definition 4.1 Let $(\mathcal{U}^n, g, \nabla)$ denote (an isometric representation of) a Local Riemannian Manifold. Let p be a point in $\mathcal{U}$ and V_0 a g-unit vector in $T_p\mathcal{U}$. The geodesic γ_{V_0} of length b issuing from p in the direction V_0 is the unique arc-length (unit speed) parametrized curve which solves the initial value problem:

$$\frac{D}{ds}\gamma'_{V_0}(s) = 0 \quad , \quad \gamma_{V_0}(0) = p \quad , \quad \gamma'_{V_0}(0) = V_0 \quad , \quad s \in I = [0, b] \quad . \tag{4.1}$$

The key object in this definition is clearly the set of n coupled differential equations, which says that $\gamma = \gamma_{V_0}$ is autoparallel – cf. chapter 3 – i.e. for all $k = 1, \cdots, n$:

$$0 = \frac{D}{ds}\gamma'(s) = \sum_k \left(\frac{d^2}{ds^2}\gamma^k(s) + \sum_{ij} (\gamma^i)'(s) \cdot (\gamma^j)'(s) \cdot \Gamma^k_{ij}(\gamma(s)) \right) \cdot e_k \quad . \tag{4.2}$$

This equation is a second order ordinary – typically nonlinear – differential equation system. We infer from the theory of such systems that there exists a unique solution to the initial value problem (4.1) for sufficiently small values of b. We refer to [19] and [4] for thorough discussions and proofs of these results.

Before going into some concrete examples it is important to note that geodesics are preserved by isometries: If γ is a geodesic in the Local Riemannian Manifold $(\mathcal{U}, g_{\mathcal{U}}, \nabla_{\mathcal{U}})$ then $\phi(\gamma)$ is also a geodesic in any other isometric representation $(\mathcal{V}, g_{\mathcal{V}}, \nabla_{\mathcal{V}})$ where $\phi(\mathcal{U}) = \mathcal{V}$, and where $g_{\mathcal{V}}$ is obtained via ϕ from $g_{\mathcal{U}}$ as in equation (1.41) in chapter 1, and where $\nabla_{\mathcal{V}}$ is defined uniquely from $g_{\mathcal{V}}$ as in chapter 3.

The first example/exercise is trivial:

EXERCISE 4.2

Let $(\mathcal{U} = \mathbb{R}^2, g_E, \nabla)$ denote the standard Euclidean plane. All Christoffel symbols vanish, so the geodesic equations are very simple. Find all geodesics γ through the point $p = (1,0) = \gamma(0)$. I.e. for every $\theta \in]-\pi, \pi[$ find the geodesic γ_θ through this point which has initial direction vector of g_E-unit speed:

$$\gamma_\theta'(0) = (\cos(\theta)\,,\, \sin(\theta)) \quad . \tag{4.3}$$

Make sure that the geodesics are parametrized by signed arc length s from $p = \gamma_\theta(0) = (1,0)$.

The next example/exercise appears to be non-trivial, but it is not:

EXERCISE 4.3

We let $(\mathcal{U}, g, \nabla)$ denote the Local Riemannian Manifold defined by $\mathcal{U} = \{(x^1, x^2) \in \mathbb{R}^2 \,|\, x^1 > 0\}$ with metric tensor g determined by the metric matrix function:

$$G(x^1, x^2) = \begin{bmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{bmatrix} \quad . \tag{4.4}$$

Find all geodesics γ through the point $p = (1,0) = \gamma(0)$. I.e. for every $\theta \in]-\pi, \pi[$ find the geodesic γ_θ through this point and with initial direction vector of g-unit speed:

$$\gamma_\theta'(0) = \frac{1}{\|(\cos(\theta)\,,\, \sin(\theta))\|_g} \cdot (\cos(\theta)\,,\, \sin(\theta)) \quad . \tag{4.5}$$

Make sure that the geodesics are parametrized by signed arc length s from $p = \gamma_\theta(0)$. A number of solution geodesics – for various choices of initial directions – are shown in figure 4.1. Hint: You may want to apply the diffeomorphism ψ on $\mathcal{U}$ defined by

$$\psi(x^1, x^2) = (x^1 \cdot \cos(x^2)\,,\, x^1 \cdot \sin(x^2)) \quad , \quad (x^1, x^2) \in \mathcal{U} \quad , \tag{4.6}$$

use coordinates $(y^1\,,\, y^2) \in \mathcal{V} = \psi(\mathcal{U})$ and the following metric matrix function in $\mathcal{V}$, which makes ψ an isometry:

$$G_{\mathcal{V}} = J^*_{\psi^{-1}} \cdot G(\psi^{-1}) \cdot J_{\psi^{-1}} \quad . \tag{4.7}$$

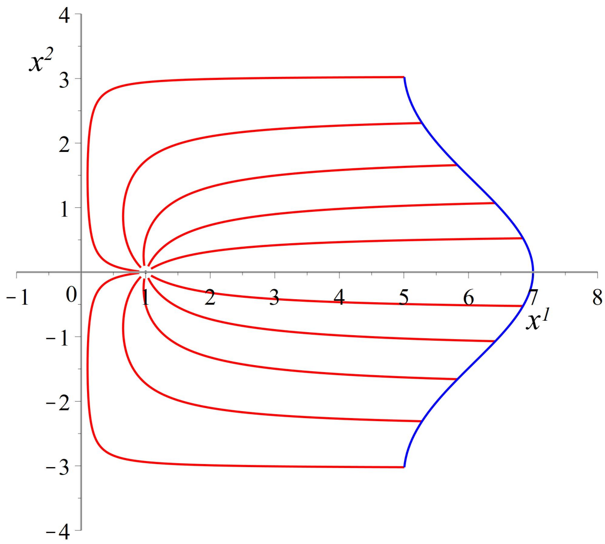


Figure 4.1: Geodesics through $p = (1,0)$ in the Local Riemannian Manifold that is studied in exercise 4.3. All the geodesics – except one – are stopped at the blue curve so as to give them the same g-length. The single one alluded to is stopped already at $(0,0)$. Why?

Example 4.4

We consider again the paraboloid of revolution represented by $(\mathcal{U}, g, \nabla)$ – as in the previous chapters – with coordinates x^1 and x^2, i.e. the paraboloid is parametrized by $r(x^1, x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2)$. In figure 4.2 we choose a point $p = (1,1) \in \mathcal{U}$ and two g-unit vectors V_0 and W_0 in the tangent space $T_p\mathcal{U}$. The two corresponding initial value problems (4.1) are solved numerically and displayed to the left in figure 4.2. The corresponding lifted geodesic curves on the paraboloid itself are indicated to the right in the figure.

4.2 The defining diffeomorphism

From the op. cit. references, [19], and [4], we can – and do without going into the details of proof – extract much more useful information about the solutions to the initial value problem (4.1):

Theorem 4.5 Let $(\mathcal{U}, g, \nabla)$ denote a Local Riemannian Manifold and let $p \in \mathcal{U}$. Then there exists an open set $\mathcal{E}_0(p)$ containing the origin (zero vector) in the tangent space $T_p\mathcal{U}$, so that the following map is *a diffeomorphism* of $\mathcal{E}_0(p)$ onto an open subset $Q_p \subset \mathcal{U}$ containing p:

$$\mathrm{Exp}_p(0) = p \quad \text{and} \quad \mathrm{Exp}_p(v) = \gamma_{V_0(v)}(\|v\|_g) \quad \text{for all non-zero } v \in \mathcal{E}_0 \subset T_p\mathcal{U}\,, \qquad (4.8)$$

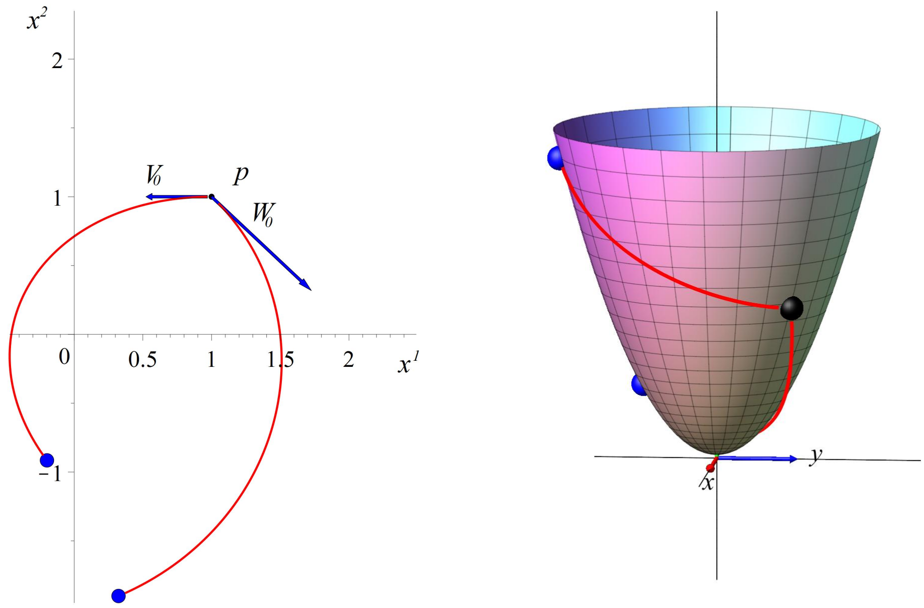


Figure 4.2: Left: Two unit speed geodesics from point $p \in \mathcal{U}$ for the paraboloid of revolution. Right: The two geodesics are lifted into the paraboloid. The two initial vectors have the same g-length. The two geodesic curves on the paraboloid also have the same curve length – they have been constructed over the same s-interval $I = [0, b]$.

where the curve γ denotes the unique geodesic starting at p with initial g-unit direction vector $V_0(v) = v/\|v\|_g$ and with total length $b = \|v\|_g$.

Definition 4.6 The (local) diffeomorphism Exp_p is called the Exponential map of $(\mathcal{U}, g, \nabla)$ from $\mathcal{E}_0$ in the tangent space $T_p\mathcal{U}$ at p into Q_p in $\mathcal{U}$.

The (local) inverse diffeomorphism $\mathrm{Exp}_p^{-1} = \mathrm{Log}_p$ is called the Logarithmic map of $(\mathcal{U}, g, \nabla)$ from $\mathcal{U}$ into the tangent space $T_p\mathcal{U}$ at p.

Note that the Exponential map Exp_p is usually *not* a diffeomorphism on *all* of $T_p\mathcal{U}$. See figure 4.5. Correspondingly, Log_p is usually *not* defined on *all* of $\mathcal{U}$.

Definition 4.7 A metric ball $B_\rho(p)$ of radius ρ in a given tangent space $T_p\mathcal{U}$ is defined as follows:

$$B_\rho(p) = \{v \in T_p\mathcal{U} \mid \|v\|_g \leq \rho\} \quad . \tag{4.9}$$

The corresponding metric sphere of radius ρ in $T_p\mathcal{U}$ is then correspondingly denoted as follows:

$$\partial B_\rho(p) = \{v \in T_p\mathcal{U} \mid \|v\|_g = \rho\} \quad . \tag{4.10}$$

A geodesic ball $D_\rho(p)$ of radius ρ in $\mathcal{U}$ is defined as follows under the assumption that $B_\rho(p)$ is contained in a domain $\mathcal{E}_0$ of the Exponential map diffeomorphism Exp_p:

$$D_\rho(p) = \mathrm{Exp}_p(B_\rho(p)) \quad . \tag{4.11}$$

The corresponding geodesic sphere of radius ρ in $\mathcal{U}$ is:

$$\partial D_\rho(p) = \mathrm{Exp}_p(\partial B_\rho(p)) \quad . \tag{4.12}$$

In dimension 2 we permit ourselves to use the more natural words 'disk' and 'circle' instead of 'ball' and 'sphere' wherever relevant.

Example 4.8

In continuation of example 4.4 we consider a ball $B_b(p)$ in $T_p\mathcal{U}$ which consists of the vectors v that have g-length less than or equal to b. The value of b is assumed to be sufficiently small, so that $B_b(p)$ is contained in the open set $\mathcal{E}$ guaranteed by theorem 4.5. Then we can display the image of $B_b(p)$ by the exponential map Exp – both in $\mathcal{U}$ and in this case also lifted into the paraboloid, see figure 4.3. All the shown geodesics have the same constant length b – which is only visually evident form the lifted image on the paraboloid. However, this same-length property can be also displayed and indicated in $\mathcal{U}$ if we superimpose the fingerprint of the metric tensor into the parameter domain as in figure 4.4.

4.3 Normal coordinates and polar coordinates

The local diffeomorphism Exp_p induces a lot of local coordinate systems in $(\mathcal{U}, g, \nabla)$ in the vicinity of any given point p, i.e. in the image $\mathcal{U}_p = \mathrm{Exp}_p(\mathcal{E}_0(p)) \in \mathcal{U}$. Indeed, suppose that $\{z^1, z^2, \cdots, z^n\}$ is any given standard coordinate system in the vector space $T_p\mathcal{U}$ with origin at the zero vector, then the corresponding coordinates of a point $q \in \mathcal{U}_p$ are $z^i(\mathrm{Log}_p(q))$, $i = 1, \cdots, n$, and $z^i(\mathrm{Log}_p(p)) = 0$ for all i.

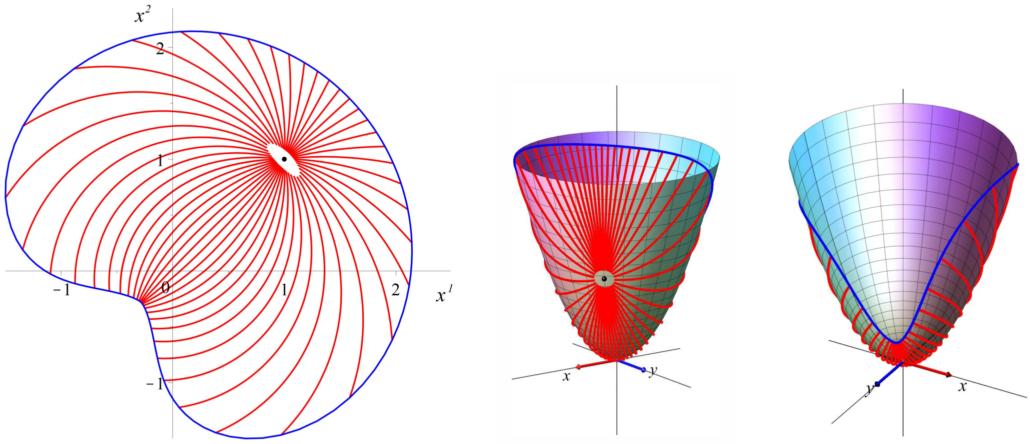


Figure 4.3: Leftmost: The exponential map (plus a few of the defining geodesics) of a metric ball $B_b(p)$ from the tangent space $T_p\mathcal{U}$ at $p = (1,0)$ into the corresponding geodesic disk $D_b(p)$ in $\mathcal{U}$. Middle and rightmost: The image of the exponential map of $D_b(0)$ lifted from $\mathcal{U}$ into the paraboloid – seen from 'front' and 'back', respectively. Note that every geodesic reaches the geodesic circle orthogonally – see lemma 4.22.

Definition 4.9 Suppose that $\{E_1, E_2, \cdots, E_n\}$ is a g-orthonormal basis of the tangent space $T_p\mathcal{U}$ at the point $p \in \mathcal{U}$ with corresponding rectilinear Cartesian coordinates $\{z^1, \cdots, z^n\}$ in $T_p\mathcal{U}$. Then $z^i(\mathrm{Log}_p(q))$, $i = 1, \cdots n$, are called the normal coordinates of q with respect to the chosen g-orthonormal basis. The corresponding coordinate induced normal basis vector fields in all of $\mathcal{U}_p$, i.e. also away from p, are still denoted by $\{E_1, E_2, \cdots, E_n\}$.

Note that in general $\{E_1, E_2, \cdots, E_n\}$ is not g-orthonormal away from p.

Normal coordinates have very nice and useful properties at the base point p:

Proposition 4.10 Let $z^i(\mathrm{Log}(q))$ denote normal coordinates at p. With respect to these coordinates we get the following evaluations at p for all i, j, and k:

$$
\begin{aligned}
&g_{ij}(p) = g(E_i(p), E_j(p)) = \delta_{ij} \\
&\nabla_{E_i} E_j = 0 \quad , \quad \text{i.e.} \quad \Gamma^k_{ij}(p) = 0 \\
&\frac{\partial}{\partial z^k} g_{ij} = E_k(g_{ij}) = E_k(g(E_i, E_j)) = 0 \quad .
\end{aligned}
\tag{4.13}
$$

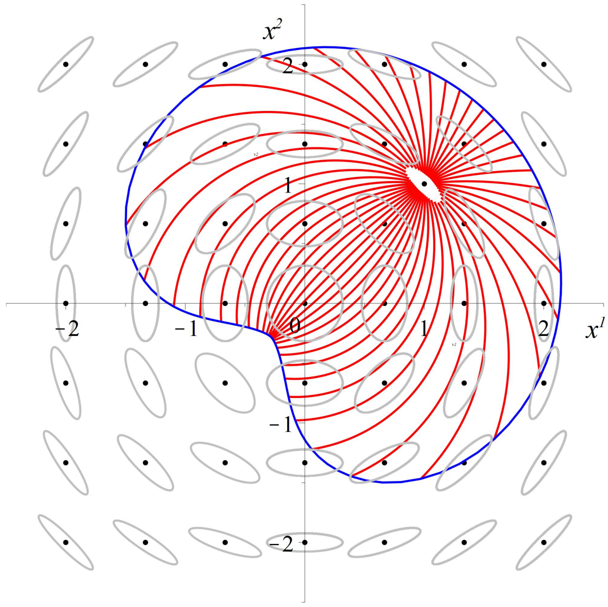


Figure 4.4: The image of the exponential map considered also in figure 4.3 – now with superimposed indicatrix field for the corresponding metric tensor. The fact that the geodesics, which mold the image geodesic disk, have the same length is intuitively plausible.

Although normal coordinates at a given base point p are in general not so easy to construct – because you need first to construct the Exponential map diffeomorphism Exp_p – they are (by their mere existence) tremendously useful. For example for establishing tensor identities at any given point p, in particular when the identities involve the metric (and its derivatives) and/or the covariant derivatives of vector fields and tensor fields. This will be illustrated by a number of examples in Chapter 7. The point is, that once you have established a tensor identity at p using normal coordinates, then that tensor identity will be true at p when you express the identity in any other coordinate system.

Proof. The first equation in (4.13) is clear by construction of the basis vector frame at p. The second follows from the fact that any geodesic γ issuing from p has the form $\gamma(s) = \mathrm{Exp}_p(s \cdot V)$ for a g-unit vector $V = \gamma'(s)$ in $T_p\mathcal{U}$ and thence in z-coordinates for each $i = 1, \cdots, n$:

$$z^i(s) = z^i(\mathrm{Log}_p(\gamma(s))) = z^i(\mathrm{Log}_p(\mathrm{Exp}_p(s \cdot V))) = z^i(s \cdot V) = s \cdot v^i \quad . \tag{4.14}$$

In consequence, the geodesic equations in normal coordinates now reads for each k:

$$0 = (z^k)''(s) + \sum_i \sum_j (z^i)'(s) \cdot (z^j)'(s) \cdot \Gamma^k_{i\,j}(\gamma(s)) = \sum_i \sum_j v^i \cdot v^j \cdot \Gamma^k_{i\,j}(\gamma(s)) \quad . \tag{4.15}$$

Since $\Gamma^k_{i\,j}(\gamma(0))$ is common for all directions V, we have for all k and for all initial vectors V:

$$0 = v^i \cdot v^j \cdot \Gamma^k_{i\,j}(p) \quad . \tag{4.16}$$

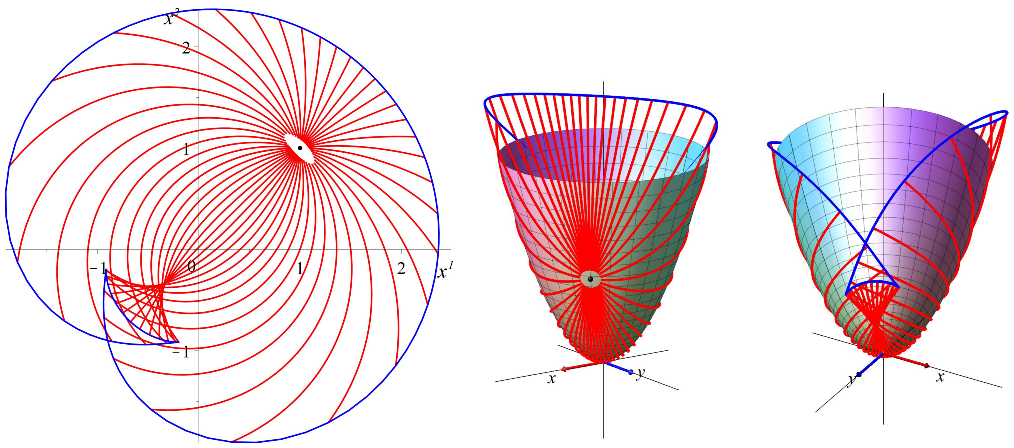


Figure 4.5: Left: An extended geodesic 'disk' *beyond* the radius up to which the Exponential map is a diffeomorphism in $\mathcal{U}$ for the paraboloid. The breaking of the diffeomorphism property is clearly visible – also on the 'back' side of the paraboloid itself which is shown in the rightmost figure. In consequence, the inverse map, Log_p, does not exist in the domain that is double covered by the geodesics issuing from p.

The symmetric matrix $\Gamma^k_{ij}(p)$ with indices i and j (and any fixed k) therefore has eigenvalues that are all zero. Hence

$$\Gamma^k_{ij}(p) = 0 \quad \text{for all } i,\ j, \text{ and } k \quad . \tag{4.17}$$

The third equation follows then immediately from this vanishing of $\nabla_{E_i} E_j$ at p and from the metric compatibility of ∇:

$$E_k(g(E_i, E_j)) = g\left(\nabla_{E_k} E_i\, ,\, E_j\right) + g\left(E_i\, ,\, \nabla_{E_k} E_j\right) = 0 \quad . \tag{4.18}$$

□

Note that the radial coordinate ρ induced by the normal coordinates is determined by the radial coordinate r in $T_p\mathcal{U}$ as follows:

$$\rho(q) = r(\mathrm{Log}_p(q)) = \sqrt{\sum_i (z^i(\mathrm{Log}_p(q)))^2} \quad , \tag{4.19}$$

which, of course, is nothing but the length of the geodesic from p to q in $\mathcal{U}_p$.

This radial coordinate may thus be completed to a system of polar, or spherical, coordinates $\{\rho, \theta^1, \theta^2, \cdots, \theta^{n-1}\}$ in $\mathcal{U}_p - \{p\}$ by *choosing* appropriate (angle-) coordinates $\{\phi^1, \cdots, \phi^{n-1}\}$ on the metric ball $B_1(p)$ – and thence on every metric ball $B_\rho(p)$ – in $T_p\mathcal{U}$ and by defining for each $i = 1, \cdots, n$:

$$\theta^i(q) = \phi^i(\mathrm{Log}_p(q)) \quad . \tag{4.20}$$

EXERCISE 4.11

We consider a smooth regular parametrized surface in 3D Euclidean space:

$$\alpha(x^1,x^2) = \left(x^1\,,\, x^2\,,\, f(x^1,x^2)\right) \quad , \tag{4.21}$$

where f is a smooth function on $\mathcal{U} = \mathbb{R}^2$. Find the induced metric matrix function on $\mathcal{U}$. Under which condition(s) on f are the properties in (4.13) satisfied by the coordinates $\{x^1,x^2\}$ at $p = (0,0)$. Show, however, that $\{x^1,x^2\}$ are in general *not normal coordinates* at $p = (0,0)$ with respect to the standard basis $\{e_1,\cdots,e_n\}$ at p? Hint: See equation (4.14).

Obviously, the coordinate curves for a given system of normal coordinates at a point p are the images via Exp_p of the straight line rectilinear Cartesian coordinate curves in the tangent space $T_p\mathcal{U}$. Specifically, if $\{E_1,E_2,\cdots,E_n\}$ is a g-orthonormal basis of the tangent space $T_p\mathcal{U}$ at the point $p \in \mathcal{U}$ with corresponding rectilinear Cartesian coordinates $\{z^1,\cdots,z^n\}$ in $T_p\mathcal{U}$. Then the k'th Cartesian coordinate line in the tangent space through the point $(z_0^1,\cdots,z_0^n)$ with corresponding position vector

$$P_0 = \sum_i z_0^i \cdot E_i \tag{4.22}$$

is parametrized by a vector function

$$C_k(t) = P_0 + t\cdot E_k \quad , \quad t \in \mathbb{R} \quad . \tag{4.23}$$

The normal coordinate lines in $\mathcal{U}$ through the point $\mathrm{Exp}_p(P_0)$ are therefore – for each $k = 1,\cdots,n$:

$$C_k^{\mathcal{U}}(t) = \mathrm{Exp}_p(C_k(t)) = \mathrm{Exp}_p(P_0 + t\cdot E_k) \quad , \tag{4.24}$$

where t should necessarily be restricted by the condition that $P_0 + t\cdot E_k \in \mathcal{E}_0(p)$ where Exp_p is guaranteed to be a diffeomorphism.

As indicated, the notion of polar coordinates are already defined above:

Definition 4.12 Let $\{E_1,E_2,\cdots,E_n\}$ denote a g-orthonormal basis of the tangent space $T_p\mathcal{U}$ at the point $p \in \mathcal{U}$ with corresponding polar coordinates $\{\rho,\theta^1,\theta^2,\cdots,\theta^{n-1}\}$ in $T_p\mathcal{U}$. Then $r(q) = \rho(\mathrm{Log}_p(q))$, and $\phi^i(q) = \theta^i(\mathrm{Log}_p(q))$, $i = 1,\cdots n-1$, are called the polar coordinates of q with respect to the chosen g-orthonormal basis and with respect to the *choice of coordinates* $\{\phi^1,\cdots,\phi^{n-1}\}$ on the metric ball $B_1(p)$ in $T_p\mathcal{U}$. The corresponding coordinate induced polar basis vector fields in $\mathcal{U}_p$, i.e. away from p, are typically denoted $e_r, e_{\phi^1},\cdots,e_{\phi^{n-1}}$.

Note that in contrast to normal coordinates there is for polar coordinates a choice to be made for the functions $\{\phi^1, \cdots, \phi^{n-1}\}$ on the metric ball $B_1(p)$ in $T_p\mathcal{U}$. In dimension $n = 2$ this corresponds to a choice of parametrization of the unit circle; in dimension $n = 3$ the parametrization of the unit 2D sphere can, as we have seen, be done in several applications dependent ways – using geographic coordinates, Mercator coordinates, stereographic coordinates, etc. Every choice must, however, be expressed in terms of the g-orthonormal basis $\{E_1, E_2, \cdots, E_n\}$, so that the orientation of the metric ball $B_1(p)$ is thereby properly 'anchored' in $T_p\mathcal{U}$.

As we will see below, both coordinate systems, normal coordinates and polar coordinates, are – each in their own way – very useful auxiliary tools for the local analysis of Riemannian manifolds. Although natural coordinates are not so easy to construct, their mere existence gives rise to considerable simplifications when establishing various tensor identities point wise. This hinges, of course, mainly on the fact that all Christoffel symbols vanish at the base point of the normal coordinates.

|||| Example 4.13

Below in the sequence of figures 4.6-4.9 and (in the last section of this chapter) 4.16-4.19 we illustrate systems of normal coordinates (and polar coordinates) on various surfaces in 3D and in the 2D Poincaré models. In the case of surfaces we show the normal and polar coordinates both for the specific choices of representations $(\mathcal{U}, g, \nabla)$ of the surfaces and on the surfaces themselves, respectively. The surface representations are by so-called Monge patches (for the paraboloid and the saddle surface) and by Mercator's parametrization (for the sphere); the corresponding coordinates will be called Monge coordinates and Mercator coordinates.

$$
\begin{aligned}
&\text{Paraboloid:} \quad \mu(x^1, x^2) = \left(x^1, x^2, (x^1)^2 + (x^2)^2\right) \quad \text{for } x^1 \in \mathbb{R} \text{ and } x^2 \in \mathbb{R} \\
&\text{Saddle:} \quad \mu(x^1, x^2) = \left(x^1, x^2, ((x^1)^2 - (x^2)^2)/2\right) \quad \text{for } x^1 \in \mathbb{R} \text{ and } x^2 \in \mathbb{R} \\
&\text{Sphere:} \quad \mu(x^1, x^2) = \left(\frac{\cos(x^2)}{\cosh(x^1)}, \frac{\sin(x^2)}{\cosh(x^1)}, \tanh(x^1)\right) \quad \text{for } x^1 \in \mathbb{R} \text{ and } x^2 \in \,]-\pi, \pi[\quad .
\end{aligned}
\tag{4.25}
$$

The figures 4.6-4.9 induce several observations concerning the properties of normal coordinates. Firstly, the two coordinate curves (red and blue) through the (yellow) base point are g-orthogonal geodesics (easily inspected in the Poincaré models, where we know them). And the normal coordinate curves do closely resemble the ordinary Cartesian coordinate grid close to the base point – but in general with possible increasing deviations away from the base, cf. proposition 4.10. Figure 4.6 also shows clearly that normal coordinates break down (to the left of the left hand display) if we extend them beyond the region $\mathcal{E}(p)$ around p where Exp is a diffeomorphism.

EXERCISE 4.14

Show or disprove the conjecture: The Euclidean spaces, the spheres and the Poincaré models have the special property that every coordinate curve for any normal coordinate system in these Riemannian manifolds has constant geodesic curvature. Hint: Presumably difficult(?)

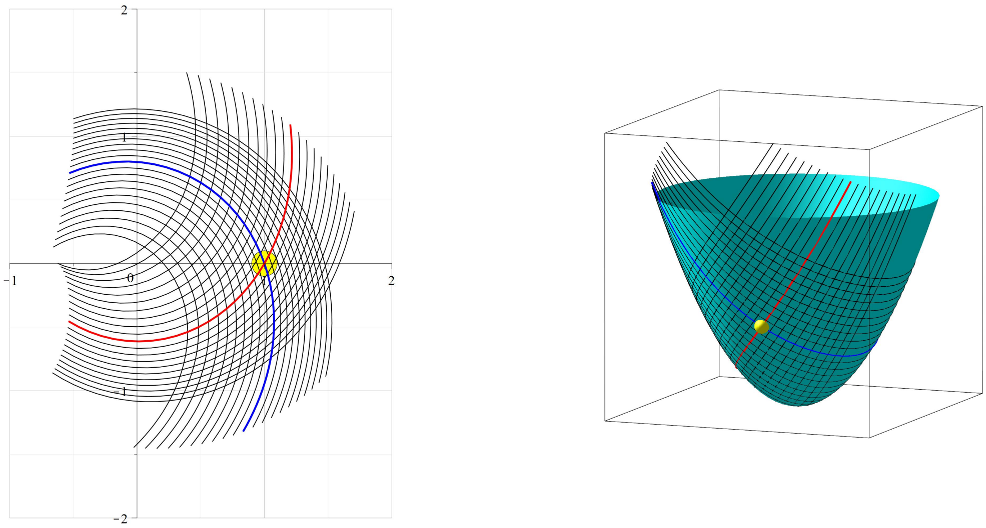


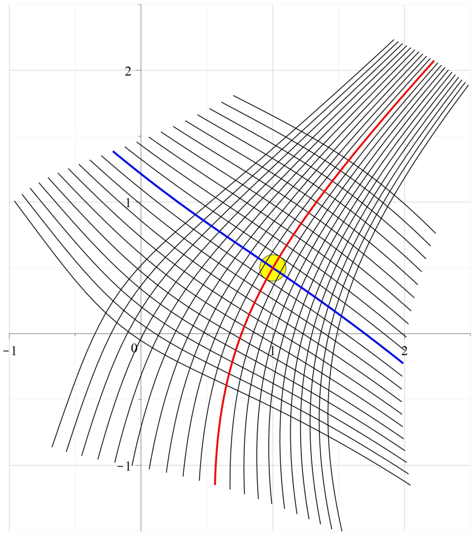
Figure 4.6: Normal coordinates on a paraboloid of revolution. Left: The coordinate curves are shown on a background of Monge patch coordinates.

4.4 Locally shortest curves are geodesics

We now show that within the image $Q_p \subset \mathcal{U}$ of the Exponential diffeomorphism Exp_p the following important statement which shows that geodesics are indeed good candidates for being shortest curves:

Theorem 4.15 Let $(\mathcal{U}^n, g, \nabla)$ denote a Local Riemannian Manifold. Let η denote a regular smooth curve from p to q in Q_p. We assume that η is g-arc-length parametrized by $s \in [0, L]$, so that the length of the curve is L. If η is the shortest curve in Q_p between p and q in the sense that all other curves with the same two end points are at least as long, then η is a geodesic.

Note that this theorem does not tell us anything about the existence of shortest curves – only that if there is one, then it must be a geodesic. The local existence is guaranteed by the existence of geodesics (via theorem 4.5 and the Exponential map), as we shall see in the next section.



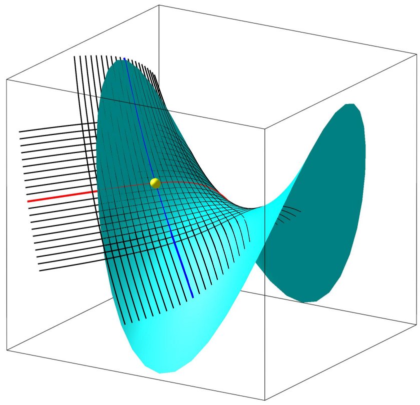


Figure 4.7: Normal coordinates on a saddle surface. Left: The coordinate curves are shown on a background of Monge patch coordinates.

Before going into the proof of this result we need a means to control nearby curves with the same endpoints as η.

Definition 4.16 In order to compare the lengths of neighboring curves to η and in order to analyze what it means to be the shortest curve between two given points, we first define a one-parameter family $H_u(s)$ – parametrized by $u \in]-\varepsilon, \varepsilon[$ – of nearby regular smooth curves that are organized as an (u,s)-parametrized smooth surface in Q_p so that:

$$H_0(s) = \eta(s) \quad \text{for all } s \in [0,L] \quad , \quad \text{and} \quad H_u(0) = \eta(0) = p \quad \text{for all } u \in]-\varepsilon, \varepsilon[. \tag{4.26}$$

The parametrized family of curves H is called a variation of the base curve η. If, moreover, we also have

$$H_u(L) = \eta(L) = q \quad \text{for all } u \in]-\varepsilon, \varepsilon[, \tag{4.27}$$

then H is called a proper variation of the base curve η.

The local linear transverse behaviour of the nearby curves is, for each s_0, obtained by the tangent vectors to the u-curves $H_u(s_0)$ at $u = 0$:

Definition 4.17 In the above setting, the vector field $V \in \mathfrak{X}(\gamma)$ defined by

$$V(s) = \frac{\partial}{\partial u} H_u(s)_{|u=0} \tag{4.28}$$

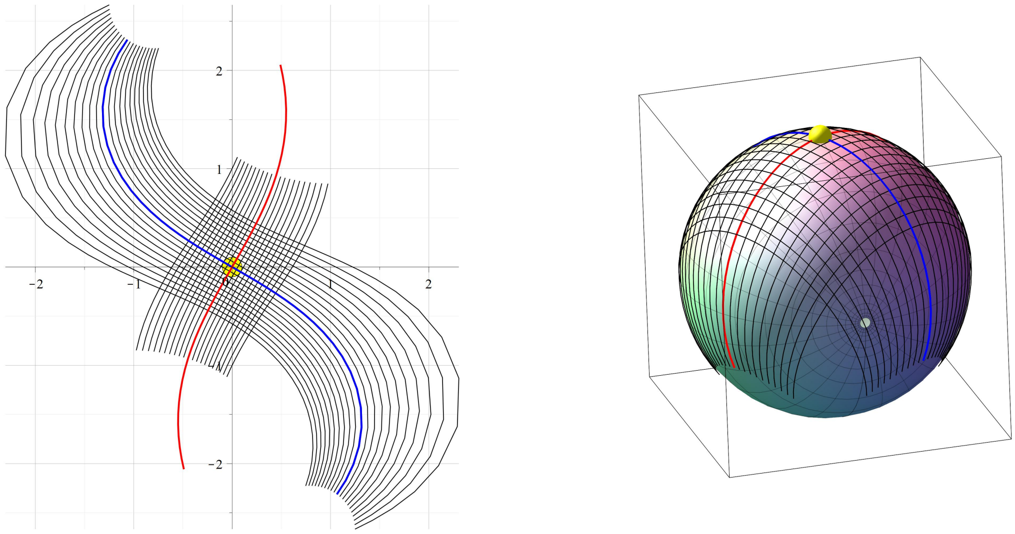


Figure 4.8: Normal coordinates on the sphere. Left: The coordinate curves are shown on a background of Mercator coordinates.

is called the variation vector field along γ induced by the variation H of γ.

Since $H_u(0) = \eta(0)$ for all u we have $V(0) = 0$. If H is a proper variation we also have $V(L) = 0$.

Note that although all the curves H_u are parametrized by s, this parameter is *not necessarily an arc length parameter* along the curve – unless, of course, $u = 0$. Thus the curves $H(u)$ do not necessarily have the same length L as γ. In fact, this is the main point in the proof of theorem 4.15.

Theorem 4.15 will follow once we have established the following preliminary supporting theorem, which itself is of independent interest. We simply consider the lengths $\mathcal{L}(u)$ of all the (competing) curves H_u in the variation, and then find the derivative of the length function with respect to u at $u = 0$. Then this derivative must be non-negative if the length of η – corresponding to $u = 0$ – is smaller or equal the length of all other curves between the same endpoints as is assumed in theorem 4.15. I.e. $\mathcal{L}$ must have a stationary point at $u = 0$. From there it then follows that η must have zero acceleration and thence zero geodesic curvature. We present the details of this argument below. In section 4.8 we spell out in all details what is going on in the case of a concrete horizontal straight line base curve η in the Poincaré half plane model.

Lemma 4.18 Let H denote a variation of η as above. Then the g-length of H_u is:

$$C = \int_0^L \sqrt{g\left(\frac{\partial}{\partial s}H_u(s)\,,\,\frac{\partial}{\partial s}H_u(s)\right)}\,ds \quad , \tag{4.29}$$

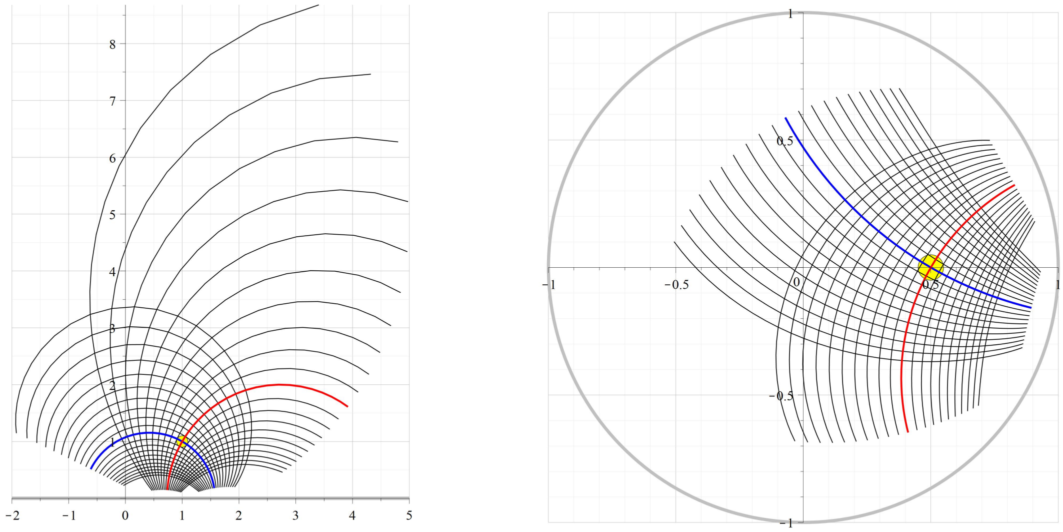


Figure 4.9: Normal coordinates in the two Poincaré models. Left: Half plane model; Right: Disk Model.

The derivative of $\mathcal{L}(u)$ with respect to u at $u = 0$ is:

$$\frac{\partial}{\partial u}_{|u=0} \mathcal{L}(u) = g\,(V(L), \eta'(L)) - \int_0^L g\left(V(s), \nabla_{\eta'(s)}\eta'(s)\right) ds \quad , \tag{4.30}$$

where V is the variation vector field of H based on η, and where

$$\nabla_{\eta'(s)}\eta'(s) = \frac{D}{ds}\eta'(s) = \mathrm{acc}_\eta(s) \tag{4.31}$$

is the acceleration of η – with norm $\mathrm{acc}_\eta(s) = \kappa^g_\eta(s)$, the geodesic curvature of η.

Suppose the endpoints of a *non-geodesic* arc-length-parametrized curve η are *both* fixed in the above variation, so that also $V(L) = 0$, then, by choosing a vector field $V = f(t) \cdot \mathrm{acc}_\eta(s)$ with a smooth positive function f with $f(0) = f(L) = 0$, will give a positive integral value in equation (4.30), and thence a negative derivative of $\mathcal{L}(u)$. This means, loosely speaking, that by pushing the curve η in the direction of the acceleration vector field (which is orthogonal to the curve) while keeping the endpoints fixed will produce shorter curves between these endpoints. The same principle can be applied for non-arc-length-parametrized curves $\gamma(t)$ if only the curve $\gamma(t)$ is not a pre-geodesic so it cannot be reparametrized to an arc-length-parametrized geodesic, i.e. if only $\mathrm{acc}_\gamma(t)$ is not everywhere proportional to $\gamma'(t)$ – see Proposition 3.47.

Proof of lemma 4.18. Taking derivatives at $u = 0$ we have:

$$\begin{aligned}\frac{\partial}{\partial u}_{|u=0}\mathcal{L}(u) &= \int_0^L \frac{\partial}{\partial u}_{|u=0}\left(\sqrt{g\left(\frac{\partial}{\partial s}H_u(s)\,,\,\frac{\partial}{\partial s}H_u(s)\right)}\right)ds \\ &= \int_0^L \frac{1}{\|\eta'(s)\|_g}\cdot g\left(\frac{D}{du}\left(\frac{\partial H_u(s)}{\partial s}\right),\,\frac{\partial H_u(s)}{\partial s}\right)ds \\ &= \int_0^L g\left(\frac{D}{ds}\left(\frac{\partial H_u(s)}{\partial u}\right),\,\frac{\partial H_u(s)}{\partial s}\right)ds \quad ,\end{aligned} \tag{4.32}$$

so that

$$\begin{aligned}\frac{\partial}{\partial u}_{|u=0}\mathcal{L}(u) &= \int_0^L g\left(\nabla_{\eta'(s)}\frac{\partial H_u(s)}{\partial u}\,,\,\eta'(s)\right)ds \\ &= \int_0^L \left(\frac{\partial}{\partial s}\left(g\left(\frac{\partial H_u(s)}{\partial u}\,,\,\eta'(s)\right)\right) - \frac{\partial H_u(s)}{\partial u}\,,\,\nabla_{\eta'(s)}\eta'(s)\right)ds \\ &= g\left(\frac{\partial H_u(s)}{\partial u}\,,\,\eta'(s)\right)_{|s=L} - \int_0^L g\left(\frac{\partial H_u(s)}{\partial u}\,,\,\nabla_{\eta'(s)}\eta'(s)\right)ds \\ &= g\left(V(L)\,,\,\eta'(L)\right) - \int_0^L g\left(V(s)\,,\,\nabla_{\eta'(s)}\eta'(s)\right)ds \quad .\end{aligned} \tag{4.33}$$

□

EXERCISE 4.19

In the proof above we used that

$$\frac{D}{du}\left(\frac{\partial H_u(s)}{\partial s}\right) = \frac{D}{ds}\left(\frac{\partial H_u(s)}{\partial u}\right) \quad . \tag{4.34}$$

Why is this interchange of derivatives allowed? Hint: The relevant Lie bracket vanishes everywhere. See also the explicit calculation of this identity in [4, p. 68].

We can now prove theorem 4.15:

Proof of theorem 4.15. Suppose η is shortest among all neighboring curves – in particular among all the neighboring curves appearing in any given *proper* variation H of γ. Then we have from equation (4.30):

$$0 = \int_0^L g\left(V(s)\,,\,\nabla_{\eta'(s)}\eta'(s)\right)ds \tag{4.35}$$

for all variational vector fields $V \in \mathfrak{X}(\gamma)$ with $V(L) = 0$. This implies that

$$\nabla_{\eta'(s)}\eta'(s) = \mathrm{acc}_\eta(s) = 0 \quad \text{and} \quad \kappa_\eta^g = 0 \quad \text{for all } s \in [0, L] \quad , \tag{4.36}$$

because otherwise the integral in equation (4.35) could be made different from 0 by choosing/constructing a variation vector field V in the direction of $\nabla_{\eta'(s)}\eta'(s)$ where this latter field is non-zero. Remember that the acceleration field is always g-orthogonal to the arc length parametrized curve η. We conclude from equation (4.36) that η is indeed a geodesic curve from p to q – as was to be proved. □

4.5 Geodesics are locally shortest curves

The converse to theorem 4.15 is also true:

Theorem 4.20 Let γ denote a geodesic from p to q in Q_p and assume that q is a point on the geodesic sphere $\partial D_\rho(p)$ centered at p and with radius ρ. Then γ is the shortest curve from p to q in Q_p in the following sense: If η is any piecewise differentiable curve joining p and q then

$$\rho = \mathrm{L}(\gamma) \leq \mathrm{L}(\eta) \quad . \tag{4.37}$$

We take this opportunity to define the notion of distance between points that are sufficiently close.

Definition 4.21 In the setting of theorem 4.20 we define the distance between p and q to be the length of the (shortest) geodesic between the points:

$$\mathrm{dist}(p,q) = \mathcal{L}(\gamma) = \rho \quad . \tag{4.38}$$

Again we need an observation – known as Gauss' lemma – before we can prove theorem 4.20:

Lemma 4.22 Suppose that H is a (non-proper) variation of a geodesic γ as defined in the previous section. If H is a variation through geodesics, so that H_u is an arc length parametrized geodesic for every $u \in]-\varepsilon, \varepsilon[$, then the variation vector field V is everywhere orthogonal to the base curve:

$$g\left(\frac{\partial}{\partial u}H_u(s)_{|u=0}\,,\,\gamma'(s)\right) = g\left(V(s)\,,\,\gamma'(s)\right) = 0 \quad \text{for all } s \in [0,L]. \tag{4.39}$$

Proof of lemma 4.22. This follows readily from the variation formula (4.30): Since all the curves in the H-variation are *geodesics*, now even of *the same length* L, then we have:

$$\begin{aligned} 0 &= \frac{\partial}{\partial u}_{|u=0} \mathcal{L}(u) \\ &= g\left(V(L), \gamma'(L)\right) - \int_0^L g\left(V(s), \nabla_{\gamma'(s)}\gamma'(s)\right) ds \\ &= g\left(V(L), \gamma'(L)\right) \quad , \end{aligned} \tag{4.40}$$

and the lemma follows, because in this argument L can be substituted by any smaller value than the one first chosen. So equation (4.39) holds for all $s \in [0, L]$. □

Corollary 4.23 It follows immediately from the Gauss' lemma that every geodesic from a point p intersects every geodesic sphere $\partial D_\rho(p)$ orthogonally.

Proof of theorem 4.20. We use the Exponential map to compare the given geodesic from p to q in $\partial D_\rho(p)$ with η, i.e. we assume that q is an endpoint in the geodesic sphere of radius ρ centered at p. The competing curve η can then be written as follows, where we use a *choice* of parametrization of η so that $\eta(0) = p$ and $\eta(1) = q$:

$$\eta(t) = \text{Exp}_p(r(t) \cdot v(t)) = f(r(t), t) \quad , \quad t \in I = [0, 1] \quad , \tag{4.41}$$

where v denotes a curve in $T_p\mathcal{U}$ with $\|v(t)\|_g = 1$ for all $t \in [0, 1]$, and where r denotes the geodesic distance of $\eta(t)$ to p, i.e. it is a piecewise differentiable function of t so that $r(t) \cdot v(t) = \text{Log}_p(\eta(t))$, see the definition of the Log map in 4.6. Then we have, at points where η is differentiable:

$$\eta'(t) = \frac{\partial f}{\partial r} \cdot r'(t) + \frac{\partial f}{\partial t} \quad . \tag{4.42}$$

But here we can now use that $\|\frac{\partial f}{\partial r}\|_g = 1$ and from equation (4.39) that $g(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}) = 0$, so that :

$$\|\eta'(t)\|_g^2 = (r'(t))^2 + \|\frac{\partial f}{\partial t}\|_g^2 \geq (r'(t))^2 \quad . \tag{4.43}$$

In consequence we have:

$$\text{L}(\eta) = \int_0^1 \|\eta'(t)\|_g \, dt \geq \int_0^1 |r'(t)| \, dt \geq \int_0^1 r'(t) \, dt = r(1) = \rho = \text{L}(\gamma) \quad . \tag{4.44}$$

□

EXERCISE 4.24

Note that we have tacitly assumed in the above proof of theorem 4.20, that η is contained in $D_\rho(p)$. What happens with the argument if this assumption is not satisfied? Is the conclusion still true?

EXERCISE 4.25

Show that if there is equality in equation 4.37, i.e. if $\mathrm{L}(\gamma) = \mathrm{L}(\eta)$ – in that setting of theorem 4.20 – then η can be reparametrized to become a geodesic curve, namely the same geodesic as γ.

4.6 The gradient of the distance function

Recall from definition 4.21 that the distance function $\rho(x) = \mathrm{dist}(p,x)$ from a fixed point p to any point $x \in Q_p$ is the length of the unique geodesic $\gamma_{p,x}$ which connects p to x. By now it should come as no surprise, that within $Q_p - \{p\}$ the gradient $\mathrm{grad}(\rho)$ of the distance function at x is precisely the unit tangent vector $\gamma'_{p,x}$ to the geodesic $\gamma_{p,x}$ at x. Indeed, it is the direction in which the distance to p increases the most and the speed with which it increases is the arc length speed of 1 away from p. Here is a precise recap of the statement together with some immediate consequences:

Proposition 4.26 With the notation above we have for all $x \in Q_p - \{p\}$:

$$\mathrm{grad}(\rho)_{|x} = \gamma'_{p,x}(\rho(x)) \quad . \tag{4.45}$$

In particular we therefore have

$$\|\mathrm{grad}(\rho)_{|x}\| = 1 \quad \text{for all } x \in Q_p - \{p\}. \tag{4.46}$$

and, since the metric spheres $\partial D_\rho(p)$ by definition 4.7 are level surfaces for the distance function, we get from exercise 3.56 that $\mathrm{grad}(\rho)_{|x}$ and thence, of course, $\gamma'_{p,x}(\rho(x))$ is g-orthogonal to $\partial D_{\rho(x)}(p)$ at the point x.

Proof. We prove equation (4.45) via the first variation formula (4.30) in lemma 4.18. We let V denote any vector in $T_x\mathcal{U}$ at x and only need to show that

$$V(\rho)_{|x} = g(V, \gamma'_{p,x}(\rho(x))) \quad , \tag{4.47}$$

because then, by definition 3.50 we get directly (4.45). So we let $\xi(u)$, $u \in]-\varepsilon, \varepsilon[$ denote any curve in $\mathcal{U}$ with $\xi(0) = x$ and $\xi'(0) = V$. Then

$$V(\rho)_{|x} = \frac{d}{du}\rho(\xi(u))_{|0} \quad . \tag{4.48}$$

For each $u \in]-\varepsilon, \varepsilon[$ the distance $\rho(\xi(u))$ is realized as the length $\mathcal{L}(u)$ of the unique geodesic from p to $\xi(u)$. These geodesics form a variation (in the sense of definition 4.16) of the base geodesic, which is $\gamma_{p,x}$, so equation (4.30) gives directly (since the base geodesic has acceleration 0):

$$V(\rho)_{|x} = \frac{d}{du}\rho(\xi(u))_{|0} = \frac{\partial}{\partial u}_{|u=0}\mathcal{L}(u) = g\left(V\,,\,\gamma'_{p,x}(\rho)\right) \quad , \tag{4.49}$$

which proves the proposition. □

EXERCISE 4.27

Let $(\mathcal{U} = \mathbb{R}^n, g_E)$ denote the usual Euclidean n-dimensional space. Verify (by direct interpretation and calculations) that the statements in proposition 4.26 hold true for that special manifold with $Q_p = \mathbb{R}^n$ for all p.

4.7 Feynman's example

This example is a Riemannian illustration of Snell's refraction law. We let our Local Riemannian Manifold be $(\mathcal{U}, g, \nabla)$ with $\mathcal{U} = \mathbb{R}^2$ and g is given by its metric matrix function

$$G(x^1, x^2) = \mu(x^1)\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \tag{4.50}$$

where μ is a step-like function of x^1 with

$$\mu_{\alpha,\beta}(x^1) = 1 + \frac{\alpha}{2}\cdot(1+\tanh(\beta\cdot x^1)) \quad . \tag{4.51}$$

The function is essentially stepping up from 1 (for very negative values of x^1) to $1+\alpha$ for very positive values of x^1, the step being located sharply at $x^1 = 0$ for large values of β. See figure 4.10. (Alternatively we could have used the error function erf instead of tanh to build such a function.) The metric tensor g gives rise to the following Christoffel symbol functions:

$$\begin{aligned}
\Gamma^1_{11}(x^1,x^2) &= \frac{\alpha\cdot\beta\cdot\left(\tanh^2(\beta\cdot x^1)-1\right)}{2\alpha\cdot\tanh^2(\beta\cdot x^1)+2\alpha+4} \\
\Gamma^2_{11}(x^1,x^2) &= 0 \\
\Gamma^1_{12}(x^1,x^2) &= \Gamma^1_{21}(x^1,x^2) = 0 \\
\Gamma^2_{12}(x^1,x^2) &= \Gamma^1_{11}(x^1,x^2) \\
\Gamma^1_{22}(x^1,x^2) &= 0 \\
\Gamma^2_{22}(x^1,x^2) &= -\Gamma^1_{11}(x^1,x^2) \quad .
\end{aligned} \tag{4.52}$$

The corresponding geodesic equations have been solved numerically for $\alpha = 4$ and $\beta = 7$, and segments of the corresponding geodesics are on display in figure 4.11.

We observe from the figure that the geodesics are almost straight lines on both sides of $x^1 = 0$. This is consistent with the fact that for large values of β the Christoffel symbols become small –

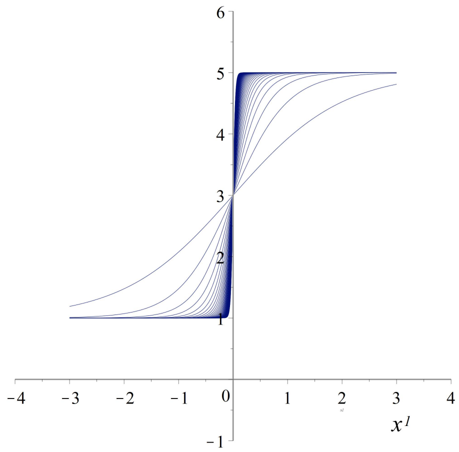


Figure 4.10: The ’step function’ $\mu_{4,\beta}(x^1)$ for various values of β.

except at $x^1 = 0$.

EXERCISE 4.28

Show that all the Christoffel symbols in (4.52) go to 0 for $x^1 \neq 0$ and $\beta \longmapsto \infty$.

We also observe from the figure that the geodesics change direction when they hit (or get close to) the line $x^1 = 0$.

EXERCISE 4.29

Assume that μ is constant 1 to the left of the line $x^1 = 0$ and that μ is constant $1 + \alpha$ to the right of the line $x^1 = 0$, so that the geodesics really are straight lines on both sides of $x^1 = 0$ in the local Riemannian manifold (μ, g, ∇) above. What is then the relation between the incoming angle of the geodesic (coming from $x^1 < 0$) to the outgoing angle of the broken geodesic (going into $x^1 > 0$) when both angles are measured with respect to the horizontal normal to the line $x^1 = 0$ at the break point? Discuss what all this has to do with Snell’s law of refraction – see Snell’s law.

4.8 Shortest geodesic curves in the 2D Poincaré models

We are now able to return to the setting of exercise 1.52 in chapter 1, which read as follows:

Let γ_2 denote the following curve in *the disk model*:

$$\gamma_2(t) = \left(\frac{1}{2}, 2 \cdot t\right) \quad , \quad t \in [-1/4, 1/4] \quad . \tag{4.53}$$

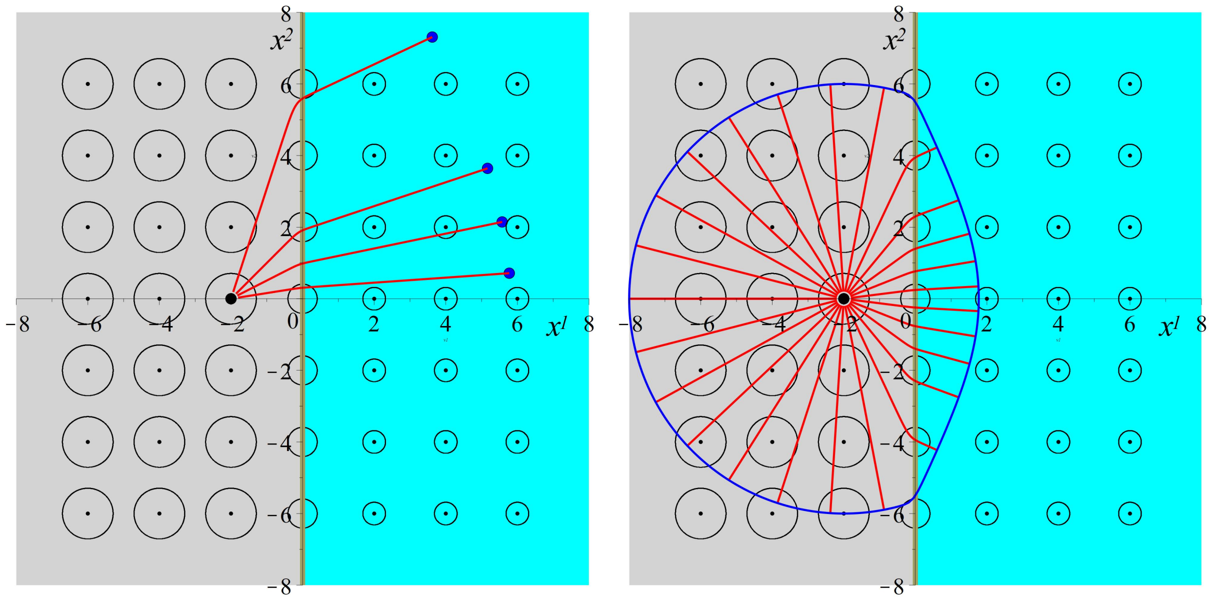


Figure 4.11: Feynman's example. Left: Four geodesics of the same length issuing from $p = (-2,0)$ in different directions towards positive x^1-values, where the indicatrices are significantly smaller than in the gray area to the left of $x^1 = 0$ due to the values of $\mu(x^1)$. Right: All points on the blue geodesic circle have constant geodesic distance from p.

The task was to find the $g_{\mathcal{U}}$ length of the curve and to construct another curve that connects the two endpoints of γ_2 but is shorter than γ_2.

In fact, we will solve this second part of that exercise in all details. Since we now have the tools to find shortest curves in any local Riemannian manifold, we can find the absolute shortest curve that connects the endpoints $p = (1/2, -1/2)$ and $q = (1/2, 1/2)$ of the straight line γ_2.

However, we will begin the construction of that shortest curve by finding all shortest curves, i.e. all geodesics, in the Poincaré *half plane model*, see figure 4.12.

We recall the metric of the half plane model in terms of its metric matrix function on $\mathcal{V}$:

$$\mathcal{V} = \{(y^1, y^2) \in \mathbb{R}^2 \,|\, y^2 > 0\,\}$$

$$G_{\mathcal{V}}(y^1, y^2) = \frac{1}{(y^2)^2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{4.54}$$

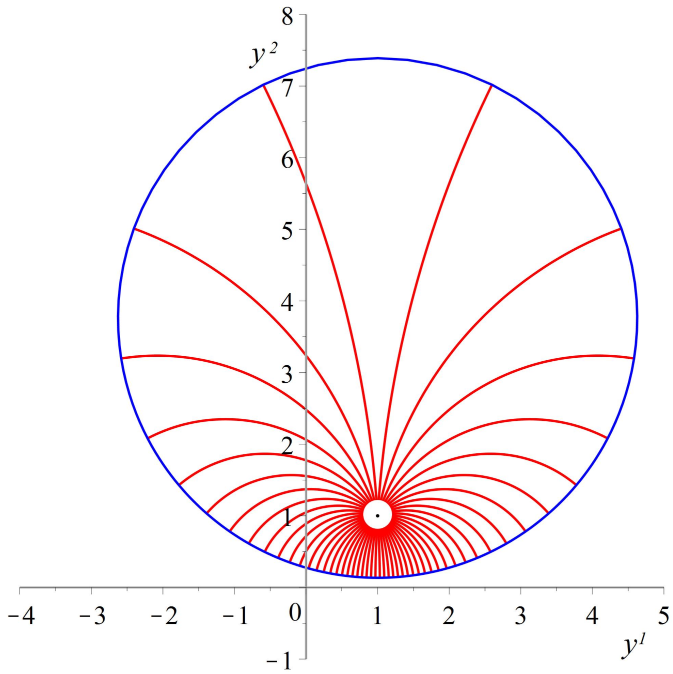


Figure 4.12: Geodesics from $p = (1,1)$ with the same length define a geodesic disk and the blue geodesic circle. All geodesics intersect the geodesic circle orthogonally, both with respect to the Euclidean metric and with respect to the g-metric tensor field of the Poincaré half plane! As is also indicated, all the geodesics – if extended past the boundary of the blue geodesic circle – intersect the boundary line $y^2 = 0$ orthogonally. The Exponential map Exp_p from *any point p* in the half plane model is a diffeomorphism from the corresponding tangent plane, $T_p\mathcal{V}$, and thence the Logarithm map Log_p is also a diffeomorphism from all of $\mathcal{V}$ onto $T_p\mathcal{V}$.

The Christoffel symbols are the following:

$$\begin{aligned}
\Gamma^1_{1\,1}(y^1,y^2) &= 0 \\
\Gamma^2_{1\,1}(y^1,y^2) &= \frac{1}{y^2} \\
\Gamma^1_{1\,2}(y^1,y^2) &= \Gamma^1_{2\,1}(y^1,y^2) = -\frac{1}{y^2} \\
\Gamma^2_{1\,2}(y^1,y^2) &= \Gamma^2_{2\,1}(y^1,y^2) = 0 \\
\Gamma^1_{2\,2}(y^1,y^2) &= 0 \\
\Gamma^2_{2\,2}(y^1,y^2) &= -\frac{1}{y^2} \quad .
\end{aligned} \tag{4.55}$$

Therefore the geodesic equations are:

$$
\begin{aligned}
(\gamma^1)''(s) &= \frac{2\cdot(\gamma^1)'(s)\cdot(\gamma^2)'(s)}{\gamma^2(s)} \\
(\gamma^2)''(s) &= \frac{((\gamma^2)'(s))^2 - ((\gamma^1)'(s))^2}{\gamma^2(s)} \\
1 &= \frac{((\gamma^1)'(s))^2 + ((\gamma^2)'(s))^2}{(\gamma^2(s))^2} \quad .
\end{aligned}
\tag{4.56}
$$

where the last equation is included to guarantee that the geodesics are unit speed pararametrized.

Note that the last unit-speed condition in (4.56) is automatically satisfied for all s if the first two autoparallel conditions are satisfied for all s *and* the unit speed condition is satisfied for just *one* value of s. This follows from a previous result saying that all autoparallel curves have constant speed.

EXERCISE 4.30

Show that the following curves are solutions to the geodesic equations (4.56) for every choice of constants C, K, and B:

$$
\begin{aligned}
\gamma_{C,K}(s) &= (C\cdot\tanh(s)+K,\, C/\cosh(s)) \quad , \quad s\in\mathbb{R} \\
\gamma_B(s) &= (B, e^s) \quad , \quad s\in\mathbb{R} \quad .
\end{aligned}
\tag{4.57}
$$

EXERCISE 4.31

Show that every geodesic solution in equation (4.57) considered as a point-set curve in the Euclidean plane $(\mathbb{R}^2, g_E)$ is either a (Euclidean) straight line g_E-orthogonal to the half plane boundary $y^2 = 0$ or a (Euclidean) half circle with center $(K,0)$ and radius C.

EXERCISE 4.32

Show that there are no other geodesics than the ones obtained in (4.57) and show that every pair of geodesics has at most one point of intersection.

Since the disk model $(\mathcal{U}, g_{\mathcal{U}}, \nabla)$ is isometric to the half plane model – via the Cayley transform ϕ^{-1}, which we will spell out below – we also know, that all the geodesics in the disk model are obtained as images of the geodesics just found in the half plane model. And they have the same property: every pair of geodesics has at most one point of intersection.

In the disk model we want to find the geodesic segment which connects the two points $p = (1/2, -1/2)$ and $q = (1/2, 1/2)$ because that segment is the absolute shortest curve connecting the two points. But instead of finding the length of that geodesic segment in the disk model we find the length of the corresponding geodesic segment in the half plane model – the two lengths are identical by the isometry ϕ, the inverse of the Cayley transform.

For this we first need to find the image of p and q by ϕ.

The Cayley transform is defined as follow – using $z = y^1 + i \cdot y^2$:

$$\phi^{-1}(z) = \frac{z-i}{z+i} \quad , \tag{4.58}$$

so that the inverse of the transform is – using $w = x^1 + i \cdot x^2$:

$$\phi(w) = \frac{1+i\cdot w}{1-w} \quad . \tag{4.59}$$

Therefore $p = (1/2, -1/2) = 1/2 - i/2$ and $q = (1/2, 1/2) = 1/2 + i/2$ are mapped into $\tilde{p}$ and $\tilde{q}$, respectively, where

$$\begin{aligned} \tilde{p} &= \phi(p) = 2 + i = (2,1) \\ \tilde{q} &= \phi(q) = -2 + i = (-2,1) \quad . \end{aligned} \tag{4.60}$$

By symmetry the geodesic half circle through these two points in the half plane model has center at $(0,0)$ and radius $R = \sqrt{5}$. The particular geodesic segment connecting the two points can thus be parametrized (by angle) as follows:

$$\eta(t) = (\sqrt{5}\cdot\cos(t),\, \sqrt{5}\cdot\sin(t)) \quad , \quad t \in [\arccos(2/\sqrt{5}),\, \pi - \arccos(2/\sqrt{5})] \quad . \tag{4.61}$$

so that

$$\eta'(t) = (-\sqrt{5}\cdot\sin(t),\, \sqrt{5}\cdot\cos(t)) \quad , \tag{4.62}$$

and

$$\|\eta'(t)\|_{g_V} = \frac{1}{\sin(t)} \quad . \tag{4.63}$$

EXERCISE 4.33

Show that the metric gives this simple expression for the length of the tangent vectors along the circle in equation (4.61).

Finally we therefore have: The shortest curve between $\tilde{p}$ and $\tilde{q}$ in the half plane model – and therefore the shortest curve between p and q in the disk model – has length:

$$L = \int_{\arccos(2/\sqrt{5})}^{\pi-\arccos(2/\sqrt{5})} \frac{1}{\sin(t)}\,dt = 2\ln\left(2+\sqrt{5}\right) \approx 2.89 \quad . \tag{4.64}$$

In other words, we have shown that the distance between $\tilde{p}$ and $\tilde{q}$ is

$$\operatorname{dist}(\tilde{p}\,,\,\tilde{q}) = \operatorname{dist}((-2,0)\,,\,(2,0)) = 2\ln\left(2+\sqrt{5}\right) \quad . \tag{4.65}$$

It is fairly obvious, that the horizontal straight line between $\tilde{p}$ and $\tilde{q}$ carries non-zero geodesic curvature κ_ξ^g – i.e. non-zero acceleration acc_ξ. In fact, let the straight line be denoted by

$$\xi(t) = (t, 1) \quad , \quad t \in [-2,\,2] \quad . \tag{4.66}$$

Then

$$\mathrm{acc}_\xi(t) = \frac{D}{dt}\xi'(t) = (0,1) \quad , \quad t \in [-2,\,2] \quad . \tag{4.67}$$

|||| EXERCISE 4.34

| | Verify by calculations that equation (4.67) gives the acceleration of ξ along ξ.

Below we want to consider a proper variation H of the line segment in order to show that a specific variation of the segment gives curves in the variation with shorter lengths than the length of the line segment. To set up the variation in the way we have done it in general in section 4.5, we first parametrize the line segment by arc length. This is easy, because it is in fact already arc length parametrized:

$$\|\xi'(t)\|_g = 1 \quad , \tag{4.68}$$

so we may, and do, replace the parameter t by s and refer to the line segment ξ as unit speed parametrized. The total length of the line segment is then clearly

$$L = \int_{-2}^{2} \|\xi'(s)\|_g \, ds = 4 \quad . \tag{4.69}$$

We know already from the analysis of the geodesic circle between the same endpoints that the shortest curve (that circle) has length approximately 2.9. We also expect that if we vary the straight line in the direction of that circle – while keeping the endpoints fixed – then we should get length values of the curves in the variation that are significantly shorter than 4 – but not shorter, of course. Such a proper variation of ξ is, for example, the one displayed in figure 4.14:

$$\begin{aligned} H_u(s) &= \xi(s) + \left(0,\, u\cdot\cos\left(s\cdot\frac{\pi}{4}\right)\right) \\ &= \left(s,\, 1 + u\cdot\cos\left(s\cdot\frac{\pi}{4}\right)\right) \quad , \quad s \in [-2, 2] \quad . \end{aligned} \tag{4.70}$$

The variation vector field along ξ is then in the direction of the acceleration vector $\mathrm{acc}_\xi(t) = (0,1)$, and that is what it takes to decrease the length of the curve:

$$V(s) = \frac{\partial H_u(s)}{\partial u}_{|u=0} = \left(0,\, \cos\left(s\cdot\frac{\pi}{4}\right)\right) \quad . \tag{4.71}$$

According to equation (4.30) the curves H_u in the variation H satisfy:

$$\begin{aligned}
\frac{\partial}{\partial u}_{|u=0}\mathcal{L}(u) &= -\int_0^L g\left(V(s)\,,\,\nabla_{\xi'(s)}\xi'(s)\right)\,ds \\
&= -\int_{-2}^{2} g\left(\left(0,\,\cos\left(s\cdot\frac{\pi}{4}\right)\right)\,,\,\nabla_{\xi'(s)}\xi'(s)\right)\,ds \\
&= -\int_{-2}^{2} g\left(\left(0,\,\cos\left(s\cdot\frac{\pi}{4}\right)\right)\,,\,(0,\,1)\right)\,ds \\
&= -\int_{-2}^{2} \cos\left(s\cdot\frac{\pi}{4}\right)\,ds \\
&= -\frac{8}{\pi}\approx -2.55 \quad .
\end{aligned} \tag{4.72}$$

The derivative of the length functional $\mathcal{L}(u)$ is negative – as expected. The variation really does give shorter curves.

We can verify the above derivative by direct estimation of the lengths $\mathcal{L}(u)$ in this concrete case. First we calculate

$$\frac{\partial}{\partial s}H_u(s) = \left(1,\,-u\cdot\frac{\pi}{4}\cdot\sin\left(s\cdot\frac{\pi}{4}\right)\right) \quad , \tag{4.73}$$

so that

$$\begin{aligned}
\mathcal{L}(u) &= \int_{-2}^{2}\sqrt{g\left(\frac{\partial}{\partial s}H_u(s)\,,\,\frac{\partial}{\partial s}H_u(s)\right)}\,ds \\
&= \int_{-2}^{2}\sqrt{g\left(\left(1,\,-u\cdot\frac{\pi}{4}\cdot\sin\left(s\cdot\frac{\pi}{4}\right)\right)\,,\,\left(1,\,-u\cdot\frac{\pi}{4}\cdot\sin\left(s\cdot\frac{\pi}{4}\right)\right)\right)}\,ds \\
&= \int_{-2}^{2}\frac{\sqrt{1+u^2\cdot(\frac{\pi}{4})^2\cdot\sin^2\left(s\cdot\frac{\pi}{4}\right)}}{1+u\cdot\cos\left(s\cdot\frac{\pi}{4}\right)}\,ds \quad .
\end{aligned} \tag{4.74}$$

The integrand in the last expression in equation (4.74) has the following u-derivative at $u=0$:

$$\frac{\partial}{\partial u}_{|u=0}\left(\frac{\sqrt{1+u^2\cdot(\frac{\pi}{4})^2\cdot\sin^2\left(s\cdot\frac{\pi}{4}\right)}}{1+u\cdot\cos\left(s\cdot\frac{\pi}{4}\right)}\right) = -\cos\left(s\cdot\frac{\pi}{4}\right) \quad , \tag{4.75}$$

so that we do indeed recover the value obtained above from the general formula that we applied in equation (4.72):

$$\frac{\partial}{\partial u}_{|u=0}\mathcal{L}(u) = -\int_{-2}^{2}\cos\left(s\cdot\frac{\pi}{4}\right)\,ds = -\frac{8}{\pi}\approx -2.55 \tag{4.76}$$

A numerical evaluation of the expression for $\mathcal{L}(u)$ in equation (4.74) shows that the smallest curve in the specific variation family H, that we have considered, is obtained for $u\approx 1.31$ and that the length of the corresponding curve is $\mathcal{L}(1.31)\approx 2.92$, which is quite close to the optimal value represented by the geodesic circle segment with the exact length $2\cdot\ln(2+\sqrt{5})\approx 2.89$, see figure 4.14.

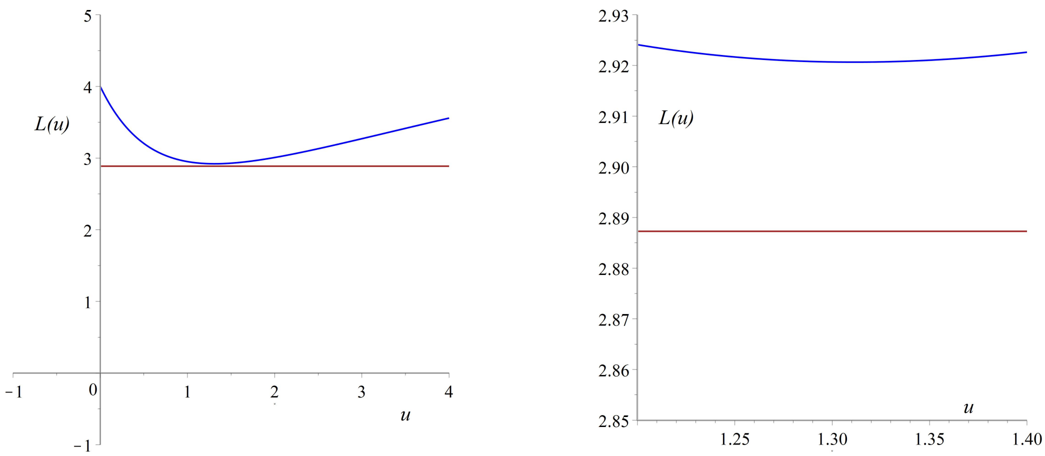
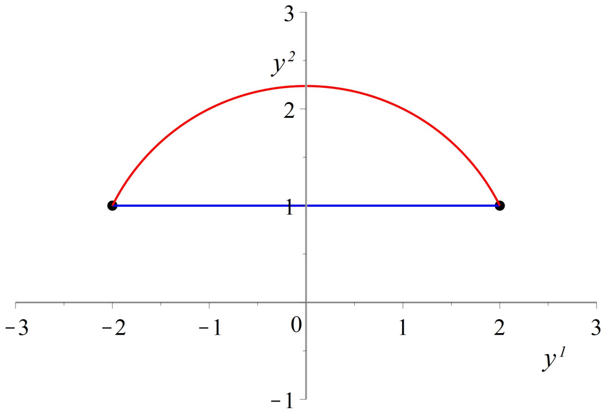


Figure 4.13: Values of curve lengths of H_u, $u \in [0, 1.5]$, in the variation H considered in (4.70). The horizontal brown line indicates the optimal shortest curve length which is (only) almost attained by the variation.

In prolongation of the concrete examples above we must mention the following general distance formula:

Proposition 4.35 Let $\tilde{p} = (\tilde{p}^1, \tilde{p}^2)$ and $\tilde{q} = (\tilde{q}^1, \tilde{q}^2)$ denote two given points in the Poincaré half plane model. Then the distance between the points is:

$$\mathrm{dist}(\tilde{p}, \tilde{q}) = \mathrm{arcosh}\left(1 + \frac{(\tilde{q}^1 - \tilde{p}^1)^2 + (\tilde{q}^2 - \tilde{p}^2)^2}{2 \cdot \tilde{p}^2 \cdot \tilde{q}^2}\right) \quad . \tag{4.77}$$



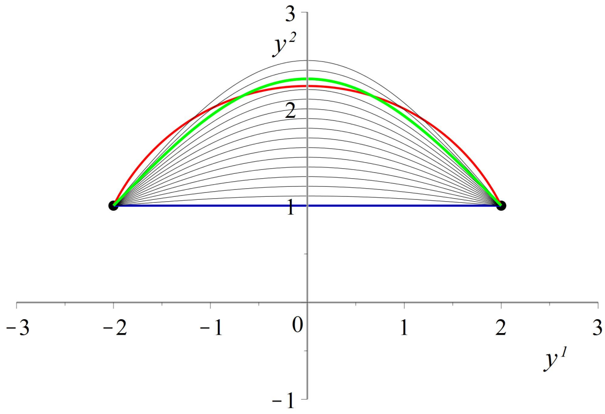


Figure 4.14: Red: The shortest geodesic between the endpoints. Right: The variation cos-curves applied for comparison with the shortest red curve. The green curve is the shortest *among the variation curves*.

EXERCISE 4.36

Verify, that this equation gives the same result as we obtained above in the case of $\tilde{p} = (2,1)$ and $\tilde{q} = (-2,1)$.

EXERCISE 4.37

Prove the general distance formula that is expressed in (4.77).

With the distance function and the explicit construction of all geodesics in hand we can express any Exponential map diffeomorphism Exp_p and Logarithmic map diffeomorphism Log_p in terms of the canonical basis vectors e_1 and e_2 in $T_p\mathcal{V}$ and in terms of the coordinates y^1 and y^2 in $\mathcal{V}$, respectively.

EXERCISE 4.38

Given p and q in $\mathcal{V}$. Find θ such that

$$\mathrm{Exp}_p(\mathrm{dist}(p,q) \cdot V_0) = q \quad , \tag{4.78}$$

where V_0 is the unit vector in the direction of $(\cos(\theta), \sin(\theta)$ in $T_p\mathcal{V}$. (Note that then we have: $\mathrm{Log}_p(q) = \mathrm{dist}(p,q) \cdot V_0$.) Hint: Trigonometry.

EXERCISE 4.39

Given p in $\mathcal{V}$, $d \in \mathbb{R}_+$, and a g-unit vector V_0 in the *direction* of $(\cos(\theta), \sin(\theta))$ in $T_p\mathcal{V}$. Find the coordinates of q in $\mathcal{V}$ so that

$$\text{Log}_p(q) = d \cdot V_0 \quad . \tag{4.79}$$

(Note that then we have: $\text{Exp}_p(d \cdot V_0) = q$ and $\text{dist}(p,q) = d$.)

4.9 The Exponential map in the 3D Poincaré half space

We consider the following 3-dimensional extension of the Poincaré half plane. Let $(\mathcal{V}^3, g_{\mathcal{V}}, \nabla)$ be defined as follows:

$$\begin{aligned} \mathcal{V} &= \{(y^1, y^2, y^3) \in \mathbb{R}^3 \mid y^3 > 0\} \\ G_{\mathcal{V}}(y^1, y^2, y^3) &= \left(\frac{1}{y^3}\right)^2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \end{aligned} \tag{4.80}$$

EXERCISE 4.40

In 3 dimensions the number of Christoffel symbol functions Γ^k_{ij} is in general 18, when symmetry in indices i and j are taken into account. Find the non-zero Christoffel symbol functions for the Poincaré half space $(\mathcal{V}^3, g_{\mathcal{V}}, \nabla)$ defined above.

EXERCISE 4.41

Show that every geodesic of $(\mathcal{V}^3, g_{\mathcal{V}}, \nabla)$ is either a straight half line orthogonal to the plane $y^3 = 0$ or a half circle with center on that plane and itself contained in a plane parallel to the y^3-axis. Use this property to show that two different geodesics have at most one intersection point.

The Exponential map Exp_p of $(\mathcal{V}^3, g_{\mathcal{V}}, \nabla)$, $p \in \mathcal{V}^3$, is therefore – in consequence of exercise 4.41 – a diffeomorphism on all of $T_p\text{m}^3$. The inverse, the Logarithm map Log_p, is thence also a diffeomorphism on all of $\mathcal{V}^3$.

Proposition 4.42 Let $\tilde{p} = (\tilde{p}^1, \tilde{p}^2, \tilde{p}^3)$ and $\tilde{q} = (\tilde{q}^1, \tilde{q}^2, \tilde{q}^3)$ denote two given points in the Poincaré half space model. Then the distance between the points is:

$$\text{dist}(\tilde{p}, \tilde{q}) = \text{arcosh}\left(1 + \frac{(\tilde{q}^1 - \tilde{p}^1)^2 + (\tilde{q}^2 - \tilde{p}^2)^2 + (\tilde{q}^3 - \tilde{p}^3)^2}{2 \cdot \tilde{p}^3 \cdot \tilde{q}^3}\right) \quad . \tag{4.81}$$

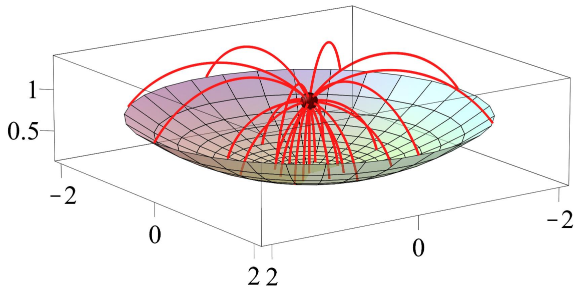


Figure 4.15: Geodesics (in red) from point $p = (0,0,1)$ of constant length in the Poincaré half space. The corresponding geodesic sphere is indicated.

EXERCISE 4.43

Formulate the exercises 4.38 and 4.39 in the 3-dimensional setting of the Poincaré half space model – and solve them.

4.10 Polar coordinates revisited

In a polar coordinate system based at p in $\mathcal{U}$, the first polar coordinate $r(q)$ of a point q in $\mathcal{U}_p$ is just the length of the unique geodesic from p to q, i.e. the geodesic distance between p and q in the manifold. Correspondingly, the first coordinate curves are the geodesics issuing from p. In the figures below the g-angle between two consecutive geodesics is constant.

The relative spread of geodesics is encoded into the respective Jacobi fields (introduced later in these notes) based on each geodesic. The difference in the present figures is the representation of the second polar coordinate curves, i.e. of the metric geodesic circles with equal distance between two consecutive circles. Obviously it is much easier to construct the polar coordinate curves than it is to construct the normal coordinate curves based at a given point.

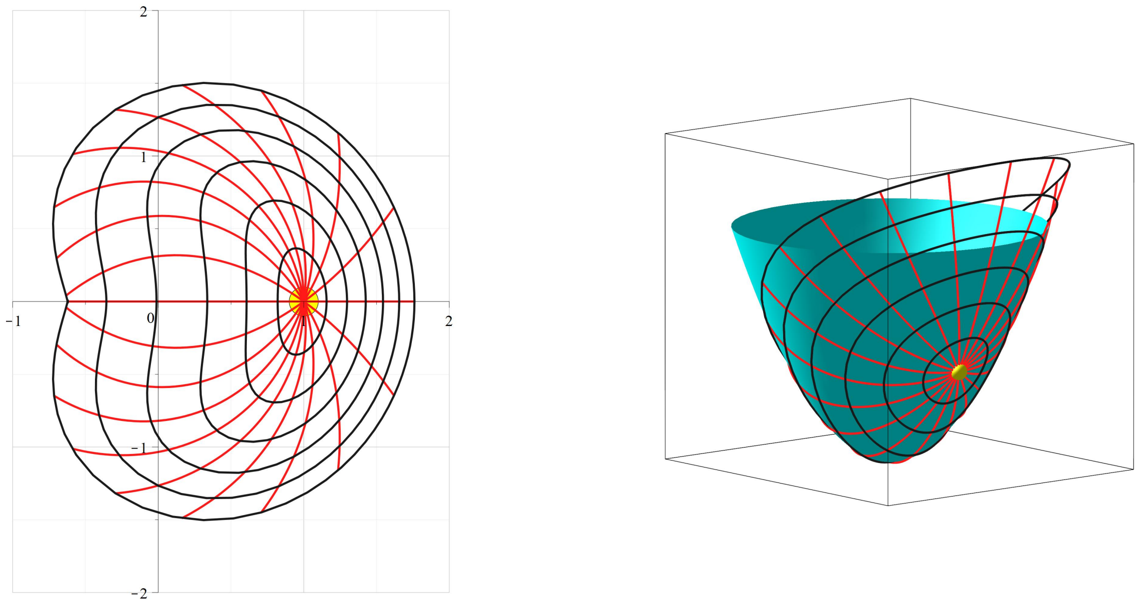


Figure 4.16: Polar coordinates on a paraboloid of revolution. Left: The coordinate curves are shown on a background of Monge patch coordinates obtained as in the defining equation 4.25.

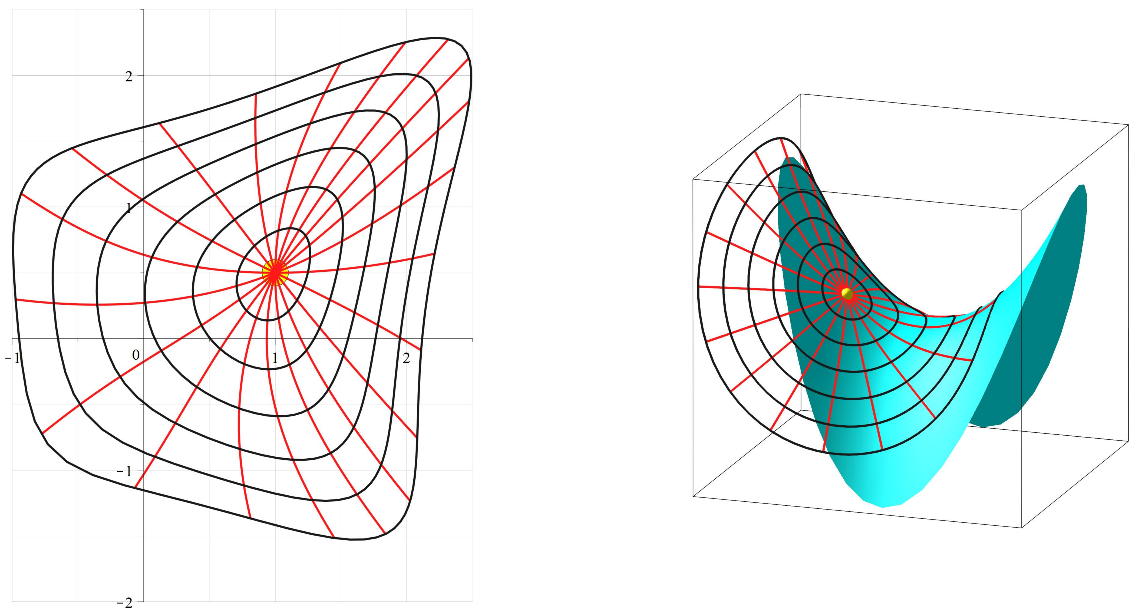


Figure 4.17: Polar coordinates on a saddle surface. Left: The coordinate curves are shown on a background of Monge patch coordinates, cf. equation 4.25.

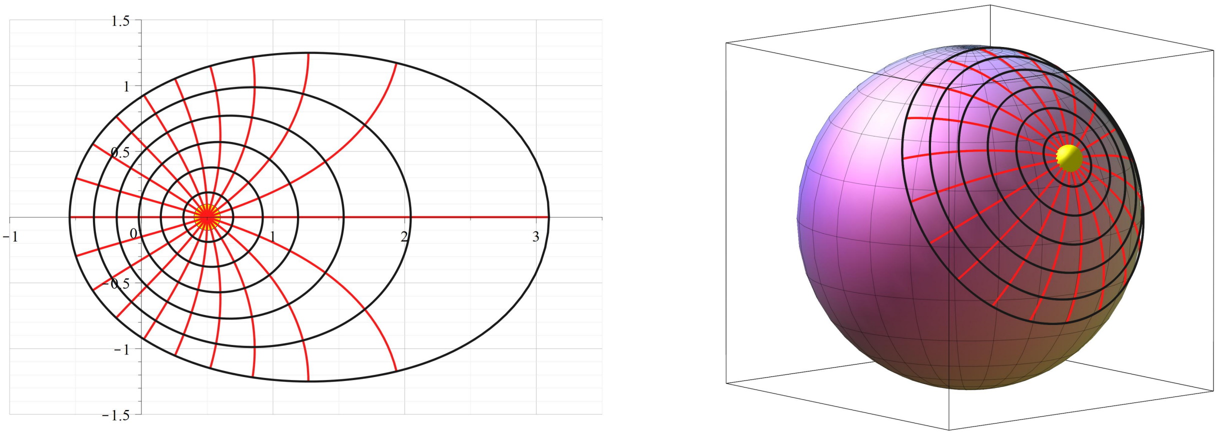


Figure 4.18: Polar coordinates on the sphere. Left: The coordinate curves are shown on a background of Mercator coordinates as in the defining equation 4.25

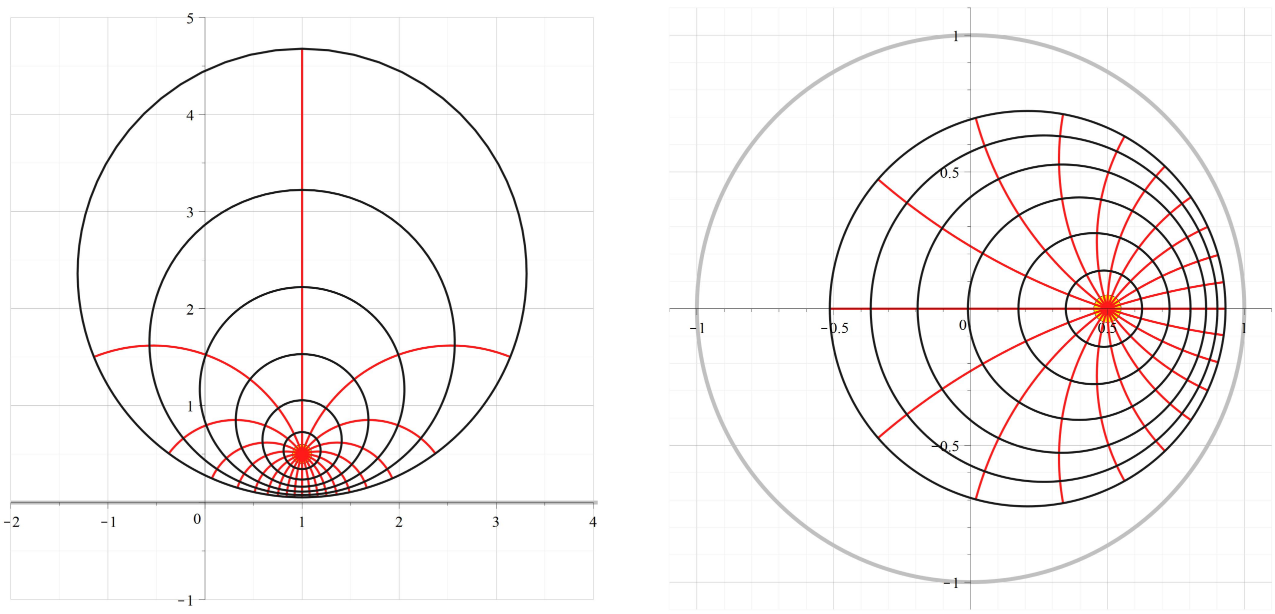


Figure 4.19: Polar coordinates in the two Poincaré models. Left: Half plane model; Right: Disk Model.

4.11 Outlook recap: Statistics on Riemannian manifolds

In chapter 1 we saw how a slightly upwards elongated distribution of points on the standard paraboloid of revolution did not have good statistical representations in the parameter plane, at least when we used (rectilinear or polar) Monge parameters for the representation of the surface. The Log map polar or normal coordinates give much better representations in the tangent plane of the surface at the base point for the Exponential map. There will always be discrepancies in between internal distances when projecting a non-flat background into the plane.

We indicate how a Riemannian center of mass can be well defined for a distribution which is not too 'wide' in the sense that it is supported in a neighbourhood $\mathcal{E}_p$ of some point p so that the exponential map is a diffeomorphism and so that we have access to normal coordinates (and polar coordinates) in the neighborhood.

The idea is the following: Suppose the point set Ω is 'catched' by the Log map Log_{p_1} into a tangent space $T_{p_1}\mathcal{U}$. Then calculate the center of mass C_1 of the image $\mathrm{Log}_{p_1}(\Omega)$ – using the standard Euclidean metric in $T_{p_1}\mathcal{U}$. We denote the corresponding position vector also by C_1 and next consider the normal coordinate system in $\mathcal{U}$ with the new base point $p_2 = \mathrm{Exp}(C_1)$; we then find the Euclidean center of mass C_2 of $\mathrm{Log}_{p_2}(\Omega)$ and continue the process. It is intuitively reasonable to expect that this construction of points $p_1, \cdots, p_n, cdots$ converges to a point p_∞ which then should be called the Riemannian center of mass. This expectation holds true and has been proved and applied in e.g. [11], [15], [25].

The center of mass is just the very first significant statistical information about a given distribution. The classical principal component analysis also suggests the determination of best fitting lines – which in the context of distributions on Riemannian manifolds should be the best fitting geodesics, see e.g. [7]. One useful generalization of normal coordinates in this respect is the concept of Fermi coordinates, see [20]. They are also rectangular and have vanishing Christoffel symbols not just at a point but along a geodesic – for example along the geodesic from the Riemannian center of mass which best describes the elongation of Ω on the paraboloid considered here.

Example 4.44

In the figures below we illustrate the images $\mathrm{Log}_{p_1}(\Omega)$ and $\mathrm{Log}_{p_2}(\Omega)$ for the beginning of the construction of the center of mass of a given point set on the paraboloid – with initial point $p_1 = (0.65, 0.65)$ (with respect to the Monge patch coordinates for the paraboloid).

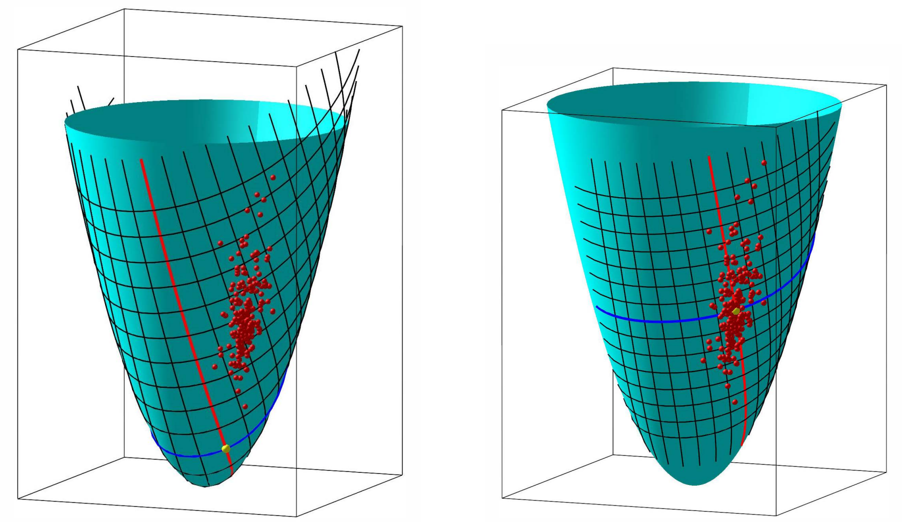


Figure 4.20: A point distribution Ω on the paraboloid and two normal coordinate systems with base points off the center of mass and at the center of mass, respectively. These two base points are given by their Monge patch coordinates $(0.65, 0.65)$ and $(1.3, 1.3)$, respectively.

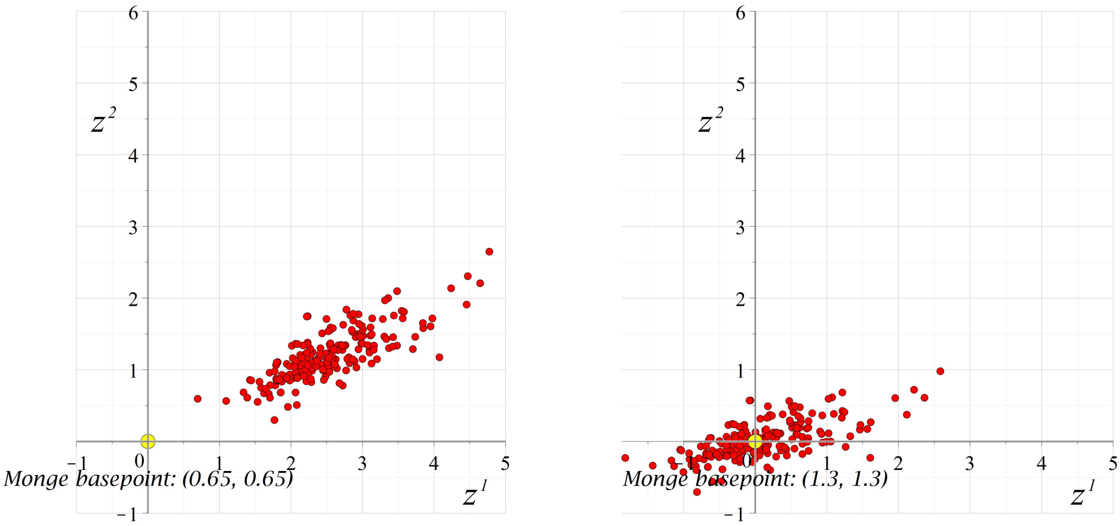


Figure 4.21: The images $\mathrm{Log}_{p_1}(\Omega)$ and $\mathrm{Log}_{p_2}(\Omega)$ in their respective tangent spaces, based at the points $(0.65, 0.65)$ and $(1.3, 1.3)$ corresponding to figure 4.20.

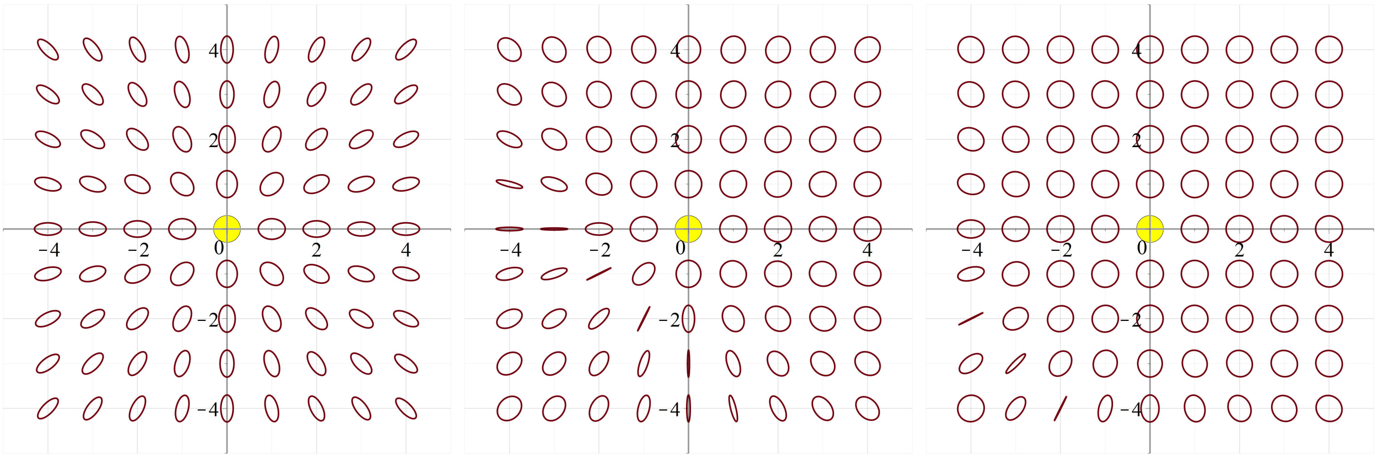


Figure 4.22: Metric fingerprints for the paraboloid in normal coordinates based at points (expressed in Monge patch coordinates) $(0,0)$, $(0.65,0.65)$, and $(1.3,1.3)$, respectively – the two last ones is for direct comparison with figure 4.21 above. From the standard finger print indicatrix field interpretation it is evident, that when all indicatrices are almost identical to the unit circle indicatrix at the base point as in the rightmost figure, then this supports a better and faster determination of the center of mass.

Chapter 5

Helices, circles, and the Frenet-Serret apparatus in 3 dimensions

We have previously discussed various types of curves in Riemannian manifolds $(\mathcal{U}, g, \nabla)$ – such as the time-parametrized tracks of Newtonian particles in a gravitational field and geodesics. And we have discussed briefly the general notion of acceleration of a given time-parametrized curve in the manifold. The key instrument for these concepts is the Levi-Civita connection ∇ and the corresponding covariant derivation together with the ensuing notion of parallel transport along curves.

In this chapter we will introduce a more involved and intricate application of parallel transport of g-orthonormal frame fields $\{E_1, E_2, E_3\}$ in 3-dimensional Riemannian manifolds and generalize what is known classically as the so-called Frenet-Serret apparatus for a given curve in Euclidean 3-space.

5.1 The classical helices in $\mathbb{R}^3$

Definition 5.1 The classical standard helix in standard position in ordinary Euclidean 3-space $(\mathbb{R}^3, g_E)$ with the usual coordinates $\{x^1, x^2, x^3\}$ and the corresponding induced basis vector fields $\{e_1, e_2, e_3\}$ is the following unit speed parametrized curve:

$$\gamma(s) = \left(a \cdot \cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right), a \cdot \sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right), b \cdot \left(\frac{s}{\sqrt{a^2 + b^2}}\right)\right) \quad , \quad s \in I \quad , \qquad (5.1)$$

where $a > 0$ and b are constants and I is any connected open interval in $\mathbb{R}$.

EXERCISE 5.2

Show that the standard helix γ in the above definition 5.1 is unit speed parametrized by s i.e. $\|\gamma'(s)\|_{g_E} = 1$ for all $s \in I$. Show that if $I =]\alpha, \beta[$, with finite (possibly negative) values of $\alpha < \beta$, then the Euclidean length of the curve $\gamma(I)$ is $\beta - \alpha$.

Two examples of helices are displayed in figure 5.1. One (to the right in the figure) has $b > 0$, it is a right handed helix, it has positive torsion and positive helicity. The other (to the left in the figure) has $b < 0$, it is a left handed helix, it has negative torsion and negative chirality. The torsion of the helix is defined via the exercises below:

EXERCISE 5.3

Show that γ has constant geodesic curvature:

$$\kappa = \kappa(s) = \kappa_\gamma(s) = \|\,\mathrm{acc}_\gamma(s)\| = \|\gamma''(s)\| = \frac{a}{a^2 + b^2} \quad , \tag{5.2}$$

where we have applied the purely Euclidean identity $\|\nabla_{\gamma'}\gamma'(s)\| = \|\gamma''(s)\|$.

Note explicitly, that here we may write $\frac{d}{ds}$ instead of $\nabla_{\gamma'}$ and $\frac{D}{ds}$ – since they are all identical operators in Euclidean space – all the Christoffel symbols vanish.

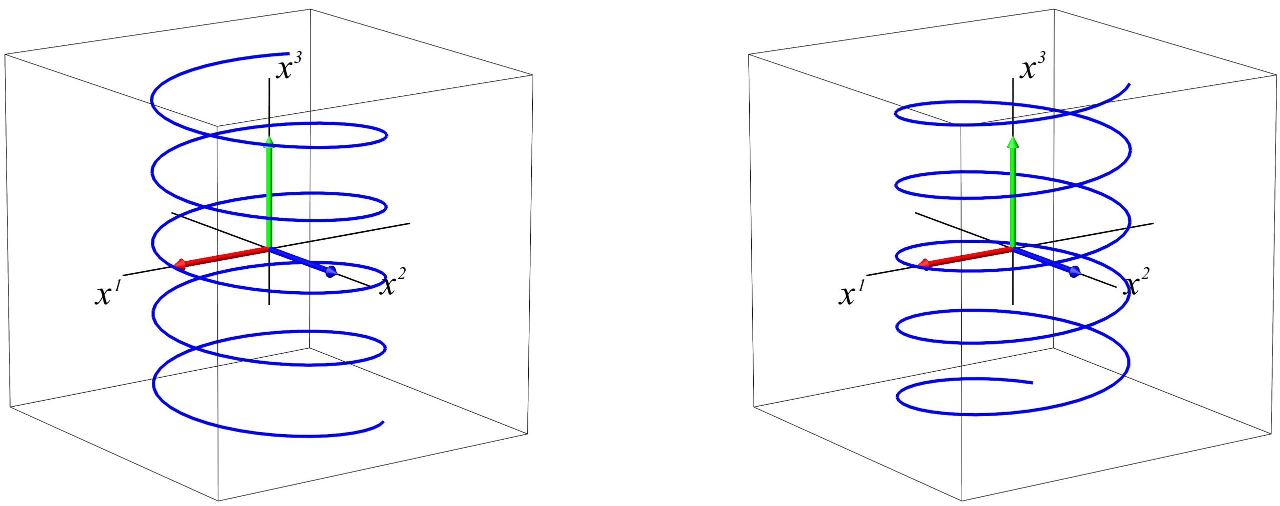


Figure 5.1: Helices; with positive torsion (right) and negative torsion (left).

We now define three unit vector fields along the helix, T, N, and B in $\mathfrak{X}(\gamma)$, as follows:

$$\begin{aligned} T(s) &= \gamma'(s) \\ N(s) &= \left(\frac{1}{\kappa}\right)\cdot\gamma''(s) = \left(\frac{1}{\kappa}\right)\cdot T'(s) \\ B(s) &= T(s)\times N(s) \quad , \end{aligned} \tag{5.3}$$

where we have used the standard Euclidean cross-product, $\times$, in $\mathbb{R}^3$.

EXERCISE 5.4

Find the explicit expressions for $T(s)$, $N(s)$, and $B(s)$ for the helix curve defined in equation (5.1), and show that $\{T(s), N(s), B(s)\}$ is a g_E-orthonormal basis for $T_{\gamma(s)}\mathbb{R}^3$ for all s.

The pairwise orthogonal unit vector fields $\{T(s), N(s), B(s)\}$ along γ constructed in this way is called the Frenet-Serret apparatus for the helix along γ in $\mathbb{R}^3$. We claim that the following identities hold true for the helix:

$$\begin{aligned} T'(s) &= \kappa\cdot N(s) \\ N'(s) &= -\kappa\cdot T(s) + \tau\cdot B(s) \\ B'(s) &= -\tau\cdot N(s) \quad , \end{aligned} \tag{5.4}$$

where κ and τ are unique curvature constants for the helix with values:

$$\begin{aligned} \tau &= \frac{b}{a^2+b^2} \\ \kappa &= \frac{a}{a^2+b^2} \quad . \end{aligned} \tag{5.5}$$

EXERCISE 5.5

Show the identities in (5.4) using the expressions for τ and κ in (5.5) for the given helix curve in (5.1).

The constant τ appearing in (5.5) is the torsion of the helix. It appears in this way as the second curvature constant for the curve.

We note the following obvious shorthand notation for the identities in equation (5.4):

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \cdot \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \quad . \tag{5.6}$$

Each one of the equivalent equations (5.4) and (5.6) is a coupled first order differential equation system which has unique solutions $T(s)$, $N(s)$, and $B(s)$ whenever κ and τ are given constants and when initial conditions are specified for the three vector fields. But once the unit vector field $T(s)$ is found as a vector function solution, then the unit speed curve $\gamma(s)$ also follows uniquely from an integration of $T(s)$ if only one point on the curve, $\gamma(s_0)$, is also given in advance.

Any solution curve to (5.6) is called a helix with curvature κ and torsion τ. Although the (initial standard positioned) helix curve γ in (5.1) is surely a solution to the differential equation system (5.6), it is clearly not the only solution, since other initial conditions $T(s_0)$, $N(s_0)$, $B(s_0)$, $\gamma(s_0)$ give different solutions with the same values of κ and τ that are positioned differently in 3-space.

EXERCISE 5.6

Show that every helix curve solution η to (5.6) with length $L = \beta - \alpha$, curvature $\kappa = a/\sqrt{a^2+b^2}$ and torsion $\tau = b/\sqrt{a^2+b^2}$ is ambient isometric to the standard curve γ defined in (5.1) in the following sense: There is an orientation preserving rotation R and a translation Λ in $\mathbb{R}^3$ so that $\eta = R(\gamma) + \Lambda$.

5.2 Riemannian helices and circles

Motivated by the classical helix constructions above we now define arc length parametrized Riemannian helices as follows:

Definition 5.7 Let $\kappa > 0$ and τ denote two constants and let γ denote a smooth regular curve parametrized by arc length. Then γ is called a Riemannian helix in $(\mathcal{U}^3, g, \nabla)$ with (constant, positive) curvature κ and (constant) torsion τ if $T = \gamma'$ is member of a positively oriented g-orthonormal frame field $\{T(s), N(s), B(s)\}$ that satisfies the following system of vector differential equations:

$$\begin{aligned}
\nabla_{\gamma'} T(s) &= \kappa \cdot N(s) \\
\nabla_{\gamma'} N(s) &= -\kappa \cdot T(s) + \tau \cdot B(s) \\
\nabla_{\gamma'} B(s) &= -\tau \cdot N(s) \quad ,
\end{aligned} \tag{5.7}$$

Obviously, we then call $\{T(s), N(s), B(s)\}$ the Riemannian Frenet-Serret frame for the Riemannian helix γ.

In particular, helices without torsion give rise to the following:

Definition 5.8 An arc length parametrized Riemannian circle in $(\mathcal{U}^3, g, \nabla)$ is an arc length parametrized Riemannian helix with torsion $\tau = 0$ (and constant curvature κ).

Note that a Riemannian circle is not necessarily a geodesic metric circle as we have defined those previously – in chapter 4 – via the Exponential map. Moreover, as we shall see by example below, a Riemannian circle is in general not even a closed curve.

Arc length parametrized Riemannian circles must then necessarily satisfy the following properties:

Observation 5.9 Let $\kappa > 0$ and let γ denote a smooth regular curve parametrized by arc length. Then γ is a Riemannian circle in $(\mathcal{U}^3, g, \nabla)$ with constant positive curvature κ if $T = \gamma'$ together with the induced orthogonal unit vector field N along γ satisfies the equations:

$$\begin{aligned} \nabla_{\gamma'} T(s) &= \kappa \cdot N(s) \\ \nabla_{\gamma'} N(s) &= -\kappa \cdot T(s) \quad . \end{aligned} \tag{5.8}$$

Note that we may now – and will – generalize the notion of circles even further and consider (5.8) as the defining properties of a Riemannian circle in any Riemannian manifold $(\mathcal{U}^n, g, \nabla)$ of any dimension $n \geq 2$.

EXERCISE 5.10

Show that a Euclidean helix in $(\mathbb{R}^3, g_E)$ with vanishing torsion and curvature $\kappa > 0$ is an ordinary Euclidean circle with radius $1/\kappa$.

The system (5.7) is translated directly into (5.4) if we apply an auxiliary *g-orthonormal parallel frame field* $\{E_1, E_2, E_3\}$ along γ in which we express the vector functions T, N, and B. Indeed, as we know from chapter 3, if V is a vector field along γ with coordinate functions $v^i(s)$ with respect to the parallel frame $\{E_1, E_2, E_3\}$, then

$$\nabla_{\gamma'} V(s) = \sum_i (v^i)'(s) \cdot E_i(\gamma(s)) \quad , \tag{5.9}$$

and thence, if we denote the coordinate columns for T, N, and B with respect to $\{E_1, E_2, E_3\}$ with the same capital letters, we get precisely the system (5.4).

The main difference, however, is that we do not in advance know a parallel frame along γ since we do not, of course, know the curve in advance. Therefore we have to solve the parallel transport problem for each E_i along γ simultaneously with the Frenet-Serret differential equations.

To be a bit more concrete, we let $E_i^k(s)$ denote the k'th coordinate function of E_i with respect to the standard fixed basis $\{e_1, e_2, e_3\}$. Then

$$\gamma'(s) = T^i(s) \cdot E_i^k(s) \cdot e_k \quad , \tag{5.10}$$

and the condition for E_i to be parallel along γ now reads for each $k = 1, 2, 3$, i.e. a total of 9 equations:

$$\frac{d}{dt} E_i^k(s) + \sum_\ell \sum_m \sum_j V_i^m(s) \cdot T^j(s) \cdot E_j^\ell(s) \cdot \Gamma_{\ell m}^k(\gamma(s)) = 0 \quad . \tag{5.11}$$

In this way we end up with a system of (nonlinear) coupled differential equations for the parallel frame fields involving both the unknown vector field $T(s)$ and the curve γ itself (via the Christoffel symbols). This system is then coupled with the Frenet-Serret differential equations for T, N, and B in the coordinate form of (5.4). In total we get a coupled system of 21 equations, i.e 9 for the coordinates of E_i and 9 for the coordinates of T, N and B respectively, and finally 3 for the coordinates of γ. They are then solved from a corresponding set of 21 initial data, including the initial starting point of γ, etc. A couple of examples are displayed in figures 5.2 and 5.3.

See also the references [23] and [6], where the latter may indicate an interesting application of Riemannian helices to the study of the van Allen belts around the Earth. These belts consists of spiralling charged particles that are trapped in the Earth's magnetic field.

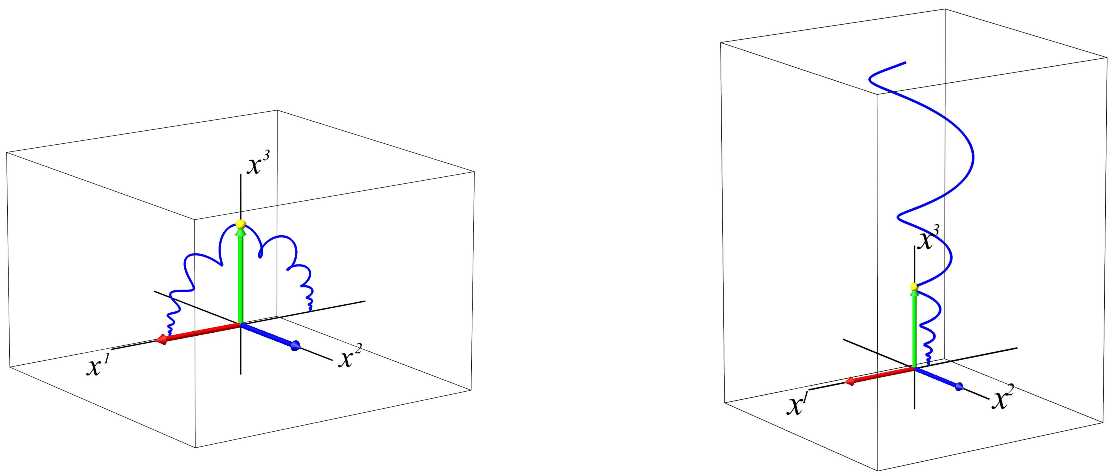


Figure 5.2: Two helices in the Poincaré half space with different initial conditions at $(0,0,1)$.

EXERCISE 5.11

We let $(\mathcal{U}^3, g, \nabla))$ denote the Riemannian manifold with metric matrix defined in the half space $\mathcal{U}^3$ of $\mathbb{R}^3$ where $x^1 > 0$ as follows:

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \tag{5.12}$$

Let γ_0 denote the following simple curve in $\mathcal{U}^3$ for any fixed choice of $R > 0$:

$$\gamma_0(s) = (R,\, s/R,\, 0) \quad , \quad s \in \mathbb{R} \quad . \tag{5.13}$$

a) Show that γ_0 is arc length parametrized by s in $(\mathcal{U}^3, g, \nabla))$.

b) Show that γ_0 is a Riemannian circle in $(\mathcal{U}^3, g, \nabla))$.

c) Determine the constant curvature κ_0 of γ_0.

More generally, let $k \in \mathbb{R}$ be a constant and let η_k denote the following curve segment in $\mathcal{U}^3$:

$$\eta_k(t) = (R\,,\, t\,,\, k \cdot t) \quad , \quad t \in]-\pi, \pi[\quad . \tag{5.14}$$

d) Find an arc length re-parametrization γ_k of η_k.

e) Show that γ_k is a Riemannian helix in $(\mathcal{U}^3, g, \nabla))$ for every $k \in \mathbb{R}$.

f) Determine the constant curvature κ_k and the constant torsion τ_k of γ_k .

g) Find an isometry ψ of an open set of $(\mathcal{U}^3, g, \nabla)$ into Euclidean space $(\mathbb{R}^3, g_E)$ and show, that $\psi(\gamma_k)$ is a segment of a standard helix in standard position in $(\mathbb{R}^3, g_E)$. Hint: You may want to have a look at the 2D isometry in example 3.3 in chapter 3.

Note that the last question in the above exercise 5.11 is but a strong token of the fact, that we have only used isometry-invariant concepts to define the Riemannian helices and circles, so it is only natural that they are themselves invariant under ambient isometries.

5.2.1 The Feynman wall

Suppose we have already constructed an interesting 2-dimensional Riemannian manifold $(\mathcal{U}^2, g, \nabla_g)$, with a given metric g with matrix function G. Then we can easily extend the manifold to a 3-dimensional Riemannian manifold $(\mathcal{U}^3, h, \nabla_h)$ by a simple extension of the old metric to the following:

$$H = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \tag{5.15}$$

EXERCISE 5.12

The above metric h induces (in principle, i.e. without using symmetry) 27 Christoffel symbol functions for ∇_h in $(\mathcal{U}^3, h, \nabla_h)$. They are clearly related to – and can be expressed in terms of – the (in principle) 8 Christoffel symbol functions for ∇_g in $(\mathcal{U}^2, g, \nabla_g)$. Find these expressions/relations, either in general or just for the simpler, more concrete, metric h given in equation (5.16) below.

One interesting and non-trivial 2-dimensional Riemannian manifold is Feynman's example in section chapter 4. Choosing the metric g from there we get the following extended metric matrix function for a metric h in $\mathcal{U} = \mathbb{R}^3$:

$$H = \begin{bmatrix} \mu(x^1) & 0 & 0 \\ 0 & \mu(x^1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \tag{5.16}$$

where μ is governed by two positive constants α and β as follows:

$$\mu(x^1) = \mu_{\alpha,\beta}(x^1) = 1 + \frac{\alpha}{2} \cdot \left(1 + \tanh\left(\beta \cdot x^1\right)\right) \quad . \tag{5.17}$$

In chapter 4 we noted how geodesics were bent/broken (according to Snell's law) when penetrating through the soft/hard (beach/water-) line $x^1 = 0$. Similarly we will refer to the *plane* $x^1 = 0$ in $\mathbb{R}^3$ as the 'soft Feynman wall'. The softness of the wall is clearly governed by the choice of constants α and β for the function $\mu_{\alpha,\beta}(x^1)$.

For the displays in figure 5.3 we have chosen values $\alpha = 4$ and $\beta = 7$.

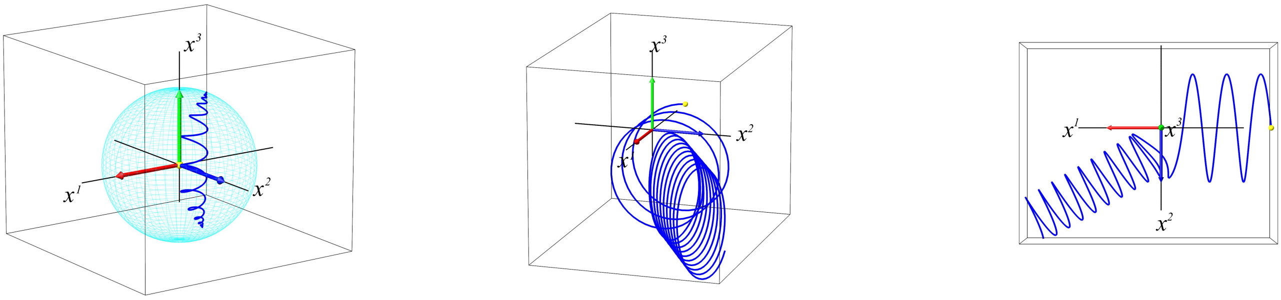


Figure 5.3: Left: A helix in a Poincaré ball. Middle and right: A standard helix approaches from $x^1 < 0$ a soft 'Feynman wall' at $x^1 = 0$ orthogonally and is then during penetration through the wall bent into an 'elliptic helix' with another angle to the wall; In the top view in the right hand display, the classical helix approaches the wall from the right $x^1 < 0$; the bending angle is clearly dependent on the direction of first impact with the wall.

EXERCISE 5.13

Explain the fact, which is clearly visible in figure 5.3, that the approaching helix (before the wall) has a standard circular (cross sectional) appearance whereas the transmitted helix (after the wall) has an elliptic (cross sectional) appearance. Hint: The metric matrix is almost constant away from the wall: $\mu(x^1) \approx 1$ for $x^1 << 0$ and $\mu(x^1) \approx 5$ for $x^1 >> 0$.

5.3 The Frenet-Serret apparatus for general curves

Suppose now that γ is a smooth arc length parametrized curve – not necessarily a helix – in a 3-dimensional Riemannian manifold $(\mathcal{U}, g, \nabla)$. And suppose that the geodesic curvature $\kappa_\gamma^g(s)$ is positive for all s. Then γ still admits a unique Frenet-Serret frame $\{T(s), N(s), B(s)\}$ which satisfies the Frenet-Serret system equations as in (5.8), but now obviously with *varying curvature function* $\kappa(s) = \kappa^g_\gamma(s)$ and *varying torsion function* $\tau(s)$:

Proposition 5.14 Let γ be a unit speed parametrized curve $(\mathcal{U}, g, \nabla)$ with positive geodesic curvature $\kappa^g_\gamma(s) = \kappa(s)$. Then, along γ, there is a positively oriented g-orthonormal frame field $\{T(s), N(s), B(s)\}$ with $T(s) = \gamma'(s)$ and a smooth torsion function $\tau(s)$, so that:

$$\begin{aligned}
\nabla_{\gamma'} T(s) &= \kappa(s) \cdot N(s) \\
\nabla_{\gamma'} N(s) &= -\kappa(s) \cdot T(s) + \tau(s) \cdot B(s) \\
\nabla_{\gamma'} B(s) &= -\tau(s) \cdot N(s) \quad ,
\end{aligned} \tag{5.18}$$

Definition 5.15 The frame field $\{T(s), N(s), B(s)\}$ together with the functions $\kappa(s)$ and $\tau(s)$ is called the Frenet-Serret apparatus for the curve γ.

Conversely, the fundamental theorem of curves in (3D) Riemannian geometry states:

Theorem 5.16 Suppose $\kappa(s)$ is a given positive smooth function of a parameter $s \in [0, \ell]$ and that $\tau(s)$ is a given smooth function of s. Let $\{T_0, N_0, B_0\}$ denote a positively oriented basis of $T_p\mathcal{U}^3$ in a 3-dimensional Riemannian manifold $(\mathcal{U}, g, \nabla)$. Then there exists (for sufficiently small value of ℓ) a unique unit speed parametrized curve $\gamma(s)$ with $\gamma(0) = p$ and with Frenet-Serret apparatus $\{\{T(s), N(s), B(s)\}, \kappa(s), \tau(s)\}$ so that $\{T(0), N(0), B(0)\} = \{T_0, N_0, B_0\}$.

The proof of this theorem clearly again hinges on the existence and uniqueness of solutions to (typically nonlinear) ordinary differential equation systems (21 equations and 21 initial conditions) – a topic which is still beyond the primary scopes of these notes.

Except from the existence and uniqueness statement in theorem 5.16 there is one more obvious question to think about: The g-orthonormality of the vector fields $\{T(s), N(s), B(s)\}$ generated along the solution curve γ is clearly satisfied by assumption at $s = 0$, since it is part of the specified initial conditions. But why and how does (5.18) guarantee by itself, that they *stay orthonormal* for all s? Hint: The structure matrix on the right hand side of (5.18) is skew symmetric. In a parallel frame field along γ this means that it is *closely related to* the s-derivative of an s-dependent rotation matrix. And rotation matrices keep orthonormality.

Example 5.17

Let $\kappa(s) = s$, $\tau(s) = 0$ and $\{T_0, N_0, B_0\} = \{e_1, e_2, e_3\}$ at $p = (0,0,0)$ in Euclidean 3-space $(\mathbb{R}^3, g_E)$. The curve with these Frenet-Serret (initial) data (and given initial point) is displayed in figure 5.4. This is the famous Euler spiral, which is much used in the design and construction of roads and other smooth pathways, see that Wikipedia posting. With the methods introduced above we can study and use similar curves and their properties in the much wider context of Riemannian manifolds, see figure 5.5.

EXERCISE 5.18

Show that the planar Euler spiral in figure 5.4 with initial data as given in example 5.17 can be expressed analytically by the following explicit parametrization:

$$\gamma_c(s) = \left(\int_0^s \cos\left(u^2/2\right) du \, , \, \int_0^s \sin\left(u^2/2\right) du \, , \, 0 \right) \quad , \quad s \in \mathbb{R} \quad . \tag{5.19}$$

EXERCISE 5.19

Explain how *negative curvature* can be allowed and well defined and used for planar curves – like for the Euler spiral in example 5.17, where we have $\kappa(s) = s$ for all $s \in \mathbb{R}$, i.e. including the negative values of s and thence of $\kappa(s)$. Hint: Only in the plane, or more generally in an oriented 2-dimensional Riemannian manifold, is it possible to define anti-clockwise (positive) turning of the tangent vector field T to a given curve corresponding to positive curvature, and clockwise (negative) turning of the tangent vector field corresponding to negative curvature of the curve.

As a converse to the construction of curves via given curvature and torsion functions we finally present the following constructive method for finding the Frenet-Serret apparatus for a given curve γ in a 3-dimensional Riemannian manifold *without assuming, that the curve is arc length parametrized.*

For this we shall use an invariant g-version of the cross product operation:

Definition 5.20 Let V and W denote two linearly independent vectors in a given tangent space $T_p\mathcal{U}^3$ for a 3D Riemannian manifold $(\mathcal{U}, g, \nabla)$. The cross product $V \times_g W$ of V and W is then defined as the unique vector in $T_p\mathcal{U}$ which satisfies the following (natural) conditions:

$$\begin{cases} g(V \times_g W \, , \, V) = 0 \\ g(V \times_g W \, , \, W) = 0 \\ \text{The triple } \{V, W, V \times_g W\} \text{ is a positively oriented basis for } T_p\mathcal{U} \\ \|V \times_g W\|_g = \|V\|_g \cdot \|W\|_g \cdot \sin\left(\sphericalangle_g(V, W)\right) \quad , \end{cases} \tag{5.20}$$

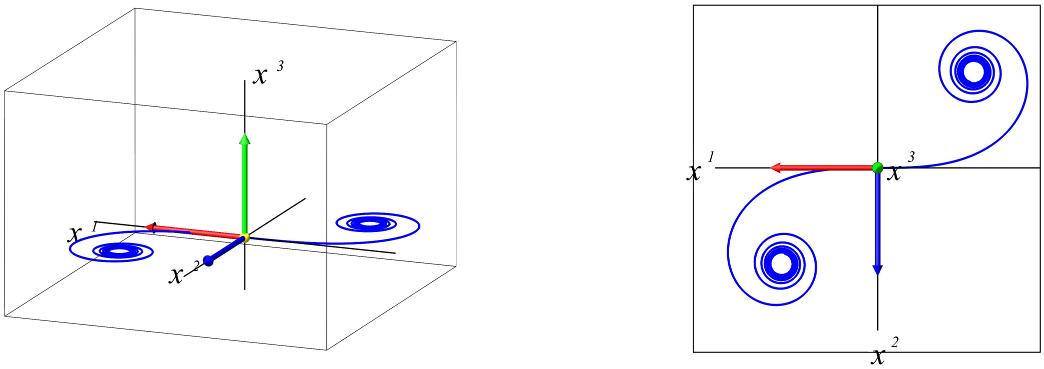


Figure 5.4: The planar Euler spiral in Euclidean space.

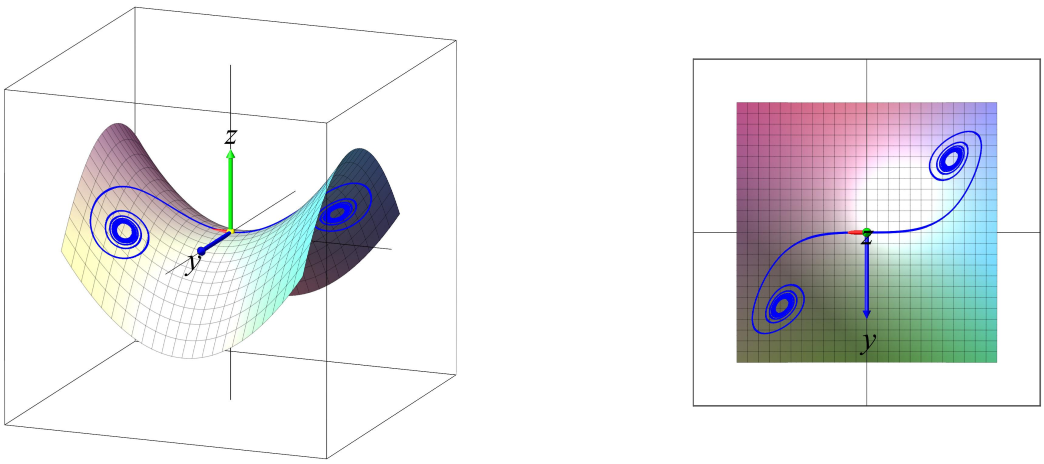


Figure 5.5: An Euler spiral with $\kappa(s) = s$ on a saddle surface.

where the angle $\sphericalangle_g(V,W)$ denotes the unique angle θ in $[0,\pi]$ with

$$\cos(\theta) = \frac{g(V,W)}{\|V\|_g \cdot \|W\|_g} \quad . \tag{5.21}$$

EXERCISE 5.21

Show that the conditions in definition 5.20 determine a unique cross product $\times_g$ in $T_p\mathcal{U}$. Hint: Introduce a g-orthonormal basis $\{E_1, E_2, E_3\}$ and apply the coordinates of V and W with respect to that basis.

Theorem 5.22 Let $\gamma(t)$, $t \in I$, be a smooth regular curve in $(\mathcal{U}^3, g, \nabla)$. In particular we need regularity, i.e. $\gamma'(t) \neq 0$. The t-parametrized Frenet-Serret apparatus for γ can then be obtained as follows:

Firstly, we define the speed of the curve parametrization:

$$v(t) = \|\gamma'(t)\|_g > 0 \quad . \tag{5.22}$$

Secondly, remember that we have previously (in chapter 3) defined the acceleration of the t-parametrized curve as the following vector field along γ:

$$\mathrm{acc}_\gamma(t) = \frac{D}{dt}\gamma'(t) = \nabla_{\gamma'}\gamma'(t) \tag{5.23}$$

Thirdly, we will also apply the double covariant derivative of $\gamma'(t)$, i.e.

$$\frac{D}{dt}\mathrm{acc}_\gamma(t) = \nabla_{\gamma'}\left(\nabla_{\gamma'}\gamma'(t)\right) \quad . \tag{5.24}$$

Fourthly, we apply the following shorthand notation for the 'space product':

$$\left[V\,,\,W\,,\,U\right]_g = g\left(V \times_g W\,,\,U\right) \quad . \tag{5.25}$$

With these ingredients we then generate the Frenet-Serret apparatus in the following way (note the appearance of the condition $\kappa(t) > 0$):

$$\begin{aligned}
T(t) &= \frac{1}{v(t)} \cdot \gamma'(t) \\
\kappa(t) &= \frac{\|T(t) \times_g \mathrm{acc}_\gamma(t)\|_g}{v^2(t)} \\
\tau(t) &= \frac{\left[T(t)\,,\, \mathrm{acc}_\gamma(t)\,,\, \frac{D}{dt}\mathrm{acc}_\gamma(t)\right]_g}{v(t)\cdot\|T(t) \times_g \mathrm{acc}_\gamma(t)\|_g^2} \quad \text{for } \kappa(t) > 0 \\
B(t) &= \frac{T(t) \times_g \mathrm{acc}_\gamma(t)}{\|T(t) \times_g \mathrm{acc}_\gamma(t)\|_g} \quad \text{for } \kappa(t) > 0 \\
N(t) &= B(t) \times_g T(t) \quad \text{for } \kappa(t) > 0 \quad .
\end{aligned} \tag{5.26}$$

Note that $T(t)$ and $B(t)$ is – most effectively – calculated *before* $N(t)$.

The Frenet-Serret vector functions $T(t)$, $N(t)$, and $B(t)$ and the curvature and torsion functions $\kappa(t)$ and $\tau(t)$ then satisfy the following system of equations, which is precisely the general t-version of proposition 5.14:

$$\begin{aligned}
\nabla_{\gamma'} T(t) &= \quad v(t)\kappa(t)N(t) \\
\nabla_{\gamma'} N(t) &= -v(t)\kappa(t)T(t) + v(t)\tau(t)B(t) \\
\nabla_{\gamma'} B(t) &= \quad -v(t)\tau(t)N(t) \quad .
\end{aligned} \tag{5.27}$$

In this way we have then constructed the general t-version of the Frenet-Serret apparatus for any given smooth regular curve parametrized by t and with $\kappa(t) > 0$ for all t, i.e. $\{\{T(t), N(t), B(t)\}\,,\ \kappa(t), \tau(t)\}$.

The statements in this theorem 5.22 are not surprising – in view of our previous discussion of helices. Moreover, the proof is also fairly straightforward – in particular if everything is expressed in coordinates with respect to a parallel frame field along γ.

We 'illustrate' the methods involved in the general construction of the Frenet-Serret apparatus via the following exercises:

EXERCISE 5.23

We consider the t-parametrized helix (without reparametrization) in Euclidean 3-space:

$$\gamma(t) = (a\cdot\cos(t)\,,\ a\cdot\sin(t)\,,\ b\cdot t) \quad , \quad t \in \mathbb{R} \quad . \tag{5.28}$$

Apply theorem 5.22 directly and re-find the Frenet-Serret apparatus for γ as functions of the given parameter t.

EXERCISE 5.24

Consider the t-parametrized curve in Euclidean 3-space:

$$\gamma(t) = (t\,,\,t^2\,,\,t^3) \quad , \quad t \in \mathbb{R} \quad . \tag{5.29}$$

Apply theorem 5.22 and find the Frenet-Serret apparatus for γ at the point $\gamma(0)$, i.e. for $t = 0$. Note that even for this simple curve it is not a simple matter to first find an arc length parametrization of the curve.

EXERCISE 5.25

We let $(\mathcal{U}^3, g, \nabla)$ denote the half-space in $\mathbb{R}^3$ with $x^1 > 0$ and with the previously encountered metric – from exercise 5.11:

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \tag{5.30}$$

Suppose now that a smooth regular t-parametrized curve is given in $\mathcal{U}^3$ by the following simple parametrization:

$$\mu(t) = (1\,,\,\cos(t)\,,\,\sin(t)) \quad , \quad t \in \mathbb{R} \quad . \tag{5.31}$$

Apply theorem 5.22 and find the curvature and torsion for μ as functions of the given parameter t.

Chapter 6

Curvature

The curvature of a Riemannian Manifold $M = (\mathcal{U}, g, \nabla)$ at a point $p \in \mathcal{U}$ is a measure of the local deformation of the tangent space $T_p\mathcal{U}$ that is performed by the Exponential map Exp_p when it maps a metric ball $B_\rho(p)$ into $\mathcal{U}$.

Remember that Exp_p maps the straight radial lines (i.e. the straight lines in $T_p\mathcal{U}$ through the origin in $T_p\mathcal{U}$) into radial geodesic curves in $\mathcal{U}$ issuing from p. In figures 6.1 and 6.2 we display two geodesic variations H in two very different 2-dimensional Riemannian manifolds. All the geodesics in both cases have the same length, ρ, so they define geodesic circles of that same radius ρ centered at the respective base points. It is visibly clear that the corresponding orthogonal variation vector fields V (in light blue) – based on the green geodesics – have different length functions $\|V(s)\|_g$, $s \in [0, \rho]$. The g-length of the variation vector field along the base geodesic is a measure of how fast the geodesics spread apart from each other, respectively how fast they re-approach each other.

The purpose of this chapter is to show how that behaviour of nearby geodesics in a geodesic variation is determined by the curvature tensor, to be defined below: If the curvature is large and positive then the geodesics in a geodesic variation issuing from p will tend to converge back to the base geodesic of the variation; if the curvature is very negative, then the geodesics will tend to diverge away from the base geodesic of the variation.

6.1 The curvature operator

Since the variation vector field V of a geodesic variation H is already a first order derivative, the development of V along the base geodesic – i.e. its convergence or divergence – must be determined by an operator which contains some combination of second order derivatives of H that are invariant under isometries. The curvature operator is precisely designed for this purpose:

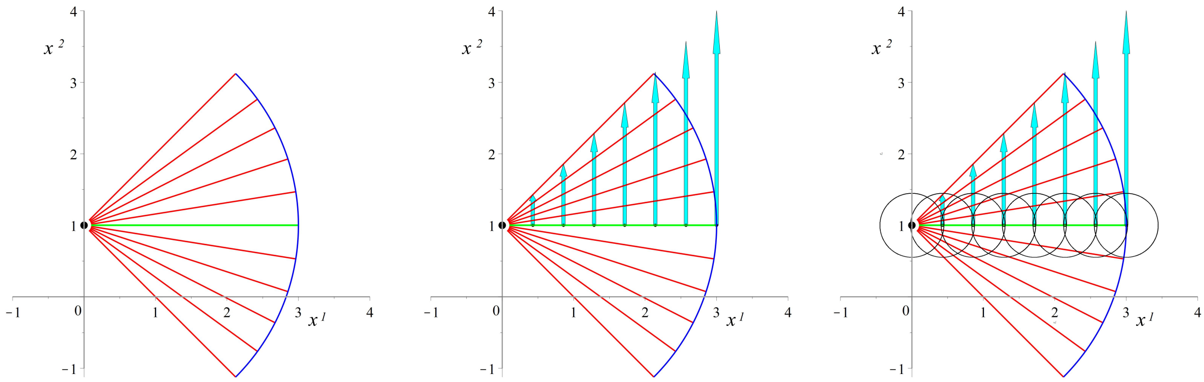


Figure 6.1: A geodesic variation in the Euclidean plane. The variation vector field is indicated in light blue along the green base geodesic of the spray. The simple (scaled) indicatrix field of the metric tensor is also indicated along the base geodesic.

Definition 6.1 Let $M^n = (\mathcal{U}^n, g, \nabla)$ denote a Riemannian manifold. Let X and Y denote smooth vector fields in $\mathfrak{X}(\mathcal{U})$. Then the curvature operator $R(X,Y)$ is a smooth mapping which acts on vector fields $Z \in \mathfrak{X}(\mathcal{U})$ and produces a fourth vector field in $\mathfrak{X}(\mathcal{U})$ as follows:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \text{for all } Z \in \mathfrak{X}(\mathcal{U}) \quad . \tag{6.1}$$

Since, as claimed, $R(X,Y)Z$ is a vector field in $\mathcal{U}$, it has coordinates with respect to the canonical basis vector fields $e_1, \cdots, e_n$. We will see how these coordinates unfold in proposition 6.3 below. For this to work we need to know first, that the curvature operator is tensorial in the following two senses:

Proposition 6.2 The curvature operator R is bilinear over $\mathfrak{F}(\mathcal{U})$ in its first two arguments, X and Y:

$$\begin{aligned} R(f \cdot X_1 + h \cdot X_2\,,\, Y)Z &= f \cdot R(X_1\,,\, Y)Z + h \cdot R(X_2\,,\, Y)Z \\ R(X\,,\, f \cdot Y_1 + h \cdot Y_2)Z &= f \cdot R(X\,,\, Y_1)Z + h \cdot R(X\,,\, Y_2)Z \end{aligned} \tag{6.2}$$

for all f and h in $\mathfrak{F}(\mathcal{U})$ and all X_1, X_2, Y_1, and Y_2 in $\mathfrak{X}(\mathcal{U})$.

And R is linear over $\mathfrak{F}(\mathcal{U})$ in its third argument, Z:

$$\begin{aligned} R(X,Y)(Z+W) &= R(X,Y)Z + R(X,Y)W \\ R(X,Y)(f \cdot Z) &= f \cdot R(X,Y)Z \end{aligned} \tag{6.3}$$

for all f in $\mathfrak{F}(\mathcal{U})$ and all X, Y, Z and W in $\mathfrak{X}(\mathcal{U})$.

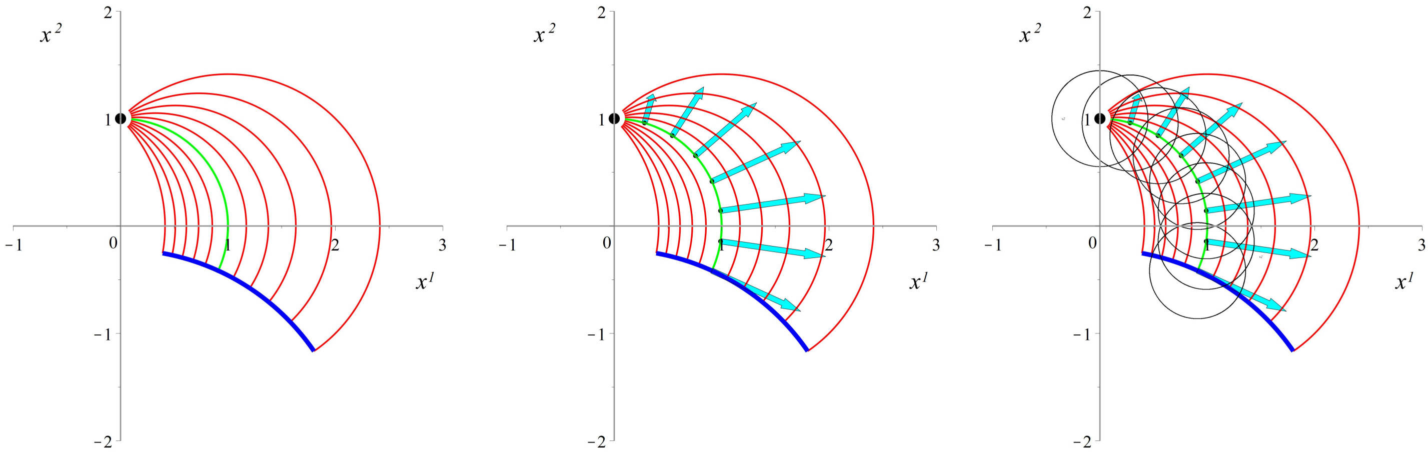


Figure 6.2: A geodesic variation in a Local Riemannian Manifold with metric given as in exercise 6.10 below. The variation vector field is indicated in light blue along the green base geodesic of the spray. The simple (scaled) indicatrix field of the metric tensor is also indicated along the base geodesic. See also figure 6.3.

Proof. We only prove the last identity, $R(X,Y)(f \cdot Z) = f \cdot R(X,Y)Z$. For this we calculate:

$$\begin{aligned} \nabla_X \nabla_Y (f \cdot Z) &= \nabla_X (f \cdot \nabla_Y Z + Y(f) \cdot Z) \\ &= f \cdot \nabla_X \nabla_Y Z + (X(f)) \cdot \nabla_Y Z + (X(f)) \cdot \nabla_Y Z + (X(Y(f))) \cdot Z \quad , \end{aligned} \tag{6.4}$$

so that

$$\begin{aligned} &\nabla_X \nabla_Y (f \cdot Z) - \nabla_Y \nabla_X (f \cdot Z) = \\ &\qquad f \cdot (\nabla_X \nabla_Y - \nabla_Y \nabla_X) Z + ((XY - YX)(f)) \cdot Z \end{aligned} \tag{6.5}$$

and thence

$$\begin{aligned} R(X,Y)(f \cdot Z) &= f \cdot \nabla_X \nabla_Y Z - f \cdot \nabla_Y \nabla_X Z + ([X,Y]f) \cdot Z - f \cdot \nabla_{[X,Y]} Z - ([X,Y]f) \cdot Z \\ &= f \cdot R(X,Y)Z \quad . \end{aligned} \tag{6.6}$$

□

Using the linearity of the curvature operator, we can now express the vector field value of $R(X,Y)Z$ in coordinates:

Proposition 6.3 With respect to the canonical basis e_i we denote the coordinate functions for X, Y, and Z as follows:

$$X = \sum_i u^i \cdot e_i \quad , \quad Y = \sum_j v^j \cdot e_j \quad , \quad Z = \sum_k w^k \cdot e_k \quad . \tag{6.7}$$

Then we have directly from the linearity of the curvature operator established in proposition 6.2 that there exist unique coefficient functions R^m_{ijk}, so that

$$R(X,Y)Z = \sum_{ijkm} R^m_{ijk} \cdot u^i \cdot v^j \cdot w^k \cdot e_m \tag{6.8}$$

The coefficient functions $R^m_{ijk} = R^m_{ijk}(x^1, \cdots, x^n)$ are determined from the Christoffel symbol functions as follows:

$$R^m_{ijk} = \frac{\partial}{\partial x^i}\Gamma^m_{jk} - \frac{\partial}{\partial x^j}\Gamma^m_{ik} + \sum_s \Gamma^s_{jk} \cdot \Gamma^m_{is} - \sum_s \Gamma^s_{ik} \cdot \Gamma^m_{js} \quad , \tag{6.9}$$

Proof. The coefficient functions can be generated by calculating $R(e_i, e_j)e_k$:

$$\begin{aligned}
R(e_i,e_j)e_k &= \sum_{ijkm} R^m_{ijk} \cdot e_m \\
&= \nabla_{e_i}\nabla_{e_j}e_k - \nabla_{\mathrm{e}_j}\nabla_{e_i}e_k - \nabla_{[e_i,e_j]}e_k \\
&= \nabla_{e_i}\nabla_{e_j}e_k - \nabla_{\mathrm{e}_j}\nabla_{e_i}e_k \\
&= \nabla_{e_i}\left(\Gamma^m_{jk} \cdot e_m\right) - \nabla_{\mathrm{e}_j}\left(\Gamma^m_{ik} \cdot e_m\right) \\
&= e_i\left(\Gamma^m_{jk}\right) \cdot e_m - e_j\left(\Gamma^m_{ik}\right) \cdot e_m + \sum_s \Gamma^s_{jk} \cdot \Gamma^m_{is} \cdot e_m - \sum_s \Gamma^s_{ik} \cdot \Gamma^m_{js} \cdot e_m \\
&= \left(e_i\left(\Gamma^m_{jk}\right) - e_j\left(\Gamma^m_{ik}\right) + \sum_s \Gamma^s_{jk} \cdot \Gamma^m_{is} - \sum_s \Gamma^s_{ik} \cdot \Gamma^m_{js}\right) \cdot e_m \\
&= \left(\frac{\partial}{\partial x^i}\left(\Gamma^m_{jk}\right) - \frac{\partial}{\partial x^j}\left(\Gamma^m_{ik}\right) + \sum_s \Gamma^s_{jk} \cdot \Gamma^m_{is} - \sum_s \Gamma^s_{ik} \cdot \Gamma^m_{js}\right) \cdot e_m \quad .
\end{aligned} \tag{6.10}$$

□

6.2 The curvature tensor

The curvature operator gives rise to the curvature tensor as follows:

Definition 6.4 Let $M^n = (\mathcal{U}^n, g, \nabla)$ denote a Riemannian manifold. Let X, Y, Z, and U denote smooth vector fields in $\mathfrak{X}(\mathcal{U})$. Then the Riemannian curvature tensor $\mathcal{R}$ is defined as follows:

$$\mathcal{R}(X,Y,Z,U) = g\left(R(X\,,\,Y)Z\,,\,U\right) \quad . \tag{6.11}$$

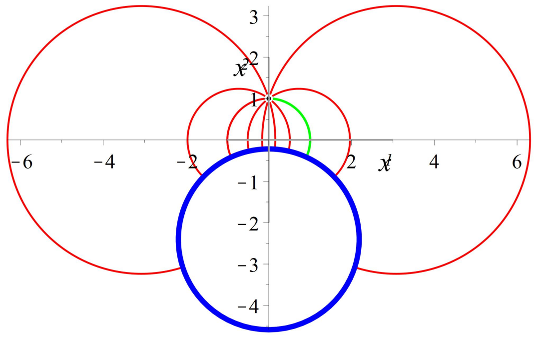


Figure 6.3: The example considered in figure 6.2, but now with a full angular spray of geodesics of length ρ in all directions. Obviously, the geodesics tend to re-focus at some point $(0,-1)$ 'outside' the blue boundary geodesic circle. This phenomenon of having geodesics refocusing at a well defined 'antipodal' point is well known from the usual sphere surface in 3D. As we shall see, this analogy is no coincidence: The surface metric and the metric used for this example are isometric. See also the zoomed-in version of this figure in figure 6.2.

In coordinates we therefore get the following expression for the coordinate functions of the curvature tensor via the coordinate functions for the curvature operator:

$$\begin{aligned}\mathcal{R}_{ijkm} &= \sum_s R^s_{ijk} \cdot g_{sm} \\ &= \sum_s \left(\frac{\partial}{\partial x^i}\Gamma^s_{jk} - \frac{\partial}{\partial x^j}\Gamma^s_{ik} + \sum_p \Gamma^p_{jk} \cdot \Gamma^s_{ip} - \sum_p \Gamma^p_{ik} \cdot \Gamma^s_{jp} \right) \cdot g_{sm} \quad .\end{aligned} \tag{6.12}$$

The curvature operator and curvature tensor carry the following symmetry properties:

Proposition 6.5 Let X, Y, Z, and U denote smooth vector fields in $\mathfrak{X}(\mathcal{U})$. Then

$$\begin{aligned} R(X,Y)Z &= -R(Y,X)Z \\ R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= 0 \\ \mathcal{R}(X,Y,Z,U) &= -\mathcal{R}(X,Y,U,Z) \\ \mathcal{R}(X,Y,Z,U) &= \mathcal{R}(Z,U,X,Y) \quad . \end{aligned} \tag{6.13}$$

EXERCISE 6.6

Prove these symmetries. Hint: See [4, p. 91], but beware of the sign difference in do Carmo's definition of the curvature operator.

The coordinate functions for the curvature tensor therefore satisfy the following symmetries:

$$\begin{aligned} \mathcal{R}_{ijkm} &= -\mathcal{R}_{jikm} = -\mathcal{R}_{ijmk} \\ \mathcal{R}_{ijkm} &= \mathcal{R}_{kmij} \\ 0 &= \mathcal{R}_{ijkm} + \mathcal{R}_{ikmj} + \mathcal{R}_{imjk} \end{aligned} \tag{6.14}$$

In particular, in dimension $n = 2$ there is essentially only one coordinate function for the curvature tensor – all other non-zero coordinate functions can be obtained from it via an application of one or more of the above symmetries; it is represented by

$$\mathcal{R}_{1221} = \mathcal{R}(e_1, e_2, e_2, e_1) = g\left(R(e_1, e_2)e_2\, ,\, e_1\right) \quad . \tag{6.15}$$

EXERCISE 6.7

Verify, that in 2D the curvature tensor is determined in the way explained by the coordinate function expressed in equation (6.15).

EXERCISE 6.8

We consider the Euclidean plane $(\mathcal{U}, g, \nabla)$ with the usual Cartesian coordinates and $g = g_E$, the Euclidean metric tensor. Find the coordinate function $\mathcal{R}_{1221}$ for this Riemannian manifold.

EXERCISE 6.9

Consider the Poincaré half plane $(\mathcal{U}, g, \nabla)$ with g represented in the usual way by the metric matrix function (we use now coordinates (x^1, x^2) instead of the previously used (y^1, y^2)):

$$G(x^1, x^2) = \left(\frac{1}{x^2}\right)^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{6.16}$$

Find the coordinate function $\mathcal{R}_{1221}(x^1, x^2)$ for this Riemannian manifold.

EXERCISE 6.10

Consider the Local Riemannian Manifold $(\mathcal{U}, g, \nabla)$ with g represented by the metric matrix function (note that it is *not* the Poincaré disk)

$$G(x^1, x^2) = \left(\frac{2}{1 + (x^1)^2 + (x^2)^2}\right)^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{6.17}$$

Find the coordinate function $\mathcal{R}_{1221}(x^1, x^2)$ for this Riemannian manifold.

6.3 Sectional curvature

In higher dimensions than 2 we can consider several two-dimensional sections in each tangent space $T_p\mathcal{U}$ of a local Riemannian manifold $(\mathcal{U}, g, \nabla)$. To each such section spanned by two linearly independent vectors in the tangent space we associate a curvature, the sectional curvature of the section as follows:

Definition 6.11 Let X_p and Y_p denote two linearly independent vectors in $T_p\mathcal{U}$. Then the squared area spanned by the two vectors is

$$\mathrm{Area}_g^2(X_p, Y_p) = \|X_p\|_g^2 \cdot \|Y_p\|_g^2 - (g(X_p, Y_p))^2 = \|X_p \times_g Y_p\|_g^2 \quad . \tag{6.18}$$

The sectional curvature of M at p is then defined to be the following real value, which is well defined since the curvature tensor only depends on the point wise values of X and Y:

$$K(X_p, Y_p) = \frac{\mathcal{R}(X_p, Y_p, Y_p, X_p)}{\mathrm{Area}_g^2(X_p, Y_p)} \quad . \tag{6.19}$$

If X and Y are everywhere linearly independent vectorfields in $\mathfrak{X}(\mathcal{U})$, then the sectional curvature function is a smooth function in $\mathfrak{F}(\mathcal{U})$:

$$K(X, Y) = \frac{\mathcal{R}(X, Y, Y, X)}{\mathrm{Area}_g^2(X, Y)} \quad . \tag{6.20}$$

EXERCISE 6.12

Show that if X_p and Y_p span the same 2-plane as U_p and V_p in $T_p\mathcal{U}$, i.e.

$$\mathrm{span}\{X_p, Y_p\} = \mathrm{span}\{U_p, V_p\} \quad , \tag{6.21}$$

then

$$K(X_p, Y_p) = K(U_p, V_p) \quad . \tag{6.22}$$

EXERCISE 6.13

We consider again the three metric tensor fields considered in exercises 6.8, 6.9, and 6.10. In these 2D examples there is only one two-plane to choose in each tangent space, namely the tangent plane itself which is spanned by e_1 and e_2. In consequence $K(X,Y) = K(e_1, e_2)$ is in each case a smooth function on $\mathcal{U}$. Find these respective sectional curvature functions associated with the given three metric tensor fields.

Example 6.14

For surfaces in $\mathbb{R}^3$ we would of course like to recover the Gaussian curvature as the sectional curvature associated with the induced metric tensor on the surface. We illustrate that this is indeed the case by considering a graph surface over $\mathcal{U} = \mathbb{R}^2$:

$$S_{a,b} \; : \; r(x^1, x^2) = (x^1\, ,\, x^2\, ,\, a \cdot (x^1)^2 + b \cdot (x^2)^2) \quad , \tag{6.23}$$

where a and b are constants in $\mathbb{R}$. The metric matrix function induced in $\mathcal{U}$ from the surface is:

$$G = \begin{bmatrix} 1 + 4 \cdot a^2 \cdot (x^1)^2 & 4 \cdot a \cdot b \cdot x^1 \cdot x^2 \\ 4 \cdot a \cdot b \cdot x^1 \cdot x^2 & 1 + 4 \cdot b^2 \cdot (x^2)^2 \end{bmatrix} \quad . \tag{6.24}$$

The Gaussian curvature function of such a surface is classically known to be the following – see e.g. [27, 32, 5]

$$K(x^1, x^2) = \frac{4 \cdot a \cdot b}{(1 + 4 \cdot a^2 \cdot (x^1)^2 + 4 \cdot b^2 \cdot (x^2)^2)^2} \quad . \tag{6.25}$$

In particular, the curvature is positive when a and b have the same sign, negative when a and b have opposite signs and K is zero if one or both of a and b is zero. See figure 6.4.

EXERCISE 6.15

Show that the function in equation (6.25) is precisely the sectional curvature function of $(\mathcal{U}, g, \nabla)$ – computed directly from the surface induced metric matrix function.

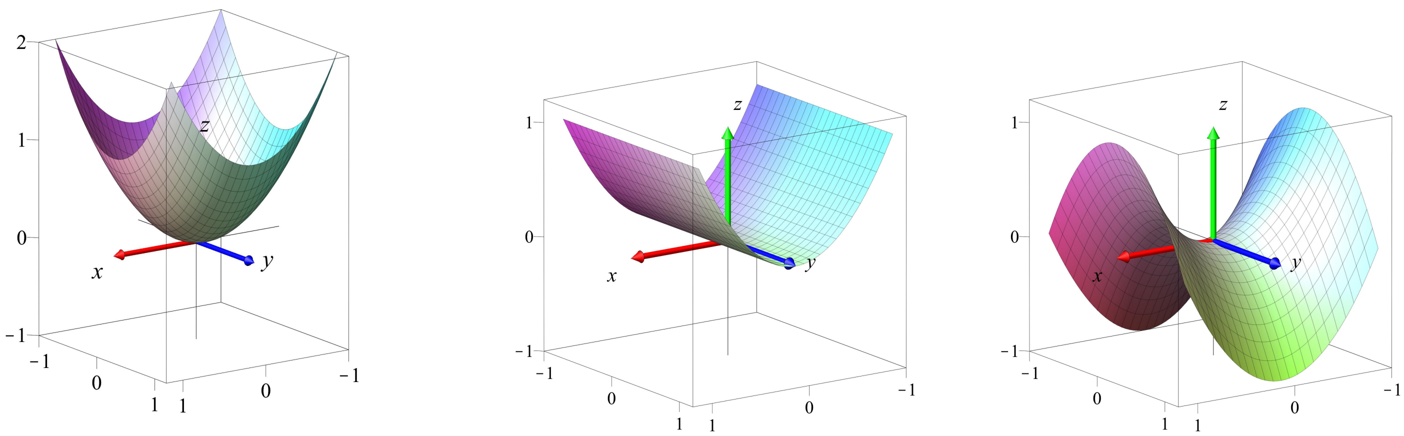


Figure 6.4: From left to right, three surfaces from example 6.14: $S_{1,1}$, $S_{1,0}$ and $S_{1,-1}$ with positive, zero, and negative sectional curvature functions, respectively.

6.4 Second derivatives of variation vector fields

As already alluded to in the introduction to this chapter, the curvature operator appears naturally in the second order analysis of a (geodesic) variation H in $(\mathcal{U}, g, \nabla)$. In order to see this, we first consider again a most general variation of an arc length parametrized base curve η, which is not necessarily a geodesic:

Definition 6.16 We define a one-parameter family H of regular smooth curves that are organized as an (u,s)-parametrized smooth surface in Q_p so that:

$$H_0(s) = \eta(s) \quad \text{for all } s \in [0, L] \quad , \quad \text{and} \quad H_u(0) = \eta(0) = p \quad \text{for all } u \in]-\varepsilon, \varepsilon[. \tag{6.26}$$

The parametrized family of curves H is then a variation of the base curve η.

The local linear transverse behaviour of the nearby curves close to η is, for each s_0, obtained by the tangent vectors to the u-curves $H_u(s_0)$ at $u = 0$:

Definition 6.17 In the above setting, the vector field $V \in \mathfrak{X}(\gamma)$ defined by

$$V(s) = \frac{\partial}{\partial u} H_u(s)_{|u=0} \tag{6.27}$$

is called the variation vector field along γ induced by the variation H of γ.

The vector field V is naturally generalized to a vector field along every (longitudinal) curve H_{u_0} in the variation as follows:

$$V(u_0, s) = \frac{\partial}{\partial u} H_u(s)_{|u=u_0} \quad . \tag{6.28}$$

In the next section we shall consider this extended field in the special case of a geodesic variation, i.e. where all the longitudinal curves in the variation are geodesics issuing from the same point $p = \eta(0)$.

The extended variation vector field V along the longitudinal curves in H is but one example of a general vector field along H itself:

Definition 6.18 We consider a Local Riemannian Manifold $M^n = (\mathcal{U}, g, \nabla)$. Let H denote a variation based on a given arc length parametrized regular smooth curve η, so that $H(u,s)$, $u \in]-\varepsilon, \varepsilon[$, $s \in]0, L[$, defines a regular smooth *surface* in $\mathcal{U}^n$. A smooth vector field W along H is then defined as a smooth choice of vector $W(u,s)$ in each tangent space $T_{H_u(s)}\mathcal{U}$ for all u and s in their respective parameter domains. We write as follows: $W \in \mathfrak{X}(H)$.

The vector field W along the variation surface H in $\mathcal{U}$ is thus to be thought of as an association of a vector (in $T_q\mathcal{U}$) to each point q on the surface *without demanding* that the vector lies in the tangent space of that surface.

The natural (coordinate) tangent vectors to the surface H along H itself are, of course, then the following two fields h_1 and h_2 in $\mathfrak{X}(H)$:

$$\begin{aligned} h_1(u,s) &= \frac{\partial}{\partial u} H_u(s) \\ h_2(u,s) &= \frac{\partial}{\partial s} H_u(s) \quad . \end{aligned} \tag{6.29}$$

We then have the following relation.

Lemma 6.19 With the notation above we get:

$$\nabla_{h_1} \nabla_{h_2} W - \nabla_{h_2} \nabla_{h_1} W = R(h_1\,,\, h_2) W \quad , \tag{6.30}$$

or, equivalently, in terms of covariant derivations:

$$\frac{D}{\partial u}\left(\frac{D}{\partial s} W\right) - \frac{D}{\partial s}\left(\frac{D}{\partial u} W\right) = R\left(\frac{\partial}{\partial u} H_u(s)\,,\, \frac{\partial}{\partial s} H_u(s)\right) W \quad . \tag{6.31}$$

This lemma may seem obvious in view of the definition of the curvature operator (plus the fact that the Lie bracket $[h_1, h_2]$ vanishes everywhere), but it is not *that* trivial because the vector fields h_1 and h_2 are not fields in $\mathfrak{X}(\mathcal{U})$, only in $\mathfrak{X}(H)$.

The proof is a long – but fairly direct – calculation. We refer to [4, pp. 98-99] for a detailed account.

6.5 The Jacobi equation

Suppose now that the variation H considered in the previous section is a *geodesic variation*, i.e. every curve H_u, $u \in]-\varepsilon, \varepsilon[$, is a geodesic issuing from the common point $p = H_u(0)$ for all u. Then we have for all u and all s:

$$\frac{D}{\partial s}\left(\frac{\partial}{\partial s} H_u(s)\right) = 0 \quad , \tag{6.32}$$

so that lemma 6.19 therefore gives

$$
\begin{aligned}
0 &= \frac{D}{\partial u}\left(\frac{D}{\partial s}\left(\frac{\partial}{\partial s}H_u(s)\right)\right) \\
&= \frac{D}{\partial s}\left(\frac{D}{\partial u}\left(\frac{\partial}{\partial s}H_u(s)\right)\right) + R\left(\frac{\partial}{\partial u}H_u(s)\,,\,\frac{\partial}{\partial s}H_u(s)\right)\frac{\partial}{\partial s}H_u(s) \\
&= \frac{D}{\partial s}\left(\frac{D}{\partial s}\left(\frac{\partial}{\partial u}H_u(s)\right)\right) + R\left(\frac{\partial}{\partial u}H_u(s)\,,\,\frac{\partial}{\partial s}H_u(\partial s)\right)\frac{\partial}{\partial s}H_u(s) \quad ,
\end{aligned}
\tag{6.33}
$$

where we have also used the previously observed fact, that we can interchange the two covariant derivations:

$$
\frac{D}{\partial u}\left(\frac{\partial}{\partial s}H_u(s)\right) = \frac{D}{\partial s}\left(\frac{\partial}{\partial u}H_u(s)\right) \quad . \tag{6.34}
$$

For geodesic variations we choose to denote and name the variation vector field as follows:

Notation 6.20 Let H be a *geodesic variation* in $\mathcal{U}$ based on a geodesic γ parametrized by arc length $s \in [0,L]$. The corresponding variation vector field along γ is then denoted as follows:

$$
J(s) = \frac{\partial}{\partial u}H_u(s)_{|u=0} \quad . \tag{6.35}
$$

Using this expression for J in (6.33) – restricted to the base geodesic γ – we get for all $s \in [0,L]$:

$$
\frac{D^2}{ds^2}J(s) + R(J(s)\,,\,\gamma'(s))\gamma'(s) = 0 \quad . \tag{6.36}
$$

Definition 6.21 Let γ be a geodesic in $\mathcal{U}$. A vector field $J \in \mathfrak{X}(\gamma)$ is called a Jacobi vector field along γ if it stems from a geodesic variation H as described above, or, equivalently, if it is a vector field along γ that is g-orthogonal to γ and satisfies the equation (6.36), with $J(0) = 0$. The equation (6.36) is called the Jacobi equation along γ.

Note the two claims contained in the definition above. Firstly, that if J stems from a variation H, then it is g-orthogonal to the base curve (we knew this already from the Gauss lemma in chapter 4), but secondly also, that if J is a vector field along γ which is g-orthogonal to γ and satisfies the Jacobi equation with $J(0) = 0$ then there exists a variation H that produces the vector field as the variation vector field of H, see e.g. [4, p. 113].

In this sense we are now on the road to actually see what was claimed in the introduction, e.g. that positive curvature will make nearby geodesics re-converge back to the base geodesic in a geodesic variation. The length function $\|J(s)\|_g$ of the Jacobi field will give us a measure of this convergence – or lack of convergence.

A first observation concerning Jacobi fields is the following, which is well-known from other *linear* second order ordinary differential equation systems:

Proposition 6.22 Any Jacobi field along a given geodesic γ is determined by its initial conditions, $J(0)$ and $\frac{DJ}{ds}(0)$ at $\gamma(0)$.

Proof. Let $\{E_1(s), \cdots, E_n(s)\}$ denote a parallel orthonormal frame field along γ, and let us define unique coefficient functions as follows for all indices i and j:

$$\begin{aligned} J(s) &= \sum_j f_j(s) \cdot E_j(s) \\ \frac{D}{ds} J(s) &= \sum_j f_j'(s) \cdot E_j(s) \\ \frac{D^2}{ds^2} J(s) &= \sum_j f_j''(s) \cdot E_j(s) \\ a_{ij} &= g\left(R(E_i(s), \gamma'(s))\gamma'(s), E_j(s)\right) = \mathcal{R}\left(E_i(s), \gamma'(s), \gamma'(s), E_j(s)\right) \quad . \end{aligned} \tag{6.37}$$

Then

$$\frac{D^2}{ds^2} J(s) = \sum_i f_i''(s) \cdot E_i(s) \tag{6.38}$$

and

$$\begin{aligned} R(J(s), \gamma'(s))\gamma'(s) &= \sum_j g\left(R(J(s), \gamma'(s))\gamma'(s), E_j(s)\right) \cdot E_j(s) \\ &= \sum_{ij} f_i(s) \cdot g\left(R(E_i(s), \gamma'(s))\gamma'(s), E_j(s)\right) \cdot E_j(s) \\ &= \sum_{ij} f_i(s) \cdot a_{ij}(s) \cdot E_j(s) \quad . \end{aligned} \tag{6.39}$$

The Jacobi equation is then equivalent to the following system of equations

$$f_j''(s) + \sum_i f_i(s) \cdot a_{ij}(s) = 0 \quad \text{for all } j = 1, 2, \cdots, n \quad , \tag{6.40}$$

which is a linear differential equation system of second order. Given initial conditions as in the proposition there is thence a unique solution to the Jacobi equation as claimed. □

A second fundamental observation relates directly the behaviour of geodesics close to a point p to the curvature tensor – in fact to the sectional curvature of two-plane sections in $T_p\mathcal{U}$ at p, see [4, pp. 114–115]:

Theorem 6.23 Let γ denote a geodesic in $\mathcal{U}$ and let $J(s)$ denote a Jacobi field along γ with initial conditions $J(0) = 0$ and $\frac{DJ}{ds}(0) = w$. Then the Taylor expansion of $\|J(s)\|^2$ at $s = 0$ is given by:

$$\|J(s)\|_g^2 = s^2 - \frac{1}{3}\mathcal{R}(w, \gamma'(0), \gamma'(0), w) \cdot s^4 + \varepsilon(s) \cdot s^4 \quad , \tag{6.41}$$

where $\varepsilon(s)$ is an epsilon function of s, i.e. $\varepsilon(s) \longmapsto 0$ for $s \longmapsto 0$.

Proof. We use shorthand notation as follows: $J' = J'(s) = \frac{D}{ds}J(s)$, $J'' = J''(s) = \frac{D^2}{ds^2}J(s)$ etc. Moreover we will also denote $g(X,Y)$ by $\langle X,Y\rangle$, so that we can write as follows – evaluating derivatives at $s = 0$ in the usual way when setting up Taylor's formula:

$$\begin{aligned}
\langle J,J\rangle' &= 2\cdot\langle J,J'\rangle = 0 \\
\langle J,J\rangle'' &= 2\cdot\langle J',J\rangle' = 2\cdot\langle J'',J\rangle + 2\cdot\langle J',J'\rangle = 2\cdot\langle w,w\rangle = 2 \\
\langle J,J\rangle''' &= 2\cdot\langle J'',J\rangle' + 2\cdot\langle J',J'\rangle' = 2\cdot\langle J''',J\rangle + 6\cdot\langle J'',J\rangle = 0 \\
\langle J,J\rangle'''' &= 2\cdot\langle J''',J\rangle' + 6\cdot\langle J'',J'\rangle' \\
&= 2\cdot\langle J'''',J\rangle + 8\cdot\langle J''',J'\rangle + 6\cdot\langle J'',J''\rangle \\
&= -8\cdot\langle (R(J,\gamma')\gamma')',w\rangle \quad ,
\end{aligned} \tag{6.42}$$

where – to get the last expression – we have used that $J'(0) = w$ and that $J(s)$ satisfies the Jacobi equation,

$$J''(s) = -R(J(s),\gamma'(s))\gamma'(s) \quad , \tag{6.43}$$

so that

$$J'''(s) = -\left(R(J(s),\gamma'(s))\gamma'(s)\right)' \quad , \tag{6.44}$$

and, at $s = 0$:

$$J''(0) = -R(J(0),\gamma'(0))\gamma'(0) = 0 \quad . \tag{6.45}$$

At this point we need the following identity, which holds for any $W \in \mathfrak{X}(\gamma)$:

$$\left\langle \frac{D}{ds}(R(J,\gamma')\gamma'),W \right\rangle = \langle (R(J,\gamma')\gamma')',W\rangle = \langle R(J',\gamma')\gamma',W\rangle \quad . \tag{6.46}$$

Indeed, we have – using the symmetries of proposition 6.5 and the rules for covariant differentiation:

$$\begin{aligned}\left\langle \frac{D}{ds}(R(J,\gamma')\gamma'),W\right\rangle &= \frac{d}{ds}\langle R(J,\gamma')\gamma',W\rangle - \langle R(J,\gamma')\gamma',W'\rangle \\ &= \frac{d}{ds}\langle R(W,\gamma')\gamma',J\rangle - 0 \\ &= \left\langle \frac{D}{ds}(R(W,\gamma')\gamma'),J\right\rangle + \langle R(J',\gamma')\gamma',W\rangle \\ &= \langle R(J',\gamma')\gamma',W\rangle \quad .\end{aligned} \tag{6.47}$$

Therefore, inserting into the last expression in equation (6.42), we get:

$$\langle J,J\rangle'''' = -8\cdot\langle (R(J',\gamma')\gamma'),w\rangle = -8\cdot\mathcal{R}\left(w,\,\gamma'(0)\,,\,\gamma'(0)\,,\,w\right) \quad . \tag{6.48}$$

Using Taylor's theorem for $h(s) = \langle J,J\rangle(s)$ at $s=0$ then gives the theorem as stated. □

Note that if w is g-orthogonal to $\gamma'(0)$ with $\|w\|_g = 1$ we get from $\|\gamma'(s)\|_g = 1$ that

$$\mathcal{R}\left(w,\,\gamma'(0)\,,\,\gamma'(0)\,,\,w\right) = K_p(\gamma'(0),w) \quad , \tag{6.49}$$

and thence the following consequence from theorem 6.23:

Corollary 6.24

$$\|J(s)\|_g^2 = s^2 - \frac{1}{3}K_p(\gamma'(0),w)\cdot s^4 + \varepsilon(s)\cdot s^4 \quad , \tag{6.50}$$

and therefore, in fact:

$$\|J(s)\|_g = s - \frac{1}{6}K_p(\gamma'(0),w)\cdot s^3 + \varepsilon(s)\cdot s^3 \quad , \tag{6.51}$$

In 'continuation' of the previous figures 6.1 and 6.2 where the (sectional) curvature functions are constant zero and constant positive, respectively, we must also show a corresponding example of constant negative curvature, see figure 6.5 below.

6.6 Constant curvature

Definition 6.25 We say that a Local Riemannian Manifold $M^n = (\mathcal{U},g,\nabla)$ has constant curvature k if all sectional curvatures of M are identical, i.e. $K_p(X,Y) = K(X,Y) = k$ for all $p\in\mathcal{U}$ and for all pairs of linearly independent vectors X and Y in every tangent space $T_p\mathcal{U}$.

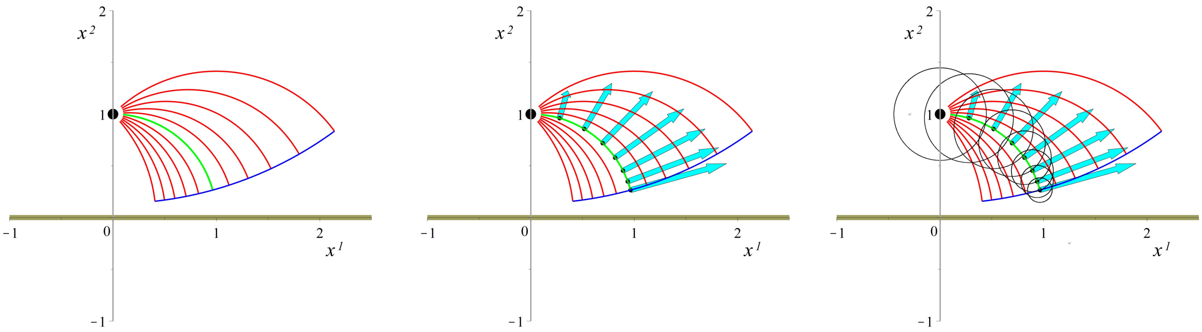


Figure 6.5: A geodesic variation in the Poincaré half plane. The variation vector field is again indicated in light blue along the green base geodesic of the spray. The simple (scaled) indicatrix field of the metric tensor is also indicated along the base geodesic. In contrast to the previous figures 6.1 and 6.2 the indicatrix circles become smaller and smaller as they approach the boundary of the half plane. Although the Euclidean length of the displayed Jacobi field goes to a constant the g-length of the field becomes exponentially large when it approaches the boundary $x^2 = 0$ – in precise accordance with the equation (6.51).

The Jacobi fields of geodesic variations in constant curvature manifolds are quite simple:

Proposition 6.26 Let M^n have constant curvature k and suppose that $J(s)$ is an orthogonal Jacobi field along a geodesic γ in M, parametrized by $s \in [0, L]$. Let $w \in \mathfrak{X}(\gamma)$ denote a parallel vector field along γ with $w(0) = \frac{DJ}{ds}(0)$ and $\|w(s)\|_g = 1$. Then

$$J(s) = \begin{cases} \frac{\sin(s\cdot\sqrt{k})}{\sqrt{k}} \cdot w(s) & \text{if } k > 0 \\ s \cdot w(s) & \text{if } k = 0 \\ \frac{\sinh(s\cdot\sqrt{-k})}{\sqrt{-k}} \cdot w(s) & \text{if } k < 0 \end{cases} \quad . \tag{6.52}$$

Proof. Since $K(X,Y) = k$ for all X and Y, we also have that $\mathcal{R}(X,Y,Y,X) = k$ for all orthogonal vectors X and Y with $\|X\| = \|Y\| = 1$ – since then $\mathrm{Area}^2(X,Y) = 1$. In the notation of the proof of proposition 6.22 we therefore have

$$a_{ij}(s) = k \cdot \delta_{ij} = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{6.53}$$

so that the Jacobi equation becomes equivalent with

$$f_j''(s) + k \cdot f_j(s) = 0 \quad \text{for all } j = 1, 2, \cdots, n \quad . \tag{6.54}$$

However, by choosing $E_1(s) = w(s)$ only $f_1(s)$ is non-zero, because

$$\frac{D}{ds} J(0) = \sum_j f_j'(0) \cdot E_j(0) = w(0) \quad . \tag{6.55}$$

Finally we therefore just need to solve

$$\begin{aligned} J(s) &= f_1(s) \cdot E_1(s) = f_1(s) \cdot w(s) \\ 0 &= f_1''(s) + k \cdot f_1(s) \end{aligned} \tag{6.56}$$

with $f_1(0) = 0$ and $f_1'(0) = 1$. The unique solutions are precisely the ones given in equation (6.52). □

EXERCISE 6.27

Suppose M^n has constant curvature k. Show by explicit calculation from (6.52) that $\|J(s)\|_g$ can be expressed as follows for small at s. Hint: Apply Taylor expansion of $\|J(s)\|_g^2$ at $s = 0$ and compare as in Corolllary 6.24.

$$\|J(s)\|_g = s - \frac{1}{6} k \cdot s^3 + \varepsilon(s) \cdot s^3 \quad . \tag{6.57}$$

6.7 Sectional geodesic circles

In a given tangent space $T_p\mathcal{U}$ of $(\mathcal{U}, g, \nabla)$ we consider a 2-plane section σ spanned by two g-orthogonal vectors v and w, $\sigma = \text{span}(v, w)$. The exponential map Exp_p restricted to σ is – in a sufficiently small 2-dimensional metric disc $B_p^\sigma(\rho)$ around the origin in σ – a diffeomorphism onto the image geodesic disc $D_p^\sigma(\rho) = \text{Exp}_p(B_p^\sigma(\rho))$ in $\mathcal{U}$. We use, for example, in accordance with previous notation:

$$B_p^\sigma(\rho) = \{V \in \sigma \,|\, \|V\|_g \leq \rho\} \quad . \tag{6.58}$$

We can– and will – consider $D_p^\sigma(\rho)$ as a geodesic variation surface based on any one of the geodesics γ_θ issuing from p in $\text{Exp}(\sigma)$ in the *direction* of $\cos(\theta) \cdot v + \sin(\theta) \cdot w$ for any given $\theta \in]-\pi, \pi]$.

The boundaries of the two discs $B_p^\sigma(\rho)$ and $D_p^\sigma(\rho)$ will be called the sectional metric circle in $T_p\mathcal{U}$ and the sectional geodesic circle in $\mathcal{U}$, respectively. They will be denoted by $\partial B_p^\sigma(\rho)$ and $D_p^\sigma(\rho)$.
We want to compare the g-length of the sectional metric circle with the g-length of the sectional

geodesic circle. First, we observe that

$$\mathcal{L}(\partial B_p^\sigma(\rho)) = 2 \cdot \pi \cdot \rho \quad . \tag{6.59}$$

EXERCISE 6.28

Show that the length of the metric circle of radius ρ in σ is indeed given by equation 6.59. Hint: The metric circle is an ellipse determined by the *constant metric* g_p in $T_p\mathcal{U}$ restricted to σ. The ellipse is the intersection of σ with the indicatrix I_p of g at p. The exercise is to show that the length of that ellipse with respect to that constant metric – which thus defines it – is precisely $2 \cdot \pi \cdot \rho$. Better hint: Express/parametrize the metric circle in a g-orthonormal basis in σ.

For each radial base geodesic γ_θ the geodesic variation H gives a Jacobi field of length $\|J_\theta(s)\|$ along γ, $s \in [0, \rho]$. Since this is the orthogonal transverse variation vector field of the variation surface, the length of the sectional geodesic circle of geodesic radius ρ from p is:

$$\mathcal{L}(D_p^\sigma(\rho)) = \int_{\theta=-\pi}^{\theta=\pi} \|J_\theta(\rho)\|_g \, d\theta \quad . \tag{6.60}$$

Inserting the Taylor series estimate of $\|J_\theta(\rho)\|_g$ from (6.51) we get, in terms of the sectional curvature at p – for small ρ:

$$\mathcal{L}(D_p^\sigma(\rho)) = 2 \cdot \pi \cdot \left(\rho - \frac{1}{6} K_p(v, w) \cdot \rho^3 + \varepsilon(\rho) \cdot \rho^3 \right) \tag{6.61}$$

In short, we have shown the following infinitesimal comparison theorem:

Theorem 6.29 Using notation as above, we can construct the value of the sectional curvature $K_p(\sigma) = K_p(v, w)$ as follows – using only the values of the lengths of small sectional geodesic circles centered at p in $\mathcal{U}$:

$$K_p(\sigma) = \lim_{\rho \longmapsto 0} \left(\frac{3}{\pi} \right) \cdot \left(\frac{2\pi\rho - \mathcal{L}(D_p^\sigma(\rho))}{\rho^3} \right) \quad . \tag{6.62}$$

EXERCISE 6.30

Suppose that $M_k^n = (\mathcal{U}, g, \nabla)$ has constant sectional curvature k. Find the exact expression for the length $\mathcal{L}(D_p^{\sigma_k}(\rho))$ of the geodesic circles in M_k^n for each k, all p, all ρ, and all two-plane sections σ_k. Show that your expression gives back the respective values of k when the expression is inserted into equation (6.62).

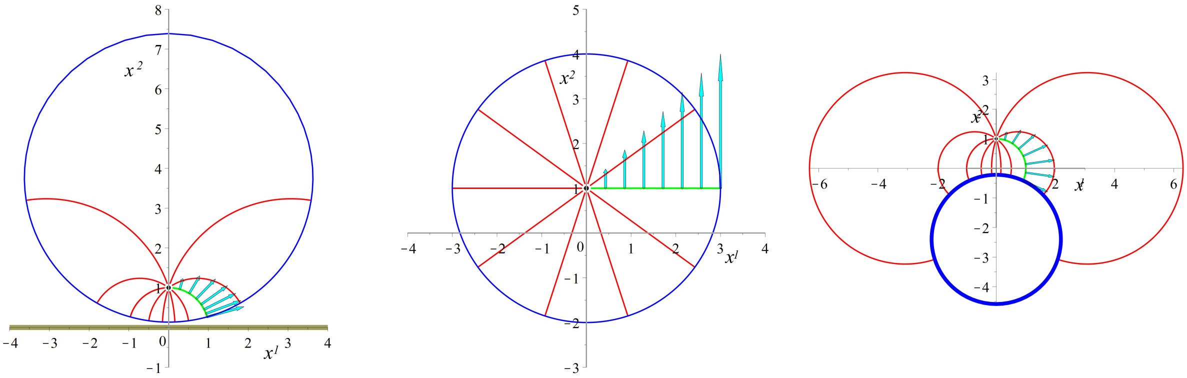


Figure 6.6: Geodesic circles (in blue) in metrics of constant curvature – from the left: Constant curvature -1, 0, and 1, respectively. Although all three geodesic circles have the same radius, $\rho = 3$, they have dramatically different total lengths, cf. equation (6.61) and exercise 6.30.

6.8 From sectional curvatures to the curvature tensor

In the previous section we have seen how all the sectional curvatures of a Local Riemannian Manifold can be obtained very concretely – simply from first sampling lengths of small geodesic circles and then comparing them with the lengths of standard metric circles of the same radii. It is now natural to ask if the sectional curvatures themselves give 'access' to more information about the full curvature tensor $\mathcal{R}$. The following result shows that indeed they do. In fact, two curvature tensors cannot have the same sectional curvatures without being identical. Moreover, we display an interesting – but somewhat lengthy – explicit formula which gives the curvature tensor in terms of sectional curvatures:

Proposition 6.31 Let $M^n = (\mathcal{U}, g, \nabla)$. Let $\widehat{\mathcal{R}}$ denote a multi-linear mapping at p with the same symmetry properties as $\mathcal{R}$, see propositions 6.2 and 6.5. Suppose that the two mappings give rise to the same sectional curvatures for all two-plane sections σ in $T_p\mathcal{U}$, i.e. $\widehat{K}(\sigma) = K(\sigma)$. Then $\widehat{\mathcal{R}} = \mathcal{R}$, i.e.

$$\widehat{\mathcal{R}}(X,Y,Z,U) = \mathcal{R}(X,Y,Z,U) \quad \text{for all } X,\ Y,\ Z, \text{ and } U \text{ in } T_p\mathcal{U} \quad . \tag{6.63}$$

In fact, the value of $\mathcal{R}(X,Y,Z,U)$ can be spelled out directly in terms of addends of sectional-curvature-like values of the following bi-quadratic function:

$$\kappa(X,Y) = \mathcal{R}(X,Y,Y,X) \quad . \tag{6.64}$$

The expansion of $\mathcal{R}(X,Y,Z,U)$ in terms of κ follows:

$$\begin{aligned}6\cdot\mathcal{R}(X,Y,Z,U) = &\ \kappa(X+U,Y+Z)-\kappa(X,Y+Z)-\kappa(U,Y+Z)-\kappa(Y+U,X+Z)\\ &+\kappa(Y,X+Z)+\kappa(U,X+Z)-\kappa(X+U,Y)+\kappa(X,Y)\\ &+\kappa(U,Y)-\kappa(X+U,Z)+\kappa(X,Z)+\kappa(U,Z)\\ &+\kappa(Y+U,X)-\kappa(Y,X)\\ &-\kappa(U,X)+\kappa(Y+U,Z)\\ &-\kappa(Y,Z)-\kappa(U,Z) \quad .\end{aligned} \tag{6.65}$$

The proofs of these results are fairly simple and purely algebraic applications of the common symmetry properties of the curvature mappings. See [4, p. 95] and [16, pp. 252–253] for direct and crisp accounts.

In consequence: If we know the lengths of all sufficiently small sectional geodesic circles in a Riemannian manifold of any dimension, then we can extract the curvature tensor from these lengths.

6.9 Two dimensions; more examples

We finish this chapter with a few illustrations, which show (in the same way as in the above figures) wedges of geodesic sprays, Jacobi fields along chosen geodesics, and the corresponding (sectional) geodesic circles for two-dimensional Local Riemannian Manifolds $M^2 = (\mathcal{U}^2, g, \nabla)$. These examples typically have variable curvature – like the surfaces $S_{a,b}$ in example 6.14, but there are nice benefits to harvest in the low dimension.

In dimension 2 the Jacobi equation is particularly simple. First we observe, that since the Jacobi fields are orthogonal to their base geodesic, they are automatically proportional to the *unique* vector field $w \in \mathfrak{X}(\gamma)$ which is everywhere orthogonal to γ' and having $g\left(\frac{DJ}{ds}(0), w\right) = 1$:

$$J(s) = f(s)\cdot w(s) \quad . \tag{6.66}$$

In this simple setting the Jacobi equation (with initial conditions) reduces to:

$$f''(s) + K(s)\cdot f(s) = 0 \quad , \quad f(0) = 0 \quad , \quad f'(0) = 1 \quad , \tag{6.67}$$

where K is the sectional curvature of M^2 along γ:

$$\begin{aligned}K(s) &= K(\gamma(s))\\ &= K(\gamma'(s), w(s))\\ &= \mathcal{R}(w(s), \gamma'(s), \gamma'(s), w(s))\\ &= \mathcal{R}_{1221}(\gamma(s)) \quad .\end{aligned} \tag{6.68}$$

EXERCISE 6.32

The surfaces $S_{a,b}$, that were studied in example 6.14, have curvature functions as given in equation (6.25). Given a and b, show that the coordinate curves $\eta_1(t) = (t, 0)$ and $\eta_2(t) = (0, t)$ can be reparametrized to geodesic curves $\gamma_1(s)$ and $\gamma_2(s)$ with $\gamma_1(0) = \gamma_2(0) = (0, 0)$ in $(\mathcal{U}, g, \nabla)$. Let J_1 and J_2 denote the corresponding Jacobi fields along γ_1 and γ_2, respectively. Show that for any given a and b there exists a positive exponent α so that $\frac{1}{s^\alpha} \cdot \|J_1(s)\|_g$ and $\frac{1}{s^\alpha} \cdot \|J_2(s)\|_g$ approach constant values when s goes to infinity. Extra: What is the smallest value of α that will work for a given pair (a, b)?

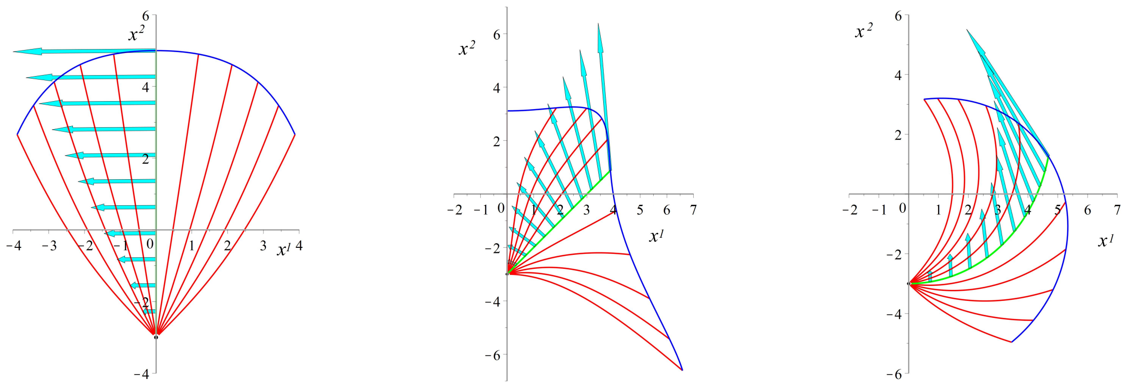


Figure 6.7: Geodesic variations and Jacobi fields in the $(\mathcal{U}, g, \nabla)$-representations of three surfaces $S_{a,b}$ from example 6.14. From the left: a positive and b zero; a positive and b negative; a and b both positive. The surfaces and the corresponding lifted geodesic variations and Jacobi fields are displayed in figure 6.8 below. Note that the Jacobi fields are *not necessarily* Euclidean-orthogonal to the base curves in $(\mathcal{U}, g, \nabla)$ – they are g-orthogonal to the base curves. Compare with figure 6.8 below.

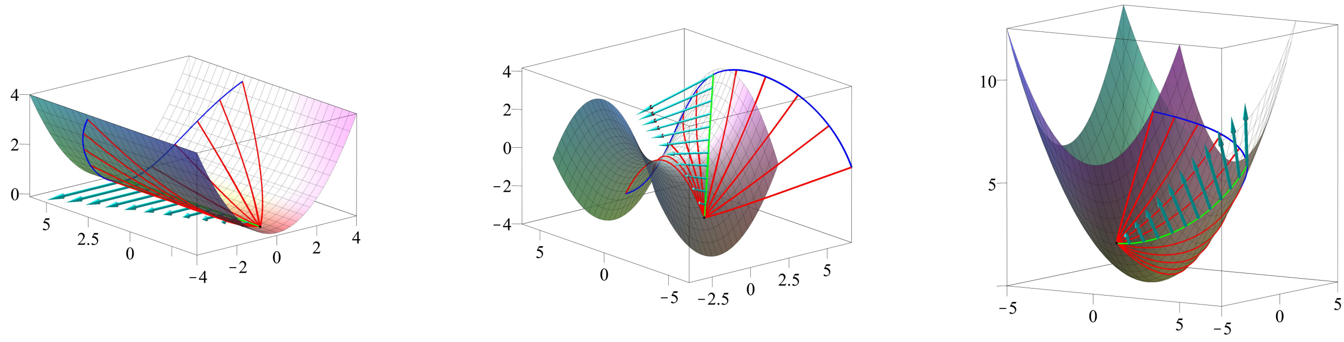


Figure 6.8: Geodesic variations and Jacobi fields on three surfaces $S_{a,b}$ from example 6.14. From the left: a positive and b zero; a positive and b negative; a and b both positive. The corresponding displays in the respective parameter domains with the surface induced metric tensor fields are shown in figure 6.7 above. Note that the Jacobi fields are visually orthogonal to the base curves in the Euclidean metric inherited by the surfaces from the ambient Euclidean space. The orthogonality is represented as g-orthogonality in $(\mathcal{U}, g, \nabla)$, see figure 6.7.

Chapter 7

Tensor fields

We have already encountered two tensor fields – and already named them so: The metric tensor field g in a Local Riemannian Manifold $(\mathcal{U}^n, g, \nabla)$, and the ensuing curvature tensor field $\mathcal{R}$. They deserve the name of tensors because they are pointwise multilinear maps of vector fields in $\mathfrak{X}(\mathcal{U})$ into $\mathfrak{F}(\mathcal{U})$ in the sense already established for these two particular tensor fields:

7.1 The tensor property

Definition 7.1 Let $X_1, X_2, \cdots, X_q$ denote $q \geq 1$ vector fields in $\mathfrak{X}(\mathcal{U})$. A tensor field T of type q is a multilinear mapping from q copies of $\mathfrak{X}(\mathcal{U})$ into $\mathfrak{F}(\mathcal{U})$:

$$T \,:\, \mathfrak{X}(\mathcal{U}) \times \mathfrak{X}(\mathcal{U}) \times \cdots \times \mathfrak{X}(\mathcal{U}) \longmapsto \mathfrak{F}(\mathcal{U}) \quad , \tag{7.1}$$

which means that $T(X_1, \cdots, X_q)$ is a smooth function on $\mathcal{U}$ which is linear at each point and in each of the q arguments as illustrated here:

$$T(X_1, \cdots, f \cdot Y + h \cdot Z, \cdots, X_q) = f \cdot T(X_1, \cdots, Y, \cdots, X_q) + h \cdot T(X_1, \cdots, Z, \cdots, X_q) \tag{7.2}$$

for all real values of f and h.

It is important to note, that once the tensor field is given, then the *evaluation* of the real value $T(X_1, \cdots, f \cdot Y + h \cdot Z, \cdots, X_q)$ at a given point $p \in \mathcal{U}$ can be obtained from knowing only the vector values of the vector fields in $T_p\mathcal{U}$ and the values of the functions f and h at the point p. It is not needed to know the vector values or the function values in a neighborhood around p – as it would have been needed if the tensor evaluation had been depending on, say covariant derivatives of the vector fields or derivatives of the functions.

Notation 7.2 The set of smooth tensor fields of type q on $\mathcal{U}$ is denoted by $\mathfrak{T}_q(\mathcal{U})$, $q > 1$.

The metric tensor field g of $(\mathcal{U}, g, \nabla)$ is indeed a tensor field of type 2, since it was already 'born' as a pointwise 2-linear mapping on each pair of vectors from the tangent space $T_p\mathcal{U}$. So we can write $g \in \mathfrak{T}_2(\mathcal{U})$.

The curvature tensor field $\mathcal{R}$ is a tensor field of type 4, i.e. $\mathcal{R} \in \mathfrak{T}_4(\mathcal{U})$. Indeed, we just have to recall the multilinearities of the curvature operator R from chapter 5:

Proposition 7.3 The curvature operator R is bilinear in its first two arguments, X and Y:

$$\begin{aligned} R(f \cdot X_1 + h \cdot X_2\,,\, Y)Z &= f \cdot R(X_1\,,\, Y)Z + h \cdot R(X_2\,,\, Y)Z \\ R(X\,,\, f \cdot Y_1 + h \cdot Y_2)Z &= f \cdot R(X\,,\, Y_1)Z + h \cdot R(X\,,\, Y_2)Z \end{aligned} \tag{7.3}$$

for all f and h in $\mathfrak{F}(\mathcal{U})$ and all X_1, X_2, Y_1, and Y_2 in $\mathfrak{X}(\mathcal{U})$.

And R is linear in its third argument, Z:

$$\begin{aligned} R(X,Y)(Z+W) &= R(X,Y)Z + R(X,Y)W \\ R(X,Y)(f \cdot Z) &= f \cdot R(X,Y)Z \end{aligned} \tag{7.4}$$

for all f in $\mathfrak{F}(\mathcal{U})$ and all X, Y, Z and W in $\mathfrak{X}(\mathcal{U})$.

It follows in particular therefore that $\mathcal{R}(X,Y,Z,U) = g(R(X,Y)Z, U)$ satisfies all the tensor properties as illustrated here:

$$\begin{aligned} \mathcal{R}(X_1, \cdots, f \cdot Y + h \cdot Z, \cdots, X_4) = \\ f \cdot \mathcal{R}(X_1, \cdots, Y, \cdots, X_4) \\ + h \cdot \mathcal{R}(X_1, \cdots, Z, \cdots, X_4) \end{aligned} \tag{7.5}$$

One could think, that we might be able to produce a tensor field of type 3 out of the covariant derivative operator as follows:

$$C(X,Y,Z) = g(\nabla_X Y\,,\, Z) \quad \text{for all } X, Y, Z \text{ in } \mathfrak{X}(\mathcal{U}). \tag{7.6}$$

But this C is not a tensor field, see exercise 7.4.

EXERCISE 7.4

Show that the mapping C defined in equation (7.6) is *not a tensor field*.

EXERCISE 7.5

Show that the mapping defined by

$$B(X,Y,Z) = g\left(g(X,Y)\cdot Z,\, X\right) \quad \text{for all } X,\, Y, \text{ and } Z \text{ in } \mathfrak{X}(\mathcal{U}) \tag{7.7}$$

is *not a tensor field*.

EXERCISE 7.6

Let W denote a fixed given vector field in $\mathfrak{X}(\mathcal{U})$. Show that the mapping defined by

$$Q(X) = g(W,X) \quad \text{for all } X \text{ in } \mathfrak{X}(\mathcal{U}) \tag{7.8}$$

is a tensor field of type 1.

Proposition 7.7 Let f denote a smooth function in $\mathfrak{F}(\mathcal{U})$. The Hessian of f was introduced in chapter 3, definition 3.51:

$$\mathrm{Hess}(f)(X,Y) = X(Y(f)) - (\nabla_X Y)(f) \quad \text{for all } X \text{ and } Y \text{ in } \mathfrak{X}(\mathcal{U}). \tag{7.9}$$

The Hessian $\mathrm{Hess}(f)$ is a symmetric tensor field of type 2, i.e. $\mathrm{Hess}(f) \in \mathfrak{T}_2(\mathcal{U})$.

Proof. We first observe, that

$$\begin{aligned}(\nabla_X Y)(f) &= (\nabla_Y X)(f) + ([X,Y])(f) \\ &= (\nabla_Y X)(f) + X(Y(f)) - Y(X(f))\end{aligned} \tag{7.10}$$

so that

$$\begin{aligned}\mathrm{Hess}(f)(X,Y) &= X(Y(f)) - (\nabla_X Y)(f) \\ &= X(Y(f)) - (\nabla_Y X)(f) - X(Y(f)) + Y(X(f)) \\ &= Y(X(f)) - (\nabla_Y X)(f) \\ &= \mathrm{Hess}(f)(Y,X)\end{aligned} \tag{7.11}$$

which means, that the operator $\mathrm{Hess}(f)$ is symmetric.
To establish the tensor property of $\mathrm{Hess}(f)$ we recall the definition of the gradient of f:

$$Y(f) = g(\mathrm{grad}(f),Y) \quad \text{for all } Y \in \mathfrak{X}(\mathcal{U}) \quad . \tag{7.12}$$

Therefore

$$X(Y(f)) = X(g(\text{grad}(f),Y)) = g(\nabla_X \text{grad}(f),Y) + g(\text{grad}(f),\nabla_X Y) \\ = g(\nabla_X \text{grad}(f),Y) + (\nabla_X Y)(f) \quad , \tag{7.13}$$

so that

$$\text{Hess}(f)(X,Y) = g(\nabla_X \text{grad}(f),Y) + (\nabla_X Y)(f) - (\nabla_X Y)(f) = g(\nabla_X \text{grad}(f),Y) \quad . \tag{7.14}$$

Since $\nabla_X Z$ is linear in the X-argument, and since g is linear in both of its arguments, we see that $\text{Hess}(f)$ is in fact linear in both of its arguments, so that it is a tensor field of type 2. □

7.2 Tensor coordinates

As was the case for the metric tensor field g and for the curvature tensor field $\mathcal{R}$, every tensor field in $\mathfrak{T}_q(\mathcal{U})$ is uniquely determined by its smooth coefficient functions with respect to the canonical basis vector fields $\{e_1, \cdots, e_n\}$ in $\mathcal{U}$:

Proposition 7.8 Let T be a tensor of type q in $\mathcal{U}^n$ and let $X_i = \sum_{j=1}^{j=n} u_i^j \cdot e_j$ for q given vector fields X_i, $i = 1, \cdots, q$. Then u_i^j, $j = 1, \cdots, n$, are the coefficient functions for X_i, and we have directly from the multilinearity of T:

$$\begin{aligned} T(X_1, \cdots, X_q) &= T\left(\sum_{j_1=1}^{j_1=n} u_1^{j_1} \cdot e_{j_1}, \cdots, \sum_{j_q=1}^{j_q=n} u_q^{j_q} \cdot e_{j_q}\right) \\ &= \sum_{j_1}\sum_{j_2}\cdots\sum_{j_q} u_1^{j_1} \cdots u_q^{j_q} \cdot T\left(e_{j_1}, \cdots, e_{j_q}\right) \\ &= \sum_{j_1}\sum_{j_2}\cdots\sum_{j_q} u_1^{j_1} \cdots u_q^{j_q} \cdot T_{j_1,\cdots,j_q} \quad . \end{aligned} \tag{7.15}$$

The functions $T\left(e_{j_1}, \cdots, e_{j_q}\right) = T_{j_1,\cdots,j_q}$, where all indices j_k run independently through all values $1, 2, \cdots, q$, are called the components of the tensor field T with respect to the canonical basis vector fields $\{e_1, \cdots, e_n\}$.

Note that the last expression in equation (7.15) is most conveniently written without the summation signs

$$u_1^{j_1} \cdots u_q^{j_q} \cdot T_{j_1,\cdots,j_q} \quad , \tag{7.16}$$

which is allowed by Einstein's summation convention, i.e. summation is tacitly active whenever the expression in question contains an upper index and a lower index with the same name. Sometimes, however, it can be relevant to use the explicit summation signs as a redundant support for the reading.

In general there are n^q component functions for a tensor of type q in a Riemannian manifold of dimension n. Compare with the number of component functions for g and $\mathcal{R}$ – in practice the number of effectively different component functions can often be reduced due to the symmetries of the tensor fields in question.

Example 7.9

The coordinate functions of the metric tensor field g in $(\mathcal{U}, g, \nabla)$ with respect to the canonical basis fields $\{e_1, \cdots, e_n\}$ are clearly the functions

$$g(e_i, e_j) = g_{ij} \quad , \tag{7.17}$$

so that we (still) have:

$$g(X,Y) = \sum_i \sum_j u^i \cdot u^j \cdot g_{ij} \quad \text{for all } X = \sum_i u^i \cdot e_i \text{ and all } Y = \sum_j v^j \cdot e_j. \tag{7.18}$$

The coordinate functions for the curvature tensor are likewise:

$$\mathcal{R}(e_i, e_j, e_k, e_m) = \mathcal{R}_{ijkm} \tag{7.19}$$

so that

$$\mathcal{R}(X,Y,Z,U) = \sum_i \sum_j \sum_k \sum_m u^i \cdot v^j \cdot w^k \cdot r^m \cdot \mathcal{R}_{ijkm} \tag{7.20}$$

for all $X = \sum_i u^i \cdot e_i$, $Y = \sum_j v^j \cdot e_j$, $Z = \sum_k w^k \cdot e_k$, and $U = \sum_m r^m \cdot e_m$.

7.3 The Ricci curvature and the scalar curvature

Definition 7.10 Let $\{E_1, \cdots, E_n\}$ denote any choice of a g-orthonormal basis in each tangent space $T_p\mathcal{U}$ in a Riemannian manfold $(\mathcal{U}, g, \nabla)$ and let X and Y denote two vector fields in $\mathfrak{X}(\mathcal{U})$. The following mapping of (X,Y) into $\mathfrak{F}(\mathcal{U})$ is then well-defined:

$$\mathrm{Ric}(X,Y) = \sum_i \mathcal{R}(X, E_i, E_i, Y) = \sum_i \mathcal{R}(E_i, X, Y, E_i) \quad \text{for all } X \text{ and } Y \text{ in } \mathfrak{X}(\mathcal{U}). \tag{7.21}$$

Proposition 7.11 The mapping Ric is a symmetric tensor field of type 2, i.e. $\mathrm{Ric} \in \mathfrak{T}_2(\mathcal{U})$. It is called the Ricci tensor field of $(\mathcal{U}, g, \nabla)$.

Proof. The symmetry and multilinearity of Ric follow directly from these properties of the curvature tensor $\mathcal{R}$. We must then also show that $\mathrm{Ric}(X,Y)$ is independent of the choice of

g-orthonormal basis, that is used for its construction. First we observe the following identities which again follow from the symmetries of $\mathcal{R}$ and from its construction from the curvature operator R – we assume that X and Y are fixed and given vector fields in $\mathfrak{X}(\mathcal{U})$:

$$\begin{aligned} \mathcal{R}(X,E_i,E_i,Y) &= \mathcal{R}(E_i,X,Y,E_i) \\ &= g(R(E_i,X)Y,E_i) \end{aligned} \tag{7.22}$$

Now consider the map (still with X and Y fixed and given):

$$\overline{R}(Z) = R(Z,X)Y \quad \text{for all } Z \text{ in } \mathfrak{X}(\mathcal{U})) \quad , \tag{7.23}$$

which, for each point $p \in \mathcal{U}$ is a linear map from the vector space $T_p\mathcal{U}$ into $T_p\mathcal{U}$. Expressing this map in *any basis* $\{a_1, \cdots, a_n\}$ of $T_p\mathcal{U}$ gives a matrix representation h_i^j of $\overline{R}$:

$$\overline{R}(a_i) = \sum_j h_i^j \cdot a_j \quad . \tag{7.24}$$

The trace of the matrix representation is thus:

$$\text{trace}(\overline{R}) = \sum_i h_i^i \quad . \tag{7.25}$$

The trace of the matrix representation h_i^j is independent of the choice of basis $\{a_1, \cdots, a_n\}$ of $T_p\mathcal{U}$. Therefore we can just write it like $\text{trace}(\overline{R})$. The invariance of the trace follows from the well-known result in matrix linear algebra: $\text{trace}(D^{-1} \cdot A \cdot D) = \text{trace}(A)$.

We begin to extract the said trace as follows:

$$g(\overline{R}(a_i), a_k) = \sum_j h_i^j \cdot g(a_j, a_k) \quad . \tag{7.26}$$

At this point we *now choose* any g-orthonormal basis $\{a_1, \cdots, a_n\} = \{E_1, \cdots, E_n\}$ as in the statement of the theorem. Then

$$g(\overline{R}(E_i), E_k) = \sum_j h_i^j \cdot \delta_{jk} = h_i^k \quad , \tag{7.27}$$

where as usual

$$g(E_j, E_k) = \delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \quad . \end{cases} \tag{7.28}$$

It follows that

$$\sum_i g(\overline{R}(E_i), E_i) = \sum_i h_i^i = \text{trace}(\overline{R}) \tag{7.29}$$

independent of choice of g-orthonormal basis $\{E_1, \cdots, E_n\}$. In conclusion we have therefore now shown that

$$\begin{aligned}\operatorname{trace}(\overline{R}) &= \sum_i g(\overline{R}(E_i), E_i) \\ &= \sum_i g(R(E_i, X)Y, E_i) \\ &= \sum_i \mathcal{R}(E_i, X, Y, E_i) \\ &= \sum_i \mathcal{R}(X, E_i, E_i, Y) \\ &= \operatorname{Ric}(X, Y)\end{aligned} \tag{7.30}$$

– independent of the choice of g-orthonormal basis, which is what we wanted to show. □

Proposition 7.12 The coordinate functions of the Ricci tensor with respect to the canonical basis vector fields $\{e_1, \cdots, e_n\}$ are determined via the coordinate functions of the curvature tensor as follows:

$$\operatorname{Ric}_{jk} = \operatorname{Ric}(e_j, e_k) = \sum_i \sum_m \mathcal{R}_{ijkm} \cdot g^{im} \quad , \tag{7.31}$$

or, alternatively, using the symmetry of $\mathcal{R}$:

$$\operatorname{Ric}_{im} = \operatorname{Ric}(e_i, e_m) = \sum_j \sum_k \mathcal{R}_{ijkm} \cdot g^{jk} \quad . \tag{7.32}$$

We have again applied the notation g^{mj} for the elements of the inverse of the metric matrix function associated to g with respect to the canonical basis.

Proof. We make use of the trace expression in equation (7.25) for $\operatorname{Ric}(X,Y)$ (from the proof of proposition 7.11):

$$\operatorname{Ric}(X, Y) = \operatorname{trace}(\overline{R}) = \sum_i h_i^i \quad , \tag{7.33}$$

which now, in the setting of this proposition, specializes to, with $X = e_w$ and $Y = e_d$:

$$\operatorname{Ric}(e_w, e_d) = \sum_k h_k^k \quad , \tag{7.34}$$

where h_m^k is now determined by the following version of equation (7.24) – choosing the canonical basis $\{e_1, \cdots, e_n\}$ for $\{a_1, \cdots, a_n\}$ and $Z = e_i$:

$$\overline{R}(e_i) = \sum_k h_i^k \cdot e_k \quad , \tag{7.35}$$

with

$$\overline{R}(e_i) = R(e_i, e_w)e_d \quad , \tag{7.36}$$

which means

$$R(e_i, e_w)e_d = \sum_k h_i^k \cdot e_k \quad , \tag{7.37}$$

and thence

$$g\left(R(e_i, e_w)e_d\,,\, e_m\right) = \mathcal{R}_{iwdm} = g\left(\sum_k h_i^k \cdot e_k\,,\, e_m\right) = \sum_k h_i^k \cdot g_{km} \quad , \tag{7.38}$$

and

$$\sum_m \mathcal{R}_{iwdm} \cdot g^{mj} = \sum_m \sum_k h_i^k \cdot g_{km} \cdot g^{mj} = h_i^j \quad , \tag{7.39}$$

so that finally:

$$\sum_j \sum_m \mathcal{R}_{jwdm} \cdot g^{mj} = \sum_j h_j^j = \mathrm{Ric}(e_w, e_d) \quad , \tag{7.40}$$

which shows the coordinate identities in the proposition. □

The Ricci tensor can be obtained from knowledge about Ric solely on identical arguments, like $\mathrm{Ric}(X,X)$, via polarization:

$$\mathrm{Ric}(X+Y, X+Y) = \mathrm{Ric}(X,X) + \mathrm{Ric}(Y,Y) + 2 \cdot \mathrm{Ric}(X,Y) \quad , \tag{7.41}$$

so that for all vector fields X and Y we have:

$$\mathrm{Ric}(X,Y) = \frac{1}{2} \cdot \left(\mathrm{Ric}(X+Y, X+Y) - \mathrm{Ric}(X,X) - \mathrm{Ric}(Y,Y)\right) \quad . \tag{7.42}$$

Definition 7.13 Let $X \in T_p\mathcal{U}$ denote a g-unit vector and let $\{E_1, \cdots, E_n\}$ denote any choice of a g-orthonormal basis. Then the following real value

$$\mathrm{Ric}(X,X) = \sum_i \mathcal{R}(X, E_i, E_i, X) \tag{7.43}$$

is called the Ricci curvature (at p) in the direction X.

The Ricci curvature $\mathrm{Ric}(X,X)$ in any g-unit vector direction X is the sum of sectional curvatures of orthogonal 2-plane sections which contain that direction:

Proposition 7.14

$$\mathrm{Ric}(X,X) = \sum_{i=2}^{n} K(X, H_i) \quad , \tag{7.44}$$

where $\{H_1 = X, H_2, \cdots, H_n\}$ is a (special) choice of g-orthonormal basis, where, as indicated, $H_1 = X$ and the other $(n-1)$ g-unit vectors in the basis are g-orthogonal to X and pairwise g-orthogonal.

Proof. The statement follows directly from the definition of sectional curvatures in chapter 5. Note that the squared area $\mathrm{Area}^2(X, H_i)$ is 1 for all $i = 2, \cdots, n$. □

In other words, the Ricci curvature in direction X is the sum of the sectional curvatures of $n-1$ pairwise g-orthogonal 2-planes which contain X.

Finally we define the scalar curvature:

Definition 7.15 The scalar curvature of a Local Riemannian Manifold $(\mathcal{U}, g, \nabla)$ is the function $S \in \mathfrak{F}(\mathcal{U})$ which is obtained as the sum of Ricci curvatures in orthonormal directions, i.e.:

$$S = \sum_j \mathrm{Ric}(E_j, E_j) = \sum_j \sum_i \mathcal{R}(E_j, E_i, E_i, E_j) \quad , \tag{7.45}$$

where again $\{E_1, \cdots, E_n\}$ is any choice of g-orthonormal basis in each tangent space $T_p\mathcal{U}$.

The scalar curvature is clearly a sum of Ricci curvatures, namely, for any choice of orthonormal basis $\{E_1, \cdots, E_n\}$:

$$S = \sum_j \mathrm{Ric}(E_j, E_j) \quad . \tag{7.46}$$

And thence the scalar curvature is also a sum of sectional curvatures:

$$S = \sum_i \sum_{j \neq i} K(E_i, E_j) = 2 \cdot \sum_{i<j} K(E_i, E_j) \quad . \tag{7.47}$$

Proposition 7.16 With respect to the canonical basis $\{e_1, \cdots, e_n\}$ we get from the expression of the Ricci curvatures in (7.31):

$$S = \sum_i \sum_j \mathrm{Ric}_{ij} \cdot g^{ij} = \sum_j \sum_k \mathrm{Ric}_{jk} \cdot g^{jk} = \sum_j \sum_k \sum_i \sum_m \mathcal{R}_{ijkm} \cdot g^{im} \cdot g^{jk} \quad . \tag{7.48}$$

Proof. We represent the Ricci tensor (quadratic form) via a linear map B as follows

$$\mathrm{Ric}(X, Y) = g(B(X), Y) \quad \text{for all } X \text{ and } Y \text{ in } \mathfrak{X}(\mathcal{U}) \tag{7.49}$$

with $B(a_i) = \sum_k t_i^k \cdot a_k$, where t_i^k are the coefficient functions of B with respect to any chosen basis $\{a_1, \cdots, a_n\}$, so that

$$\mathrm{Ric}(a_i, a_j) = g\left(\sum_j t_i^k \cdot a_k \,,\, a_j\right) = \sum_j t_i^k \cdot g_{kj} \quad . \tag{7.50}$$

Consequently

$$\sum_i \sum_j \mathrm{Ric}(a_i, a_j) \cdot g^{ij} = \sum_i \sum_j \sum_k t_i^k \cdot g_{kj} \cdot g^{ij} = \sum_k t_k^k = \mathrm{trace}(B). \tag{7.51}$$

Since the trace is independent of the chosen basis we get:

$$\sum_i \sum_j \mathrm{Ric}(e_i, e_j) \cdot g^{ij} = \sum_i \mathrm{Ric}(E_i, E_i) = S \quad , \tag{7.52}$$

□

Obviously, in dimension 2 the Ricci curvature and the scalar curvature reduce to the sectional curvature function:

Proposition 7.17 Let $(\mathcal{U}^2, g, \nabla)$ be a 2-dimensional Riemannian manifold. Then

$$\begin{aligned} \mathrm{Ric}(X,X) &= K(Y,Z) \quad \text{for all } g\text{-unit } X \text{ and any } g\text{-orthonormal pair } Y \text{ and } Z \\ S &= 2 \cdot K(Y,Z) \quad \text{for any } g\text{-orthonormal pair } Y \text{ and } Z \end{aligned} \tag{7.53}$$

Similarly we have:

Proposition 7.18 Let $(\mathcal{U}^n, g, \nabla)$ be an n-dimensional Riemannian manifold of *constant (sectional) curvature* k. Then

$$\begin{aligned} \mathrm{Ric}(X,X) &= (n-1) \cdot k \quad \text{for all } g\text{-unit vector fields } X \\ S &= n \cdot (n-1) \cdot k \quad , \quad \text{i.e. constant on all of } \mathcal{U} \quad . \end{aligned} \tag{7.54}$$

7.4 Covariant derivatives of tensor fields

In this section we consider the obvious question of how to use the Levi-Civita connection to define and study the covariant derivatives of tensor fields – as we have previously done only for vector fields.

Definition 7.19 We let T denote a tensor field on $(\mathcal{U}, g, \nabla)$ of type r and let X denote a vector field in $\mathfrak{X}(\mathcal{U})$. The covariant derivative $\nabla_X T$ of T with respect to X is then a tensor of the same type r determined by its operation on r vector fields $Y_1, \cdots, Y_r$ as follows:

$$(\nabla_X T)(Y_1, \cdots, Y_r) = \nabla T(Y_1, \cdots, Y_r, X) \quad , \tag{7.55}$$

where ∇T is shorthand for the following tensor field of type $r+1$, called the total covariant derivative of T:

$$\begin{aligned}(\nabla T)(Y_1,\cdots,Y_r,X) = X\left(T(Y_1,\cdots,Y_r)\right) - T(\nabla_X Y_1,\cdots,Y_r)\\ - \cdots - T(Y_1,\cdots,\nabla_X Y_r) \quad .\end{aligned} \tag{7.56}$$

Note that with this definition we get immediately the standard derivation properties including:

Proposition 7.20 Let $f \in \mathfrak{F}(\mathcal{U})$. Then

$$(\nabla(f\cdot T))(Y_1,\cdots,Y_r,X) = X(f)\cdot T(Y_1,\cdots,Y_r) + f\cdot(\nabla T)(Y_1,\cdots,Y_r,X) \quad . \tag{7.57}$$

EXERCISE 7.21

Show the claim, i.e. equation (7.57).

This invariant definition is somewhat complicated, but it reduces considerably to something quite reasonable if we calculate it in the following natural setting: Let $\alpha(t)$ denote a smooth curve in $(\mathcal{U},g,\nabla)$ with $\alpha(0)=p$ and $\alpha'(t)=X(\alpha(t))$, so that – in effect – we assume α is an integral curve of the vector field X through p. As another hugely simplifying assumption we will use $\{E_1(t),\cdots,E_n(t)\}$, a parallel frame field (of g-orthonormal vector fields) along α, for the expression of the covariant derivative of T. The restriction of the tensor field T to the curve α has tensor coordinate functions along α, that we denote by shorthand as follows:

$$\begin{aligned}T(E_{j_1}(\alpha(t)),\cdots,E_{j_r}(\alpha(t))) &= T(E_{j_1}(t),\cdots,E_{j_r}(t))\\ &= T_{j_1\cdots j_r}(t)\end{aligned} \tag{7.58}$$

Then – by the definition of $\nabla_X T$ – we get:

$$\begin{aligned}(\nabla_X T)(E_{j_1}(t),\cdots,E_{j_r}(t)) = X(T_{j_1\cdots j_r}(t)) - T(\nabla_X E_{j_1}(t),\cdots,E_{j_r}(t))\\ - \cdots - T(E_{j_1}(t),\cdots,\nabla_X E_{j_r}(t)) \quad .\end{aligned} \tag{7.59}$$

Since $E_{j_k}(t)$ are all parallel along γ we have $\nabla_X E_{j_k}(t) = 0$ and therefore:

$$\begin{aligned}(\nabla_X T)(E_{j_1}(t),\cdots,E_{j_r}(t)) &= X(T_{j_1\cdots j_r}(t)\\ &= \frac{d}{dt}T_{j_1\cdots j_r}(t) \quad ,\end{aligned} \tag{7.60}$$

which thus constitutes a 'reasonability check' for the definition 7.19.

The general coordinate expression for the covariant derivative of T with respect to a vector field follows directly from the definition 7.19 and it naturally involves the Christoffel symbols Γ^{ℓ}_{km}:

Proposition 7.22 Let $T\left(e_{j_1},\cdots,e_{j_q}\right)=T_{j_1,\cdots,j_q}$ denote the component functions of a tensor field $T\in\mathfrak{T}_q(\mathcal{U})$ with respect to the standard basis vector fields $\{e_1,\cdots,e_n\}$ in $\left(\mathcal{U}^n,g,\nabla\right)$. Then the component functions of the covariant derivative of T are:

$$
\begin{aligned}
(\nabla_{e_i}T)\,(e_{j_1},\cdots,e_{j_r}) = {} & e_i\,(T_{j_1,\cdots,j_r}) \\
& -T\,(\nabla_{e_i}e_{j_1}\,,\,e_{j_2}\,,\cdots,\,e_{j_r}) \\
& -T\,(e_{j_1}\,,\,\nabla_{e_i}e_{j_2}\,,\cdots,\,e_{j_r}) \\
& \cdots \\
& -T\,(e_{j_1}\,,\,e_{j_2}\,,\cdots,\,\nabla_{e_i}e_{j_r}) \\
\\
= {} & \frac{\partial}{\partial x^i}\,(T_{j_1,\cdots,j_r}) \\
& -\sum_{m_1}\Gamma^{m_1}_{i\,j_1}\cdot T\,(e_{m_1}\,,\,e_{j_2}\,,\cdots,\,e_{j_r}) \\
& -\sum_{m_2}\Gamma^{m_2}_{i\,j_2}\cdot T\,(e_{j_1}\,,\,e_{m_2}\,,\cdots,\,e_{j_r}) \\
& \cdots \\
& -\sum_{m_r}\Gamma^{m_r}_{i\,j_r}\cdot T\,(e_{j_1}\,,\,e_{j_2}\,,\cdots,\,e_{m_r}) \\
\\
= {} & \frac{\partial}{\partial x^i}\,(T_{j_1,\cdots,j_r}) \\
& -\sum_{m_1}\Gamma^{m_1}_{i\,j_1}\cdot T_{m_1\,j_2\cdots j_r} \\
& -\sum_{m_2}\Gamma^{m_2}_{i\,j_2}\cdot T_{j_1\,m_2\cdots j_r} \\
& \cdots \\
& -\sum_{m_r}\Gamma^{m_r}_{i\,j_r}\cdot T_{j_1\,j_2\cdots m_r}
\end{aligned}
\tag{7.61}
$$

Notation 7.23 A super-shorthand notation for the coordinates of the component functions of the covariant derivative of T with respect to e_i is often used in the literature:

$$(\nabla_{e_i}T)\,(e_{j_1},\cdots,e_{j_r}) = T_{j_1\cdots j_r\,;\,i} \quad . \tag{7.62}$$

Note the position of the semicolon ; and the position of the index i.

In particular, if $T \in \mathfrak{T}_2(\mathcal{U})$ we get:

$$(\nabla_{e_i} T)(e_k, e_\ell) = T_{k\ell;i} = \frac{\partial}{\partial x^i} T_{k\ell} - \sum_m \Gamma^m_{ik} \cdot T_{m\ell} - \sum_q \Gamma^q_{i\ell} \cdot T_{kq} \quad . \tag{7.63}$$

We illustrate the covariant derivative of tensor fields by stating two key results concerning the derivative of the metric tensor field and of the curvature tensor field, respectively:

Proposition 7.24 The metric tensor g always has vanishing covariant derivative in $(\mathcal{U}, g, \nabla)$.

Proof. For all X, Y, and Z in $\mathfrak{X}(\mathcal{U})$ we get:

$$(\nabla_Z g)(X,Y) = (\nabla g)(X,Y,Z) = Z(g(X,Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = 0 \quad , \tag{7.64}$$

because the Levi-Civita connection is – by definition – *compatible* with the metric. □

EXERCISE 7.25

In exercise 7.6 we defined a tensor field Q of type 1 via a fixed vector field $W \in \mathfrak{X}(\mathcal{U})$ as follows: $Q(X) = g(W,X)$ for all $X \in \mathfrak{X}(\mathcal{U})$. Show that the total covariant derivative ∇Q of Q is the following tensor field of type 2:

$$(\nabla Q)(Y,X) = g(\nabla_X W, Y) \quad \text{for all } Y \text{ and } X \text{ in } \mathfrak{X}(\mathcal{U}) \quad . \tag{7.65}$$

The following key result about the covariant derivatives of the curvature tensor is known in the literature as Bianchi's second (differential) equality. Note that we now permit ourselves to drop the somewhat redundant parentheses around the tensor, that is being considered, and just write $\nabla\mathcal{R}$ for $(\nabla\mathcal{R})$:

Proposition 7.26 The covariant derivative of the curvature tensor field $\mathcal{R}$ satisfies the following identity for all vector fields X, Y, Z, U, and W in $\mathfrak{X}(\mathcal{U})$:

$$\nabla\mathcal{R}(X,Y,Z,U,W) + \nabla\mathcal{R}(X,Y,U,W,Z) + \nabla\mathcal{R}(X,Y,W,Z,U) = 0 \tag{7.66}$$

or, equivalently,

$$\nabla_W \mathcal{R}(X,Y,Z,U) + \nabla_Z \mathcal{R}(X,Y,U,W) + \nabla_U \mathcal{R}(X,Y,W,Z) = 0 \quad . \tag{7.67}$$

Interestingly, (7.66) can be used to show the following fact, which is otherwise not so obvious:

Theorem 7.27 Let M^n denote a Local Riemannian Manifold of dimension $n \geq 3$ which has isotropic sectional curvatures in the sense that $K(X,Y)$ is constant for all linearly independent X and Y in $T_p\mathcal{U}$ for each $p \in \mathcal{U}$. Then $K(X,Y)$ does not depend on the point either, i.e. M^n has locally constant (sectional) curvature.

7.5 Proofs concerning derivatives of the curvature tensor

In the proofs below we will several times use the existence – and the nice properties of – normal coordinates. They were thoroughly introduced in Chapter 4. For convenience we repeat the main result about normal coordinates – using the notations and the setting from Chapter 4:

Proposition 7.28 Let $z^i(\mathrm{Log}(q))$ denote normal coordinates at p. With respect to these coordinates we get the following evaluations at p for all i, j, and k:

$$\begin{aligned} g_{ij}(p) &= g(E_i(p), E_j(p)) = \delta_{ij} \\ \nabla_{E_i} E_j &= 0 \quad , \quad \text{i.e.} \quad \Gamma^k_{ij}(p) = 0 \\ \frac{\partial}{\partial z^k} g_{ij} &= E_k(g_{ij}) = E_k(g(E_i, E_j)) = 0 \quad . \end{aligned} \tag{7.68}$$

Proof of proposition 7.26. The equation

$$\nabla_W \mathcal{R}(X,Y,Z,U) + \nabla_Z \mathcal{R}(X,Y,U,W) + \nabla_U \mathcal{R}(X,Y,W,Z) = 0 \quad . \tag{7.69}$$

is proved pointwise, i.e at a given point p in $\mathcal{U}$, and with all vectors X, Y, U, W, and Z individually equal to any choice of one vector from the g-orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ of a normal coordinate system for $(\mathcal{U}, g, \nabla)$ at p. This is sufficient since we are proving a tensor-identity. Then the first term in (7.69) is expanded as follows, using first a symmetry of the curvature tensor:

$$\begin{aligned} \nabla_W \mathcal{R}(X,Y,Z,U) &= \nabla_W \mathcal{R}(Z,U,X,Y) \\ \nabla_W g\,(R(Z,U)X\,,\,Y) &= g\,(\nabla_W \nabla_Z \nabla_U X - \nabla_W \nabla_U \nabla_Z X\,,\,Y) \quad . \end{aligned} \tag{7.70}$$

Now we write this equation three times with a cyclic permutation of W, Z, and U and sum the result:

$$\begin{aligned}
&\nabla_W \mathcal{R}(X,Y,Z,U) + \nabla_Z \mathcal{R}(X,Y,U,W) + \nabla_U \mathcal{R}(X,Y,W,Z) \\
&= g\left(\nabla_W \nabla_Z \nabla_U X - \nabla_W \nabla_U \nabla_Z X\,,\, Y\right) \\
&\quad + g\left(\nabla_Z \nabla_U \nabla_W X - \nabla_Z \nabla_W \nabla_U X\,,\, Y\right) \\
&\qquad + g\left(\nabla_U \nabla_W \nabla_Z X - \nabla_U \nabla_Z \nabla_W X\,,\, Y\right) \\
&= g\left(R(W,Z)\nabla_U X + R(Z,U)\nabla_W X + R(U,W)\nabla_Z X\,,\, Y\right) \\
&= 0 \quad ,
\end{aligned} \tag{7.71}$$

where the last equality follows from $\nabla_U X = \nabla_W X = \nabla_Z X = 0$ at the point p – which is again a consequence of the choice of normal coordinates. □

Proof of theorem 7.27. We have already seen in chapter 5 Proposition 5.31, that the sectional curvatures determine the curvature tensor. If the sectional curvatures at p are independent of the sections (the two-planes) at each point, it follows that the curvature tensor at p is a constant $K(p)$ times the following standard curvature-like and squared-area like tensor $\mathcal{R}_1$ at p:

$$\mathcal{R}(X,Y,Z,U) = K(p)\cdot\mathcal{R}_1(X,Y,Z,U) = K(p)\cdot g(g(Y,Z)\cdot X - g(X,Z)\cdot Y\,,\,U) \quad , \tag{7.72}$$

because only then can we get

$$K(X,Y) = \frac{\mathcal{R}(X,Y,Y,X)}{\mathrm{Area}^2(X,Y)} = K(p) \quad \text{for all linearly independent } X \text{ and } Y \text{ in } T_p\mathcal{U} \quad . \tag{7.73}$$

EXERCISE 7.29

Show the claim that $\mathcal{R} = K(p)\cdot\mathcal{R}_1$ under the given conditions.

In consequence we therefore have – now with K as a smooth function on $\mathcal{U}$ – and using again normal coordinates z^i based at p and $X = E_j$, $Y = E_i$, $Z = E_k$, $U = E_\ell$, and $W = E_h$ for any choices of indices:

$$\begin{aligned}
\nabla_W \mathcal{R}(X,Y,Z,U) &= \nabla_W \left(K\cdot\mathcal{R}_1(X,Y,Z,U)\right) \\
&= \nabla_{E_h}\left(K\cdot g(g(E_i,E_k)\cdot E_j - g(E_j,E_k)\cdot E_i\,,\,E_\ell)\right) \\
&= \frac{\partial}{\partial z^h}\left(K\cdot(\delta_{ik}\cdot\delta_{j\ell} - \delta_{jk}\cdot\delta_{i\ell})\right) \\
&= \frac{\partial K}{\partial z^h}\cdot(\delta_{ik}\cdot\delta_{j\ell} - \delta_{jk}\cdot\delta_{i\ell}) \quad .
\end{aligned} \tag{7.74}$$

Now we use the identity (7.69) above and get:

$$\begin{aligned} 0 &= \nabla_W \mathcal{R}(X,Y,Z,U) + \nabla_Z \mathcal{R}(X,Y,U,W) + \nabla_U \mathcal{R}(X,Y,W,Z) \\ &= \frac{\partial K}{\partial z^h} \cdot (\delta_{ik} \cdot \delta_{j\ell} - \delta_{jk} \cdot \delta_{i\ell}) \\ &\quad + \frac{\partial K}{\partial z^k} \cdot (\delta_{i\ell} \cdot \delta_{jh} - \delta_{ih} \cdot \delta_{j\ell}) \\ &\quad\quad + \frac{\partial K}{\partial z^\ell} \cdot (\delta_{ih} \cdot \delta_{jk} - \delta_{ik} \cdot \delta_{jh}) \quad . \end{aligned} \tag{7.75}$$

Now, since $n \geq 3$, if h is given we can find i and j so that i, j, and h are all distinct. Set $k = i$ and $\ell = j$. Then it follows from (7.75) that

$$\frac{\partial K}{\partial z^h} = 0 \quad \text{for all indices } h \quad . \tag{7.76}$$

So all directional derivatives of the function K vanish at every point. The function, i.e. the sectional curvature, is therefore locally constant. □

7.6 Divergence of type-2 tensors

We first recall the definition of the divergence of vector fields as was presented in chapter 3:

Definition 7.30 Let $V \in \mathfrak{X}(\mathcal{U})$ be a smooth vector field in $(\mathcal{U}, g, \nabla)$ and let $\{E_1, \cdots, E_n\}$ denote a g-orthonormal basis in the tangent space $T_p\mathcal{U}$ at the point $p \in \mathcal{U}$. The divergence of V at p is then

$$\operatorname{div}(V) = \sum_i g(\nabla_{E_i} V, E_i) \quad . \tag{7.77}$$

The divergence of tensor fields of type 2 is similarly defined as follows:

Definition 7.31 Let $A \in \mathfrak{T}_2(\mathcal{U})$ be a smooth tensor field of type 2. The divergence of A is then the following tensor field of type 1, $\operatorname{div}(A) \in \mathfrak{T}_1(\mathcal{U})$, obtained as follows via any g-orthonormal frame field $\{E_1, \cdots, E_n\}$ in $\mathcal{U}$:

$$(\operatorname{div}(A))(V) = \sum_i (\nabla_{E_i} A)(V, E_i) \quad . \tag{7.78}$$

In general standard coordinates with basis field $\{e_1, \cdots, e_n\}$ the sum on the right hand side can be expressed as follows:

Proposition 7.32 Let $A \in \mathfrak{T}_2(\mathcal{U})$ with coordinates A_{ij} with respect to $\{e_1, \cdots, e_n\}$. Then the divergence of A has coordinates

$$
\begin{aligned}
(\mathrm{div}(A))(e_k) &= \sum_i \sum_\ell g^{i\ell} \cdot A_{ik;\ell} \\
&= \sum_i \sum_\ell g^{i\ell} \cdot \left(\frac{\partial A_{ik}}{\partial x^\ell} - \sum_m \Gamma^m_{\ell i} \cdot A_{mk} - \sum_q \Gamma^q_{\ell k} \cdot A_{iq} \right) \quad ,
\end{aligned} \tag{7.79}
$$

so that when $\mathrm{div}(A)$ is evaluated on the vector field $V = \sum_k v^k \cdot e_k \in \mathfrak{X}(\mathcal{U})$ we get the function $(\mathrm{div}(A))(V) \in \mathfrak{F}(\mathcal{U})$:

$$
(\mathrm{div}(A))(V) = \sum_k v^k \cdot \left(\sum_i \sum_\ell g^{i\ell} \cdot \left(\frac{\partial A_{ik}}{\partial x^\ell} - \sum_m \Gamma^m_{\ell i} \cdot A_{mk} - \sum_q \Gamma^q_{\ell k} \cdot A_{iq} \right) \right) \quad . \tag{7.80}
$$

We have already encountered a number of type-2 tensors, and note here their respective divergences:

Since the metric has covariant derivatives 0 we get immediately for all V:

$$
(\mathrm{div}(g))(V) = 0 \quad . \tag{7.81}
$$

Moreover, if we let $f \in \mathfrak{F}(\mathcal{U})$ we then get from proposition 7.20:

$$
(\mathrm{div}(f \cdot g))(V) = V(f) \quad . \tag{7.82}
$$

Next to the metric, the Ricci tensor field Ric is the most prominent tensor field of type 2 – not least because classical general relativity flows from the Einstein equation, which is formulated in terms of both of these tensor fields – see section 7.8 below.

We shall need the divergence of the Ricci tensor. It is quite simply expressed via the *scalar curvature function* S in $\mathcal{U}$ as follows:

Proposition 7.33

$$
(\mathrm{div}(\mathrm{Ric}))(V) = \frac{1}{2} \cdot V(S) \quad . \tag{7.83}
$$

Proof. We prove this identity most conveniently (again) by using normal coordinates – including the choice of $V = E_k$. We have – in particular from the symmetries of the curvature tensor

(Proposition 5.5):

$$
\begin{aligned}
(\mathrm{div}(\mathrm{Ric}))(E_k) &= \sum_i (\nabla_{E_i}\mathrm{Ric})(E_k, E_i) \\
&= \sum_i \nabla_{E_i}\mathrm{Ric}(E_k, E_i)) \\
&= \sum_i \sum_j \nabla_{E_i}\mathcal{R}(E_j, E_k, E_i, E_j) \\
&= \frac{1}{2}\cdot\left(\sum_i \sum_j \nabla_{E_i}\mathcal{R}(E_j, E_k, E_i, E_j) + \sum_i \sum_j \nabla_{E_j}\mathcal{R}(E_i, E_k, E_j, E_i)\right) \\
&= -\frac{1}{2}\cdot\left(\sum_i \sum_j \nabla_{E_i}\mathcal{R}(E_j, E_k, E_j, E_i) + \sum_i \sum_j \nabla_{E_j}\mathcal{R}(E_k, E_i, E_j, E_i)\right) \\
&= \frac{1}{2}\cdot\left(\sum_i \sum_j \nabla_{E_k}\mathcal{R}(E_i, E_j, E_j, E_i)\right) \quad ,
\end{aligned}
\tag{7.84}
$$

where we have also used the identity in proposition 7.26. It now follows that

$$
\begin{aligned}
(\mathrm{div}(\mathrm{Ric}))(E_k) &= \frac{1}{2}\cdot\nabla_{E_k}\sum_i \sum_j \mathcal{R}(E_i, E_j, E_j, E_i) \\
&= \frac{1}{2}\cdot\nabla_{E_k}S \\
&= \frac{1}{2}\cdot E_k(S) \quad ,
\end{aligned}
\tag{7.85}
$$

which was to be proved. □

7.7 Einstein metrics

From the definition and properties of the Ricci tensor, that tensor field is formally comparable with the metric tensor field in any given Riemannian Manifold $(\mathcal{U}, g, \nabla)$ – they are both symmetric tensor fields of type 2. This, and several other interesting properties of the Ricci tensor – motivates the following definition:

Definition 7.34 Suppose that the following condition is satisfied in a Local Riemannian Manifold $M^n = (\mathcal{U}, g, \nabla)$ for some constant (positive, zero, or negative) $\lambda \in \mathbb{R}$:

$$
\mathrm{Ric}(X,Y) = \lambda \cdot g(X,Y) \quad \text{for all } X \text{ and } Y \text{ in } \mathfrak{X}(\mathcal{U}) \quad . \tag{7.86}
$$

Then M^n is called an Einstein manifold.

In all previous chapters we have always assumed, that the metric g of M^n was a *given* tensor field in $\mathcal{U}$ from which we have then extracted the Levi-Civita connection and curvatures etc. An equation like (7.86) opens up the possibility of finding and using specific metrics that are 'balanced' by its own curvature. This is, in a rather precise sense, what general relativity is all about. Of course, if equation (7.86) is expressed and spelled out in local coordinates $\{x^1, \cdots, x^n\}$ in $\mathcal{U}$, the resulting equation is a (complicated) second order partial differential equation system for the elements g_{ij} of the metric tensor field.

In spite of the comment above, we *can*, however, already say something:

Proposition 7.35 If M^n has constant (sectional) curvature, then M^n is Einstein.

Proof. In constant (sectional) curvature k we have

$$\mathrm{Ric}(X,X) = (n-1)\cdot k\cdot g(X,X) \quad , \tag{7.87}$$

so that via polarization:

$$\mathrm{Ric}(X,Y) = (n-1)\cdot k\cdot g(X,Y) \quad . \tag{7.88}$$

□

Proposition 7.36 In low-dimensions (meaning $n = 2$ or $n = 3$) $(\mathcal{U}^n, g, \nabla)$ is Einstein if and only if M^n has constant (sectional) curvature.

Proof. Since in particular

$$\mathrm{Ric}(E_i,E_i) = \lambda\cdot g(E_i,E_i) = \lambda \quad \text{for any orthonormal basis } \{E_1,\cdots,E_n\} \quad , \tag{7.89}$$

we get (in dimension 2, leaving dimension 3 for the exercise below):

$$S = 2\cdot K = \sum_i \mathrm{Ric}(E_i,E_i) = 2\cdot\lambda \quad , \tag{7.90}$$

so the sectional curvature is constant $K = \lambda$. □

EXERCISE 7.37

Show that $(\mathcal{U}^3, g, \nabla)$ is Einstein if and only if it has constant (sectional) curvature.

7.8 The Einstein tensor

An important combination of the metric tensor field g, the Ricci tensor field Ric, and the scalar curvature function S is the following:

Definition 7.38 Let $M^n = (\mathcal{U}, g, \nabla)$ be a local Riemannian manifold. Then the tensor field

$$\mathcal{G} = \mathrm{Ric} - \left(\frac{S}{2}\right) \cdot g \tag{7.91}$$

is called the Einstein tensor field on M^n.

The Einstein tensor field is obviously of type 2. Moreover, it has zero divergence:

Proposition 7.39

$$\mathrm{div}(\mathcal{G}) = 0 \quad . \tag{7.92}$$

Proof. This is a direct consequence of proposition 7.33. □

As previously indicated, the Einstein tensor represents the geometric entrance to general relativity. Indeed, the simplest Riemannian version of Einstein's field equations reads – modulo suitable universal constants:

$$\mathcal{T} = \mathcal{G} \quad , \tag{7.93}$$

where $\mathcal{T} \in \mathfrak{T}_2$ is the physical stress-energy tensor field, which therefore tells the manifold how to curve – via the field equation. Since stress-energy tensors in physics and applications are typically divergence free, the challenge (at the early days of the development of general relativity) was to find a purely geometric interpretation and modelling of the stress-energy tensor. In Einstein's own words (see [16, p. 330]):

1. *The tensor in question should contain no higher than second derivatives of g_{ij}.*
2. *The tensor should depend linearly on the second derivatives of g_{ij}.*
3. *The divergence of the tensor should vanish identically.*

EXERCISE 7.40

We know that $\mathcal{G}$ satisfies the third condition. Show that the two other conditions are also satisfied.

The Einstein tensor satisfies all three conditions and is in a sense uniquely determined by them.

We refer to the following seminal mathematical monographs for further studies of the differential geometry of Lorentzian manifolds, relativistic cosmology, and general relativity: [12, 24, 2, 30].

7.9 The volume of geodesic balls

We have seen in chapter 5, theorem 5.29, how the length of small sectional geodesic circles determine the sectional curvatures of $(\mathcal{U}, g, \nabla)$ and eventually therefore also the full curvature tensor. In the same vein it is reasonable to expect that the *area* of small geodesic spheres $\partial D_\rho(p)$ as well as the *volume* of small geodesic balls $D_\rho(p)$ (as defined in chapter 4, Definition 4.7) could give information about the curvature tensor at the point p – at least in some mean value sense. And indeed, to complete the picture we indicate below how the scalar curvature of the Riemannian manifold can be locally re-constructed in both of these ways:

Theorem 7.41 Let $(\mathcal{U}, g, \nabla)$ denote a Local Riemannian Manifold and p a point in $\mathcal{U}^n$. The *volumes* of the metric ball $B_\rho(p)$ of radius ρ in $T_p\mathcal{U}$ and of the geodesic ball $D_\rho(p) = \mathrm{Exp}_p(B_\rho(p))$ of radius ρ in $\mathcal{U}$ satisfy the Taylor expansion formula:

$$\mathrm{Vol}(D_\rho(p)) = \mathrm{Vol}(B_\rho(p)) \cdot \left(1 - \frac{S(p)}{6 \cdot (n+2)} \cdot \rho^2 + \varepsilon(\rho) \cdot \rho^2\right) \quad . \tag{7.94}$$

The areas (i.e. the $(n-1)$-dimensional volumes) of the metric sphere $\partial B_\rho(p)$ of radius ρ in $T_p\mathcal{U}$ and of the geodesic sphere $\partial D_\rho(p)$ of radius ρ in $\mathcal{U}$ satisfy correspondingly the expansion formula:

$$\mathrm{Area}(\partial D_\rho(p)) = \mathrm{Area}(\partial B_\rho(p)) \cdot \left(1 - \frac{S(p)}{6 \cdot n} \cdot \rho^2 + \varepsilon(\rho) \cdot \rho^2\right) \quad . \tag{7.95}$$

In effect, we recover the same phenomenon as we have previously encountered : When the curvature – in this case the scalar curvature S – is positive, then the Exponential map is locally *contracting the metric spheres* when mapping them into $\mathcal{U}$; and when the curvature is negative, then the Exponential map is locally *expanding the metric spheres*.

As an immediate consequence of theorem 7.41 we can read off the scalar curvature $S(p)$ as limits of volume (and area) fractions for $\rho \longmapsto 0$:

EXERCISE 7.42

Apply the expansions in theorem 7.41 to express $S(p)$ as a second order derivative (at $\rho = 0$) of a volume fraction involving the volumes of metric balls and geodesic balls. Express $S(p)$ as a second order derivative (at $\rho = 0$) of an area fraction involving the areas of metric spheres and geodesic spheres.

Sketch of proof of theorem 7.41. For the proof of the theorem we obviously need a measure of volumes and areas of domains (and their boundaries) in a Local Riemannian manifold. These general notions are both motivated by the classical calculations of volumes and areas of parametrized domains and surfaces in 3D Euclidean space, i.e. via the Jacobians of the respective vector functions. In our case the vector functions in question are defined by the Exponential map and the respective Jacobians are correspondingly organized by the (orthogonal) Jacobi fields along the radial geodesics in the distance balls.

The Exponential map is by standard assumption a diffeomorphism of $B_\rho(p)$ onto $D_\rho(p)$. Along each radial geodesic γ from p we consider $(n-1)$ Jacobi fields $J_i(s)$, $i = 1, \cdots, n-1$, along γ with initial conditions $J_i(0) = 0$, $J_i'(0) = E_i$, where $\{E_1, \cdots, E_n = \gamma'(0)\}$ is an orthonormal basis at $p = \gamma(0)$. These Jacobi fields determine the Jacobian matrix of type $n \times n$ along γ for the Exponential map:

$$\mathcal{J}_{ij}(s) = g(J_i(s), J_j(s)) \quad . \tag{7.96}$$

$$\mathrm{Vol}(D_\rho(p)) = \int_{\partial B_1(p)} \int_{t=0}^{t=\rho} \sqrt{\mathrm{Det}(\mathcal{J}_{ij}(t))}\, dt\, d\mu \tag{7.97}$$

and

$$\mathrm{Area}(D_\rho(p)) = \int_{\partial B_1(p)} \sqrt{\mathrm{Det}(\mathcal{J}_{ij}(\rho))}\, d\mu \quad , \tag{7.98}$$

where $d\mu$ is the canonical measure (of area) in the tangent space $T_p\mathcal{U}$. We have now:

$$\int_{\partial B_\rho(p)} d\mu = \mathrm{Vol}(B_\rho(p)) \tag{7.99}$$

$$\mathrm{Vol}(D_\rho(p)) = \int_{t=0}^{t=\rho} \mathrm{Area}(\partial D_t(p))\, dt \quad , \tag{7.100}$$

and correspondingly:

$$\mathrm{Vol}(B_\rho(p)) = \int_{\partial B_1(p)} \int_{t=0}^{t=\rho} t^{n-1}\, dt\, d\mu = \frac{1}{n} \cdot \rho^n \cdot \mathrm{Area}(\partial B_1(p)) \tag{7.101}$$

and

$$\mathrm{Area}(B_\rho(p)) = \int_{\partial B_1(p)} \rho^{n-1}\, d\mu = \rho^{n-1} \cdot \mathrm{Area}(\partial B_1(p)) \quad , \tag{7.102}$$

In particular we already have:

$$\mathrm{Area}(D_\rho(p)) = \frac{d}{d\rho} \mathrm{Vol}(D_\rho(p)) \quad . \tag{7.103}$$

Thence we only need to establish the Taylor expansion formula for the volume fraction. For this we can apply the previous findings for the Jacobi fields J_i with respect to a parallel frame field obtained by parallel transport of $\{E_1, \cdots, E_n = \gamma'(0)\}$ along γ:

$$J_i(s) = s \cdot E_i(s) - \frac{s^3}{6} \cdot R(E_i(s), \gamma'(s))\gamma'(s) + \varepsilon(s) \cdot s^3 \quad . \tag{7.104}$$

The Jacobian matrix $\mathcal{J}$ is just a matrix function of s and satisfies therefore the following identity:

$$\frac{d}{ds}\mathrm{Det}(\mathcal{J}(s)) = \mathrm{Det}(\mathcal{J}(s)) \cdot \mathrm{trace}\left(\mathcal{J}^{-1}(s) \cdot \frac{d}{ds}\mathcal{J}(s)\right) \quad . \tag{7.105}$$

The above ingredients can now be put together to give a detailed proof of the theorem, see e.g. [8, p. 168]. □

EXERCISE 7.43

Assume that $M^n = (\mathcal{U}, g, \nabla)$ has constant (sectional) curvature in the sense of definition 5.25 in chapter 5, i.e. all sectional curvatures are identical and the same constant k everywhere. We recall from equation (7.47) that in such cases we have $S = 2 \cdot \sum_{i<j} k = n \cdot (n-1) \cdot k$. Apply the explicit expressions for the Jacobi fields in proposition 5.26 and the volume and area formulas developed above to verify the Taylor series expansions for the scalar curvature S in theorem 7.41, and thence also the limit formulas for S as they appear in exercise 7.42.

7.10 Examples in dimension 3 and beyond

The curvature tensors are obviously most interesting in dimensions greater than 2, because, according to proposition 7.17, in 2D all the various notions of curvature are essentially equivalent to one another – via the sectional curvature function K on the Riemannian manifold. In this section we construct some simple examples, which illustrate some of the simplest curvature features in 3 and higher dimensions.

Example 7.44

We consider $\mathcal{U}^3 \subset \mathbb{R}^3$ with the usual coordinates $\{x^1, x^2, x^3\}$ and the corresponding canonical basis $\{e_1, e_2, e_3\}$. The Euclidean metric g_E in $\mathcal{U}^3$ has the trivial metric matrix function:

$$G_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.106}$$

with all its curvatures equal to 0 – all Christoffel symbols vanish. As a first generalization of this metric we consider the following (so-called conformal) modification of the Euclidean metric:

$$G(x^1, x^2, x^3) = f^2(x^1, x^2, x^3) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \tag{7.107}$$

where f is a smooth function on $\mathcal{U}^3$.

EXERCISE 7.45

We have already seen the geodesics of one of the metrics expressed in equation (7.107), namely with $f(x^1,x^2,x^3) = 1/x^3$ with $x^3 > 0$. Show that this choice of f gives a metric g which has constant (sectional) curvature $K = -1$.

EXERCISE 7.46

Suppose that we instead modify the above metric as follows:

$$G(x^1,x^2,x^3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & f^2(x^1,x^2,x^3) & 0 \\ 0 & 0 & f^2(x^1,x^2,x^3) \end{bmatrix} \quad , \tag{7.108}$$

again with $f(x^1,x^2,x^3) = 1/x^3$, $x^3 > 0$. Find the scalar curvature S of the corresponding metric. Find the Ricci curvatures of the metric in the *directions* of e_1, e_2, and e_3. Find the sectional curvatures of the metric for the 3 two-planes that are spanned by the three pairs of basis vectors e_1, e_2, and e_3, i.e. $K(e_1,e_2)$, $K(e_1,e_3)$, and $K(e_2,e_3)$.

EXERCISE 7.47

Suppose that we modify the metric as follows:

$$G(x^1,x^2,x^3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f^2(x^1,x^2,x^3) \end{bmatrix} \quad , \tag{7.109}$$

with $f(x^1,x^2,x^3) = 1/x^3$, $x^3 > 0$. Find the scalar curvature S of the corresponding metric. Find the Ricci curvatures of the metric in the *directions* of e_1, e_2, and e_3. Find the sectional curvatures of the metric for the 3 two-planes that are spanned by the three pairs of basis vectors e_1, e_2, and e_3, i.e. $K(e_1,e_2)$, $K(e_1,e_3)$, and $K(e_2,e_3)$.

EXERCISE 7.48

Let g be determined by the following (so-called warped product) metric matrix for $x^3 > 0$:

$$G(x^1,x^2,x^3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (h(x^1)/x^3)^2 & 0 \\ 0 & 0 & (h(x^1)/x^3)^2 \end{bmatrix} \quad . \tag{7.110}$$

Find the scalar curvature S of the corresponding metric for each function h with $h(x^1) > 0$ for all x^1. Show that the metric has constant (sectional) curvature -1 for $h(x^1) = \cosh(x^1)$.

It is a simple matter to construct and to consider higher dimensional examples of $M^n = (\mathcal{U}^n, g, \nabla)$

along the lines of the examples above. Alternatively, the smooth regular vector functions of $\mathbb{R}^k$ into $\mathbb{R}^n$, $n \geq k$, also produce a plethora of examples:

Example 7.49

Let r denote the vector function which maps $\mathbb{R}^4$ into $\mathbb{R}^6$ via the expression:

$$r(x^1,x^2,x^3,x^4) = (x^1,x^2,x^3,x^4,f(x^1,x^2),1) \quad , \tag{7.111}$$

where f is a smooth function on $\mathbb{R}^2$. The Jacobian of this (regular and smooth) vector function gives the induced metric matrix in the parameter space $(\mathcal{U}^4, g, \nabla)$ – in the same way as for surfaces in Euclidean 3-space $\mathbb{R}^3$:

$$G = \text{Jacobi}_r^* \cdot \text{Jacobi}_r = \begin{bmatrix} 1+\left(\frac{\partial f}{\partial x^1}\right)^2 & \left(\frac{\partial f}{\partial x^1}\right)\cdot\left(\frac{\partial f}{\partial x^2}\right) & 0 & 0 \\ \left(\frac{\partial f}{\partial x^1}\right)\cdot\left(\frac{\partial f}{\partial x^2}\right) & 1+\left(\frac{\partial f}{\partial x^2}\right)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.112}$$

EXERCISE 7.50

Verify the above expression for the metric matrix function that is associated with the metric tensor field g induced in this way for the 4-dimensional Riemannian manifold $M^4 = (\mathbb{R}^4, g, \nabla)$, representing the image of $\mathbb{R}^4$ via r in $\mathbb{R}^6$.

The determinant of G is

$$\text{Det}(G) = 1 + \left(\frac{\partial f}{\partial x^1}\right)^2 + \left(\frac{\partial f}{\partial x^2}\right)^2 \quad . \tag{7.113}$$

Suppose we let H denote the (classical Euclidean Hessian) matrix (for f in $\mathbb{R}^2$):

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial (x^1)^2} & \frac{\partial^2 f}{\partial x^1 \partial x^2} \\ \frac{\partial^2 f}{\partial x^2 \partial x^1} & \frac{\partial^2 f}{\partial (x^2)^2} \end{bmatrix} \tag{7.114}$$

EXERCISE 7.51

Show that the scalar curvature of M^4 is

$$S = 2 \cdot \frac{\text{Det}(H)}{(\text{Det}(G))^2} \quad . \tag{7.115}$$

7.11 Curvature bounds, Laplacian, and distance functions

We are now able to prove and apply a theorem, which is known as Bochner's formula (after Salomon Bochner). To state the theorem we need to recall a few notations from previous chapters:

We consider a smooth function $f \in \mathfrak{F}(\mathcal{U})$ in a Riemannian manifold $(\mathcal{U}, g_{\mathcal{U}}, \nabla)$, where we use standard coordinates $\{x^i\}$, $i = 1, \cdots, n$, and the corresponding basis vector fields $\{e_i\}$. Then the gradient, the Hessian, and the Laplacian of f are, respectively (we refer to definitions and results in Chapter 3):

$$\begin{aligned} \mathrm{grad}(f) &= g\left(\mathrm{grad}(f)\,,\,X\right) = X(f) \quad \text{for all } X \in \mathfrak{X}(\mathcal{U}) \\ \mathrm{Hess}(f)(X,Y) &= \mathrm{Hess}(f)(Y,X) = g\left(\nabla_X \,\mathrm{grad}(f)\,,\,Y\right) \\ \Delta(f) &= \mathrm{div}(\mathrm{grad}(f)) = \sum_i \sum_j \mathrm{Hess}(f)(e_i, e_j) \cdot g^{ij} \quad . \end{aligned} \tag{7.116}$$

The Hessian of f is a symmetric quadratic form – a symmetric tensor of type 2, with coordinates $H_{ij} = \mathrm{Hess}(f)(e_i, e_j)$ w.r.t. the given basis. The eigenvalues of the associated matrix $\widehat{H}_i^j = \sum_k H_{ik} \cdot g^{kj}$ are denoted by $\lambda_1, \cdots, \lambda_n$, so that $\Delta(f) = \mathrm{trace}(\widehat{H}) = \sum_i \lambda_i$ (the trace is independent of the basis). The so-called Frobenius norm of the Hessian operator is defined via the sum of the squares of the eigenvalues of $\widehat{H}$ (which is also basis-independent). We denote the norm squared as follows:

$$\|\mathrm{Hess}(f)\|^2 = \mathrm{trace}(\widehat{H}^* \cdot \widehat{H}) = \sum_i \sum_j \left(\widehat{H}_i^j\right)^2 = \sum_i \lambda_i^2 \quad . \tag{7.117}$$

Theorem 7.52 (Bochner's formula)

$$\frac{1}{2}\Delta(\|\mathrm{grad}(f)\|^2) = \|\mathrm{Hess}(f)\|^2 + g(\mathrm{grad}(f), \mathrm{grad}(\Delta(f))) + \mathrm{Ric}(\mathrm{grad}(f), \mathrm{grad}(f)) \quad . \tag{7.118}$$

Proof. We fix a point p and let z^i denote normal coordinates (i.e. a new but isometric parameter domain representation of $(\mathcal{U}, g, \nabla)$, in a neighborhood of p with induced base vector frame fields E_i. Below we will several times use the fact that at p (using these coordinates) we have $g_{ij} = \delta_{ij}$ and $\nabla_{E_i} E_j = 0$, which in particular also means that e.g. $\nabla_{\mathrm{grad}(f)} E_j = 0$. Computing everything at p then gives:

$$\begin{aligned} \frac{1}{2}\Delta\left(\|\mathrm{grad}(f)\|^2\right) &= \frac{1}{2}\sum_i E_i\left(E_i(g(\mathrm{grad}(f), \mathrm{grad}(f)))\right) \\ &= \sum_i E_i(g(\nabla_{E_i} \mathrm{grad}(f), \mathrm{grad}(f))) \\ &= \sum_i E_i(\mathrm{Hess}(f)(E_i, \mathrm{grad}(f))) \\ &= \sum_i E_i(\mathrm{Hess}(f)(\mathrm{grad}(f), E_i)) \\ &= \sum_i E_i(g(\nabla_{\mathrm{grad}(f)} \mathrm{grad}(f)\,,\,E_i)) \\ &= \sum_i g(\nabla_{E_i}(\nabla_{\mathrm{grad}(f)} \mathrm{grad}(f)\,,\,E_i)) \end{aligned} \tag{7.119}$$

From this last expression we can now extract the curvature content as follows:

$$\begin{aligned}\frac{1}{2}\Delta\left(\|\operatorname{grad}(f)\|^2\right) = \sum_i g(R(E_i, \operatorname{grad}(f))\operatorname{grad}(f)\,,\,E_i) \\ + \sum_i g\left(\nabla_{\operatorname{grad}(f)}\nabla_{E_i}\operatorname{grad}(f)\,,\,E_i\right) \\ + \sum_i g\left(\nabla_{[E_i\,,\,\operatorname{grad}(f)]}\operatorname{grad}(f)\,,\,E_i\right)\end{aligned} \tag{7.120}$$

The first term in (7.120) is just $\operatorname{Ric}(\operatorname{grad}(f), \operatorname{grad}(f))$ as needed for the theorem. The second term is

$$\begin{aligned}&\sum_i(\operatorname{grad}(f))\left(g(\nabla_{E_i}\operatorname{grad}(f)\,,\,E_i)\right) - \sum_i g(\nabla_{E_i}\operatorname{grad}(f)\,,\,\nabla_{\operatorname{grad}(f)}E_i) \\ &= (\operatorname{grad}(f))\left(\sum_i g(\nabla_{E_i}\operatorname{grad}(f)\,,\,E_i)\right) - 0 \\ &= (\operatorname{grad}(f))(\Delta(f)) \\ &= g(\operatorname{grad}(f)\,,\,\operatorname{grad}(\Delta(f))) \quad .\end{aligned} \tag{7.121}$$

The third term in (7.120) finally reduces to $\|\operatorname{Hess}(f)\|^2$ via the following steps:

$$\begin{aligned}\sum_i g\left(\nabla_{[E_i\,,\,\operatorname{grad}(f)]}\operatorname{grad}(f)\,,\,E_i\right) &= \sum_i \operatorname{Hess}(f)([E_i\,,\,\operatorname{grad}(f)]\,,\,E_i) \\ &= \sum_i \operatorname{Hess}(f)\left(\nabla_{E_i}\operatorname{grad}(f) - \nabla_{\operatorname{grad}(f)}E_i\,,\,E_i\right) \\ &= \sum_i \operatorname{Hess}(f)\left(\nabla_{E_i}\operatorname{grad}(f)\,,\,E_i\right) - \operatorname{Hess}(f)\left(\nabla_{\operatorname{grad}(f)}E_i\,,\,E_i\right) \\ &= \sum_i \operatorname{Hess}(f)\left(\nabla_{E_i}\operatorname{grad}(f)\,,\,E_i\right) - 0 \\ &= \sum_i \operatorname{Hess}(f)(E_i\,,\,\nabla_{E_i}\operatorname{grad}(f)) \\ &= \sum_i g\left(\nabla_{E_i}\operatorname{grad}(f)\,,\,\nabla_{E_i}\operatorname{grad}(f)\right) \quad ,\end{aligned} \tag{7.122}$$

In the chosen normal coordinates based at p the last sum is precisely $\|\operatorname{Hess}(f)\|^2$. Indeed, we have in terms of $\widehat{H}_i^j$ with respect to these (orthonormal) coordinates:

$$\widehat{H}_i^j = g\left(\nabla_{E_i}\operatorname{grad}(f), E_j\right) \quad , \tag{7.123}$$

so that

$$\sum_i g\left(\nabla_{E_i}\operatorname{grad}(f), \nabla_{E_i}\operatorname{grad}(f)\right) = \sum_i \left(\widehat{H}_i^j\right)^2 = \sum_i \lambda_i^2 = \|\operatorname{Hess}(f)\|^2 \quad , \tag{7.124}$$

and this finishes the proof of the theorem. □

According to proposition 4.26 in chapter 4, the distance function $\rho(x) = \text{dist}(p,x)$ from p to points x in the domain Q_p (where Log_p is a diffeomorphism), has $\|\,\text{grad}(\rho)\| = 1$, so with Bochner's formula we get:

Corollary 7.53

$$0 = \|\,\text{Hess}(\rho)\|^2 + g(\text{grad}(\rho), \text{grad}(\Delta(\rho))) + \text{Ric}(\text{grad}(\rho), \text{grad}(\rho)) \quad . \tag{7.125}$$

The Laplacian of the distance function is the divergence of the gradient field $\text{grad}(\rho)$, which is identical to the tangent vector field given by the geodesics issuing from p. Since such geodesics that are close to any base geodesic tend to converge back to the base geodesic when they experience positive curvature along the base geodesic (as we have observed already in terms of the behaviour of Jacobi fields), it is reasonable to expect, that $\Delta(\rho)$ is relatively small when the curvature of the manifold is relatively large.

Recall, for example, that the 'source' and 'sink' vector fields in the Euclidean plane $(\mathbb{R}^2, g_E)$

$$\begin{aligned} V(x^1,x^2) &= (x^1,x^2)/\sqrt{(x^1)^2+(x^2)^2} \quad \text{has } \text{div}(V) > 0 \text{ and} \\ W(x^1,x^2) &= -(x^1,x^2)/\sqrt{(x^1)^2+(x^2)^2} \quad \text{has } \text{div}(V) < 0. \end{aligned} \tag{7.126}$$

Of course, $V(x^1,x^2)$ is exactly the gradient of the distance function $\rho(x) = \text{dist}(O,x)$ from $(0,0)$ in the Euclidean plane $(\mathbb{R}^2, g_E)$, and in that special case we get:

$$\Delta(\rho)_{|x} = \frac{1}{\rho(x)} \quad , \quad \text{for all } x \in \mathbb{R}^2. \tag{7.127}$$

The two vector fields V and W are thence also roughly the gradient vector fields for the distance function from the north pole on a standard unit 2-sphere close to the north pole, and close to the south pole, respectively – see exercise 7.54 below.

EXERCISE 7.54

Show that the distance function ρ from the north pole on a standard sphere $\mathbb{S}^2_R$ with radius R (and thence constant sectional curvature $K = 1/R^2$) in Euclidean 3-space has

$$\Delta(\rho)_{|x} = \left(\frac{1}{R}\right) \cdot \cot\left(\frac{\rho(x)}{R}\right) \quad , \quad \text{for all } x \in \mathbb{S}^2_R. \tag{7.128}$$

Show that $\Delta(\rho)_{|x}$ has the Laurent expansion at $\rho = 0$:

$$\Delta(\rho)_{|x} = \left(\frac{1}{\rho}\right) - \left(\frac{\rho}{3 \cdot R^2}\right) + \varepsilon(\rho) \cdot \rho^2 \quad . \tag{7.129}$$

The following proposition shows that the above indicated relation between curvature and $\Delta(\rho)$ is indeed true – even on the (mean value, trace) level of the Ricci curvature:

Proposition 7.55 Let $M^n = (\mathcal{U}^n, g, \nabla)$ denote a Riemannian manifold with Ricci curvatures bounded from below as follows:

$$\mathrm{Ric}(X,X) \geq (n-1)\cdot k \quad \text{for all unit tangent vectors } X \text{ and for some constant } k \in \mathbb{R} \quad . \tag{7.130}$$

Then the following inequality holds true for the distance function $\rho(x) = \mathrm{dist}(p,x)$ at all points x in the domain Q_p (where Log_p is a diffeomorphism):

$$\Delta(\rho) \leq \begin{cases} (n-1)\cdot\sqrt{k}\cdot\cot(\rho\cdot\sqrt{k}) & \text{if} \quad k>0 \\ (n-1)/\rho & \text{if} \quad k=0 \\ (n-1)\cdot\sqrt{-k}\cdot\coth(\rho\cdot\sqrt{-k}) & \text{if} \quad k<0 \quad . \end{cases} \tag{7.131}$$

Note that the curvature assumption $\mathrm{Ric}(X,X) \geq (n-1)\cdot k$ for all unit tangent vectors X and for some constant $k \in \mathbb{R}$ is clearly satisfied if all sectional curvatures $K(\sigma)$ satisfy $K(\sigma) \geq k$ for all two-planes σ in all tangent spaces for M^n. On this note, compare proposition 7.55 with the dual proposition 7.58 below.

Proof. At least one of the eigenvalues (λ_1, say) is 0 because $\widehat{H}$ has a null space:

$$\mathrm{Hess}(\rho)(X,\mathrm{grad}(\rho)) = g\left(\nabla_X \mathrm{grad}(\rho)\,,\,\mathrm{grad}(\rho)\right) = \left(\frac{1}{2}\right)\cdot X(\|\mathrm{grad}(\rho)\|^2) = 0 \quad . \tag{7.132}$$

Using the Cauchy-Schwarz inequality we then get:

$$\|\mathrm{Hess}(\rho)\|^2 = \sum_{i=2}^{i=n}\lambda_i^2 \leq \frac{(\sum_i \lambda_i)^2}{n-1} = \frac{\left(\mathrm{trace}(\widehat{H})\right)^2}{n-1} = \frac{(\Delta(\rho))^2}{n-1} \quad . \tag{7.133}$$

Now writing shorthand $\frac{\partial}{\partial\rho}$ for $\mathrm{grad}(\rho)$, so that $g(\mathrm{grad}(\rho),\mathrm{grad}(\Delta(\rho))) = \frac{\partial}{\partial r}\Delta(\rho)$, and inserting the assumption on the Ricci curvature into the (distance-)Bochner formula (7.125) we get:

$$\frac{(\Delta(\rho))^2}{n-1} + \frac{\partial}{\partial\rho}\Delta(\rho) + (n-1)\cdot k \leq 0 \quad . \tag{7.134}$$

Letting $\psi(\rho) = (n-1)/\Delta(\rho)$ then gives:

$$\frac{\psi'(\rho)}{1+k\cdot\psi^2(\rho)} \geq 1 \quad . \tag{7.135}$$

We then integrate the inequality on both sides and use (for the lower integral bound) that $\psi(\rho) = \rho + \varepsilon(\rho)$ for small ρ and obtain for example for $k = 1$:

$$\arctan(\psi(\rho)) \geq \rho \quad , \quad \psi(\rho) \geq \tan(\rho) \quad , \quad \Delta(\rho) \leq (n-1)\cdot\cot(\rho) \quad , \tag{7.136}$$

and similar for the other values of k, which proves the proposition.

□

EXERCISE 7.56

Verify the claim used in the proof, that $\psi(\rho) = \rho + \varepsilon(\rho)$ for small ρ. Hint: Verify first in $(\mathbb{R}^n, g_E)$.

EXERCISE 7.57

Show that if $M^n = (\mathcal{U}^n, g, \nabla)$ is a Riemannian manifold with constant curvature k, then the equality holds in equation (7.131).

At this point it is highly appropriate to also mention the following fact, which is dual to proposition 7.55 in the sense that the opposite inequality for the Laplacian of the distance function is obtained when the sectional curvatures (not the Ricci curvatures) are *bounded from above*:

Proposition 7.58 Let $M^n = (\mathcal{U}^n, g, \nabla)$ denote a Riemannian manifold with sectional curvatures bounded from above:

$$K(\sigma) \leq k \quad \text{for all two-planes } \sigma \text{ in all tangent spaces and for some constant } k \in \mathbb{R} \quad . \tag{7.137}$$

Then the following inequality holds true for the distance function $\rho(x) = \mathrm{dist}(p, x)$ at all points x in the domain $\mathcal{Q}_p$ (where Log_p is a diffeomorphism) – compare with equation (7.131):

$$\Delta(\rho) \geq \begin{cases} (n-1)\cdot\sqrt{k}\cdot\cot(\rho\cdot\sqrt{k}) & \text{if} \quad k > 0 \\ (n-1)/\rho & \text{if} \quad k = 0 \\ (n-1)\cdot\sqrt{-k}\cdot\coth(\rho\cdot\sqrt{-k}) & \text{if} \quad k < 0 \quad . \end{cases} \tag{7.138}$$

The proof can be obtained from the so-called Riccati equation or from analysis of the second variation of geodesics. We refer to [19, p. 332] for a thorough presentation.

Chapter 8

Global manifolds and submanifolds

In Chapter 1 we have (indirectly) defined a local manifold to be just an open set of $\mathbb{R}^n$. And then a local Riemannian manifold $(\mathcal{U}^n, g)$ to be such a local manifold (open set) $\mathcal{U}$ equipped with a metric g together with the equivalence class of all its isometric equivalents given by diffeomorphisms $\phi : \mathcal{U}^n \to \mathcal{V}^n$ that preserve the metric in the general sense of (1.50) and (1.51) in Chapter 1.

The purpose of the present chapter is to set up the most general notion/definition of a global manifold M^n so that locally a manifold is still represented (via so-called charts) by open sets in $\mathbb{R}^n$. However, we must also guarantee that all constructions and results from the previous chapters can be extended to such manifolds – and make sure that they hold true across charts and are independent of the charts chosen for the manifold representation. We refer to [17] and in particular [18] for thorough introductions to the notion of differentiable manifolds and submanifolds..

8.1 Abstract manifolds

Definition 8.1 Let M be a set. A chart (ψ_α, V_α) of M is a bijective map ψ_α of a subset $V_\alpha \subseteq M$ onto an *open subset* $\psi_\alpha(V_\alpha) = U_\alpha$ of $\mathbb{R}^n$ (this defines the dimension n of M and must be the same for all the charts considered below).

Two charts (ψ_1, V_1) and (ψ_2, V_2) are compatible charts if, whenever $V_1 \cap V_2 \neq \emptyset$, $\psi_1(V_1 \cap V_2)$ and $\psi_2(V_1 \cap V_2)$ are open sets in $\mathbb{R}^n$ and the change of charts $\psi_2 \circ \psi_1^{-1} : \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2)$ is a diffeomorphism between the two open sets in $\mathbb{R}^n$.

An atlas of M is a family $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \,|\, \alpha \in \Lambda\}$ of pairwise compatible charts that together cover all of M, i.e. $M = \bigcup_{\alpha \in \Lambda} V_\alpha$.

Two atlasses $\mathcal{A}$ and $\mathcal{B}$ are called equivalent atlasses if $\mathcal{A} \cup \mathcal{B}$ is itself an atlas of M, i.e. if all charts in $\mathcal{A} \cup \mathcal{B}$ are pairwise compatible.

A differentiable manifold – or just manifold – M^n of dimension n is then, finally, such a set M with such an equivalence class of atlasses.

Remark 8.2 Note that *at the outset* M is just a set, i.e. a collection of elements, that could be anything, e.g. toys in a toy shop, stars in the universe, points on a sphere, etc. However, most sets are *not* differentiable manifolds.

EXERCISE 8.3

Show that following sets are not differentiable manifolds:

1. M is the union of the (x,y)-plane and the z-axis in $\mathbb{R}^3$.
2. M is a closed unit disc in $\mathbb{R}^2$: $M = \{(x,y) \,|\, x^2 + y^2 \leq 1\}$.

On the positive side we have, see [18, Chapter 1]:

Proposition 8.4 Let M^n be a differentiable manifold with atlas $\mathcal{A}$. Then there is a unique maximal atlas on M which contains $\mathcal{A}$.

EXERCISE 8.5

Let M denote the set of points on the graph of the continuous, but not differentiable, function $f(x) = |x|$ in the (x,y)-plane. Construct an atlas on M consisting of just two compatible charts (ψ_1, V_1) and (ψ_2, V_2) on M, and construct a third chart (ψ_3, V_3) on M that is not compatible with neither (ψ_1, V_1) nor (ψ_2, V_2). The maximal atlas on M that contains the atlas (ψ_1, V_1) (and the atlas (ψ_2, V_2)) is thence different from the maximal atlas on M that contains the atlas (ψ_3, V_3). In general, for a differentiable manifold M it is therefore important to note which equivalence class of atlasses that are used for its definition. For ease of mind and intuition at this point: We shall see below that according to our definition of a submanifold, the above graph of f is not a *differentiable submanifold* of $\mathbb{R}^2$.

The connectedness of a differentiable manifold is defined via the charts of the (maximal) atlas. Again we refer to [18, Chapter 1]:

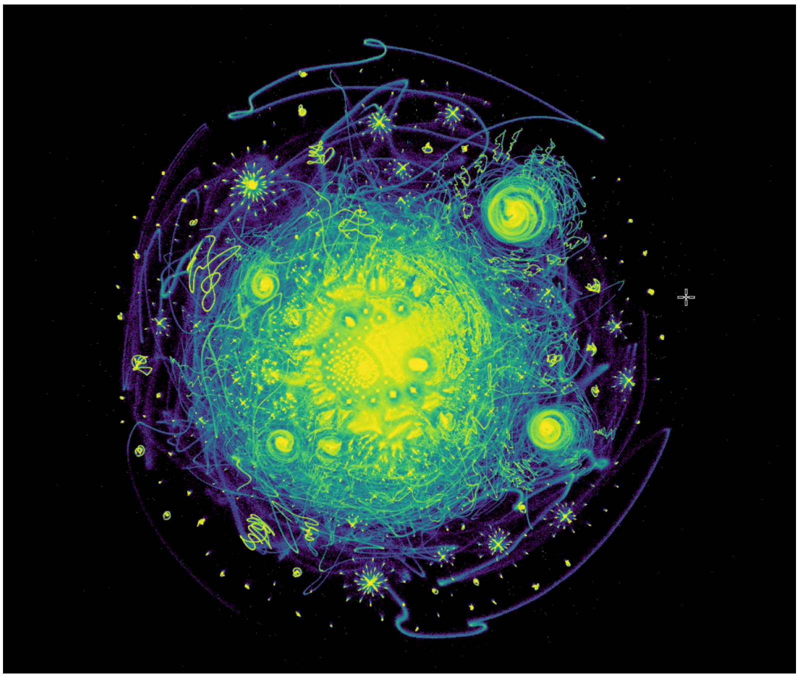


Figure 8.1: Visualization of 30 million integers as represented by binary vectors of prime divisibility, colored by density of points. This subset of $\mathbb{R}^3$, does not comply with the definition of a smooth submanifold as we will define the notion below. That does not rule out, of course, the very interesting question of approximating such sets by smooth submanifolds of $\mathbb{R}^n$. From [22]: McInnes et al., *UMAP: Uniform Manifold Approximation and Projection for Dimension Reduction*.

Definition 8.6 An open set U in $\mathbb{R}^n$ is path-connected if there exists a continuous curve $\gamma.\,[0,1] \to U$ which connects every pair of points p and q in U, i.e so that $\gamma(0) = p$ and $\gamma(1) = q$.

We will assume without lack of generality that the chart images V_α in the considered atlasses of differentiable manifolds are all path-connected.

However, the actual topology of M itself is not directly settled by the definition of a manifold. But we can use the charts to define a natural topology for M and thus, for example, define continuous maps between manifolds:

Proposition 8.7 Let M be a manifold with a maximal atlas $\mathcal{A} = \{(\psi_\alpha, V_\alpha)\,|\,\alpha \in \Lambda\}$. Then the collection $B = \{V_\alpha \subseteq M\,|\,\alpha \in \Lambda\}$ is a basis for a topology for M; it is the so-called natural topology for M.

The natural topology of a manifold is *not* necessarily Hausdorff – see example 8.8 below. So this condition must often be assumed as an extra property of a given manifold in order to well-define and analyze many 'natural' global geometric concepts and results for the manifold, once it has also been globally equipped with a metric tensor field – as locally introduced in the previous chapters for local Riemannian manifolds. Such natural concepts that are supported by the Hausdorff property are e.g. well-defined convergence of sequences, the metric space structure of Riemannian manifolds, and the possibility to represent the manifold as a submanifold in some $\mathbb{R}^n$.

Example 8.8

A simple example of a 1-dimensional manifold with a natural topology that is not Hausdorff is the following: In $\mathbb{R}^2$ we consider the set M consisting of the x-axis and the point $(0,1)$. Then let

$$
\begin{aligned}
V_1 &= \{(s,0) \,|\, s \in \mathbb{R}\} \\
V_2 &= \{(s,0) \,|\, s \in \mathbb{R} - 0\} \cup \{(0,1)\} \\
\psi_1(s,0) &= s \\
\psi_2(s,0) &= s \quad \text{for} \quad s \neq 0 \\
\psi_2(0,1) &= 0 \quad .
\end{aligned}
\tag{8.1}
$$

Then

$$
\begin{aligned}
\psi_2 \circ \psi_1^{-1} &: \mathbb{R} - 0 \to \mathbb{R} - 0 \\
\psi_2 \circ \psi_1^{-1}(s) &= s \quad ,
\end{aligned}
\tag{8.2}
$$

so that $\mathcal{A} = \{(\psi_1, V_1), (\psi_2, V_2)\}$ is an atlas for M. However, with the associated naturl topology, M is not Hausdorff, since the two elements $(0,0)$ and $(0,1)$ in M cannot be separated by open sets in M. Indeed, let V and W be open sets in M with $(0,0) \in V$ and $(0,1) \in W$. Then $\psi_1(V_1 \cap V)$ and $\psi_2(V_2 \cap W)$ are open in $\mathbb{R}$ and both contain 0. Therefore they also contain some $a \neq 0$ so that $\psi_1^{-1}(a) = (a,0) = \psi_2^{-1}(a) \in V_1 \cap V \cap V_2 \cap W$. Hence $V \cap W \neq \emptyset$, so M is not Hausdorff.

Once we have such a topology for M we obtain automatically the continuity of the chart maps in the atlas and of their inverses.

Proposition 8.9 Suppose M^n (with a maximal atlas) is endowed with its natural topology. Then every chart map ψ for a chart (ψ, V) from the maximal atlas for M^n is a homeomorphism of the open set $V \subseteq M$ onto the open subset $\psi(V) \subseteq \mathbb{R}^n$.

The assumption that the chart images V_α in the considered atlasses of differentiable manifolds are all path-connected then guarantees that we can find a continuous curve – a path – from any point to any other point in a given manifold.

Moreover, with the natural topology of a manifold M^n and the natural topology of another manifold N^n (of the same dimension n) a map f from M^n into N^n is continuous (by the usual definition of continuous maps between topological spaces, see definition 0.13) if for any open set $U \subseteq N^n$ the pre-image $f^{-1}(U) \subseteq M^n$ is also open in M^n. And f is by definition a homeomorphism if it is bijective and continuous with a continuous inverse f^{-1}.

To define a diffeomorphism betweeen manifolds we simply use the well defined notion of diffeomorphisms between open sets in $\mathbb{R}^n$:

Definition 8.10 Let M^n and N^n be manifolds of the same dimension n and let $f : M \to N$ be a map. Then f is called smooth if it is continuous and for all $p \in M$ there exists a chart map ϕ of M around p and a chart map ψ of N around $f(p)$ such that the composition $\psi \circ f \circ \phi^{-1}$ is a smooth map between the respective corresponding open sets in $\mathbb{R}^n$. Of course, $f : M \to N$ is called a diffeomorphism if it is bijective and f and f^{-1} are both smooth – M and N are then said to be diffeomorphic.

Proposition 8.11 Suppose M^n can be covered by *one* chart (ψ, M) so that $\psi : M^n \to \psi(M) = U^n \subseteq \mathbb{R}^n$. Then ψ is a diffeomorphism and M^n and U^n are diffeomorphic manifolds.

Example 8.12

The paraboloid of revolution $\mathcal{P}$, which was a key example during in the first chapters of these notes, is a 2-dimensional manifold. Indeed, as we have seen and applied there, P can be covered by a single chart (ψ, V) as follows:

$$\begin{aligned} V = \mathcal{P} &= \{(x,y,z) \in \mathbb{R}^3 \,|\, z = x^2 + y^2\} \\ \psi(V) &= \mathbb{R}^2 \\ \psi(p) &= (x,y) \quad \text{for} \quad p = (x,y,z) \end{aligned} \tag{8.3}$$

Note that the chart map ψ is in this case exactly the inverse of the parametrization map:

$$r(x^1,x^2) = \left(x^1\,,\, x^2\,,\, (x1)^2 + (x^2)^2\right) \quad , \quad (x^1,x^2) \in \mathbb{R}^2 \quad . \tag{8.4}$$

Another chart map η for a part W of the paraboloid is obtained from the following parametrization of that part – also well known from previous examples and exercises:

$$\sigma(y^1,y^2) = \left(y^1\cos(y^2)\,,\, y^1\sin(y^2)\,,\, (y^1)^2\right) \quad , \quad (y^1,y^2) \in \mathbb{R}\times]-\pi,\pi[\quad . \tag{8.5}$$

EXERCISE 8.13

Which subset of $\mathcal{P}$ is not covered by the parametrization σ in example 8.27? The chart (ψ, V) is an atlas for $\mathcal{P}$. Show that the chart (η, W) is contained in the maximal atlas for $\mathcal{P}$ defined by the atlas (ψ, V).

8.2 Abstract submanifolds

In the sense of the previous exercise 8.27 the paraboloid is a manifold in its own right. It makes perfect sense to consider it also as a submanifold of the ambient space $\mathbb{R}^3$ as was already done in Chapter 0. More generally, we will now begin to define submanifolds of any (ambient) manifold as follows:

Definition 8.14 Let M^n denote an n-dimensional manifold with maximal atlas $\mathcal{A}$ and let N be a subset of M. Then N is an k-dimensional *submanifold* of M with $k \leq n$ if for any point $p \in N$ there exists a chart (ψ, W) around p (from the atlas of M) such that $\psi(W \cap N)$ is a k-dimensional submanifold of the open set (itself an n-dimensional manifold) $\psi(W)$ in $\mathbb{R}^n$. Submanifolds of $\mathbb{R}^n$ are already defined in Chapter 0, Definition 0.55. They will be further characterized concretely in Theorem 8.25 below. In particular, $\psi(W \cap N)$ is locally diffeomorphic to an open set $U = \widehat{\psi}(W \cap N)$ in $\mathbb{R}^k$. Thus $(\widehat{\psi}, W \cap N)$ is a chart (around p) in the maximal atlas for N. In particular, $\widehat{\psi}^{-1} : U \to \psi(W \cap N)$ is a *regular parametrization* (by the k coordinate variables in U) of $\psi(W \cap N)$ in $\mathbb{R}^n$. The dimension difference $n - k$ is called the co-dimension of N^k in M^n; if the co-dimension is 1, i.e. if $k = n - 1$, then N^{n-1} is called a hypersurface in M^n.

Via the mentioned *local* diffeomorphisms from M and N to subsets of $\mathbb{R}^n$ and of $\mathbb{R}^m$, respectively, we can now perform all the *local* calculus on manifolds and submanifolds as we have done for Local Riemannian Manifolds in the previous chapters, once the manifold has been equipped with a metric g and a correspondning Levi-Civita connection ∇ that are consistent across charts. We must just make sure, that the results and concepts are all likewise natural in the sense that they agree with each other modulo diffeomorphic shifts of coordinates, i.e. modulo shift of charts. For example, as already emphasized, any regular curve on a manifold M (with regular segments of pre-images in the charts that it meets) must have a well-defined length and acceleration determined by the given metric g and connection ∇ on M – independent of the charts that are applied and combined to find these properties.

Proposition 8.15 The natural topology of a submanifold N^k of M^n is (by the construction of the maximal atlas for N in the definition above) identical to the trace topology of M^n on N^k – the trace topology being defined as follows: $V \cap N$ is declared open in N if V is open in M. The natural topology of a submanifold in a Hausdorff ambient manifold is therefore itself Hausdorff. In particular, every submanifold of $\mathbb{R}^n$ is Hausdorff.

Example 8.16

The punctured figure eight manifold.
We let M_1 denote the following set of points in $\mathbb{R}^2$: $M_1 = \{(\sin(2s), , \sin(s)) \,|\, s \in]0, 2\pi[\}$. See figure 8.2 to the left. The map $\phi : s \to (\sin(2s), , \sin(s))$ is injective and regular in all of the parameter domain $]0, 2\pi[$. But M_1 is not a submanifold of $\mathbb{R}^2$. Suppose for contradiction that there is a parametrization $\psi :]-\varepsilon, \varepsilon[\to B_r(0,0)$ of M_1 around $p = (0,0)$ such that $\psi :]-\varepsilon, \varepsilon[\to B_r(0,0) \cap M_1$ is a homeomorphism for some $r < 1$. Then, since $]-\varepsilon, \varepsilon[- \{0\}$ has two connected components, while $(M_1 \cap B_r(0,0)) - (0,0)$ has four connected components, we have arrived at the desired contradiction.
A similar reasoning shows, of course, that $M_2 = \{(\sin(2t)\,,\, \sin(t)) \,|\, t \in]-\pi, \pi[\}$ is not a submanifold of $\mathbb{R}^2$, see figure 8.2 to the right.

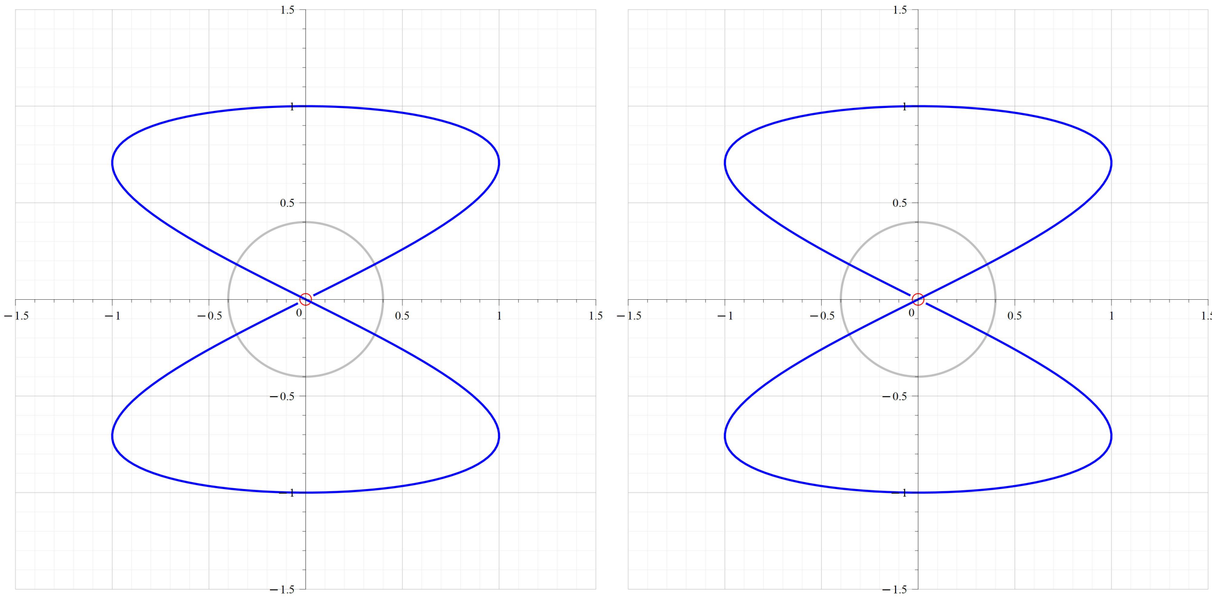


Figure 8.2: The subsets M_1 and M_2 in $\mathbb{R}^2$ discussed in example 8.16. Note the tiny difference in the neighborhood around $(0,0)$ and note that neither of the subsets is identical to the set $\widetilde{M}$ displayed in figure 8.3 below.

Example 8.17

The full self-intersecting figure 8 manifold
We let $\widetilde{M}$ denote the set $\widetilde{M} = \{(\sin(2s)\,,\,\sin(s))\,|\,s \in \mathbb{R}\}$. See figure 8.3. We see that $\widetilde{M}$ still consists of an open upper loop curve, a lower loop curve and the point $(0,0)$. Let $V_\alpha = \widetilde{M}$ and $\psi_\alpha : V_\alpha \to]0,2\pi[$ by $\psi_\alpha(\sin(2s),\sin(s)) = s$. Then ψ_α is a chart on $\widetilde{M}$ and $\mathcal{A}_\alpha = \{(\psi_\alpha, V_\alpha)\}$ is an atlas on (all of) $\widetilde{M}$ which defines $\widetilde{M}$ as a 1-dimensional manifold. Another atlas can be defined by $V_\beta = \widetilde{M}$ and $\psi_\beta : V_\beta \to]-\pi,\pi[$ by $\psi_\beta(\sin(2s),\sin(s)) = s$. Then ψ_β is again a chart on $\widetilde{M}$ and $\mathcal{A}_\beta = \{(\psi_\beta, V_\beta)\}$ is again an atlas on $\widetilde{M}$ which defines $\widetilde{M}$ as a 1-dimensional manifold. However, the two atlasses $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are *not* equivalent: In fact, the combination $\psi_\alpha \circ \psi_\beta^{-1} :]0,2\pi[\to]-\pi,\pi[$ is

$$\psi_\alpha \circ \psi_\beta^{-1}(s) = \begin{cases} s & \text{on the upper loop, where } 0 < s < \pi \\ s - \pi & \text{at the origin, where } s = \pi \\ s - 2\pi & \text{on the lower loop, where } \pi < s < 2\pi \end{cases} \tag{8.6}$$

Hence, $\psi_\alpha \circ \psi_\beta^{-1}$ is not even continuous. In other words, we have found two different non-equivalent atlasses on $\widetilde{M}$ both of which give the figure eight the structure of a manifold although it is not a submanifold of $\mathbb{R}^2$.

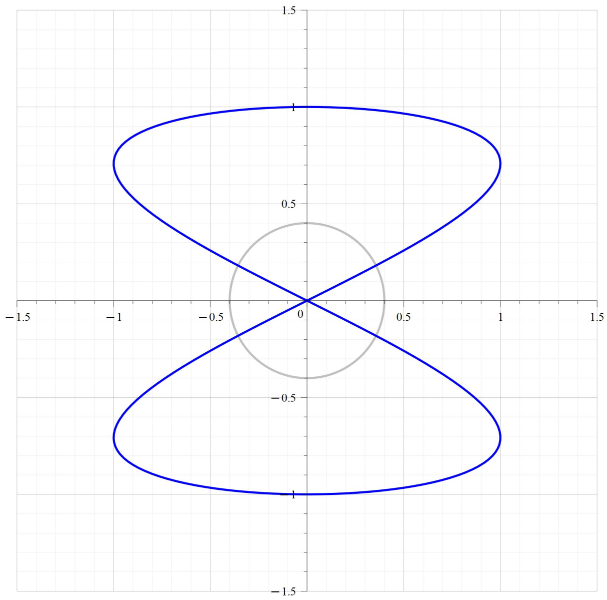


Figure 8.3: The figure eight with a double cover of the point $(0,0)$.

Example 8.18

The spheres $\mathbb{S}^n$ are manifolds. For simplicity let us consider the standard unit sphere in $\mathbb{R}^3$ as the point set:

$$\mathbb{S}^2_1 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\} \quad . \tag{8.7}$$

One obvious chart of the upper hemisphere V_1 of the sphere (i.e. for $z > 0$) is obtained by orthogonal projection into the (x,y)-plane, see figure 8.4. Thence, if $p \in \mathbb{S}^2_1$ with coordinates $p = (x,y,z)$ in *mathbbR*^{3}, then $\psi_1(p) = (x,y) = (x^1,x^2)$, so $\psi_1(V_1) = U_1$, where

$$U_1 = \{(x^1,x^2) \in \mathbb{R}^2 \,|\, (x^1)^2 + (x^2)^2 < 1\} \quad . \tag{8.8}$$

Similarly, we can cover the right hemisphere V_2 of the sphere (i.e. for $y > 0$) which is obtained by orthogonal projection into the (x,z)-plane, i.e. if $p \in \mathbb{S}^2_1$ with coordinates $p = (x,y,z)$ in *mathbbR*^{3}, then $\psi_2(p) = (x,z) = (y^1,y^2)$, so $\psi_2(V_2) = U_2$, where

$$U_2 = \{(y^1,y^2) \in \mathbb{R}^2 \,|\, (y^1)^2 + (y^2)^2 < 1\} \quad . \tag{8.9}$$

For the overlap region $V_1 \cap V_2$, which is displayed to the right in figure 8.4, we have: $y > 0$ and $z > 0$ so that

$$\begin{aligned} \psi_1(V_1 \cap V_2) &= \{(x^1,x^2) \in \mathbb{R}^2 \,|\, (x^1)^2 + (x^2)^2 < 1\,,\, x^2 > 0\} \subset U_1 \\ \psi_2(V_1 \cap V_2) &= \{(y^1,y^2) \in \mathbb{R}^2 \,|\, (y^1)^2 + (y^2)^2 < 1\,,\, y^2 > 0\} \subset U_2 \end{aligned}$$

$$\begin{aligned} \psi_2 \circ \psi_1^{-1}(x^1,x^2) &= (x^1\,,\, \sqrt{1-(x^1)^2-(x^2)^2} \\ \psi_1 \circ \psi_2^{-1}(y^1,y^2) &= (y^1\,,\, \sqrt{1-(y^1)^2-(y^2)^2} \quad . \end{aligned} \tag{8.10}$$

so that $\psi_2 \circ \psi_1^{-1}$ is a diffeomorphism of $\psi_1(V_1 \cap V_2)$ onto $\psi_2(V_1 \cap V_2)$. We conclude, that (ψ_1, V_1) and (ψ_2, V_2) are compatible charts and that they can be supplemented with 4 similar hemispherical charts to form an atlas for the full sphere – see the caption for figure 8.4.

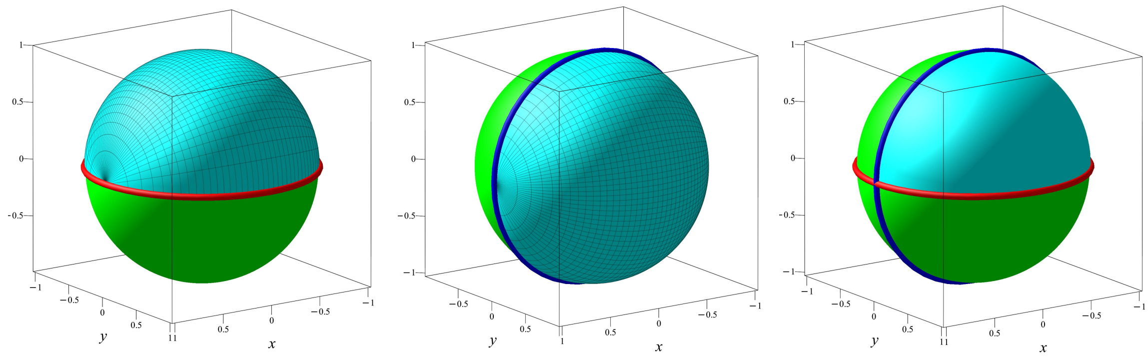


Figure 8.4: Illustration of two overlapping but not covering hemispheres (the blue ones) on the unit sphere. They overlap in the blue region shown to the right – covering roughly a quarter of the sphere. A total of 6 such overlapping hemispheres can be used to cover the full sphere. They define correspondingly 6 compatible charts and thence a maximal atlas for the sphere.

Example 8.19

Suppose we consider a 'blob' in the overlap domain as shown in figure **??**. The parametrization of the blob with parameters x^1 and x^2 is simply:

$$r(x^1, x^2) = \left(x^1, x^2, \sqrt{1-(x^1)^2-(x^2)^2}\right) \quad , \quad a < x^1 < b \, , \, c < x^2 < d \tag{8.11}$$

for suitable boundary values a, b, c, and d for the parameter rectangle.
Then the pre-images of the blob in the two domains for the two charts considered are as shown in figure 8.5. Note that by construction of the hemispheres we have $x^1 = y^1 = x$ and $x^2 = y$, $y^2 = z$ with reference to the ambient coordinates $\{x, y, z\}$ in $\mathbb{R}^3$.

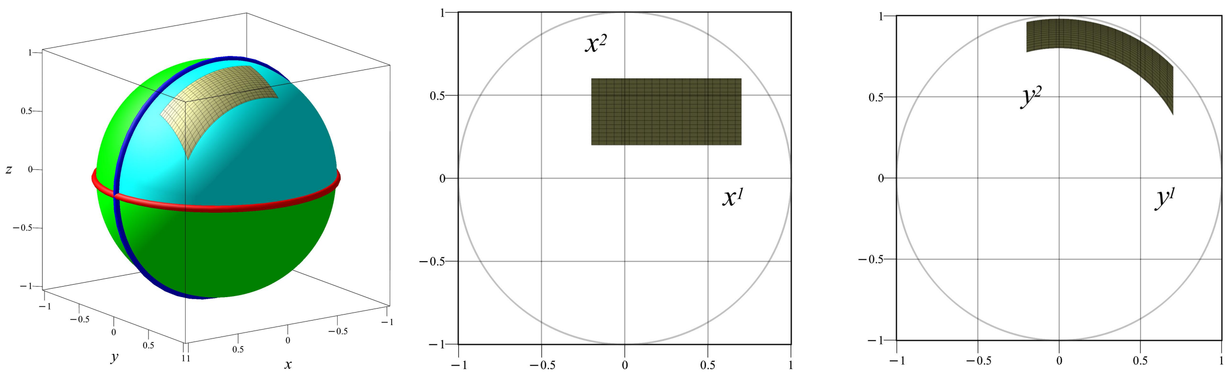


Figure 8.5: A domain in the overlap of the two hemispheres V_1 and V_2 considered in example 8.18 plus its two images by the chart maps ψ_1 and ψ_2, respectively.

EXERCISE 8.20

Use standard geographic coordinates to show that $\mathbb{S}^2_1$ can be covered by the two charts indicated in the displays in figure 8.6. Show that these charts are compatible and compatible with all the charts obtained and discussed in example 8.18. The two charts (and all their compatible charts) then define a maximal atlas for $\mathbb{S}^2_1$.

EXERCISE 8.21

Show that for $\mathbb{S}^2_1$ we need at least two charts for the construction of a covering atlas.

EXERCISE 8.22

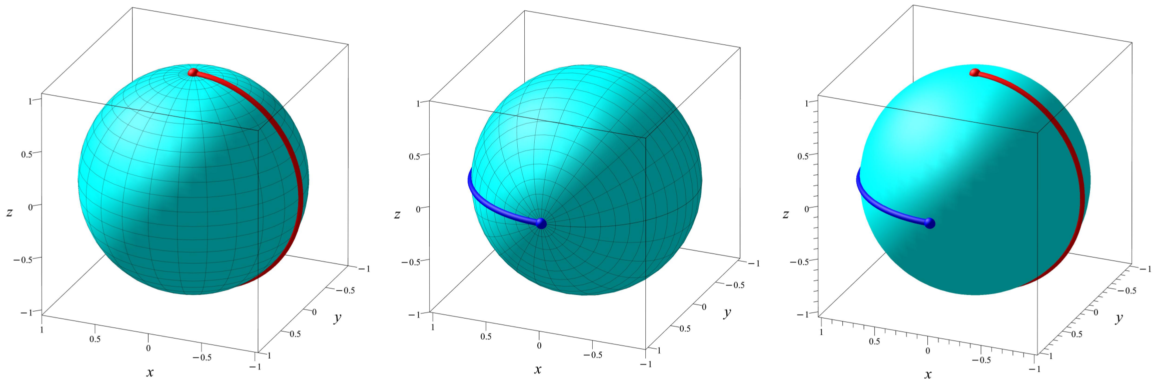


Figure 8.6: Illustration of two covering charts for an atlas on $\mathbb{S}^2_1$.

What is the minimum number of charts needed to form an atlas for an infinite cylinder in $\mathbb{R}^3$ – like the one (partly) shown to the left in figure 8.7. Note that all the cylinder can, of course, be covered in toto by the parametrization

$$r(u,v) = (\cos(v)\,,\, \sin(v)\,,\, u) \quad , \quad -\infty < u < \infty\,,\, -\pi \leq v < \pi \quad , \tag{8.12}$$

but this is not a chart! Why not?

EXERCISE 8.23

What is the minimum number of charts needed to form an atlas for an infinite catenoid in $\mathbb{R}^3$ – like the one (partly) shown in the midddle in figure 8.7.

EXERCISE 8.24

What is the minimum number of charts needed to form an atlas for a torus in $\mathbb{R}^3$ – like the one shown to the right in figure 8.7. Hint: look up standard parametrizations of tori.

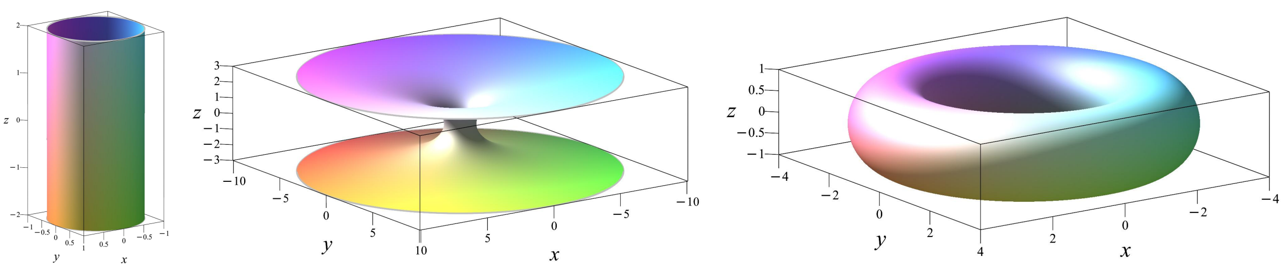


Figure 8.7: A cylinder, a catenoid, and a torus in $\mathbb{R}^3$.

8.3 Submanifolds of $\mathbb{R}^n$

In the following we will apply a constructive characterization of submanifolds of $\mathbb{R}^n$ which complies with the figure eight non-example and the non-example consisting of the graph of $f(x) = |x|$ that we have considered above.

The next theorem is in this sense very constructive and will be the main source for obtaining many examples of submanifolds that we shall need for the introduction of Riemannian submanifolds in the next chapter. The proof of the theorem follows more or less directly from the inverse function theorem and the implicit function theorem, see e.g. [18].

Theorem 8.25 Let N be a subset of $\mathbb{R}^n$. Then the following statements are equivalent:

1. N is a k-dimensional submanifold of $\mathbb{R}^n$.
2. For each point $p \in N$ there exist an open neighborhood V of p in $\mathbb{R}^n$ and a smooth regular map $f : V \to \mathbb{R}^{n-k}$ such that

$$N \cap V = f^{-1}(0) = \{x \in V \,|\, f(x) = 0\} \quad . \tag{8.13}$$

3. For each $p \in N$ there exist (after possible re-numbering of coordinates, if needed) open neighborhoods $\widehat{U} \subseteq \mathbb{R}^k$ of $\widehat{p} = (p_1, \cdots, p_k)$ and $\overline{U} \subseteq \mathbb{R}^{n-k}$ of $\overline{p} = (p_{k+1}, \cdots, p_n)$ and a smooth map $g : \widehat{U} \to \overline{U}$ such that

$$N \cap (\widehat{U} \times \overline{U}) = \{(\widehat{x}, \overline{x}) \in \widehat{U} \times \overline{U} \,|\, \overline{x} = g(\widehat{x})\} = \mathrm{graph}(g) \quad . \tag{8.14}$$

4. For each $p \in N$ there exist an open neighborhood U of p in $\mathbb{R}^n$, an open set V in $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, and a diffeomorphism $\psi : U \to V$ such that

$$\psi(N \cap U) = V \cap (\mathbb{R}^k \times \{0\}) \subseteq \mathbb{R}^k \times \{0\} \quad . \tag{8.15}$$

It follows that we can represent not only the n-dimensional manifold M^n locally by an open set $\mathcal{V}$ in $\mathbb{R}^n$ but also the submanifold locally in $\mathcal{V}$ by a regular map (given even as a graph map if we want) from an open set $\mathcal{U}$ in $\mathbb{R}^k$ by a smooth regular parametrization $\phi : \mathcal{U} \to \mathcal{V}$. This observation – which, of course – has perfect anchorage in our previous discussions (in the first chapters of these notes) of parametrized surfaces in $\mathbb{R}^3$ will be the (notational) basis for our next chapter on Riemannian (sub)manifolds.

EXERCISE 8.26

Let M denote the figure eight in $\mathbb{R}^2$ considered in examples 8.16 and 8.16. Show directly (without assuming the stated equivalence in Theorem 8.25) that M does not satisfy any of the properties 2, 3, or 4, in that theorem.

Example 8.27

Let P be the paraboloid of revolution, that we have considered before as the subset of $\mathbb{R}^3$ given for example by the regular parametrization:

$$r(u^1,u^2) = (u^1,u^2,(u^1)^2+(u^2)^2) \quad , \quad (u^1,u^2) \in \mathbb{R}^2 \quad . \tag{8.16}$$

The surface P is a submanifold of dimension 2 in $\mathbb{R}^3$ – it is a hypersurface in $\mathbb{R}^3$. Indeed, let f denote the following smooth function on $\mathbb{R}^3$:

$$f(x^1,x^2,x^3) = x^3-(x^1)^2-(x^2)^2 \tag{8.17}$$

Then the Jacobian matrix for f is

$$J_f(x^1,x^2,x^3) = (-2x^1,-2x^2,1) \neq (0,0,0) \quad \text{for all } (x^1,x^2,x^3) \in \mathbb{R}^3, \tag{8.18}$$

so that f is regular in all of $\mathbb{R}^3$. Moreover, $P = f^{-1}(0)$, so P is a submanifold of $\mathbb{R}^3$ according to statement 2 in theorem 8.25.
Statement 3 applies directly because P is already 'born' as a graph via a smooth function g:

$$P = \{(x^1,x^2,x^3) \in \mathbb{R}^3 \,|\, x^3 = g(x^1,x^2) = (x^1)^2+(x^2)^2\} \quad . \tag{8.19}$$

Also statement 4 can be applied by finding a diffeomorphism of (in this case all of) $\mathbb{R}^3$ into (in fact all of) $\mathbb{R}^3$ which will 'flatten' P to the (u^1,u^2)-plane in the target space. Indeed, let

$$\psi(x^1,x^2,x^3) = (u^1,u^2,u^3) = (x^1,x^2,x^3-(x^1)^2-(x^2)^2) \quad \text{for all } (x^1,x^2,x^3) \in \mathbb{R}^3 \quad . \tag{8.20}$$

Then ψ is a diffeomorphism and the restriction of ψ to P is the (u^1,u^2)-plane.

EXERCISE 8.28

Prove that the map ψ defined in 8.20 is a diffeomorphism, find its inverse map ψ^{-1} and the two Jacobians J_ψ and $J_{\psi^{-1}}$.

Not every manifold M^n allows such an 'embedding' in toto as a hypersurface into $\mathbb{R}^{n+1}$ as does the paraboloid (case $n=2$) in the example 8.27 above. However, it was shown by H. Whitney in 1936 that if the topology of M^n is second countable and Hausdorff (in the sense defined in chapter 0), then M^n can be 'embedded' as a submanifold in $\mathbb{R}^{2n}$. A non-orientable surface (like for example the Klein bottle and the Boy's surface) can only be 'immersed', i.e. with self-intersections, into $\mathbb{R}^3$.

We must also note, that the notions of manifold and submanifold as defined above can be extended to 'manifolds and submanifolds with boundaries' if the boundaries are sufficiently well behaved. This is discussed and implemented thoroughly in e.g. the monograph by J. Lee, [18]. For example the closed unit disc in exercise 8.3 is an example of a manifold with boundary as well as it is an example of a submanifold with boundary in $\mathbb{R}^2$. Moreover, as hinted in the above paragraph, it is also possible to define and study immersed submanifolds with self-intersections – if the intersections are sufficiently well behaved. The figure eight curve with its self-intersection at $(0,0)$ as displayed in figure 8.3 is an example of an immersed 1-dimensional submanifold of $\mathbb{R}^2$.

Example 8.29

Any k-dimensional sphere $\mathbb{S}^k$ which is centered at the origin in the punctured space $V = \mathbb{R}^n - \{0\}$, $n > k$, is a submanifold of $\mathbb{R}^n$. Indeed, the map $f : \mathbb{R}^n - \{0\} \to \mathbb{R}^{n-k}$ defined by

$$f(x) = \left(1 - \sum_{i=1}^{k+1} (x^i)^2\,,\, x^{k+2}, \cdots, x^{k+(n-k)}\right) \tag{8.21}$$

is a smooth regular map (the Jacobian has everywhere maximal rank k) with

$$\mathbb{S}^k \cap V = f^{-1}(0) = \{x \in \mathbb{R}^n \,|\, f(x) = 0\} \quad , \tag{8.22}$$

so that $\mathbb{S}^k$ is a submanifold of $\mathbb{R}^n$ according to property 2. in Theorem 8.25.

EXERCISE 8.30

Why can't we use $V = \mathbb{R}^n$ (without puncturing) for the argument in example 8.29?

EXERCISE 8.31

Show – e.g. in a similar way as in example 8.29 – that any torus in $\mathbb{R}^3$, e.g. as the one on display in figure 8.7 is a submanifold of $\mathbb{R}^3$.

EXERCISE 8.32

Show that the graph G of the function $f(x) = \sin(1/x)$ for $x > 0$ is a 1-dimensional submanifold of $\mathbb{R}^2$. Show that $G \cup \{(0,y) \in \mathbb{R}^2 \,|\, y \in [-1,1]\}$ is not a submanifold of $\mathbb{R}^2$.

Chapter 9

Riemannian manifolds and submanifolds

9.1 The metrics

We recall from definition 8.14 that an abstract submanifold N^k of an ambient manifold M^n with $n > k$ has a local representation as a submanifold in $\mathbb{R}^n$ via the charts (ψ_α, V_α) in any atlas $\mathcal{A}$ for M. In other words, $\psi_\alpha(V_\alpha \cap N)$ is a submanifold of $\mathbb{R}^n$ in the sense of Theorem 8.25 in section 8.3. Thus the submanifold representations of N in $\mathbb{R}^n$ simply 'follows' the chart maps for the local representation of the ambient manifold M as an open set in $\mathbb{R}^n$.

Now suppose that M is equipped with a globally well-defined metric tensor h in the sense that $(\psi_\alpha V_\alpha, h)$ is a local Riemannian manifold for every chart (ψ_α, V_α) in any atlas for M. Note in particular, that this implies that g is invariant in the sense of 1.50 and 1.51 under every local diffeomorphism $\phi = \psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(V_\alpha \cap V_\beta) \to \psi_\beta(V_\alpha \cap V_\beta)$.

Definition 9.1 When h satisfies this invariance under shift of charts, (M, h) is then called a global Riemannian manifold.

The submanifold now becomes a global Riemannian manifold (N, g) in its own right, simply by inheriting its metric tensor g from h as follows:

Definition 9.2 Let X and Y denote two vectors in the tangent space T_pN to N at $p \in N$. Since T_pN is a vector subspace of T_pM, we simply define:

$$g(X,Y) = h(X,Y) \quad \text{for all } X \text{ and } Y \text{ in } T_pN \quad . \tag{9.1}$$

With reference to definition 8.14 in Chapter 8 we have the following chart images at our disposal $\mathcal{U}^k \subset \mathbb{R}^k$ and $\mathcal{V} \subset \mathbb{R}^n$. In effect we get two local Riemannian manifolds, $(\mathcal{U}^k, g)$ and $(\mathcal{V}^n, h)$, and, not to forget, a regular parametrization $\phi : \mathcal{U} \to \mathbb{R}^n$ of the k-dimensional submanifold S in $\mathcal{V}^n$. Note that g acts on vectors in TN by inheritance from the action of h on vectors in TM according to Definition 9.1. Below we will, however, always represent g purely in the domain $(\mathcal{U}^k, g)$ – by its matrix representation G there – as we have previously done for local Riemannian manifolds. This is done via the Jacobian of the map ϕ as spelled out in the next section.

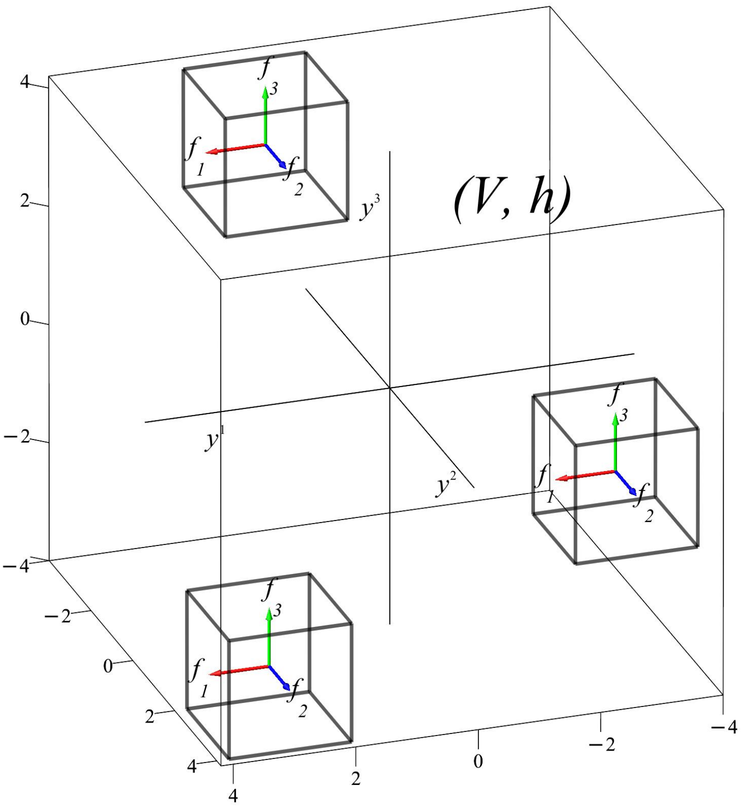


Figure 9.1: A general illustration of a 3-dimensional chart image $\mathcal{V}$ with standard coordinates $\{y^1, \cdots, y^n\}$ and corresponding basis vector fields $\{f_1, \cdots, f_n\}$ in each of indicated tangent spaces.

9.2 The lift of vector (fields) to a submanifold

The main purpose of the present chapter is to investigate the local interplay between the geometry of (N^k, g) and (M^n, h) via our previous findings for local Riemannian manifolds, i.e. in this case $(\mathcal{U}^k, g)$ and $(\mathcal{V}^n, h)$ represented in open sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{R}^k$ and $\mathbb{R}^n$, respectively.

In the following we will therefore refer explicitly to coordinates $\{x^1, \cdots, x^k$ together with the canonical basis $\{e_1, \cdots, e_k\}$ in the domain $\mathcal{U} \in \mathbb{R}^k$, and coordinates $\{y^1, \cdots, y^n\}$ together with the canonical basis $\{f_1, \cdots, f_n\}$ in the domain $\mathcal{V} \in \mathbb{R}^n$.

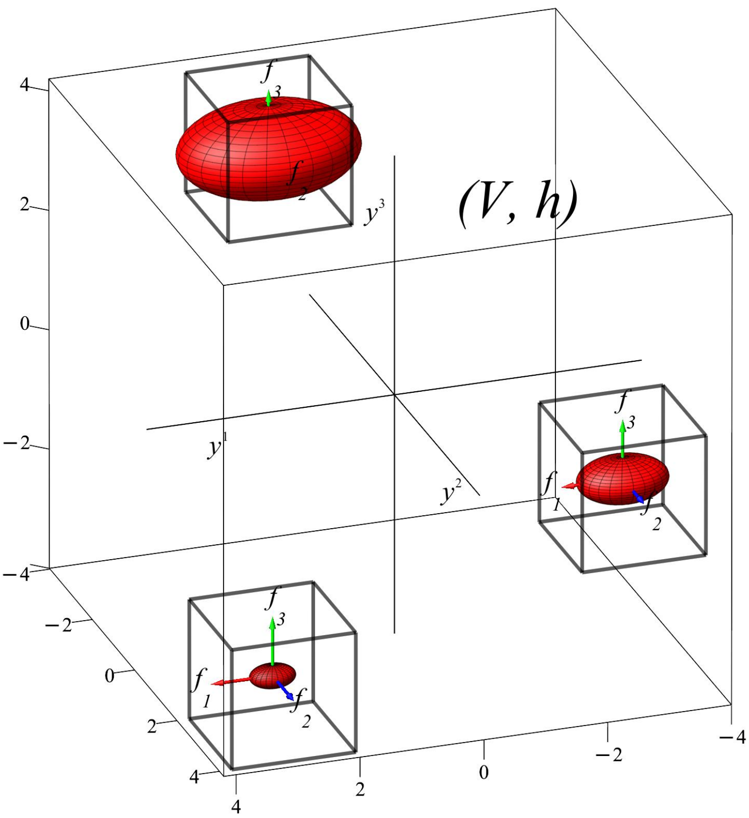


Figure 9.2: A general 3-dimensional local Riemannian chart image $(\mathcal{V}, h)$ with ellipsoidal indicatrices for h displayed in each tangent space.

As always, the $(n \times k)$ Jacobian matrix of the map ϕ plays a key role for lifting vector fields from $\mathfrak{X}(U)$ to $\mathfrak{X}(S)$. At a point $p = (x^1, \cdots, x^k)$ we have

$$J_\phi = \begin{bmatrix} \frac{\partial \phi^1(x^1,\cdots,x^k)}{\partial x^1} & . & \frac{\partial \phi^1(x^1,\cdots,x^k)}{\partial x^k} \\ . & . & . \\ \frac{\partial \phi^n(x^1,\cdots,x^k)}{\partial x^1} & . & \frac{\partial \phi^n(x^1,\cdots,x^k)}{\partial x^k} \end{bmatrix} \quad . \tag{9.2}$$

Proposition 9.3 Let $\widetilde{X} \in \mathfrak{X}(\mathcal{U})$ be a vector field on $\mathcal{U}$. Then for each $p \in \mathcal{U}$ we have $X_{\phi(p)} = J_\phi(\widetilde{X}_p)$, and X is thence a vector field on $S = \phi(\mathcal{U})$ in $\mathcal{V}$.

Note, that we still use the shorthand notation $J_\phi(\widetilde{X})$ for the (horizontal) coordinate vector expression $(J_\phi \cdot \widetilde{X}^*)^*$ as was introduced in the notation box 2.4 in Chapter 2.

Suppose $X \in T_{\phi(p)}S$ is the image by J_ϕ of $\widetilde{X} \in T_p\mathcal{U}$, then we will often write $J_\phi(\widetilde{X})$ instead of

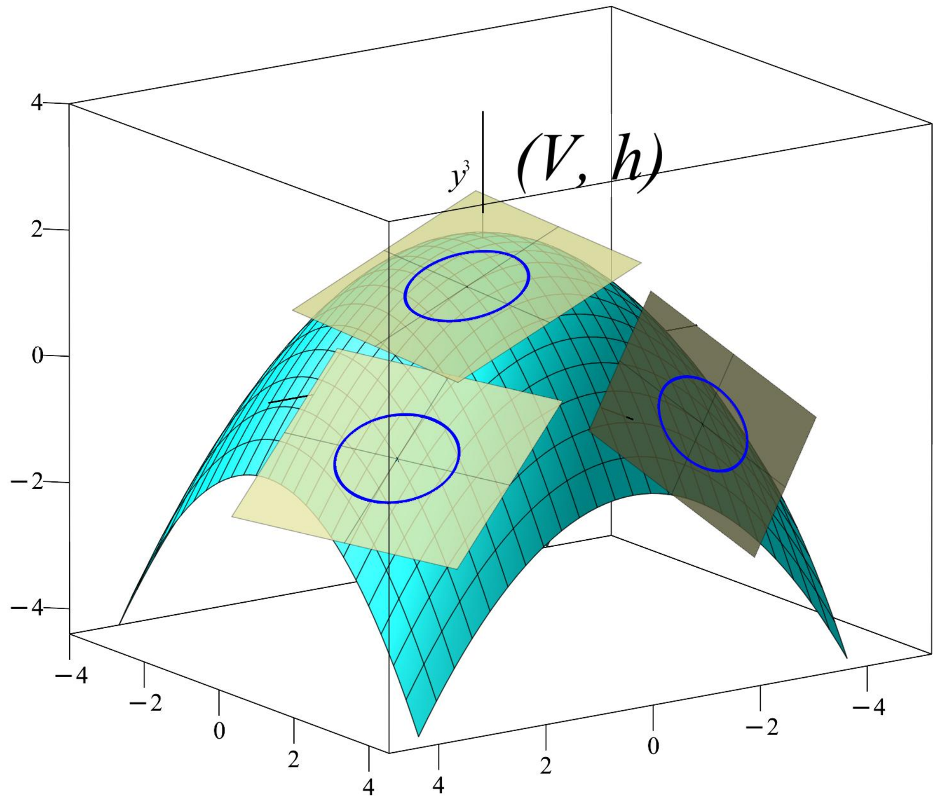


Figure 9.3: A submanifold (surface) S in $(\mathcal{V}, h)$ with inherited indicatrices in each of its surface tangent planes. The construction of the restricted ellipses is by cutting the ambient ellipsoid indicatrix by the tangent plane at the respective points in S, as illustrated in figure 9.5 below.

the much shorter X. Similarly, since there is – or should be – no confusion concerning the basis vectors $\{e_1, \cdots, e_k\}$ in $T\,\mathcal{U}$, we just write $J_\phi(e_i)$ (without the $\widetilde{*}$) for the image vectors in TS.

In the same vein we also briefly repeat here how the matrix expression for a linear map $A : T_{\phi(p)}S \to T_{\phi(p)}S$ is obtained with respect to the (natural) basis $\{J_\phi(e_1), \cdots, J_\phi(e_k)\}$ in $T_{\phi(p)}S$.

We shall need these observations when considering the generalized Weingarten map for submanifolds below. Remember that $\{J_\phi(e_1), \cdots, J_\phi(e_k)\}$ is a basis because ϕ is everywhere regular. So

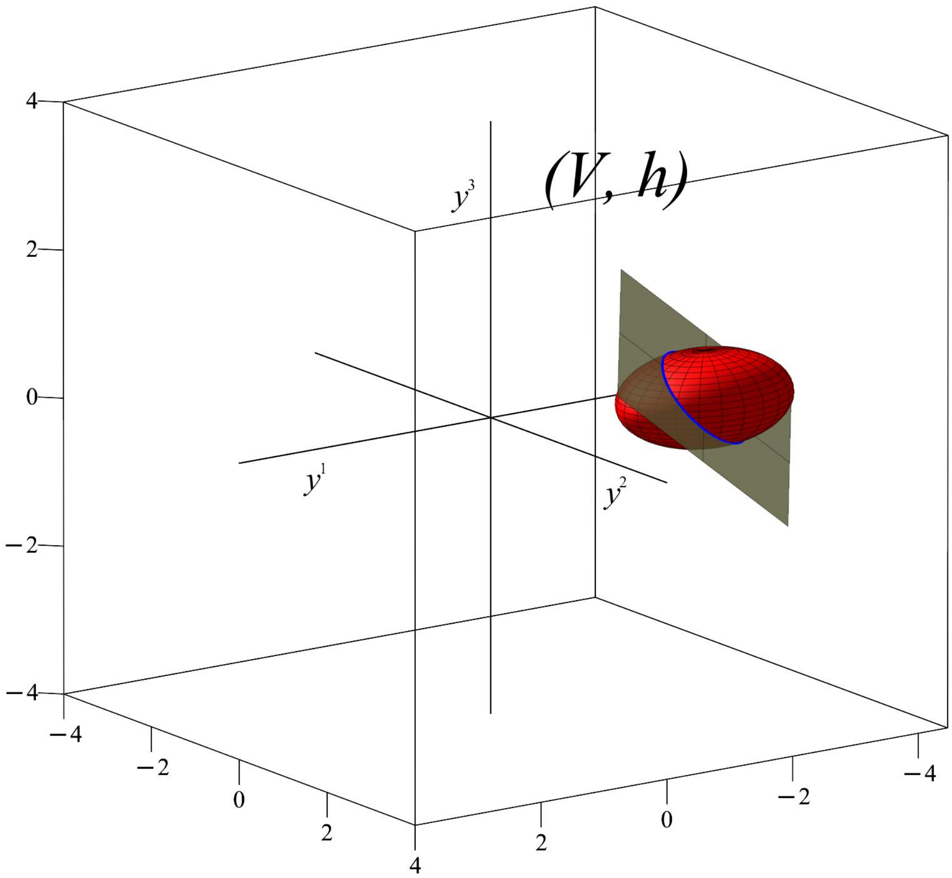


Figure 9.4: The construction of the restricted ellipse indicatrix in a tangent plane from the ambient ellipsoidal indicatrix for h at the point in question.

we let $U \in T_{\phi(p)}S$ and obtain the matrix expression at $\phi(p)$ as follows:

$$\begin{aligned} U &= \sum_{i=1}^{k} u^i \cdot J_\phi(e_i) \\ A(U) &= \sum_{i=1}^{k} u^i \cdot A(J_\phi(e_i)) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} u^i \cdot A_i^j(J_\phi(e_j)) \\ &= u^i \cdot A_i^j(J_\phi(e_j)) \quad \text{in shorthand index summation notation.} \end{aligned} \tag{9.3}$$

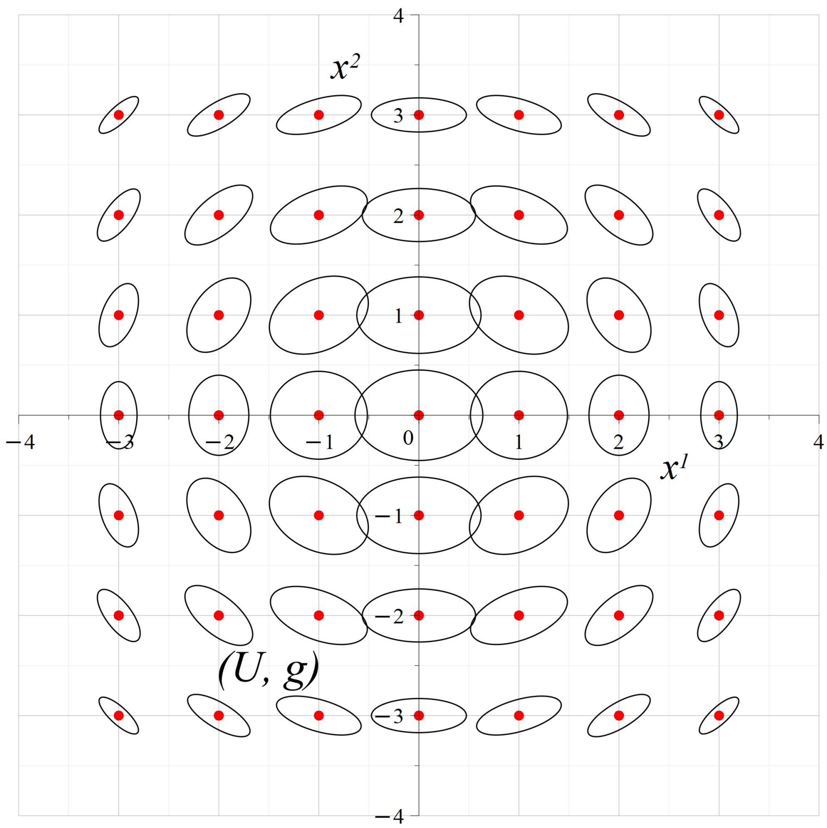


Figure 9.5: The indicatrix field of the inherited metric g in $\mathcal{U}$ from the surface S in $(\mathcal{V}, h)$.

In $T_p\mathcal{U}$ we have correspondingly from $J_\phi \widetilde{U} = U$ and $\widetilde{A}$ with respect to the basis $\{e_1, \cdots, e_k\}$:

$$\begin{aligned}\widetilde{U} &= \sum_{i=1}^{k} u^i \cdot e_i \\ \widetilde{A}(\widetilde{U}) &= \sum_{i=1}^{k} u^i \cdot \widetilde{A}(e_i) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} u^i \cdot \widetilde{A}_i^j e_j \\ &= u^i \cdot \widetilde{A}_i^j e_j \quad \text{in shorthand index summation notation}\end{aligned} \tag{9.4}$$

and, since $J_\phi(\widetilde{U}) = U$ we get

$$\begin{aligned} J_\phi(\widetilde{U}) &= U \\ J_\phi(u^i \cdot \widetilde{A}_i^j e_j) &= u^i \cdot A_i^j (J_\phi(e_j)) \\ u^i \cdot \widetilde{A}_i^j J_\phi(e_j) &= u^i \cdot A_i^j (J_\phi(e_j)) \quad , \end{aligned} \tag{9.5}$$

so that the two matrix expressions are identical:

$$A_i^j(\phi(p)) = \widetilde{A}_i^j(p) \quad . \tag{9.6}$$

In coordinates, the metric matrices $H_{\mathcal{V}}$ and $G_{\mathcal{U}}$ for metric tensor h in $(\mathcal{V}, h)$ and for the inherited metric tensor g in $(\mathcal{U}, g)$ are then at $\phi(p)$ and at p via the inheritance expressed by (9.1):

$$\begin{aligned} H_{\mathcal{V}}(\phi(p))_{ij} &= h_{\phi(p)}(f_i\,,\, f_j) \quad \text{for } i, j \in \{1, \cdots, n\} \\ G_{\mathcal{U}}(p)_{\ell m} &= h_{\phi(p)}(J_{\phi(p)}(e_\ell)\,,\, J_{\phi(p)}(e_m)) \quad \text{for } \ell, m \in \{1, \cdots, k\} \quad , \end{aligned} \tag{9.7}$$

so that explicitly $G = J_\phi^* \cdot H \cdot J_\phi$ in the following sense:

$$G_{\mathcal{U}}(p) = J_{\phi(p)}^* \cdot H_{\mathcal{V}}(\phi(p)) \cdot J_{\phi(p)} \quad \text{for } \ell \text{ and } m \text{ in } \{1, \cdots, k\} \quad . \tag{9.8}$$

With the metrics in $(\mathcal{U}, g)$ and $(\mathcal{V}, h)$ we also obtain unique and well-defined Levi-Civita connections ∇^g and ∇^h in the two local Riemannian manifolds, and thence the corresponding geodesics, curvature tensors, etc. as well.

Why do we bother to use the much more cumbersome $J_\phi(\widetilde{X})$ for the same vector X? There are at least two reasons: Firstly, it is good to remind ourselves by this explicit notation that a vector field X along the submanifold S is in $\mathfrak{X}(S)$ and not in $\mathfrak{X}(\mathcal{V})$. Secondly, in order to *calculate* for example a covariant derivative $\nabla_X^h Y$ for $X \in T_{\phi(p)}S$ and $Y \in \mathfrak{X}(\mathcal{V})$ we need to apply the explicit expression $J_\phi(\widetilde{X})$ anyway as follows:

$$\nabla_X^h Y = \nabla_{J_\phi(\widetilde{X})}^h Y \quad . \tag{9.9}$$

We now want to compare the geometry of $(\mathcal{U}, g, \nabla^g)$ and $(\mathcal{V}, h, \nabla^h)$ when g is inherited from h in the way we have described above.

9.3 The second fundamental form

With the aid of the metric h in $(\mathcal{V}, h)$ we can decompose every given vector $Z \in T_{\phi(p)}\mathcal{V}$ at a point $\phi(p)$ of $\phi(\mathcal{U}) = S$ uniquely into two orthogonal components as follows:

$$Z = Z^\top + Z^\perp \quad . \tag{9.10}$$

Here

$$Z^\perp = Z - \sum_{i=1}^k h(Z, E_i) \cdot E_i \quad , \tag{9.11}$$

where $\{E_1, \cdots, E_k\}$ denotes any h-orthonormal basis of $T_{\phi(p)}S \subset T_{\phi(p)}\mathcal{V}$.

In particular we can apply this decomposition to covariant derivatives along S:

Definition 9.4 The second fundamental form II of S in $(\mathcal{V}, h, \nabla^h)$ is defined as the following operator on $\mathfrak{X}(S) \times \mathfrak{X}(S)$ (with the J_ϕ interpretation also inserted):

$$\begin{aligned} II(X,Y) &= (\nabla^h_X Y)^\perp \\ &= (\nabla^h_{J_\phi(\widetilde{X})} J_\phi(\widetilde{Y}))^\perp \quad \text{for } X = J_\phi(\widetilde{X}),\, Y = J_\phi(\widetilde{Y}) \in \mathfrak{X}(S) \quad . \end{aligned} \tag{9.12}$$

Note that in general $\nabla^h_X Y$ is not necessarily tangent to S – even though X and Y are tangent vector fields in $\mathfrak{X}(S)$. So $(\nabla^h_X Y)^\perp$ is usually (but not always) a non-zero vector.

The second fundamental form has nice properties, which follow from properties of the connection ∇^h, see [19, Proposition 8.1]:

Proposition 9.5

1. $II(X,Y)$ is symmetric in X and Y
2. $II(X,Y)$ at $p \in S$ depends only on X_p and Y_p
3. $II(f \cdot X, Y) = f \cdot II(X,Y)$ for all $f \in \mathfrak{F}(S)$, i.e. the second fundamental form is bilinear in X and Y.
4. It follows that II can be 'polarized' as follows

$$II(X,Y) = \left(\frac{1}{2}\right) \cdot \left(II(X+Y, X+Y) - II(X,X) - II(Y,Y)\right) \quad . \tag{9.13}$$

The second fundamental form is a measure of the difference between covariant differentiation on S, i.e. in $(\mathcal{U}, g, \nabla^g)$, and covariant differentiation in $(\mathcal{V}, h, \nabla^h)$:

Theorem 9.6 (Gauss' formula.) Suppose that $X = J_\phi(\widetilde{X})$ and $Y = J_\phi(\widetilde{Y})$ are two vector fields on S stemming from the two vector fields $\widetilde{X}$ and $\widetilde{Y}$ in $(\mathcal{U}, g, \nabla^g)$. Then the lift to S of $\nabla^g_{\widetilde{X}} \widetilde{Y}$ by J_ϕ is precisely the tangential component of $\nabla^h_X Y$:

$$\nabla^h_X Y = J_\phi\left(\nabla^g_{\widetilde{X}} \widetilde{Y}\right) + II(X,Y) \quad . \tag{9.14}$$

Proof. We only need to show that

$$\left(\nabla^h_X Y\right)^\top = J_\phi\left(\nabla^g_{\widetilde{X}}\widetilde{Y}\right) \tag{9.15}$$

at all points in $\mathcal{U}$. For this we define an auxiliary map

$$\nabla^\top : \mathfrak{X}(\mathcal{U}) \times \mathfrak{X}(\mathcal{U}) \to \mathfrak{X}(\mathcal{U}) \tag{9.16}$$

as follows:

$$J_\phi\left(\nabla^\top_{\widetilde{X}}\widetilde{Y}\right) = \left(\nabla^h_X Y\right)^\top \quad . \tag{9.17}$$

A check then shows that $\nabla^\top$ is a symmetric connection on $\mathcal{U}$ which is compatible with the metric g, so it is a Levi-Civita connection on $(\mathcal{U}, g)$. But since there is precisely only one such connection, it must be identical to ∇^g, i.e.

$$\left(\nabla^h_X Y\right)^\top = J_\phi\left(\nabla^g_{\widetilde{X}}\widetilde{Y}\right) \quad . \tag{9.18}$$

□

In the study of the classical differential geometry for surfaces in Euclidean space the second fundamental form usually appears in connection with the study of curves on surfaces. We now study this in the more general setting of Riemannian submanifolds.

Recall, that the covariant derivative of a vector field X along a curve γ in $(\mathcal{V}, h, \nabla^h)$, i.e. $X \in \mathfrak{X}(\gamma)$, is the well-known $\nabla^h_{\gamma'}(X)$ – see section 3.7 in Chapter 3. We shall also now need and compare with $\nabla^g_{\eta'}(\cdot)$ for covariant derivation along curves η in $(\mathcal{U}, g, \nabla^g)$ – in particular those curves η that are pre-images of the curves γ, i.e. $\gamma(t) = \phi(\eta(t))$.

Corollary 9.7 (Gauss' formula for curves.) In the general setting above, i.e. with $\phi(\mathcal{U}) = S$ a submanifold of $(\mathcal{V}, h, \nabla^h)$, suppose that $\gamma : I \to S$ is a smooth curve in $\phi(\mathcal{U})$ with pre-image η in $\mathcal{U}$, i.e. $\gamma(t) = \phi(\eta(t))$ for all $t \in I$. Then we have for every smooth vector field $X \in \mathfrak{X}(\gamma)$ along γ with $J_\phi(\widetilde{X}) = X$:

$$\nabla^h_{\gamma'}(X) = J_\phi\left(\nabla^g_{\eta'}(X)\right) + II(\gamma', X) \quad . \tag{9.19}$$

Proof. Not surprisingly, this follows directly from the general Gauss formula 9.6 upon noting that $J_\phi(\eta'(t)) = \gamma'(t)$. □

Corollary 9.8 It follows in particular, that for any smooth curve γ in in a hypersurface $S^{k=n-1}$ with a well-defined unit normal vector field N we get:

$$h\left(\nabla^h_{\gamma'}\gamma', N\right) = h\left(II(\gamma', \gamma'),, N\right) \quad , \tag{9.20}$$

and this becomes useful for obtaining information about $II(X,Y)$ in general via the polarization mentioned in Proposition 9.5, (9.13).

Proposition 9.9 (**The Weingarten equation**) For every $\widetilde{X} \in \mathfrak{X}(\mathcal{U})$ and normal vector field N to $S = \phi(\mathcal{U})$ we have, again with $X = J_\phi(\widetilde{X})$:

$$\left(\nabla^h_X N\right)^\top = \left(\nabla^h_{J_\phi(\widetilde{X})} N\right)^\top = -W_N(X) \quad . \tag{9.21}$$

Proof. The following shows a useful relation between $\nabla^h_X N$ and $W_N X$ for every $Y = J_\phi \widetilde{Y}$:

$$\begin{aligned} 0 &= X\left(h(N,Y)\right) \\ &= \left(\nabla^h_X N, Y\right) + h\left(N, \nabla^h_X Y\right) \\ &= \left(\nabla^h_X N, Y\right) + h\left(N, J_\phi(\nabla^g_{\widetilde{X}} \widetilde{Y}) + II(X,Y)\right) \\ &= \left(\nabla^h_X N, Y\right) + h\left(N, II(X,Y)\right) \\ &= \left(\nabla^h_X N, Y\right) + h\left(W_N, Y\right) \\ &= \left(\nabla^h_X N + W_N(X), Y\right) \quad , \end{aligned} \tag{9.22}$$

so that

$$0 = \left(\nabla^h_X N + W_N(X)\right)^\top = \left(\nabla^h_X N\right)^\top + W_N(X) \quad . \tag{9.23}$$

□

The curvature (intrinsic and extrinsic) of smooth curves in S can now be interpreted via the second fundamental form of S in $\mathcal{V}$ – as in the case of surfaces in Euclidean 3-space:

Proposition 9.10 Suppose $p \in \mathcal{U}$ and $\widetilde{V}$ a g-unit vector in $T_p\mathcal{U}$ and let $\eta(s)$ denote the g-geodesic in $(\mathcal{U}, g, \nabla^g)$ with $\eta(0) = p$ and $\eta'(0) = \widetilde{V}$. Then, for $V = J_\phi(\widetilde{V})$, $II(V,V)$ is the h-curvature at $\phi(p)$ of the curve $\gamma = \phi(\eta)$ in $(\mathcal{V}, h, \nabla^h)$, i.e. at $s = 0$:

$$II(V,V)_{s=0} = \kappa_\gamma(0) \cdot N = \nabla^h_{\gamma'(0)}(\gamma'(s)) \quad . \tag{9.24}$$

Proof. This follows from an application of the Gauss formula to the tangent vector field γ' along γ:

$$\nabla^h_{\gamma'}(\gamma') = J_\phi\left(\nabla^g_{\eta'}(\eta')\right) + II(\gamma', \gamma') \quad , \tag{9.25}$$

which, since η is a g-geodesic in $(\mathcal{U}, g, \nabla^g)$, reduces to

$$\nabla^h_{\gamma'}(\gamma') = II(\gamma', \gamma') \quad . \tag{9.26}$$

□

9.4 Riemannian hypersurfaces

When $k = n - 1$ the second fundamental form of S^{n-1} in $(V^n, h, \nabla^h$ produces vectors that are h-orthogonal to S. Suppose we have chosen a smooth unit normal vector field N on S. Then we can replace the vector valued form II by a simpler scalar valued form q as follows:

$$\begin{aligned} q(X,Y) &= h(N, II(X,Y)) = h(N, \nabla^h_X Y) \\ II(X,Y) &= q(X,Y) \cdot N \quad . \end{aligned} \tag{9.27}$$

Definition 9.11 The quadratic form q is a symmetric tensor field of type 2 which gives rise to the so-called Weingarten map (or shape operator) $W_N : \mathfrak{X}(S) \to \mathfrak{X}(S)$, with respect to the chosen normal vector field N as follows:

$$h(W_N(X), Y)) = q(X,Y) = h(X, W_N(Y))) \quad . \tag{9.28}$$

The Weingarten map was (in its matrix form) mentioned already in Chapter 0, subsection 0.5.5, where it was instrumental for the definition and calculation of the *intrinsic* Gaussian curvature and the mean curvature of surfaces in Euclidean 3-space. Before we generalize this to general hypersurfaces in Riemannian manifolds we observe the following:

Proposition 9.12 Let N denote a smooth unit normal vector field along a Riemannian hypersurface S in $(\mathcal{V}, h, \nabla^h)$. Then for all $\widetilde{X} \in \mathcal{U}$ and $X = J_\phi(\widetilde{X})$:

$$\nabla^h_X N = -W_N(X) \tag{9.29}$$

Proof. The general Weingarten equation (9.21) is

$$\left(\nabla^h_X N\right)^\top = -W_N(X) \quad , \tag{9.30}$$

and since $h(N, N) = 1$ we get

$$h\left(\nabla^h_X N \, , \, N\right) = \left(\frac{1}{2}\right) X(h(N, N)) = 0 \quad , \tag{9.31}$$

it follows that $\nabla^h_X N$ is h-orthogonal to N so

$$\left(\nabla^h_X N\right)^\top = \nabla^h_X N \quad , \tag{9.32}$$

and thence the claim. □

At each point $p \in S$, the generalized W_N defined by (9.28) is a selfadjoint linear operator (endomorphism) of the tangent space T_pS and as such it has precisely (with multiplicities) $k = n - 1$ real eigenvalues $\{\lambda_1, \cdots, \lambda_k\}$ with corresponding h-orthonormal eigenvectors $\{b_1, \cdots, b_k\}$ which span the tangent space T_pS. As in the classical (hyper-)surface cases in Euclidean 3-space, the eigenvalues are called the extrinsic principal curvatures of S at p, and the corresponding h-unit eigenvectors are called the principal directions.

9.5 Extrinsic curvature expressions

The above observations lead us directly to the following general curvature definitions for Riemannian hypersurfaces:

Definition 9.13 The extrinsic Gaussian curvature of the Riemannian hypersurface $S^{k=n-1} \subset (\mathcal{V}^n, h, \nabla^h)$ is defined as

$$K = \det(W_N) = \lambda_1 \cdot \lambda_2 \cdots \lambda_k \quad , \quad k = n - 1 \quad . \tag{9.33}$$

The mean curvature of S is similarly

$$H = \left(\frac{1}{k}\right) \cdot \mathrm{trace}(W_N) = \left(\frac{1}{k}\right) \cdot (\lambda_1 + \cdots \lambda_k) \quad , \quad k = n-1 \quad . \tag{9.34}$$

The signs of K and H are sensitive to change of normal $N \to -N$. The mean curvature changes sign and the Gaussian curvature is multiplied by $(-1)^k$.

It is important to note, that the extrinsic Gaussian curvature K of S defined above is in general *not* equal to any of the intrinsic curvatures of S (with its induced metric g), i.e sectional-, Ricci-, or scalar curvatures. This holds true even for 2-dimensional surfaces in 3-dimensional ambient Riemannian spaces. We refer to the examples below, where simple surfaces in the 3-dimensional Poincaré half space illustrates this claim.

We now establish the formal anchorage to the classical surface theory by expressing the second fundamental form II – and thence W_N – of a Riemannian hypersurface in terms of the second partial derivatives of its parametrization map $\phi : \mathcal{U} \to \mathcal{V}$, i.e. expressed in the coordinates $\{x^1, \cdots, x^{n-1}\}$ of $\mathcal{U}$:

Proposition 9.14 Let $S^{n-1} = \phi(\mathcal{U}^{n-1})$ be a parametrization of a Riemannian hypersurface in $(\mathcal{V}, h, \nabla^h)$, i.e. using explicitly the coordinates in $\mathcal{U}$ and in $\mathcal{V}$ we have:

$$S : \phi\left(x^1, \cdots, x^{n-1}\right) = \left(\phi^1\left(x^1, \cdots, x^{n-1}\right), \cdots, \phi^n\left(x^1, \cdots, x^{n-1}\right)\right) \quad . \tag{9.35}$$

Then the matrix version Q (in the $\mathcal{U}$-coordinates) of the scalar second fundamental form q of S is obtained as follows at each point in $\mathcal{U}$:

$$Q_{ij} = q\left(J_\phi(e_i), J_\phi(e_j)\right) = h\left(\nabla^h_{J_\phi(e_i)} J_\phi(e_j), N\right) \quad , \tag{9.36}$$

As a consequence of the bilinearity of q we therefore have the following expression:

Corollary 9.15 Let $\widetilde{V} = \sum_{i=1}^{n-1} \widetilde{v}^i \cdot e_i$ and $\widetilde{W} = \sum_{j=1}^{n-1} \widetilde{w}^j \cdot e_j$ so that:

$$\begin{aligned} V &= J_\phi(\widetilde{V}) = J_\phi\left(\sum_{i=1}^{n-1} \widetilde{v}^i \cdot e_i\right) = \sum_{i=1}^{n-1} \widetilde{v}^i \cdot J_\phi(e_i) \in \mathfrak{X}(S) \\ Z &= J_\phi(\widetilde{Z}) = J_\phi\left(\sum_{j=1}^{n-1} \widetilde{z}^j \cdot e_i\right) = \sum_{j=1}^{n-1} \widetilde{z}^i \cdot J_\phi(e_j) \in \mathfrak{X}(S) \quad . \end{aligned} \tag{9.37}$$

Then

$$q(V,Z) = \sum_{ij} Q_{ij} \cdot \tilde{v}^i \cdot \tilde{z}^j \quad . \tag{9.38}$$

Note, that in a Euclidean space $(\mathcal{V}, h, \nabla^h)$ we have

$$\nabla^h_{J_\phi(e_i)} J_\phi(e_j) = \frac{\partial^2 \phi}{\partial x^i \partial x^j} \quad , \tag{9.39}$$

so that Q_{ij} then becomes the classical expression for the second fundamental form matrix $\mathcal{F}_{II}$ for hypersurfaces in such ambient spaces – in particular in the well known cases of surfaces in 3D as discussed in Chapter 0, subsection 0.5.4, since h is in that case just the usual scalar (dot) product.

Proof. (Proof of 9.14) We have

$$h(J_\phi(e_j), N) = 0 \quad , \tag{9.40}$$

and thence from covariant differentiation in direction $J_\phi(e_i)$:

$$\begin{aligned} 0 &= \nabla^h_{J_\phi(e_i)} \left(h(J_\phi(e_j), N)\right) \\ &= h\left(\nabla^h_{J_\phi(e_i)} J_\phi(e_j), N\right) + h\left(J_\phi(e_j), \nabla^h_{J_\phi(e_i)} N\right) \\ &= h\left(\nabla^h_{J_\phi(e_i)} J_\phi(e_j), N\right) - h\left(J_\phi(e_j), W_N(J_\phi(e_i))\right) \\ &= h\left(\nabla^h_{J_\phi(e_i)} J_\phi(e_j), N\right) - q\left(J_\phi(e_i), J_\phi(e_j)\right) \quad , \end{aligned} \tag{9.41}$$

where we have used the Weingarten equation for hypersurfaces and the symmetry of the second fundamental form q in the two last steps. □

The matrix version of the Weingarten map is now:

Proposition 9.16 The shape operator, the Weingarten map, W_N for a Riemannian hypersurface (with respect to a chosen orthogonal unit vector field N) can now be represented by a matrix function, which we will also denote by W_N, of the coordinates $\{x^1, \cdots, x^{n-1}\}$. We let Q be the $(k \times k)$-matrix version of the second fundamental form as obtained in Proposition 9.14:

$$Q_{ij} = q(J_\phi(e_i), J_\phi(e_j)) \quad , \quad i,j \in \{1, \cdots, k = n-1\} \quad , \tag{9.42}$$

and where – as usual – G is the inherited metric matrix from h in $(\mathcal{V}, h, \nabla^h)$:

$$G_{ij} = h(J_\phi(e_i), J_\phi(e_j)) \quad , \quad i,j \in \{1, \cdots, k = n-1\} \quad . \tag{9.43}$$

$$W_N = G^{-1} \cdot Q \quad , \tag{9.44}$$

where G^{-1} as usual denotes the inverse of the regular $(k \times k)$ matrix G.

Proof. This follows directly from the definition of the Weingarten map in terms of the second fundamental form:

$$q(V,Z) = h(W_N(V), Z) \quad , \tag{9.45}$$

while remembering, that g is inherited from h:

$$Q = G \cdot \mathcal{W}_N \tag{9.46}$$

which, of course, resembles the well-known relation for surfaces in Euclidean space $(\mathbb{R}^3, g_E, \nabla^{g_E}$ – as spelled out in section 0.5.5 in Chapter 0:

$$\mathcal{F}_{II} = \mathcal{F}_I \cdot \mathcal{W} \quad . \tag{9.47}$$

□

The (computationally) simplest and direct way to find the matrix Q for S in $(\mathcal{V}, h, \nabla^h)$ is by way of Gauss' formula for curves $\gamma(t)$ in S, i.e.

$$h\left(\nabla^h_{\gamma'}\gamma',\, N\right) = q(\gamma', \gamma') \quad , \tag{9.48}$$

from which we can obtain Q_{ii} by calculating the h-acceleration at $\phi(p)$ of the curve $\gamma(t) = \phi(p + t \cdot e_i)$ for each i and then polarize and get all values Q_{ij} for all $i,\, j \in \{1, \cdots, k = n-1\}$. See the example 9.18 below.

Corollary 9.17 The *extrinsic* Gauss curvature of the hypersurface S with respect to the unit normal vector field N in $(\mathcal{V}, h, \nabla^h)$ is then

$$K = \det(W_N) = \frac{\det(Q)}{\det(G)} \quad . \tag{9.49}$$

We will now elaborate by example/exercise on the previously mentioned 'warning' that even for 2-dimensional hypersurfaces $(\mathcal{U}, g, \nabla^g)$ in a 3-dimensional ambient Riemannian space $(\mathcal{V}, h, \nabla^h)$ we do *not* have that $K = \det(W_N)$ is equal to the intrinsic sectional curvature of $(\mathcal{U}, g, \nabla^g)$.

Example 9.18

We let $(\mathcal{V}, h, \nabla^h)$ denote the Poincaré upper half space in $\mathbb{R}^3$ with metric h represented by the matrix

$$H(y^1, y^2, y^3) = \left(\frac{1}{y^3}\right)^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (y^1, y^2, y^3) \in \mathcal{V} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \quad . \tag{9.50}$$

The surface S in this ambient space will be the simple plane through $p = (0, 0, 1)$ which is tilted by the Euclidean angle θ with respect to the y^3-axis:

$$\phi(x^1, x^2) = (x^1 \,,\, x^2 \cdot \cos(\theta) \,,\, 1 + x^2 \cdot \sin(\theta)) \quad , \quad (x^1, x^2) \in \mathcal{U} = \mathbb{R} \times]-1, \infty[\quad . \tag{9.51}$$

EXERCISE 9.19

Show that the induced metric g in $\mathcal{U}$ has the matrix representation in $\mathcal{U}$:

$$G(x^1, x^2) = \left(\frac{1}{1 + x^2 \cdot \sin(\theta)}\right)^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{9.52}$$

EXERCISE 9.20

Show that $(\mathcal{U}, g, \nabla^g)$ has the constant *intrinsic sectional curvature* function

$$\sec = -\sin^2(\theta) \tag{9.53}$$

EXERCISE 9.21

Show that $S = \phi(\mathcal{U})$ has the following constant *extrinsic curvature* function in $(\mathcal{V}, h, \nabla^h)$:

$$K = \det(W_N) = \cos^2(\theta) \quad . \tag{9.54}$$

Hints:

1. Any choice of normal vector field N to S is everywhere Euclidean orthogonal to S in $(\mathcal{V}, h)$. Why? Note, however, that if the h-length of N is 1 then that is not usually the case for the Euclidean length of N.

2. Let $\gamma_1(t) = \phi(x^1 + t \,,\, x^2)$, $\gamma_2(t) = \phi(x^1 \,,\, x^2 + t)$ and $\gamma_3(t) = \phi(x^1 + t \,,\, x^2 + t)$, $t \in]-\varepsilon, \varepsilon[$ denote three parametrized curve segments through $\phi(p) = \phi(x^1, x^2)$ in $\phi(\mathcal{U})$.

3. Find the $h-$covariant derivatives $\nabla^h_{\gamma'(0)} \gamma'(t)$ for each one of the three curves γ at $\phi(p)$.

4. Then find $II(\gamma'(0) \,,\, \gamma'(0))$ and use polarization of II to construct the second fundamental form matrix Q for the hypersurface S with respect to the chosen normal vector field N.

5. Construct the Weingarten matrix W_N and find its determinant K.

EXERCISE 9.22

In the same vein – and using the same strategy – as in example 9.18 above, instead of the plane we consider a vertical cylinder in the same upper half space model $(\mathcal{V}, h, \nabla^h)$:

$$\phi(x^1,x^2) = \left(R\cos(x^2)\,,\, R\sin(x^2)\,,\, x^1\right) \in \mathcal{U} =]0,\infty[\times]-\pi,\pi[\quad . \tag{9.55}$$

1. Show that the induced metric g in $\mathcal{U}$ for the cylinder has the following matrix representation in $\mathcal{U}$:

$$G(x^1,x^2) = \left(\frac{1}{x^1}\right)^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & R^2 \end{bmatrix} \quad . \tag{9.56}$$

2. Show that the intrinsic curvature of $(\mathcal{U}, g, \nabla^g)$ is $\sec = -1$

3. Show that the Weingarten matrix for S is (for a suitable choice/sign of N):

$$W_N = \begin{bmatrix} 0 & 0 \\ 0 & \frac{x^1}{R} \end{bmatrix} \quad , \tag{9.57}$$

so that the extrinsic curvature of S in the Poincaré half space is $K = 0$ and the mean curvature function is $H = \frac{x^1}{2R}$.

As yet another glimpse into the relationship between the K-curvature and the sec-curvature of 2-surfaces in ambient Riemannian 3-spaces:

EXERCISE 9.23

Let $(\mathcal{V}, h, \nabla^h)$ denote $\mathbb{R}^3$ with metric h and metric matrix H as follows:

$$H(y^1,y^2,y^3) = \begin{bmatrix} 1+\left(y^1\right)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \tag{9.58}$$

We let S be the surface in $(\mathcal{V}, h, \nabla^h)$ given by the following parametrization:

$$\phi(x^1,x^2) = \left(x^1\,,\, x^2\,,\, \left(\frac{1}{2}\right)\left(a\cdot\left(x^1\right)^2 + b\cdot\left(x^2\right)^2\right)\right) \quad , \quad (x^1,x^2) \in \mathcal{U} = \mathbb{R}^2 \quad . \tag{9.59}$$

1. Show that the induced metric g in $\mathcal{U}$ for that surface in $(\mathcal{V}, h, \nabla^h)$ has the following matrix representation in $\mathcal{U}$:

$$G(x^1,x^2) = \begin{bmatrix} (a^2+1)\cdot\left(x^1\right)^2+1 & a\cdot b\cdot x^1\cdot x^2 \\ a\cdot b\cdot x^1\cdot x^2 & b^2\cdot\left(x^2\right)^2+1 \end{bmatrix} \quad . \tag{9.60}$$

2. Show that the intrinsic curvature of $(\mathcal{U}, g, \nabla^g)$ is

$$\sec\left(x^1,x^2\right) = \frac{a\cdot b}{\left(\left(b^2\cdot(x^2)^2+a^2+1\right)\cdot(x^1)^2+b^2\cdot(x^2)^2+1\right)^2} \quad . \tag{9.61}$$

3. Show that the extrinsic curvature of $(\mathcal{U}, g, \nabla^g)$ is

$$K(x^1, x^2) = \sec(x^1, x^2) \quad . \tag{9.62}$$

EXERCISE 9.24

Let $S = \phi(\mathcal{U})$ be a Riemannian hypersurface of $(\mathcal{V}, h, \nabla^h)$ and let $F \in \mathfrak{F}(\mathcal{V})$ be a smooth function in $\mathcal{V}$ such that S is a level set of F in $\mathcal{V}$ in the usual sense:

$$S = \mathcal{K}_c = \{(y^1, \cdots, y^n) \in \mathcal{V} \,|\, F\left(y^1, \cdots, y^n\right) = c\} \quad . \tag{9.63}$$

Define an h-unit normal vector field N for S by

$$N = \frac{\operatorname{grad}_h(F)}{\|\operatorname{grad}_h(F)\|_h} \tag{9.64}$$

Show that the scalar second fundamental form of S with respect to the normal vector field N is

$$q(X, Y) = -\frac{\operatorname{Hess}_h(F)(X, Y)}{\|\operatorname{grad}_h(F)\|_h} \quad . \tag{9.65}$$

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