The internal structure of the two-soliton solution to nonlinear evolution equations of a certain class

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Abstract

We consider the class of nonlinear evolution equations that have N-soliton solutions for the dependent variable $u(x,t)$, where $u = 2(\ln f)_{xx}$ and f is obtainable by Hirota's method. The N-soliton solution is decomposed into a sum $\sum_{i=1}^{N} u_i$, where, in the limits $t \to \pm\infty$, each u_i is a 1-soliton solution to the original governing equation. During interaction 'mass' is conserved for each u_i. Our formulation of the decomposition does not use the inverse scattering technique and is similar to that used for the KdV equation by Yoneyama (1984b) and Moloney & Hodnett (1989). Focusing on the case $N = 2$, we discuss the properties of u_1 and u_2, and our results are illustrated by considering an extended KdV equation and the Sawada-Kotera equation. Also, for each of these equations, the corresponding 'interacting soliton' equations are derived for general N.

1 Introduction

The KdV 2-soliton solution regarded as the interaction of two single solitons has been investigated by several authors [1] - [5]. In [1] - [3] the decomposition of the 2-soliton solution was achieved via inverse scattering transform theory. In [4, 5], however, the Hirota formalism was used. The results in [3] and [4, 5] were later extended to the KdV N-soliton solution; see [6] and [7] respectively.

The decomposition in [6] and [7] is, in fact, applicable to a wide class of nonlinear evolution equations of which the KdV equation is a member, and it is this class of equations that is considered here. In §2 we summarise the formulation of the decomposition of the N-soliton solution. In §3 we focus on the case $N = 2$. In §4 and §5 we illustrate our results by considering an extended KdV equation and the Sawada-Kotera equation respectively. Future work is outlined in §6.

2 General Formulation

Let $u(x,t)$ be the N-soliton solution to a given nonlinear evolution equation. Suppose that u may be expressed in the form

$$u = 2(\ln f)_{xx}$$

where

$$f = f(\eta_1, \ldots, \eta_N), \qquad \eta_i = k_i(x - c_i t) + \delta_i,$$

and k_i, c_i and δ_i are constants. We express u in the form

$$u = \sum_{i=1}^{N} u_i, \tag{2.1}$$

where

$$u_i = (w_i)_x \qquad \text{and} \qquad w_i = 2k_i(\ln f)_{\eta_i}. \tag{2.2}$$

Hirota's method leads to an expression for f as a series involving $f_j := e^{2\eta_j}$, $(j = 1, \ldots, N)$. For a given i the series may be written in the form

$$f = h_i + \bar{h}_i f_i,$$

where h_i and $\bar{h}_i$ do not involve f_i. It now follows that

$$w_i = 2k_i(1 + \tanh g_i)$$

where

$$g_i = \eta_i + \frac{1}{2}\ln\left(\frac{\bar{h}_i}{h_i}\right) \tag{2.3}$$

and so

$$u_i = 2k_i(g_i)_x \operatorname{sech}^2 g_i. \tag{2.4}$$

It is clear from the following results that (2.1) with (2.2) is the desired decomposition:

(i) On using (2.4) we have $\int_{-\infty}^{\infty} u_i \, dx = 4k_i$ so the 'mass' of u_i is conserved.

(ii) From (2.3), as $t \to \pm\infty$ with η_i fixed, $g_i \to \eta_i + \text{constant}$ and so

$$u_i \to 2k_i^2 \operatorname{sech}^2[k_i(x - c_i t) + \text{constant}],$$

namely a 1-soliton solution to the original evolution equation.

Maloney & Hodnett [7] discuss the trajectories of the 'centres of the masses'. We prefer to consider the trajectories of the 'centres of mass'. The 'centre of mass' of u_i, $x_G^{(i)}$, is defined by

$$4k_i x_G^{(i)} = \int_{-\infty}^{\infty} x u_i \, dx. \tag{2.5}$$

From (2.2) and (2.5) it may be shown that the velocity $v_G^{(i)}$ of the 'centre of mass' of u_i is given by

$$v_G^{(i)} := \frac{dx_G^{(i)}}{dt} = c_i + \frac{1}{4k_i}\int_{-\infty}^{\infty} x T_i \, dx, \tag{2.6}$$

where T_i is the 'transfer function' defined by

$$T_i = \sum_{j=1}^{N} \frac{\partial W_{ij}}{\partial x}$$

and W_{ij} is given by

$$W_{ij} = (c_i - c_j)k_j(w_i)_{\eta_j}. \tag{2.7}$$

From (2.6) it can be seen clearly that the trajectory of u_i is affected by the presence of the other u_j, $(j \neq i)$.

Finally, we note an additional useful property of u_i. If $u = w_x$ then

$$u_i = k_i w_{\eta_i}. \tag{2.8}$$

3 The Case $N = 2$

Without loss of generality we assume that $k_2 > k_1$. Hirota's method gives

$$f = 1 + f_1 + f_2 + b^2 f_1 f_2, \quad b > 0,$$

where b is a function of k_1 and k_2 that depends on the nonlinear evolution equation being considered. The trajectories of the centres of mass of u_1 and u_2 intersect at the origin in x-t space if $2\delta_1 = 2\delta_2 = -\ln b$, and then, as $t \to \pm\infty$,

$$\begin{aligned} u_1 &\to 2k_1^2 \operatorname{sech}^2[k_1(x - c_1 t) \mp \tfrac{1}{2}\epsilon \ln b], \\ u_2 &\to 2k_2^2 \operatorname{sech}^2[k_2(x - c_2 t) \pm \tfrac{1}{2}\epsilon \ln b], \end{aligned}$$

where $\epsilon = \operatorname{sgn}(c_2 - c_1)$.

We wish to study the dynamical evolution of the interaction between u_1 and u_2 for $-\infty < t < \infty$. To do this the following results are useful. Here the subscript zero denotes evaluation at $x = 0$, $t = 0$, and

$$b_{KdV} := \frac{k_2 - k_1}{k_2 + k_1} = \frac{r - 1}{r + 1}, \quad \text{with} \quad r := \frac{k_2}{k_1} > 1,$$

where b_{KdV} is the expression for b associated with the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

- $u_{10} < 0$ for $0 < b < b_{KdV}$ and $u_{10} = 0$ for $b = b_{KdV}$, otherwise $u_{10} > 0$; $u_{20} > 0$ and $u_0 > 0$.
- For $0 < b < b_{KdV}$, u_1 has two zeros at $x - c_2 t = \pm p/k_2$, where p is the root of

  $$\cosh 2p = [(1 - b^2)r - (1 + b^2)]/2b. \tag{3.1}$$

  For $b = b_{KdV}$, $p = 0$ and u_{10} has one zero at $x - c_2 t = 0$. For $b > b_{KdV}$, u_1 has no zeros.
- $u_{1x0} = 0$, $u_{2x0} = 0$ and $u_{x0} = 0$.
- $u_{1xx0} \geq 0$ for $0 < b \leq b_{c1}$, otherwise $u_{1xx0} < 0$, where

  $$b_{c1} = \frac{3r^2 - 1 + r\sqrt{r^4 + 6r^2 - 3}}{(1 + r)^3}.$$

  $u_{2xx0} \geq 0$ for $b_{c2-} \leq b \leq b_{c2+}$ with $1 < r < \sqrt{1 + 2/\sqrt{3}}$, otherwise $u_{2xx0} < 0$, where

  $$b_{c2\pm} = \frac{r(3 - r^2) \pm \sqrt{-3r^4 + 6r^2 + 1}}{(1 + r)^3}.$$

  $u_{xx0} \geq 0$ for $b_{c-} \leq b \leq b_{c+}$ with $1 < r < \sqrt{2 + \sqrt{3}}$, otherwise $u_{xx0} < 0$, where

  $$b_{c\pm} = \frac{-r^4 + 6r^2 - 1 \pm r\sqrt{8(-r^4 + 4r^2 - 1)}}{(1 + r)^4}.$$

- From (2.7) we have

  $$W_{12} = 8(c_1 - c_2)k_1 k_2 f_1 f_2 (b^2 - 1)/f^2 = -W_{21}.$$

4 Example: The extended KdV Equation

The extended KdV (eKdV) equation [8] is

$$u_t + \mu(3u^2 + u_{xx})_x + \sigma(10u^3 + 5u_x^2 + 10uu_{xx} + u_{xxxx})_x = 0, \tag{4.1}$$

where μ and σ are arbitrary constants. (4.1) reduces to the KdV equation when $\mu = 1$, $\sigma = 0$, and to the 5th-order KdV equation (Lax hierarchy) when $\mu = 0$, $\sigma = 1$.

Write $u = w_x$ in (4.1) and integrate with respect to x with the conditions that w_t and x derivatives of w vanish as $x \to \pm\infty$. On applying the operator $k_i\partial/\partial\eta_i$ to the resulting equation and using (2.8) we obtain the 'interacting soliton equations' for $i = 1, \ldots, N$, namely

$$u_{it} + \mu(6uu_{ix} + u_{ixxx}) + \sigma(30u^2u_{ix} + 10u_{ix}u_{xx} + 10uu_{ixxx} + 10u_xu_{ixx} + u_{ixxxxx}) = 0.$$

The Hirota form for (4.1) is

$$[D_x(D_t + \mu D_x^3 + \frac{\sigma}{6}D_x^5) - \frac{5\sigma}{6}D_x^3D_\tau](f \cdot f) = 0,$$

$$D_x(D_\tau + D_x^3)(f \cdot f) = 0$$

from which it follows that

$$c_i = 4\mu k_i^2 + 16\sigma k_i^4 \quad \text{and} \quad b_{eKdV} = b_{KdV}.$$

As $b = b_{KdV}$, the dynamics of the two soliton interaction are similar to those given in [2] and [5] for the KdV equation.

We deduce the following features during interaction.

- u_1 has one zero and its trajectory is $x - c_2t = 0$.
- $u_{1xx0} > 0 \Rightarrow u_1$ has two maxima.
- For $1 < r < \sqrt{2}$, $u_{2xx0} > 0 \Rightarrow u_2$ has two maxima;

  for $r > \sqrt{2}$, $u_{2xx0} < 0 \Rightarrow u_2$ has one maximum.
- For $1 < r < \sqrt{3}$, $u_{xx0} > 0 \Rightarrow u$ has two maxima;

  for $r > \sqrt{3}$, $u_{xx0} < 0 \Rightarrow u$ has one maximum.

As an illustrative example, take $\mu = \sigma = 1$ in (4.1) and soliton parameters $k_1 = 1$ and $k_2 = 1.6$. In this case $r = 1.6$ so that $\sqrt{2} < r < \sqrt{3}$. Figure 1 illustrates the evolution of u_1 and u_2 in a frame of reference moving with speed c_2. In this figure the evolution is symmetric about the origin since our choice of δ_1 and δ_2 ensures symmetry about the origin in x-t space. In particular we have chosen $t = 0$, ± 0.01 and ± 0.05. Figure 1 shows clearly that the zero of u_1 is stationary in the frame of reference that moves with speed c_2. Figure 2 shows the behaviour of $v_G^{(1)}$ and $v_G^{(2)}$ as functions of time during the interaction as calculated from (2.6). The figure shows that, as u_2 accelerates, u_1 decelerates and vice versa and, for a while, the velocity of u_1 is negative. Figure 3 shows the trajectories of the centres of mass of u_1 and u_2 in the x-t plane as calculated from (2.5) and illustrates clearly the temporary backward motion of u_1.

We used *Mathematica* to perform all the calculations to produce the figures.

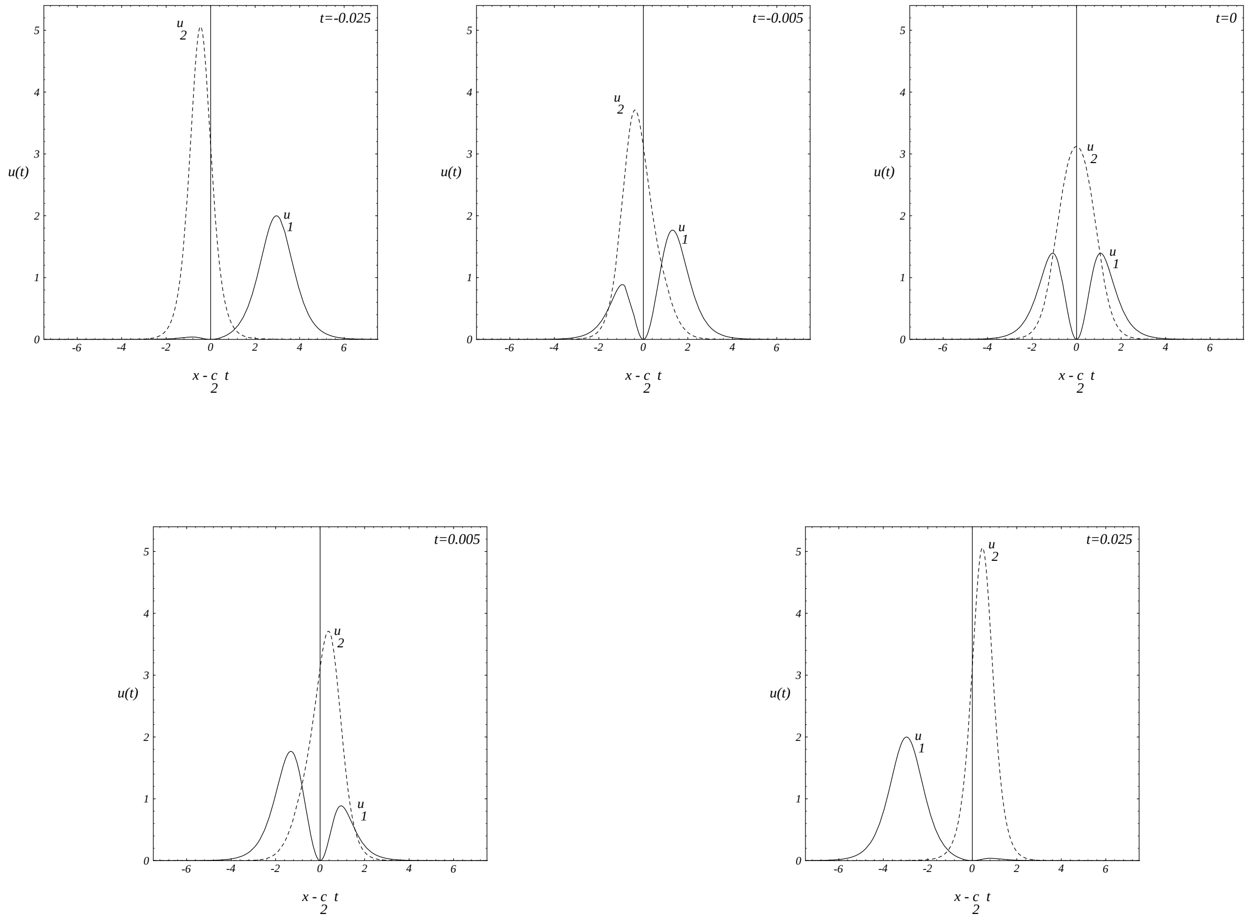
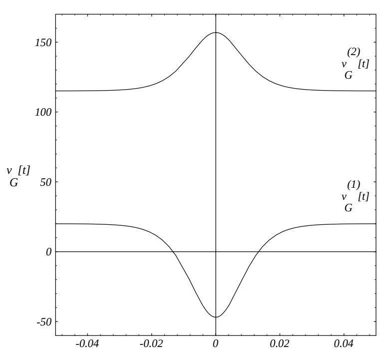


Figure 1: u_1 & u_2 for the eKdV Equation



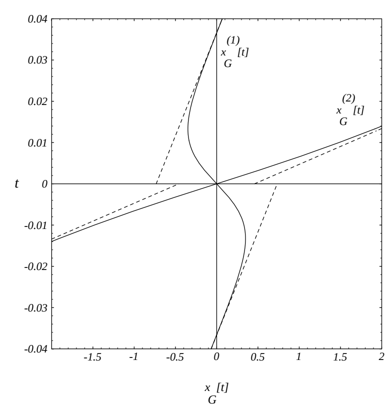


Figure 2: $v_G^{(1)}(t)$ & $v_G^{(2)}(t)$ for the eKdV Equation

Figure 3: $x_G^{(1)}(t)$ & $x_G^{(2)}(t)$ for the eKdV Equation

5 Example: The Sawada-Kotera Equation

The Sawada-Kotera (SK) equation [9] is

$$u_t + (15u^3 + 15uu_{xx} + u_{xxxx})_x = 0, \tag{5.1}$$

By using the method outlined in §4 we obtain the 'interacting soliton equations' for $i = 1, \ldots, N$, namely

$$u_{it} + 45u^2 u_{ix} + 15u_{ix}u_{xx} + 15uu_{ixxx} + u_{ixxxxx} = 0.$$

The Hirota form for (5.1) is

$$D_x(D_t + D_x^5)(f \cdot f) = 0$$

from which it follows that

$$c_i = 16k_i^4 \quad \text{and} \quad b_{SK} = \sqrt{\frac{1 + r^2 - r}{1 + r^2 + r}}\left(\frac{r-1}{r+1}\right).$$

We deduce the following features during interaction.

- u_1 has two zeros and their trajectories are $x - c_2 t = \pm p/k_2$ respectively, where p (> 0) is given by (3.1) with $b = b_{SK}$.
- $u_{1xx0} > 0 \Rightarrow u_1$ has two maxima.
- For $1 < r < 1.4596$, $u_{2xx0} > 0 \Rightarrow u_2$ has two maxima;

  for $r > 1.4596$, $u_{2xx0} < 0 \Rightarrow u_2$ has one maximum.
- For $1 < r < 1.8566$, $u_{xx0} > 0 \Rightarrow u$ has two maxima;

  for $r > 1.8566$, $u_{xx0} < 0 \Rightarrow u$ has one maximum.

As an illustrative example, take soliton parameters $k_1 = 1$ and $k_2 = 1.3$. In this case $r = 1.3$ so that $1 < r < 1.4596$. Figures 4 to 6 for the SK equation are analogous to Figures 1 to 3 for the eKdV equation. The profiles in Figure 4 are plotted for $t = 0$, ± 0.025 and ± 0.15. This figure shows clearly that, as predicted, the two zeros of u_1 are stationary and disposed symmetrically about the origin in the frame of reference that moves with speed c_2.

6 Future Work

We are considering other evolution equations such as the Boussinesq equation [10]. This equation has soliton solutions that propagate in either direction and so head-on collisions may be investigated. We aim to study interactions for the $N = 3$ case and apply the results to a variety of equations; this would generalise the results of Moloney & Hodnett [11] who considered only the KdV equation.

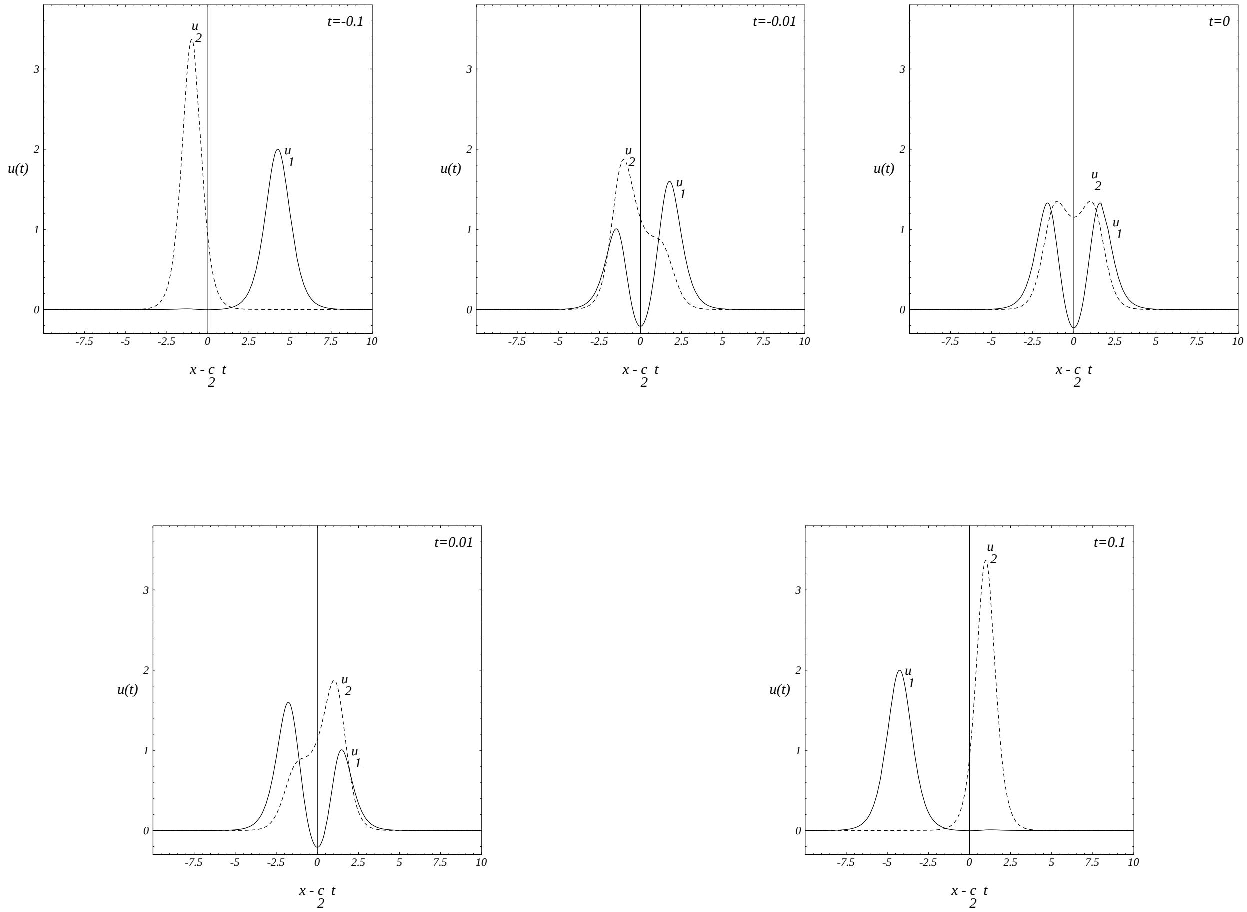


Figure 4: u_1 & u_2 for the SK Equation

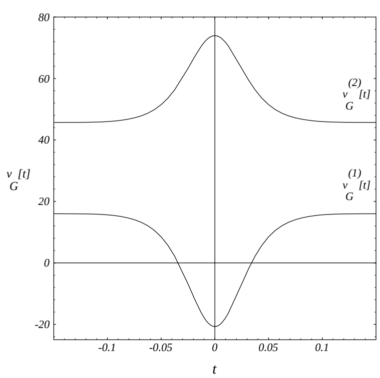


Figure 5: $v_G^{(1)}(t)$ & $v_G^{(2)}(t)$ for the SK Equation

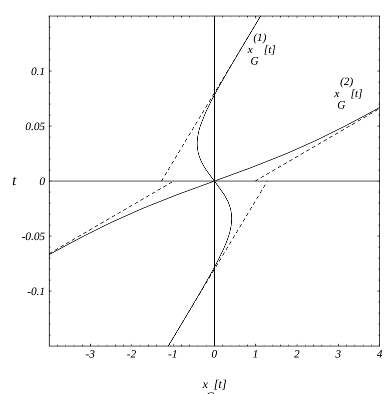


Figure 6: $x_G^{(1)}(t)$ & $x_G^{(2)}(t)$ for the SK Equation

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