FINITE-DIMENSIONAL SYSTEMS INTEGRABLE VIA INVERSE SCATTERING*

F. Pempinelli† and M. Boiti

Dipartimento di Fisica dell'Università and
Sezione INFN, 73100 Lecce, Italy‡

Abstract

The discrete spectral problem of Ablowitz–Ladik is considered in the case in which the potential has a finite support of length L. The spectral transform is explicitly computed and a recurrence relation on the length L for computing it in L algebraic step is given. This spectral transform can be used to generate via the scattering method a finite dimensional version of the dynamical systems associated to the Ablowitz–Ladik spectral problem. A special case is considered in which the potential is constraint to evolve in time on a semi-line. It is shown that the time evolution of the corresponding spectral data is given by a Riccati equation and that, consequently, the system is integrable. The truncated soliton, i.e. the potential obtained by putting to zero the one soliton outside an interval of length L is examined in detail. The sufficient and necessary condition for having a soliton contained in the truncated soliton solution is derived.

1 Introduction

It is well known that a number of nonlinear differential–difference evolution equations, that model physically interesting problems, can be solved by the inverse scattering (spectral) transform method (see [1, 2] and references quoted therein).

In general a whole class of solvable equations can be related to the same spectral problem, their spectral transforms differing only in the time evolution. We are here interested in the spectral problem of Ablowitz–Ladik [3, 4]. This spectral problem can be considered a discretization of the Zakharov–Shabat spectral problem and can be written in matrix form as

$$\mu(k, n+1) - \Lambda^{-1}\mu(k, n)\Lambda = Q\,(n+1)\,\mu(k, n+1), \tag{1.1}$$

with μ a 2×2 matrix,

$$\Lambda(k) = \begin{pmatrix} 1/z & 0 \\ 0 & z \end{pmatrix}, \quad z^2 = k, \quad Q(n) = \begin{pmatrix} 0 & q(n) \\ r(n) & 0 \end{pmatrix} \tag{1.2}$$

k is the spectral parameter and $Q(n)$ the potential.

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†e-mail: pempi@le.infn.it; phone: +39 832 320450; fax: +39 832 320505.
‡WWW address: www.fisica.unile.it

It has been used for linearizing the discrete versions of the nonlinear Schrödinger (NLS), the modified KdV, the KdV, the sine-Gordon equations, and the Toda lattice, the nonlinear self-dual network equations.

The direct and inverse problems have been first studied and solved by Ablowitz and Ladik. In this paper we use the spectral problem (1.1) differing from that originally proposed by Ablowitz and Ladik. It originates the same class of integrable differential–difference equations but the spectral analysis results to be simplified. Moreover it has been successfully used in [5] for solving an integro-differential–difference equation with non analytic dispersion relation, in [6] for deriving Bäcklund and Darboux transformations and (an equivalent version) in [7] for studying the periodic case. Its spectral transform theory as been revisited in [5] where the more modern $\overline{\partial}$–formulation has been used.

If the potential $Q(n)$ in (1.1) is zero outside the finite interval $[0, L]$

$$Q(n) = 0 \qquad \text{for } -\infty < n \leq -1 \text{ and } L+1 \leq n < +\infty \tag{1.3}$$

its spectral transform $\{\alpha^+(k),\, \alpha^-(k)\}$ can be explicitly computed and it results to be a rational function of the spectral parameter k. Solitons are present if $\alpha^+(k)$, $\alpha^-(k)$ (in $\mathcal{D} = \mathbb{C} - \{0, \infty\}$) have poles, respectively, inside and outside the unit circle $|k| = 1$. This fact can be used for studying the existence of nonlinear localized excitations in experimental data which are assumed to represent a train of envelope wave pulses [8].

As soon as the time evolution is switched on according to one of the solvable evolution equations associated to the spectral problem the potential $Q(n)$ now $Q(n,t)$ becomes in general different from zero all along the discrete line $\mathbb{Z}$. Only for the special class of evolution equations considered in [5] the dispersion relation can be chosen in such a way that $Q(n,t)$ evolves in time on the semi-line $(-\infty, L)$ and stay equal to zero outside. In this case the reflection coefficient evolves non linearly but obeying a Riccati equation. Anyway, the spectral transform $\{\alpha^+(k,t),\, \alpha^-(k,t)\}$ also at time $t \neq 0$ continues to depend only on the finite number of constants $\{Q(n)\}$, now $\{Q(n,0)\}$, the only additional constants appearing in the spectral transform being, of course, the parameters fixing the dispersion relation of the considered evolution equation.

Therefore these systems are finite-dimensional and in the cases (as in the NLS case) in which one can define a complete set of constants of motion, they are integrable in the Liouville sense.

The spectral transform $\{\alpha^+,\, \alpha^-\}$ can be considered to depend on k and on the length L of the support of $Q(n)$ and, consequently, we note $\alpha^+ = \alpha_L^+(k)$, $\alpha^- = \alpha_L^-(k)$. Taking advantage of the discrete and finite character of the potential we obtain *the explicit analytic expression of the Jost solutions and of the spectral data* and then we show that the α's satisfy a recurrence relation in L, which gives the spectral transform in L step [8]. In addition this recurrence relation is linearized furnishing a simple very efficient algebraic algorithm for computing the spectral transform.

In particular, we provide a simple computational method for characterizing completely the nonlinear coherent structures present in numerical or experimental data representing the solution, at a given time, of a nonlinear evolution equation (e.g. of the nonlinear Schrödinger family).

As an interesting example we compute explicitly the spectral transform of a truncated discrete soliton, i.e. of the solution obtained by putting to zero a soliton solution outside the interval $[0, L]$. It results that the spectrum of the truncated soliton has a continuous part corresponding to a dispersive wave and has or has not a discrete part corresponding to a soliton according to the localization of the truncation. More precisely if n_0 is the center of the soliton and b is its amplitude the truncated soliton preserves its solitonic character if and only if

$$\frac{\sinh[b(L-n_0+1/2)]\sinh[b(n_0+1/2)]}{\cosh^2 b/2} > 1. \tag{1.4}$$

In particular this means that $-1/2 \leq n_0 \leq L+1/2$ and that in order to detect a soliton the number of points $L+1$ to measure is higher for solitons of smaller amplitude (see [9] for more details).

2 The Direct Problem

Let us introduce

$$\nu(k,n) \equiv \begin{pmatrix} \nu_{11}(k,n) & \nu_{12}(k,n) \\ \nu_{21}(k,n) & \nu_{22}(k,n) \end{pmatrix} = \begin{pmatrix} \mu_{11}(k,n) & k^{-n}\mu_{12}(k,n) \\ k^n\mu_{21}(k,n) & \mu_{22}(k,n) \end{pmatrix} \tag{2.1}$$

and

$$P(k,n) = \begin{pmatrix} 0 & p(k,n) \\ s(k,n) & 0 \end{pmatrix} = \begin{pmatrix} 0 & k^{-n}q(n) \\ k^n r(n) & 0 \end{pmatrix}. \tag{2.2}$$

Then the spectral problem (1.1) reads

$$\nu(k,n+1) - \nu(k,n) = P(k,n+1)\nu(k,n+1).$$

We will make use also of the notation

$$\nu_1^{\pm} = \begin{pmatrix} \nu_{11}^{\pm} \\ \nu_{21}^{\pm} \end{pmatrix}, \quad \nu_2^{\pm} = \begin{pmatrix} \nu_{12}^{\pm} \\ \nu_{22}^{\pm} \end{pmatrix}. \tag{2.3}$$

The solution of the direct problem, i.e. the computation of the spectral transform of the potential P, goes through the construction of some well chosen solutions (the Jost solutions) of the spectral equation with some particular boundary values. These solutions are denoted by $\nu^{\pm}$ and are defined by the following "discrete integral equations"

$$\begin{pmatrix} \nu_{11}^{-}(k,n) \\ \nu_{21}^{-}(k,n) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\sum\limits_{i=n+1}^{L} p(k,i)\nu_{21}^{-}(k,i) \\ \sum\limits_{i=0}^{n} s(k,i)\nu_{11}^{-}(k,i) \end{pmatrix} \tag{2.4}$$

$$\begin{pmatrix} \nu_{11}^{+}(k,n) \\ \nu_{21}^{+}(k,n) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\sum\limits_{i=n+1}^{L} p(k,i)\nu_{21}^{+}(k,i) \\ -\sum\limits_{i=n+1}^{L} s(k,i)\nu_{11}^{+}(k,i) \end{pmatrix} \tag{2.5}$$

$$\begin{pmatrix} \nu_{12}^{-}(k,n) \\ \nu_{22}^{-}(k,n) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\sum\limits_{i=n+1}^{L} p(k,i)\nu_{22}^{-}(k,i) \\ -\sum\limits_{i=n+1}^{L} s(k,i)\nu_{12}^{-}(k,i) \end{pmatrix} \tag{2.6}$$

$$\begin{pmatrix} \nu_{12}^{+}(k,n) \\ \nu_{22}^{+}(k,n) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \displaystyle\sum_{i=0}^{n} p(k,i)\nu_{22}^{+}(k,i) \\ \displaystyle -\sum_{i=n+1}^{L} s(k,i)\nu_{12}^{+}(k,i) \end{pmatrix} \tag{2.7}$$

where $n = -1, 0, 1, \ldots, L$ and $\sum_{i=0}^{n} \equiv 0$ for $n = -1$, $\sum_{i=n+1}^{L} \equiv 0$ for $n = L$, while for $-\infty < n \leq -2$

$$\nu(k,n) = \nu(k,-1) \tag{2.8}$$

and for $L + 1 \leq n < +\infty$

$$\nu(k,n) = \nu(k,L). \tag{2.9}$$

Since the Jost solutions ν^{+} and ν^{-} are defined as solutions of a system of $L+2$ inhomogeneous linear equations they are defined for all k in the complex plane and they result to be rational functions in k. This fact is peculiar of the finite system we are considering and will be exploited in the following. Here, in order to define the spectral transform and to formulate the inverse problem as a Hilbert–Riemann problem, we have to consider a Jost solution ν defined as ν^{+} inside the unit circle and as ν^{-} outside the unit circle and then look to the discontinuity across the circle. We proceed through direct computation of the difference of the two column vectors ν_1^{+} and ν_1^{-} and obtain the integral equation for this difference. It obeys the same equation as the quantity $-\alpha^{-}(k)\,\nu_2^{-}(k,n)$ (the quantity α^{-} is defined below) and, based on the uniqueness of the solution of such equations, we conclude

$$\nu_1^{+}(k,n) - \nu_1^{-}(k,n) = -\alpha^{-}(k)\,\nu_2^{-}(k,n), \quad |k| = 1. \tag{2.10}$$

The same approach is applied to $\nu_2^{\pm}$ and we get

$$\nu_2^{+}(k,n) - \nu_2^{-}(k,n) = \alpha^{+}(k)\,\nu_1^{+}(k,n), \quad |k| = 1. \tag{2.11}$$

The *reflection coefficients* $\alpha^{-}(k)$ and $\alpha^{+}(k)$ are defined (still for $|k| = 1$) as

$$\alpha^{-}(k) = \sum_{0}^{L} s(i)\nu_{11}^{-}(k,i), \quad \alpha^{+}(k) = \sum_{0}^{L} p(i)\nu_{22}^{+}(k,i). \tag{2.12}$$

Here again the reflection coefficients result to be rational functions of k and can therefore be defined for all k in the complex plane. Then the knowledge of the reflection coefficients in the complex plane allows one to determine completely the spectral data, which consist of a continuous part (*radiation*) related to the values of $\alpha^{\pm}$ on the unit circle $|k| = 1$, and of a discrete part (*solitons*) constituted by the poles k_j^{-} of α^{-} in $|k| > 1$ and k_j^{+} of α^{+} in $|k| < 1$.

In the reduced case

$$r(n) = -\overline{q}(n) \quad \Leftrightarrow \quad \overline{Q}(n) = \sigma_2 Q(n) \sigma_2, \tag{2.13}$$

we have for the Jost solutions, which now are more conveniently written in terms of μ,

$$\sigma_2 \overline{\mu}^{-}(1/\overline{k}, n)\sigma_2 = \mu^{+}(k,n), \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.14}$$

and for the spectral data

$$\overline{\alpha}^{-}(1/\overline{k}) = -\alpha^{+}(k). \tag{2.15}$$

3 Explicit Solution of the Direct Problem

Jost solutions are defined via the finite system of linear equations (2.4)–(2.7). Thanks to the off-diagonal character of the matrix potential $Q(n)$ in the spectral equation (1.1) this system can be solved furnishing an explicit algebraic expression in terms of the potentials $q(n)$, $r(n)$ for the Jost solutions and, consequently, for the spectral data $\alpha^{\pm}$.

We rewrite (2.4) as

$$\nu_1^-(n) = \nu_1^-(L) - \sum_{i=n+1}^{L} A(i)\nu_1^-(i) \tag{3.1}$$

with

$$\nu_1^-(L) = \begin{pmatrix} 1 \\ \alpha^- \end{pmatrix} \tag{3.2}$$

$$A(i) = \begin{pmatrix} 0 & p(i) \\ s(i) & 0 \end{pmatrix}. \tag{3.3}$$

The solution is given by

$$\nu_1^-(n) = X(n)\nu_1^-(L), \qquad n = -1, 0, \ldots, L \tag{3.4}$$

where

$$X(n) = \prod_{i=n+1}^{L} (1 - A(i)) \equiv (1 - A(n+1))(1 - A(n+2)) \ldots (1 - A(L)) \tag{3.5}$$

for $n = -1, 0, \ldots, L-1$ and

$$X(L) = 1. \tag{3.6}$$

Taking into account that

$$\alpha^- = \sum_{i=0}^{L} s(i)\nu_{11}^-(i) \tag{3.7}$$

we have

$$\alpha^- = \frac{\sum_{i=0}^{L} s(i)X_{11}(i)}{1 - \sum_{i=0}^{L} s(i)X_{12}(i)}. \tag{3.8}$$

In an analogous way one can compute the other matrix elements of $\nu^{\pm}$ and α^+. We get

$$\nu^-(n) = X(n) \begin{pmatrix} 1 & 0 \\ \alpha^- & 1 \end{pmatrix} \tag{3.9}$$

$$\nu^+(n) = X(n) \begin{pmatrix} 1 & \alpha^+ \\ 0 & 1 \end{pmatrix} \tag{3.10}$$

and

$$\alpha^+ = \frac{\sum_{i=0}^{L} p(i)X_{22}(i)}{1 - \sum_{i=0}^{L} p(i)X_{21}(i)}. \tag{3.11}$$

4 Recurrence Relations and their Linearization

Let us consider the following sets of data defined recursively as follows

$$S_0 = \{(q(0), r(0))\tag{4.1}$$
$$S_L = S_{L-1} \cup \{(q(L), r(L))\}, \quad L = 1, 2, \ldots\tag{4.2}$$

and let us denote by $X^L(n)$ the matrix defined in (3.5) that corresponds to the choice of data $q(n)$, $r(n)$ which are zero for any $n < 0$ and $n > L$ and coincide with the elements $q(n)$, $r(n)$ of the set S_L for $0 \leq n \leq L$. Then one can get a recurrence relation on L for $X^L(n)$ and, consequently, a recurrence relation for α_L^+

$$\begin{aligned} \alpha_0^- &= s_0 \\ \alpha_{L+1}^- &= \frac{s_{L+1} + \alpha_L^-}{1 + p_{L+1}\alpha_L^-} \end{aligned}\tag{4.3}$$

and α_L^-

$$\begin{aligned} \alpha_0^+ &= p_0 \\ \alpha_{L+1}^+ &= \frac{p_{L+1} + \alpha_L^+}{1 + s_{L+1}\alpha_L^+}. \end{aligned}\tag{4.4}$$

Note that the obtained recurrence equations are quadratic in the $\alpha_L^\pm$'s. However, they can be linearized by using the elementary theory of the continued fractions. Precisely, one can easily derive the following formula for α_L^+

$$\alpha_L^+ = \frac{1}{s_L} - a_L \frac{C_{L-2}}{C_{L-1}}\tag{4.5}$$

with

$$\begin{aligned} &C_{-1} = 1, \qquad C_0 = b_0 - \left(p_0 - \frac{1}{s_0}\right) \\ &C_m = b_m C_{m-1} + a_m C_{m-2}, \qquad\qquad m = 1, 2, \cdots, L-1, \end{aligned}\tag{4.6}$$

which linearizes the recurrence formula (4.4) for α_L^+. Analogously for α_L^-.

5 Inverse Problem

From the system (1.1) we get

$$q(n+1) = -\mu_{12}^-(n)^{(-1)}, \quad r(n+1) = -\mu_{21}^+(n)^{(1)},\tag{5.1}$$

where $\mu_{12}^-(n)^{(-1)}$ is the coefficient of k^{-1} in the Laurent expansion for $k \to \infty$ of $\mu_{12}^-(k, n)$, and $\mu_{21}^+(n)^{(1)}$ the coefficient of k in the Taylor expansion for $k \to 0$ of $\mu_{21}^+(k, n)$.

Therefore in order to reconstruct the potentials $\{r(n), q(n)\}$ from the spectral data $\{\alpha_L^+(k), \alpha_L^-(k)\}$ we need to reconstruct the Jost solutions $\mu^\pm(k, n)$. Their analytic properties are fixed by the reflection coefficients which are obtained by evaluating $\alpha_L^+(k), \alpha_L^-(k)$ on the circle $|k| = 1$ and by the discrete part of the spectrum, which can be recovered by looking (in $\mathcal{D} = \mathbb{C} - \{0, \infty\}$) to the poles k_j^+ and k_j^- of $\alpha_L^+(k)$ and $\alpha_L^-(k)$ inside and outside, respectively, the unit circle. Consequently, this problem can be considered as a

$\overline{\partial}$–problem. It has been solved in [5] where the interested reader can find details. The solution is given by the following set of linear integral equations

$$
\begin{aligned}
&\begin{pmatrix} \mu^{\pm}_{11}(k) \\ \mu^{\pm}_{21}(k) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2\pi i} \oint_C d\zeta \frac{\zeta^{-n}\alpha^-_L(\zeta)}{\zeta-(1\mp 0)k} \; \frac{k}{\zeta} \begin{pmatrix} \mu^-_{12}(\zeta) \\ \mu^-_{22}(\zeta) \end{pmatrix} \\
&+ \sum_j (k^-_j)^{-n} \underset{k^-_j}{\mathrm{Res}}\{\alpha^-_L\} \frac{k}{k^-_j} \frac{1}{k^-_j - k} \begin{pmatrix} \mu^-_{12}(k^-_j) \\ \mu^-_{22}(k^-_j) \end{pmatrix}, \qquad (5.2)\\
&\begin{pmatrix} \mu^{\pm}_{12}(k) \\ \mu^{\pm}_{22}(k) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i} \oint_C d\zeta \frac{\zeta^{n}\alpha^+_L(\zeta)}{\zeta-(1\mp 0)k} \; \begin{pmatrix} \mu^+_{11}(\zeta) \\ \mu^+_{21}(\zeta) \end{pmatrix} \\
&- \sum_j (k^+_j)^{n} \underset{k^+_j}{\mathrm{Res}}\{\alpha^+_L\} \frac{1}{k^+_j - k} \begin{pmatrix} \mu^+_{11}(k^+_j) \\ \mu^+_{21}(k^+_j) \end{pmatrix}, \qquad (5.3)
\end{aligned}
$$

where the integrals on ζ run counterclockwise on the unit circle. The discrete part is fixed by requiring that these poles cancel in the $\overline{\partial}$–equation the contribution of the continuous part coming from the poles of $\alpha^+_L(k)$ and $\alpha^-_L(k)$ inside and outside the unit circle.

It is interesting to note that the requirement that $\alpha^-(k)k^{-L}$ is meromorphic outside the unit circle and analytic at $k=\infty$ and that $\alpha^+(k)k^L$ is meromorphic inside the unit circle and analytic at $k=0$ implies that for $n \geq L$

$$
\mu^+(k,n) = 1 + \begin{pmatrix} 0 & \alpha^+(k)k^n \\ 0 & 0 \end{pmatrix} \qquad (5.4)
$$

$$
\mu^-(k,n) = 1 + \begin{pmatrix} 0 & 0 \\ \alpha^-(k)k^{-n} & 0 \end{pmatrix} \qquad (5.5)
$$

and therefore, thanks to (5.1), that $q(n)$, $r(n)$ are zero for $n > L$.

Formulae (5.4) and (5.5) can be obtained directly from (5.2) and (5.3) by noting that for any function $\Phi^+(k)$ analytic for $|k|<1$

$$
\left.\begin{aligned}
&\frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta-(1-0)k}\Phi^+(\zeta)\zeta^m = \Phi^+(k)k^m \\
&\frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta-(1+0)k}\Phi^+(\zeta)\zeta^m = 0
\end{aligned}\right\} \quad \text{for } m \geq 0. \qquad (5.6)
$$

and for any function $\Phi^-(k)$ analytic for $|k|>1$

$$
\left.\begin{aligned}
&\frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta-(1-0)k}\Phi^-(\zeta)\zeta^m = 0 \\
&\frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta-(1+0)k}\Phi^-(\zeta)\zeta^m = -\Phi^-(k)k^m
\end{aligned}\right\} \quad \text{for } m \leq -1. \qquad (5.7)
$$

and that the contributions coming from the poles of $\alpha^+_L(k)$ and $\alpha^-_L(k)$ inside and outside the unit circle are canceled by the discrete part in (5.2) and (5.3).

The fact that the reconstructed potentials $q(n)$, $r(n)$ are zero also for $n<0$ is, instead, related to the specific algebraic characterization of $\alpha^+_L(k)$ and $\alpha^-_L(k)$, which can be obtained via a specific recurrence procedure described in the previous sections.

As soon as the time evolution of the spectral data is switched on according to the specific evolution equation one chooses among those related to the spectral problem (1.1) these specific properties of $\alpha^+_L(k)$ and $\alpha^-_L(k)$ are destroyed and, consequently, the potentials $q(n,t)$, $r(n,t)$ become in general different from zero all along the discrete line $\mathbb{Z}$. Only in the case of the evolution equations with a non analytic dispersive relation considered in [5] one can choose a specific time evolution in such a way that $q(n,t)$, $r(n,t)$ stay equal to zero at any time for $n > L$. In this case the evolution of $\alpha^{\pm}$ is described by a Riccati equation. The evolution becomes linear only for a special choice of the coupling factor of the fields involved. In the physical application this corresponds to the case in which the total photon number is conserved.

6 The Truncated Soliton

In the case of the one soliton solution we have (reduction $r = -\overline{q}$)

$$q_m = \frac{1}{\tau_m} e^{i\phi_m} \tag{6.1}$$

where

$$\phi_m = \eta_0 m + \theta_0 \tag{6.2}$$

$$\tau_m = \frac{1}{\sinh b} \cosh\left(b\,(m - n_0)\right), \qquad b > 0 \tag{6.3}$$

with η_0, θ_0, n_0 and b real constants. It is convenient to introduce the phase shifted spectral variable

$$\xi = k\, e^{-i\eta_0}. \tag{6.4}$$

Then the recurrence equation (4.6) have coefficients

$$a_m = -e^{2i\theta_0} \xi^{-2m} \tau_{m-1} \tau_{m+1} \tag{6.5}$$

$$b_m = e^{i\theta_0} \xi^{-m-1} (\tau_{m+1} + \xi \tau_m) \tag{6.6}$$

and can be explicitly solved by induction getting

$$C_m = \frac{e^{i(m+1)\theta_0}}{\tau_0} \prod_{k=1}^{m} \xi^{-k} \tau_k \left[\xi \tau_0 \tau_{-1} + (\tau_0 - \xi \tau_{-1}) \sum_{j=0}^{m+1} \xi^{-j} \tau_j \right]. \tag{6.7}$$

We get

$$\alpha_L^{+} = -e^{i\theta_0} \xi^{-L} \frac{N_L}{D_L} \tag{6.8}$$

with

$$D_L = \xi^{L+2} - (\xi \tau_{L+1} - \tau_L)(\xi \tau_{-1} - \tau_0) \tag{6.9}$$

$$N_L = \xi^{L+2} \tau_L - \xi^{L+1} \tau_{L+1} - (\xi \tau_{-1} - \tau_0). \tag{6.10}$$

It results that both the numerator and the denominator of α_L at any L have simple zeros at

$$\xi = e^{-b} \text{ and } \xi = e^{b}. \tag{6.11}$$

This remark has to be taken into account in studying the zeros of D_L and in relating them to the presence of solitons in the solution.

The number of zeros of $D_L(\xi)$, or equivalently of $-iD_L(\xi)$, inside the unit circle is given by the integral

$$\frac{1}{2\pi i} \oint_C d\xi \frac{d}{d\xi} \log(-iD_L(\xi)) = \frac{1}{2\pi} \arg(-iD_L(e^{i\theta}))\Big|_{\theta=-\pi}^{\theta=\pi}. \tag{6.12}$$

We introduce

$$c_L = \frac{\sinh[b(L-n_0+1/2)]\sinh[b(n_0+1/2)]}{\cosh^2 b/2}. \tag{6.13}$$

We can distinguish two cases.

Case $c_L > 1$

In this case $D_L(\xi)$ has an even number of zeros, that is at least two. Since we know that $D_L(\xi)$ has always a zero inside the unit circle in the complex ξ–plane at $\xi = e^{-b}$ we conclude from (6.12) that the spectral transform α_L^+ has at least one pole and the solution contains a soliton. In fact one soliton because the truncation can kill the solitonic component of the solutions but not generate new solitons.

Case $c_L < 1$

In this case the number of zeros of $D_L(\xi)$ inside the unit circle must be odd and the solution does not contain a soliton.

The soliton pole is a zero of $D_L(\xi)$ (different from the zero $\xi = e^{-b}$) located inside the unit circle centered at the origin of the complex ξ–plane.

Recall that

$$D_L(\xi) = \xi^{L+2} - (\xi\tau_{L+1} - \tau_L)(\xi\tau_{-1} - \tau_0). \tag{6.14}$$

We want to prove that if $c_L > 1$ then $D_L(\xi)$ has two poles in the interval $[0,1]$. If $c_L > 1$ we have

$$D_L(0) < 0, \qquad D_L(1) < 0 \tag{6.15}$$

and

$$\frac{dD_L}{d\xi}(0) > 0, \qquad \frac{dD_L}{d\xi}(1) < 0. \tag{6.16}$$

By looking to the behaviour of $d^2D_L/d\xi^2$ and $d^3D_L/d\xi^3$ it is easy to prove that $dD_L/d\xi$ has only one zero in the interval $[0,1]$. Then $D_L(\xi)$ has two zeros in the interval $[0,1]$. Note that for $c_L \to 1$ the location of the pole of α_L^+ moves along the interval $[0,1]$ towards the right up to 1.

We conclude that the solution obtained by truncating a soliton at $n = 0$ and at $n = L$ contains a soliton if and only if

$$c_L \equiv \frac{\sinh[b(L-n_0+1/2)]\sinh[b(n_0+1/2)]}{\cosh^2 b/2} > 1 \tag{6.17}$$

and that the truncation of the soliton changes the modulus of the location of the pole in the complex k–plane but not its phase.

It is interesting to compute the location and residuum of the pole and the reflection coefficient $\alpha_L^+(k)$ ($|k| = 1$) at large L. For $L \gg n_0$ the reflection coefficient $\alpha_L^+(k)$ has a pole in the disk of radius 1 located at

$$k \simeq \frac{\cosh(n_0 b)}{\cosh(n_0 b + b)} e^{i\eta_0} \tag{6.18}$$

if and only if

$$n_0 > -\frac{1}{2} \tag{6.19}$$

in agreement with the general formula (6.17).

The residuum of α_L at the pole at large L is given by

$$\text{Res } \alpha_L^+ \simeq -2\sinh b \frac{e^{2n_0+b}\cosh(n_0 b)}{(\cosh[(n_0+1)b])^2} e^{i\theta_0 + i\eta_0}. \tag{6.20}$$

Note that for large n_0 we recover the known formulae for the full soliton (see e.g. [5]).

The continuous part of the spectrum at large L

$$\alpha_L^+ \simeq \frac{k e^{i\theta_0 - i\eta_0} e^b \sinh b}{k e^{-i\eta_0} \cosh[(n_0+1)b] - \cosh(n_0 b)} \frac{(1 - k e^{-b-i\eta_0})}{(1 - k e^{b-i\eta_0})}, \qquad |k| = 1, \tag{6.21}$$

is obtained directly from (6.8). Note that in the limit $L \to \infty$ the discrete values of the spectrum cannot be obtained any more by continuing analytically in k the reflection coefficient α in the interior of the disk of radius 1 and then by looking at the poles. This is in agreement with the well known fact that, for generic not finite initial data, the reflection coefficient can be defined only on the unit circle.

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