The stability of solitary wave solutions to a modified Zakharov-Kuznetsov equation

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Abstract

The Zakharov-Kuznetsov equation governs the behaviour of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. We consider the more realistic situation in which the electrons are non-isothermal. With an appropriate modified form of the electron number density proposed by Schamel (1973), we show that the reductive perturbation procedure leads to a modified Zakharov-Kuznetsov equation. In suitable non-dimensionalised variables and with a convenient scaling the equation is

$$16u_t + 30u^{1/2}u_x + \nabla^2 u_x = 0,$$

where the magnetic field is in the x-direction. A possible solution to the equation is the plane solitary wave $u = \operatorname{sech}^4(x-t)$ that propagates along the magnetic field. We show that this solution is unstable to small transverse sinusoidal perturbations of wavenumber k such that $0 < k < 3$. Following the method of Allen & Rowlands (1993) we use a multiple-scale perturbation method to determine consistent expansions for the growth rate γ about $k = 0$ and $k = 3$ respectively. By considering a two-point Padé approximant, we obtain an analytical expression for γ valid for $0 \le k \le 3$. We also calculate γ numerically. The Padé approximant is in very good agreement with the numerical result.

1 Introduction

We consider the propagation of ion-acoustic waves in a plasma comprising cold ions and hot electrons in the presence of a uniform magnetic field. In non-dimensional form, the ion dynamics are governed by the following system of equations:

$$n_t + \nabla.(n\mathbf{v}) = 0, \tag{1.1}$$

$$\mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v} = -\nabla\phi + \mathbf{v}\times(\Omega\hat{\mathbf{x}}), \tag{1.2}$$

$$\nabla^2\phi = n_e - n, \tag{1.3}$$

where n, n_e, $\mathbf{v}$, ϕ and Ω are respectively the non-dimensional ion number density, the electron number density, the ion fluid velocity, the electrostatic potential, and the external magnetic field whose direction is specified by the unit vector $\hat{\mathbf{x}}$.

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If the electrons are free and isothermal

$$n_e = \exp\phi. \tag{1.4}$$

In this case, for weakly nonlinear waves, an application of the reductive perturbation procedure reduces the system (1.1)-(1.4) to the Zakharov-Kuznetsov (ZK) equation [1]. Allen & Rowlands [2] studied this equation in the form

$$u_t + uu_x + \nabla^2 u_x = 0,$$

where the magnetic field is in the x-direction. They investigated the stability of a plane solitary wave that propagates along the magnetic field to small transverse sinusoidal perturbations of wavelength k.

Here our aim is to pursue a similar investigation for the more realistic situation in which some of the electrons are resonant. In this case, for small amplitudes, (1.4) is replaced by

$$n_e = \exp\phi - \frac{4}{3}b\phi^{3/2}, \quad b > 0, \tag{1.5}$$

where $b = (1 - T_{ef}/T_{et})\pi^{-1/2}$, T_{ef} is the constant temperature of the free electrons and T_{et} is the constant temperature of the trapped electrons [3]. Thus b measures the deviation from isothermality. In §2 we derive the resulting modified form of the ZK equation (the MZK equation). In §3 an ordinary perturbation analysis suggests that there is instability for $0 < k < 3$. It is shown that an attempt to find a consistent expansion for the growth rate γ near $k = 0$ fails. In §4 we use the multiple-scale method devised by Allen & Rowlands [2] to overcome this inconsistency. A consistent expansion for the growth rate near $k = 3$ is also given. In §5 a two-point Padé approximant is used to give an approximate expression for γ valid for $0 \le k \le 3$. We compare this approximation with a numerical calculation of γ. In §6 we mention further work.

2 The MZK Equation

Let the magnetic field be in the x-direction. Introduce the scalings

$$X = \epsilon^{1/4}(x - t), \quad Y = \epsilon^{1/4}y, \quad Z = \epsilon^{1/4}z, \quad T = \epsilon^{3/4}t, \tag{2.1}$$

$$\begin{bmatrix} n \\ \phi \\ v_x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} n^{(1)} \\ \phi^{(1)} \\ v_x^{(1)} \end{bmatrix} + \epsilon^{3/2} \begin{bmatrix} n^{(2)} \\ \phi^{(2)} \\ v_x^{(2)} \end{bmatrix} + \cdots \tag{2.2}$$

$$\begin{bmatrix} v_y \\ v_z \end{bmatrix} = \epsilon^{5/4} \begin{bmatrix} v_y^{(1)} \\ v_z^{(1)} \end{bmatrix} + \epsilon^{3/2} \begin{bmatrix} v_y^{(2)} \\ v_z^{(2)} \end{bmatrix} + \cdots \tag{2.3}$$

Here ϵ is a small parameter. This choice of scalings is determined by considering the linear dispersion relation, and from balancing the effects of nonlinearity and dispersion (cf. Appendix 1 in [1], for example). We assume that $b \gg O(\epsilon^{1/2})$. Then substitution of (2.1)-(2.3) into (1.1)-(1.3) and (1.5) leads to a hierarchy of equations from which we obtain

$$n_T^{(1)} + bn^{(1)1/2}n_X^{(1)} + \frac{1}{2}n_{XXX}^{(1)} + \frac{1}{2}(1 + \Omega^{-2})(n_{YY}^{(1)} + n_{ZZ}^{(1)})_X = 0. \tag{2.4}$$

Now, for convenience, we introduce into (2.4) the scalings

$$\tilde{x} = X, \quad \tilde{y} = Y(1+\Omega^{-2})^{-1/2}, \quad \tilde{z} = Z(1+\Omega^{-2})^{-1/2}, \quad \tilde{t} = 8T, \quad u = b^2 n^{(1)}/225.$$

On dropping the tildes we obtain the MZK equation, namely

$$16u_t + 30u^{1/2}u_x + \nabla^2 u_x = 0. \tag{2.5}$$

3 Ordinary Perturbation Analysis

A plane solitary wave solution to (2.5) is

$$u = u_0(\xi) = \operatorname{sech}^4 \xi, \quad \text{where} \quad \xi = x - t.$$

To apply a sinusoidal perturbation perpendicular to the x-axis we write

$$u = u_0(\xi) + \{\delta u(\xi) \exp[ik_2 y + ik_3 z + \gamma t] + \mathrm{cc}\}, \quad k^2 = k_2^2 + k_3^2. \tag{3.1}$$

On substituting (3.1) into (2.5) and linearising with respect to δu we obtain

$$(L\delta u)_\xi = f, \tag{3.2}$$

where

$$L := \frac{\partial^2}{\partial \xi^2} + (30u_0^{1/2} - 16), \quad f := -16\gamma\delta u + k^2 \delta u_\xi.$$

We require $\delta u \to 0$ as $|\xi| \to \infty$. When $\gamma = 0$ there are two marginally stable solutions, namely $\delta u \propto TS^4$ with $k = 0$, and $\delta u \propto S^5$ with $k = 3$, where $T := \tanh \xi$ and $S := \operatorname{sech} \xi$. This suggests that there is instability for $0 < k < 3$.

In order to find an expansion for γ near $k = 0$ we write

$$\begin{aligned} \delta u &= \delta u_0 + k\delta u_1 + k^2 \delta u_2 + \cdots, \\ \gamma &= k\gamma_1 + k^2\gamma_2 + \cdots, \\ f &= f_0 + kf_1 + k^2 f_2 + \cdots \end{aligned} \tag{3.3}$$

and obtain the hierarchy of equations

$$(L\delta u_i)_\xi = f_i, \quad i = 0, 1, 2, \ldots. \tag{3.4}$$

The solution to (3.4) is given formally by

$$\delta u_i = Au_{0\xi} + Bu_{0\xi} \int \frac{d\xi}{u_{0\xi}^2} + u_{0\xi} \int \frac{1}{u_{0\xi}^2} \left(\int u_{0\xi} F_i \, d\xi \right) d\xi, \tag{3.5}$$

where A and B are arbitrary constants and $F_{i\xi} = f_i$. Without loss of generality we may take $A = 0$ for $i \geq 1$. At zero order we use (3.5) together with an appropriate choice of integration constants to find a δu_0 that vanishes as $|\xi| \to \infty$, namely

$$\delta u_0 = Au_{0\xi} = -4ATS^4. \tag{3.6}$$

We may take $A = 1$ without loss of generality. Similarly, at first order, we find that

$$\delta u_1 = -\frac{\gamma_1}{2}(\xi u_{0\xi} + 4u_0).$$

However, at next order, to remove exponentially divergent terms in δu_2 it is found that we must take

$$\gamma_1^2 = 4/63. \tag{3.7}$$

The resulting expression for δu_2 is

$$\delta u_2 = C\chi - \frac{\gamma_2}{2}(\xi u_{0\xi} + 4u_0) - 2\Big(\frac{1}{9} - \frac{\gamma_1^2}{4}\Big)TS^2 - 2\Big(\frac{1}{21} - \frac{\gamma_1^2}{4}\Big)T + \gamma_1^2(\xi u_0 + \frac{1}{8}\xi^2 u_{0\xi}), \tag{3.8}$$

where

$$\chi = 2 + 5S^2 + 15\xi TS^4 - 15S^4$$

and C is an integration constant. Due to the presence of the T term in (3.8), it is not possible to choose C so as to make δu_2 vanish both as $\xi \to -\infty$ and $\xi \to +\infty$. Thus the present derivation of (3.7) is inconsistent. (3.7) can be obtained consistently by taking u_0 to be a plane periodic solution to (2.5), finding a dispersion relation for the perturbation, and then taking the solitary wave limit [4]. However, this method is very tedious and is not easily extended to higher orders.

4 Multiple-Scale Perturbation Analysis

The derivation of (3.7) can be made consistent by applying the multiple-scale perturbation analysis devised by Allen & Rowlands [2].

For large $|\xi|$, (3.2) reduces to

$$\delta u_{\xi\xi\xi} - (16 + k^2)\delta u_\xi + 16\gamma\delta u = 0. \tag{4.1}$$

It follows that a bounded solution to (4.1) has the asymptotic behaviour

$$\delta u \sim A_1 e^{p_1\xi} + A_3 e^{p_3\xi} \quad \text{as} \quad \xi \to -\infty, \tag{4.2}$$

$$\delta u \sim A_2 e^{-p_2\xi} \quad \text{as} \quad \xi \to +\infty, \tag{4.3}$$

where, for small k and γ, $p_1 \approx 4 - \gamma/2$, $p_2 \approx 4 + \gamma/2$ and $p_3 \approx \gamma$.

Multiple scales are introduced by taking δu in (3.3) to be a function of the variables ξ and $\xi_n := k^n\xi$, $n = 1, 2\ldots$ which are to be regarded as independent.

The zeroth-order equation in the resulting hierarchy is the same as in the ordinary perturbation analysis and so (3.6) is a solution. However, we now replace the arbitrary constant in (3.6) by an arbitrary function of the ξ_n, $n = 1, 2\ldots$, so that

$$\delta u_0 = h(\xi_1, \xi_2, \ldots)TS^4.$$

From first order onwards the following procedure is adopted to make δu have the asymptotic form (4.2) or (4.3).

- Choose the integration constants so that $\delta u_i \to 0$ as $\xi \to +\infty$.
- Remove the exponentially divergent terms as $\xi \to -\infty$. [This step may give information about γ_{i-1}.]
- Remove the so-called 'ghost' secularities, namely terms of the form $\xi T S^4$. [This step gives a differential equation for h in terms of ξ_i.]
- Regroup the 'algebraic' secular terms, namely terms involving ξ^n or $\xi^n e^{2\xi}$ as $\xi \to -\infty$, so that each group of terms behaves like $e^{p_3\xi}$ as $\xi \to -\infty$.

The above procedure gives the following information:

$$h \propto \exp[-\gamma_1 \xi_1/2], \tag{4.4}$$

$$\gamma_1 = 2/\sqrt{63}, \quad \gamma_2 = -5/126, \tag{4.5}$$

$$\delta u_1 = \tfrac{1}{2}\gamma_1 h S^4,$$

$$\delta u_2 = \left(\frac{5}{42} + \frac{\gamma_2}{2}\right) h S^4 + \frac{h}{21} T S^2 - \frac{5h}{126} S^2 + \frac{\mathcal{A}_0 h}{63}, \tag{4.6}$$

$$\delta u_3 = \left(\frac{79\gamma_1}{504} + \frac{\gamma_3}{2} - \frac{5\gamma_1 \mathcal{A}_1}{28}\right) h S^4 + \gamma_1 h \left(\frac{5\mathcal{A}_2}{28} T S^4 + \frac{5\mathcal{A}_1}{84} S^2 - \frac{23}{504} S^2 + \frac{5}{126} T S^2 + \frac{\mathcal{A}_1}{42}\right). \tag{4.7}$$

In (4.6) and (4.7), the $\mathcal{A}_n$ are defined by

$$\mathcal{A}_n(\xi) = \int_\infty^\xi \mathcal{A}_{n-1}(\xi)\, d\xi, \quad \text{where} \quad \mathcal{A}_0(\xi) = T - 1.$$

$\mathcal{A}_n$ has the following asymptotic behaviour:

$$(\mathcal{A}_n)_{\xi\to\infty} \simeq (-2)^{1-n} e^{-2\xi} + O(e^{-4\xi}) \tag{4.8}$$

$$(\mathcal{A}_n)_{\xi\to-\infty} \simeq -2\xi^n/n! + O(e^{2\xi}). \tag{4.9}$$

On using (4.4) and (4.8) we see that, for $\xi \to +\infty$, δu_i $(i = 0, \ldots, 3)$ decays like $h e^{-4\xi} \propto e^{-p_2\xi}$ as required. Similarly, on using (4.4) and (4.9), δu_i $(i = 0, \ldots, 3)$ has terms that decay like $h e^{4\xi} \propto e^{p_1\xi}$ for $\xi \to -\infty$ as required. However, from second-order onwards there are other terms that, taken separately, do not decay like $e^{p_3\xi}$; it is these terms that have to be regrouped to give the desired behaviour. For example, all the relevant terms from δu_2 can be combined with some of the relevant terms from δu_3 to give the following contribution to δu:

$$-\frac{2hk^2}{63}(1 + 10e^{2\xi})(1 + \tfrac{3}{2}\gamma_1 k \xi). \tag{4.10}$$

Our presumption is that this process can be continued to higher order so that the content of the final brackets in (4.10) is simply the first two terms in the expansion of $\exp[3\gamma_1 k\xi/2]$. Thus the overall contribution to δu decays like $h \exp[3\gamma_1 k \xi/2] \propto e^{p_3\xi}$ as required. Similarly, we presume that the remaining relevant terms in δu_3 can be grouped with terms from higher orders to give the correct overall decay.

A similar lengthy calculation shows that near $k=3$

$$\gamma = \bar{\gamma}_2 \bar{k}^2 + O(\bar{k}^3), \quad \text{where} \quad \bar{\gamma}_2 = \frac{2048\bar{k}^2}{2835\pi^2} \quad \text{and} \quad \bar{k} = \sqrt{9-k^2}. \tag{4.11}$$

5 Analytical & Numerical Results for $0 \le k \le 3$

A numerical determination of γ for the ZK equation using a spectral method was given by Infeld & Frycz [5]. Allen & Rowlands [2] devised a much more efficient scheme. We have incorporated an adaptation of their method into a simple *Mathematica* program and calculated a numerical approximation to γ. The growth rate curve obtained by using this method is shown as the solid lines in Figures 1 and 2.

The growth rates given by the expansions about $k=0$ and $k=3$ respectively are also shown in Figure 1. As is to be expected, close agreement with the numerical calculation is limited to the vicinities of $k=0$ and $k=3$ respectively.

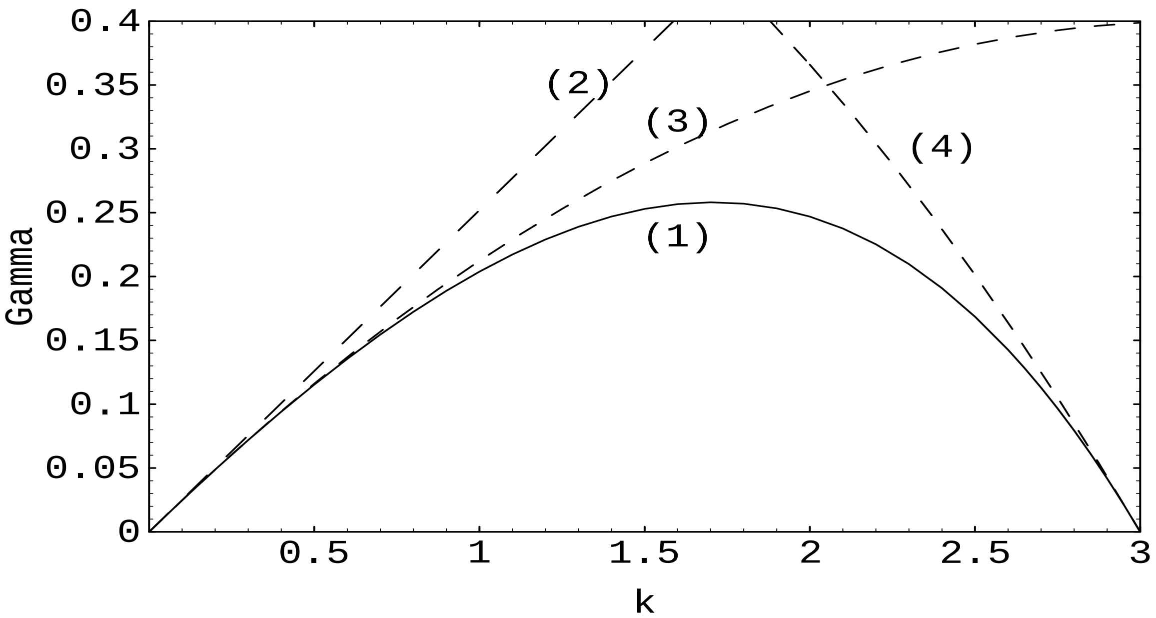

Figure 1: The growth rate $\gamma(k)$: numerical (1), $\gamma = \frac{2}{\sqrt{63}}k$ (2), $\gamma = \frac{2}{\sqrt{63}}k - \frac{5}{126}k^2$ (3), $\gamma = \frac{2048}{2835\pi^2}(9-k^2)$ (4).

The expressions (4.5) and (4.11) for γ_1, γ_2 and $\bar{\gamma}_2$ may be combined as a two-point Padé approximant to give an expression γ_P that approximates γ valid for $0 \le k \le 3$. We take

$$\gamma_P = \gamma_1 k \left(1 - \frac{k^2}{9}\right) \frac{1+ak}{1+bk}. \tag{5.1}$$

We insist that $\gamma_P = \bar{\gamma}_2(9-k^2)$ at $k=3$ and that, for small k, the expansion of (5.1) agrees with $\gamma_1 k + \gamma_2 k^2$. These requirements give

$$a = 0.734198, \quad b = 0.891683.$$

As can be seen in Figure 2, the Padé approximant and numerical results agree very closely. The maximum error is less than 0.77%.

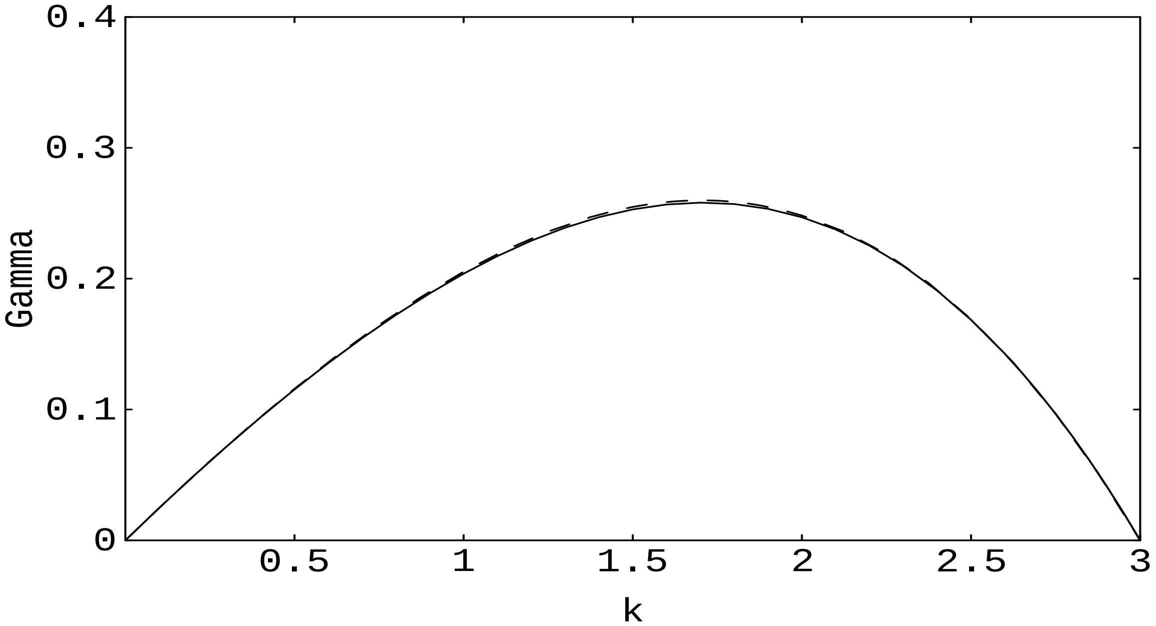

Figure 2: The growth rate $\gamma(k)$: Padé approximant (dashed), numerical (solid).

6 Future Work

For the ZK equation, Allen & Rowlands [6] considered the stability of a plane solitary wave solution that propagates at an angle to the magnetic field. We are currently investigating the corresponding problem for the MZK equation.

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