Transverse Stability of Solitary Waves on Coupled Nonlinear Transmission Lines

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Abstract

We study the transverse stability of solitary waves on a system of coupled nonlinear electrical transmission lines. In the continuum limit, and in suitably scaled coordinates, the voltage on the system is described by a modified Zakharov-Kuznetsov equation. Exact results for the growth rate of linear transverse perturbations are obtained by applying the technique of multiple-scale analysis due to Allen and Rowlands. Comparisons are made with numerical results obtained with the aid of a commercial circuit simulation package.

1 Introduction

The phenomenon of pulse compression in systems of coupled optical fibres was reported by S. K. Turitsyn at the previous conference in this series. The nonlinear dispersive propagation of light in an array of linearly coupled optical fibres can be described by the following set of coupled PDE's:

$$\mathrm{i}\frac{\partial A_m}{\partial z} - \beta_2 \frac{\partial^2 A_m}{\partial t^2} + 2\gamma |A_m|^2 A_m - \delta(A_{m+1} + A_{m-1} - 2A_m) = 0. \tag{1.1}$$

Here, $m = 1, 2, \ldots, M$ is the fibre index, t is the retarded time, δ is the linear coupling, β_2 is the group velocity dispersion coefficient, and γ is the nonlinearity coefficient (Aceves *et al.*, 1995). An initial pulse of Gaussian amplitude, injected into an array of $M = 15$ fibres at $z = 0$, was seen to 'collapse' into the central fibre $m_c = 8$ with compression ratios of about 6 over distances of the order of 200m. This effect arises from the fact that equation (1.1) has a continuum limit corresponding to the two-dimensional nonlinear Schrödinger equation in which 'pulse blow-up' can occur. The work presented here is an investigation to see whether similar instabilities arise in the analogous *electrical* system of coupled nonlinear transmission lines (NTL's).

In the next section we write down the circuit equations governing small-amplitude pulses on systems of NTL's coupled via constant capacitors. After scaling coordinates and taking a continuum limit, we reduce them to a well-known nonlinear PDE which possesses a 'line' soliton solution which can be unstable under transverse perturbation. We undertake an analytic and numerical study of the transverse stability of these line solitons. Use is made of the method of multiple-scale perturbation theory (Allen and Rowlands, 1993) in order to obtain information about the growth rate of the instability. This is then compared with numerical simulations performed with the *HSpice* circuit simulator.

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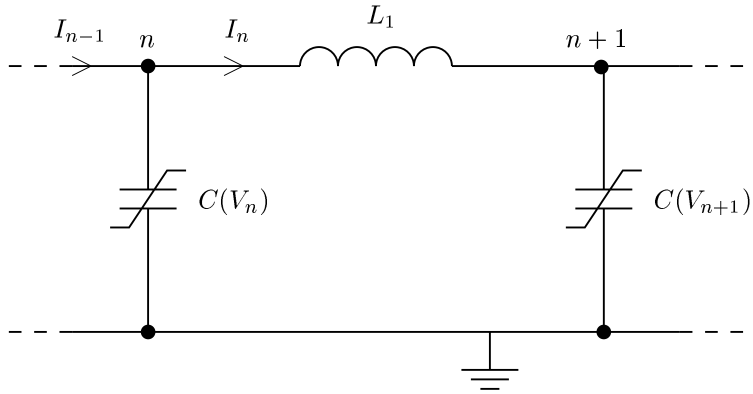

Figure 1: A typical section of a nonlinear transmission line.

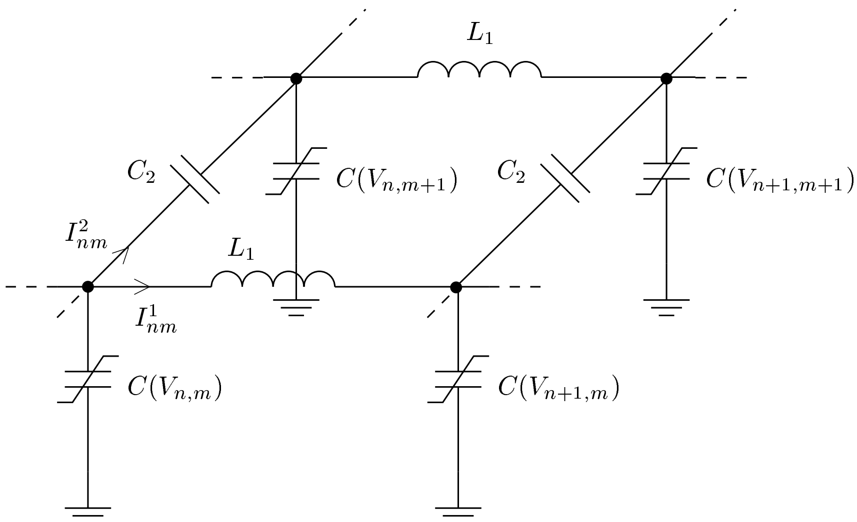

Figure 2: Part of a system of NTL's coupled by capacitors C_2.

2 Circuit Equations

Before examining coupled systems, we introduce our notation by means of a *single* nonlinear transmission line, illustrated in Figure 1. Each section of the line consists of a (constant) inductor L_1 in parallel with a nonlinear capacitance $C(V_n)$, dependent on the voltage at the n'th node of the line. In this loop we may write down Kirchoff's laws:

$$L_1 \frac{\partial I_n}{\partial t} = V_{n-1} - V_n\,, \quad \frac{\partial Q_n}{\partial t} = I_n - I_{n+1}$$

where $Q_n \equiv Q(V_n) = \int^{V_n} C(V)\,\mathrm{d}V$ is the charge stored in the n'th capacitor. We may use the first of these equations to eliminate I_n from the time-derivative of the second. The result is

$$L_1 \frac{\partial^2 Q_n}{\partial t^2} \equiv L_1 \frac{\partial}{\partial t}\left[C(V_n)\frac{\partial V_n}{\partial t}\right] = V_{n+1} + V_{n-1} - 2V_n. \tag{2.1}$$

This is the circuit equation describing the voltage $V_n(t)$ on a single line.

Now we imagine coupling many identical lines such as this by means of capacitors C_2 at each node. Such a configuration is shown in Figure 2. The nodes in the system are labelled with two discrete coordinates: n specifies the nodes in the direction of propagation of the pulse, and m labels the lines in the transverse direction. We apply Kirchoff's laws again, this time in orthogonal loops:

$$L_1 \frac{\partial I^1_{nm}}{\partial t} = V_{n,m} - V_{n+1,m}\,, \quad I^2_{nm} = C_2\,\frac{\mathrm{d}}{\mathrm{d}t}(V_{n,m} - V_{n,m+1})\,,$$
$$\frac{\partial Q_{nm}}{\partial t} = I^1_{n-1,m} - I^1_{nm} + I^2_{n,m-1} - I^2_{nm}.$$

The circuit equation for this system is therefore

$$\frac{\partial^2 Q_{nm}}{\partial t^2} = \frac{1}{L_1}(V_{n+1,m} + V_{n-1,m} - 2V_{n,m}) + C_2 \frac{\partial^2}{\partial t^2}(V_{n,m+1} + V_{n,m-1} - 2V_{n,m}). \tag{2.2}$$

3 Continuum Limit

We shall take the continuum limit of (2.2) in both the n and m directions, along with the additional provisos that the amplitude of the voltage pulse is small and its wavelength is much greater than the lattice spacings. In this analysis, we shall take the nonlinear capacitance $C(V_{n,m})$ to be of the form

$$C(V) = \frac{C_0}{1 + \left(\frac{V}{V_0}\right)^p} \tag{3.1}$$

where C_0 and V_0 are arbitrary capacitance and voltage scales, respectively, and $p > 0$. Similar forms for $C(V)$ have been used in the past to model the capacitance of certain varactor diodes as part of comparisons with experimental measurements of solitary waves on NTL's (Hicks *et al.*, 1996).

If we treat n and m as continuous variables and V as a function of n, m and t, and assume that $V \ll V_0$, then (2.2) becomes

$$L_1 C_0 \frac{\partial^2}{\partial t^2}\left[V - \frac{V^{p+1}}{(p+1)V_0^p}\right] = \frac{\partial^2}{\partial n^2}\left(V + \frac{1}{12}\frac{\partial^2 V}{\partial n^2} + \cdots\right) + L_1 C_2 \frac{\partial^4}{\partial t^2 \partial m^2}\left(V + \frac{1}{12}\frac{\partial^2 V}{\partial m^2} + \cdots\right). \tag{3.2}$$

Next, we transform to 'stretched' coordinates which reflect the fact that the dominant motion is in the n direction:

$$\xi \equiv \epsilon^{1/2}(n - vt)\,, \quad \eta \equiv \epsilon^{3/2} t\,, \quad \chi \equiv \epsilon^{1/2} m\,, \quad V(n,m,t) \equiv \epsilon^{1/p} w(\xi, \eta, \chi) \tag{3.3}$$

where ϵ is a small parameter. On using these expressions, (3.2) becomes, to lowest order in ϵ,

$$\frac{24 w_\eta}{v} + \frac{12 (w^{p+1})_\xi}{(p+1)V_0^p} + w_{\xi\xi\xi} + \frac{12 C_2 w_{\chi\chi\xi}}{C_0} = 0 \tag{3.4}$$

where $v^2 \equiv (L_1 C_0)^{-1}$. Finally, define

$$x \equiv \xi\,, \quad y \equiv \sqrt{\frac{C_0}{12 C_2}}\,\chi\,, \quad \tau \equiv \frac{v\eta}{24}\,, \quad u(x,y,\tau) \equiv \frac{12^{1/p}}{V_0} w(\xi,\eta,\chi) \tag{3.5}$$

and we arrive at the *modified Zakharov-Kuznetsov (mZK) equation*

$$u_\tau + u^p u_x + (u_{xx} + u_{yy})_x = 0. \tag{3.6}$$

This equation possesses a y-independent (*i.e.* plane, or 'line') solitary wave solution for all p. We shall be interested in the stability of this plane wave under transverse perturbations.

4 Multiple-scale Analysis

The plane soliton solution of the mZK equation (3.6) is

$$u_0(x,\tau) = (A\beta^2)^{1/p}\,\mathrm{sech}^{2/p}[\tfrac{1}{2}p\beta(x - \beta^2\tau)] \tag{4.1}$$

where $A \equiv \frac{1}{2}(p+1)(p+2)$ and β is a constant. We shall impose on this line soliton a sinusoidal perturbation in the y direction and study the growth of this perturbation as the soliton propagates in the x direction. Numerical studies have been performed before (Frycz and Infeld, 1989) which demonstrate the instability of the solitary wave (4.1) in the case $p = 1$ as long as the wavenumber k is below a certain threshold value. In that study, the decay of the pulse into separate localised lumps was accompanied by an increase in the overall amplitude. In this work we aim to discover whether similar phenomena exist in the (discrete) precursor system of coupled transmission lines, *before* the continuum limit is taken. The existence of pulse amplification especially may have relevance to the design of data transmission systems.

It is convenient to transform to the moving frame of the soliton (4.1):

$$x' = \tfrac{1}{2}p\beta(x - \beta^2\tau)\,, \quad y' = \tfrac{1}{2}p\beta y\,, \quad t' = \beta^3\tau\,, \quad n = \beta^{-2/p}u. \tag{4.2}$$

On dropping primes, the mZK equation and its plane solitary wave solution become, respectively,

$$8n_t + 4p(n^p - 1)n_x + p^3(n_{xxx} + n_{xyy}) = 0 \tag{4.3}$$

$$n_0(x) = A^{1/p}\,\mathrm{sech}^{2/p}\,x. \tag{4.4}$$

We now apply a transverse sinusoidal perturbation to n_0, and we make an *ansatz* for its x- and t-dependence:

$$n(x, y, t) = n_0(x) + \rho \mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{\gamma t}\Phi(x). \tag{4.5}$$

Here, ρ is a (small) parameter controlling the perturbation strength, and $\gamma(k)$ is the growth rate and k the wavenumber of the perturbation. Both $\gamma(k)$ and $\Phi(x)$ will be found presently. On substituting (4.5) into (4.3) we find to lowest order in ρ that Φ and γ must satisfy

$$(\mathcal{L}\Phi)' + \frac{8\gamma\Phi}{p^3} = k^2\Phi' \tag{4.6}$$

where $\mathcal{L} \equiv \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + \dfrac{4(n_0^p - 1)}{p^2}$.

Equation (4.6) is solved by means of *multiple-scale perturbation theory* (Allen and Rowlands, 1993). The central idea of this method is to introduce a family of distance scales $x_n \equiv k^n x$, to be treated as independent variables, and then to expand the growth rate γ and perturbation eigenfunction Φ in powers of k as follows:

$$\Phi = \Phi_0(x, x_1, x_2, \dots) + k\Phi_1(x, x_1, x_2, \dots) + k^2\Phi_2(x, x_1, x_2, \dots) + \cdots \tag{4.7}$$

$$\gamma = k\gamma_1 + k^2\gamma_2 + \cdots \tag{4.8}$$

Then (4.6) yields a hierarchy of equations which must be solved at each order in k for the quantities Φ_i. A bounded solution is required, and the appropriate overall behaviour can be ensured at each order by the removal or regrouping of so-called 'secular' (divergent) terms. Doing this provides values for the γ_i. Note that in Allen and Rowlands' paper, the case $p = 1$ was considered. The expansions of $\gamma(k)$ about its two zeroes were then combined into a two-point Padé approximant which agreed extremely well with numerical results. The case $p = \frac{1}{2}$ has been studied in the same way (Munro and Parkes, 1997).

In this work, we shall concentrate on the case $p = 2$. Then it turns out that the multiple-scale expansions *terminate*, giving an exact solution to (4.6). Also, this value of p is the threshold between stability and instability of two-dimensional lump solitons in the mZK equation (Pelinovsky and Grimshaw, 1996). We shall not reproduce the multiple-scale calculation here; the reader is referred instead to Allen and Rowlands' paper for a detailed account of the method.

When $p = 2$ the growth rate $\gamma(k)$ and perturbation eigenfunction $\Phi(x)$ are found to be

$$\gamma(k) = \frac{2k(3 - k^2)}{3\sqrt{3}} \tag{4.9a}$$

$$\Phi(x) = \mathrm{e}^{-kx/\sqrt{3}}(\sqrt{3}\tanh x + k)\,\mathrm{sech}\,x. \tag{4.9b}$$

That these expressions solve (4.6) when $p = 2$ is easily checked. A plot of γ^2 as a function of k^2 is given in Figure 3.

5 Discrete Transverse Coordinate

The expression (4.5) for the perturbed soliton will be used to provide an initial voltage profile for numerical simulations of the system of coupled NTL's. This input profile will be allowed to propagate along the lattice and the amplitude of the perturbation will be measured at regular intervals and compared with the predicted behaviour from the multiple-scale analysis.

The system we have simulated is 'long and thin', *i.e.* it consists of many more sections in the direction of propagation than in the transverse direction. In fact, there are 15 transmission lines ($m = 1, \dots, 15$)

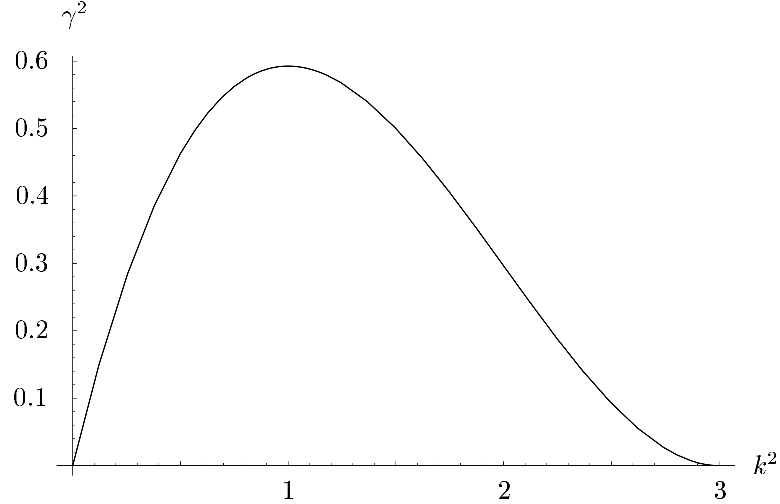

Figure 3: Growth rate (squared) of transverse perturbation as a function of wavenumber (squared) for the case $p = 2$.

each consisting of 301 sections ($n = 0, \ldots, 300$). The width of the voltage pulse is such that it covers several (~ 5) sections in the direction of propagation, meaning the continuum limit in n is an acceptable approximation. But strictly speaking the discrete nature of the transverse coordinate m cannot be dealt with in the same way. The above analysis needs to be modified to take this discreteness into account. Instead of taking the continuum limit of (2.2) in both coordinates, we shall repeat the manipulations but leaving m unchanged. The end result is what may be called a 'discrete' mZK equation

$$u_\tau + u^p u_x + u_{xxx} + \sigma(u_+ + u_- - 2u)_x = 0 \tag{5.1}$$

where $u_\pm \equiv u(x, m \pm 1, \tau)$ and $\sigma \equiv \dfrac{12C_2}{\epsilon C_0}$. Equation (5.1) takes the place of (3.6). Obviously, it possesses the same plane solitary wave solution (4.1). As before, we transform to the moving frame (4.2) and we seek perturbed solutions of the form of (4.5), where the 'y' in that expression is related to the 'laboratory' coordinate m by

$$y = \tfrac{1}{2} p\beta \sqrt{\frac{12\epsilon C_0}{C_2}}\, m \equiv (\Delta y)\, m. \tag{5.2}$$

Instead of (4.6) we arrive at

$$(\mathcal{L}\Phi)' + \frac{8\gamma\Phi}{p^3} = \tilde{k}^2\Phi \tag{5.3}$$

where

$$\tilde{k} \equiv \frac{4\sqrt{\sigma}}{\beta p} \sin\frac{k\Delta y}{2}. \tag{5.4}$$

We shall set $p = 2$ again, so that (5.3) possesses an exact solution of the form of (4.9) but with k replaced by $\tilde{k}$:

$$\gamma(\tilde{k}) = \frac{2\tilde{k}(3 - \tilde{k}^2)}{3\sqrt{3}} \tag{5.5a}$$

$$\Phi(x) = \mathrm{e}^{-\tilde{k}x/\sqrt{3}}(\sqrt{3}\tanh x + \tilde{k})\,\mathrm{sech}\, x. \tag{5.5b}$$

In the case of a continuous transverse coordinate, the growth rate γ is positive (*i.e.* the soliton is unstable) if $k < k_{\mathrm{crit}} = \sqrt{3}$. In the discrete case, this critical value is instead given by $\tilde{k}_{\mathrm{crit}} = \sqrt{3}$, which, on substituting for $\tilde{k}$, implies that we should expect transverse instability for wavenumbers below

$$k_{\mathrm{crit}} = \frac{2\sin^{-1}(\frac{1}{2}\mu\sqrt{3c})}{\mu c} \tag{5.6}$$

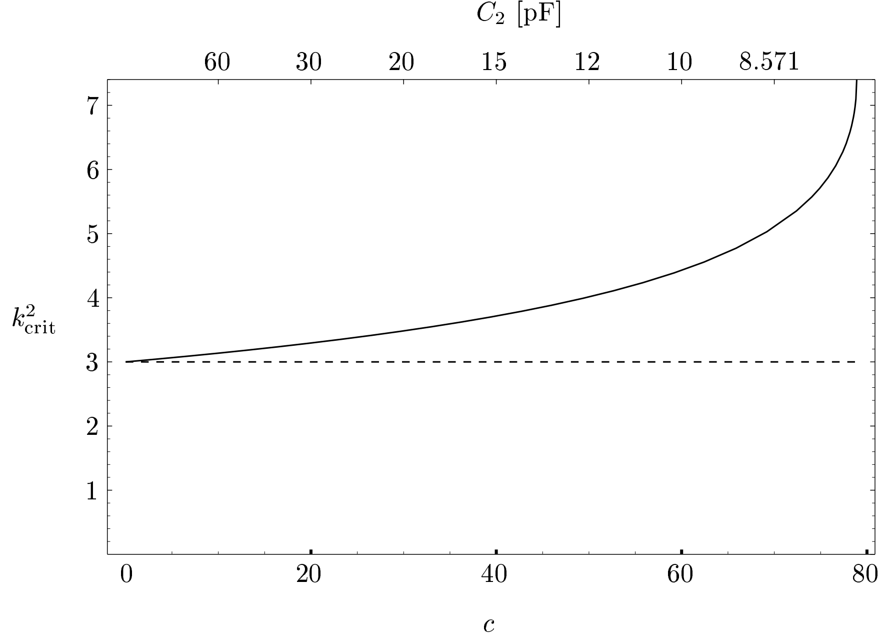

Figure 4: Critical wavenumber (squared) as a function of coupling capacitance C_2 in the case of a discrete transverse coordinate. The continuum result is indicated by the dotted line. Parameter values: $C_0 = 600$pF, $\mu = 0.13$.

where we have defined for convenience

$$\mu \equiv \frac{\beta \epsilon^{\frac{1}{2}}}{2\sqrt{3}} \quad \text{and} \quad c \equiv \frac{C_0}{C_2}.$$

In other words, if the transverse coordinate is discrete, the critical value of k *depends on the coupling capacitance* C_2 between the transmission lines. The dependence of $k_{\rm crit}$ on C_2 is indicated in Figure 4.

We have performed simulations of solitary waves for a range of values of C_2 and k, covering a large part of the graph in Figure 4, and have compared the measured growth rate of the transverse instability with the values predicted by multiple-scale analysis.

6 Simulation Results

The behaviour of transversely-perturbed solitary waves on a system of 15 coupled NTL's, each of 301 sections, was simulated by the *HSpice* circuit simulator, running under Solaris 2.5.1 on a Sun Ultra Enterprise Server 2. The voltage-dependence of the nonlinear capacitances in the system was given by equation (3.1) with $p = 2$, $C_0 = 600$pF and $V_0 = 5$V, and the inductance L_1 was chosen to be 1μH.

An initial voltage profile was required to be input at one end (*i.e.* at section $n = 0$) of the system, and this was obtained from expression (4.5) for the perturbed solitary wave. When the transverse coordinate is treated as discrete, it follows from results in the previous section that (4.5) may be written as

$$n(x, y, t) = \left[\sqrt{6} + \rho(\sqrt{3}\tanh x + \tilde{k}) \exp\left(-\frac{\tilde{k}x}{\sqrt{3}} + \mathrm{i}\tilde{k}y + \gamma t\right)\right] \operatorname{sech} x. \tag{6.1}$$

In order to obtain an input pulse from this expression we must revert to 'laboratory' coordinates n, m and t by reinstating the primes on x, y and t and applying transformations (4.2), (3.5) and (3.3) in reverse. After this is done, the input voltage as a function of m and t is obtained simply by setting $n = 0$, specifying values for the perturbation wavenumber k and coupling capacitance C_2, and taking the real part. In addition, we set $\rho = 0.1$ and $\mu = 0.13$. Figure 5 shows a typical input pulse.

The pulse then propagates in the direction of increasing n and the amplitude of the transverse sinusoidal perturbation is measured through the pulse maximum at every 10th section. As illustrations, three such transverse profiles, taken at $n = 50$, 150 and 250, are given in Figure 6. The growth of the perturbation amplitude is clearly evident. The perturbation is expected to grow in this example, rather than to be damped, because the parameters chosen ($C_2 = 10$pF, $k^2 = 2$) are well within the region of

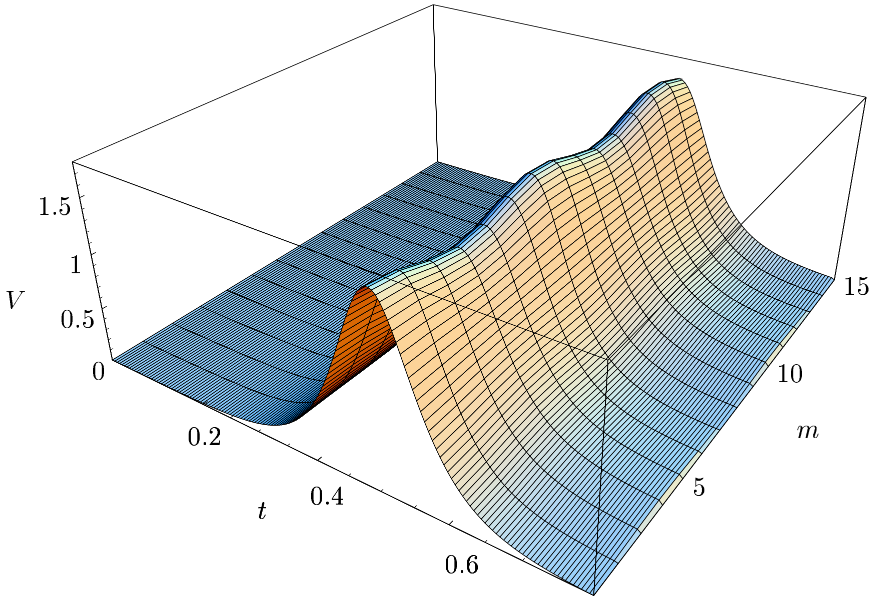

Figure 5: An initial voltage profile, to be input to the system at section $n = 0$. The coupling capacitance in this case was $C_2 = 10$pF and the perturbation wavenumber was $k = \sqrt{2}$. Units of time are microseconds and those of V are volts.

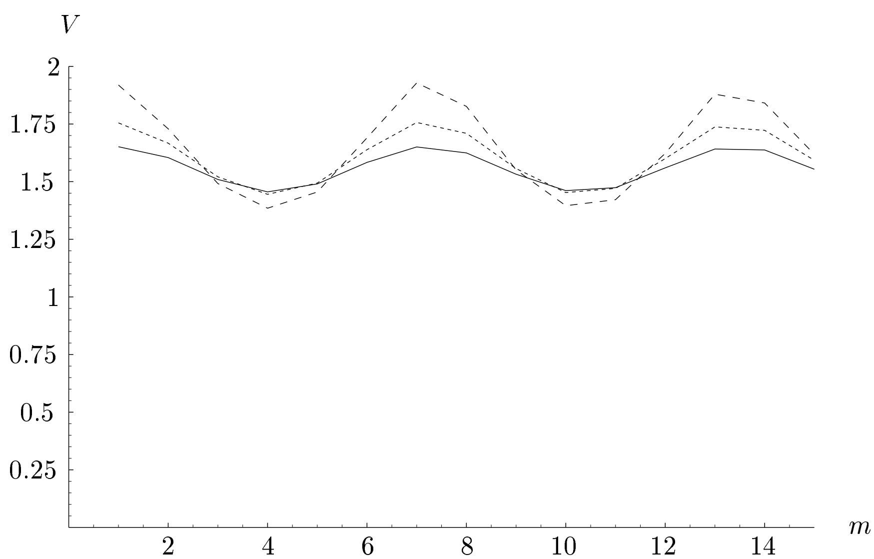

Figure 6: Transverse profiles through the pulse maximum. The solid, dotted and dashed lines represent the profile at the time when the maximum reached the 50th, 150th and 250th section respectively. The input pulse used was that shown in Figure 5.

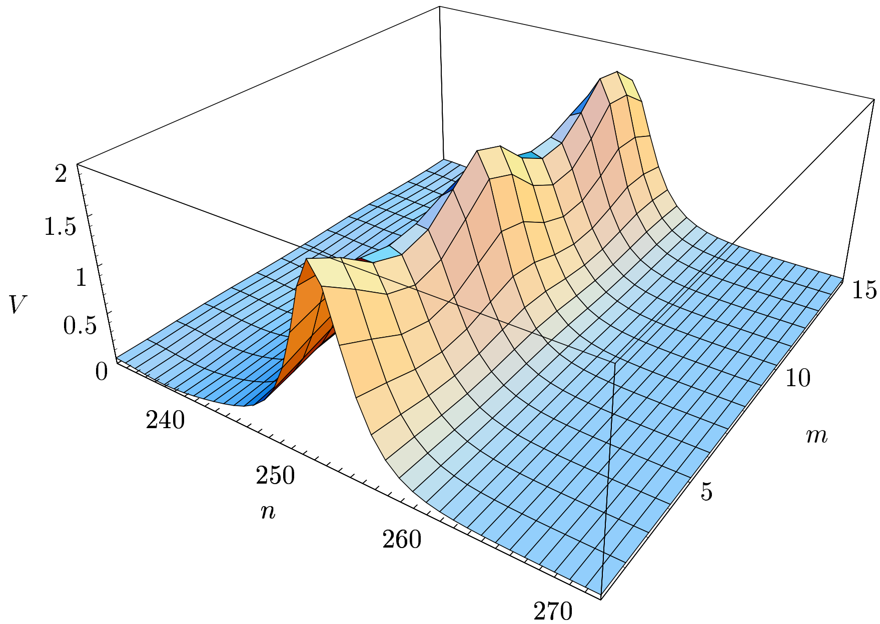

Figure 7: The pulse of Figure 5 after having travelled 250 sections along the system. The time elapsed is about 6.5μs.

instability of Figure 4. Figure 7 shows the voltage pulse itself as a function of n and m after about 6.5μs.

Since we are interested in the growth of the perturbation amplitude as a function of time, we need to compare the measured growth with the exponential factor in (6.1). Reverting to 'laboratory' time gives a predicted dependence of the form e^{gt} where

$$g = \sqrt{3} v \mu^3 \gamma(\tilde{k}). \tag{6.2}$$

A graph of the natural logarithm of the perturbation amplitude ΔV against time should therefore be a straight line of gradient g. From the simulation data, such a graph is constructed and the gradient of a line of best fit is obtained. This gradient is then divided by $\sqrt{3} v \mu^3$ to give the measured growth rate. The predicted value of γ is obtained from (5.5a), with $\tilde{k}$ being given by (5.4). Table 1 shows the measured and predicted values of γ for a range of wavenumbers k and coupling capacitances C_2.

7 Discussion and Conclusion

It is clear from Table 1 that there is general qualitative agreement between the predicted and measured growth rates. However, there is an obvious systematic discrepancy: the measured values are all too large. This may arise from the approximation incurred by retaining the continuum limit in the n direction, or perhaps (although this is unlikely) from numerical artifacts generated by *HSpice*.

Also, the decay of the pulse into separate lumps is by no means as pronounced as that seen in numerical studies of the (unmodified) ZK equation (Frycz and Infeld, 1989). The underlying discreteness of the system is presumably preventing the growth of the perturbation from proceeding beyond a certain limit.

This last point could be investigated by simulating longer lines (*i.e.* with more than 300 sections) but this will lead to greatly increased simulation times and data storage requirements. A way round these problems may be to store the shape of the voltage pulse when it has reached, say, the 280th section of a 300-section system and then to use it as the input pulse in a subsequent simulation of the *same* system.

The increase in pulse amplitude, although evident from the voltage profiles in Figure 6, as the solitary wave breaks up is also not as great as that seen in the continuum studies of Frycz and Infeld. Nevertheless, this phenomenon has been shown to be present in systems of capacitively-coupled nonlinear transmission lines and therefore they may prove useful in achieving pulse compression in data transmission by electrical means.

	Coupling capacitance C_2 [pF]					
	30	20	15	12	10	8.571
$k^2 = 6$	−0.882	−0.666	−0.580	−0.329	0.055	0.334
	−1.775	−1.345	−0.971	−0.648	−0.369	−0.132
$k^2 = 5$	−0.596	0.009	0.054	0.214	0.286	0.498
	−1.069	−0.795	−0.552	−0.337	−0.148	0.018
$k^2 = 4$	0.301	0.319	0.466	0.623	0.592	0.707
	−0.414	−0.260	−0.121	0.005	0.118	0.218
$k^2 = 3$	0.700	0.788	0.849	0.996	0.927	0.989
	0.157	0.226	0.290	0.349	0.402	0.451
$k^2 = 2$	1.106	1.123	1.135	1.151	1.177	1.228
	0.587	0.607	0.625	0.642	0.657	0.672

Table 1: Measured and predicted growth rates of transverse perturbation amplitude. For each value of k^2, the upper line of numbers comprises the measured growth rates, and the lower the predicted ones obtained from equation (5.5a).

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