On the boundary problem for a linear impulse system of the second order and the Green function of this linear impulse system

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Abstract
Conditions of existence of a solution to a two-point boundary problem for the system of linear differential equations of the second order under the conditions of impulse action are presented in this paper.

1 Introduction

Many problems from practice lead to investigation of differential equations under the condition of the impulse effect which occur in mathematical models in medicine and biology, mathematical economy, chemical technology, metallurgy etc.

One of the important problems of the modern theory of ordinary differential equations under the conditions of impulse effect is the research of the boundary problem for ordinary differential equations under the conditions of impulse effect [2, 3]

The article considers the problem of the Green function construction for a two-point boundary problem under the condition of impulse effect, besides, conditions under which the given problem has solutions which are written by means of the Green function.

2 Mane Results

The problem of twice sectionally continuous differentiated solutions to the system of linear differential equations

$$(P(t)x'(t))' - Q(t)x(t) = f(t), \ t \in J_0 = [0, l] \setminus \cup_{i=1}^{r} \{t_i\} \qquad (1)$$

subjected to the impulse effect condition

$$P(t_i+0)x'(t_i+0) - P(t_i)x'(t_i) + \alpha_i x(t_i) + \beta_i x'(t_i) = \gamma_i,\ i = 1, \dots, r, \quad (2)$$

and to two-point boundary conditions of the type

$$a_1 x(0) + b_1 x'(0) = 0, \quad (3)$$

$$a_2 x(l) + b_2 x'(l) = 0, \quad (4)$$

where $t_i, i = 1, \dots, r$, are points of impulse disturbance: $0 \equiv t_0 < t_1 < \dots < t_r < t_{r+1} \equiv l$; $f(t)$ is a vector-function which is continuous on every interval $(t_i, t_{i+1}], i = 0, 1, \dots, r; \alpha_i, \beta_i, i = 1, \dots, r; a_j, b_j, j = 1, 2$ are constant square matrices of the dimension $n \times n$ such that $\det(a_j b_j) \neq 0, j = 1, 2$; $\gamma_i,\ i = 1, \dots, r$, are n-dimension vectors.

$n \times n$ matrix-functions $P(t)$ and $Q(t)$ satisfy the following conditions:

1) on every interval $(t_i, t_{i+1}), i = 0, 1, \dots, r$ the matrix-function $P(t)$ is continuously differentiable and the matrix-function $Q(t)$ is continuous;

2) at any point $t_i,\ i = 1, \dots, r$ the matrix- function $P(t)$ is continuous from the left, that is

$$P(t_i) = P(t_i - 0) = \lim_{t \to t_i - 0} P(t);$$

3) for $P(t)$ the following inequality is true

$$\inf_{t \in (t_i, t_{i+1})} \det P(t) > 0,\ i = 0, 1, \dots, r. \quad (5)$$

Definition 1. The solution of the boundary problem with the impulse effect (1)-(4) is a sectionally twice continuous differentiated vector- function $x(t) \in R^n$ with the discontinuity of the first order at the points $t = t_i$, $i = 1, \dots, r$ for which the following conditions are correct:

1) on every interval $(t_i, t_{i+1}), i = 0, 1, \dots, r$ the vector-function $x(t)$ is twice continuous differentiated and satisfies the differential equation (1);

2) at points $t_i, i = 1, \dots, r$ the vector-function $x(t)$ is continuous from the left

$$x(t_i) = x(t_i - 0) = \lim_{t \to t_i - 0} x(t),\ i = 1, \dots, r;$$

3) at points $t_i, i = 1, \dots, r$ for $x(t)$ the condition of the impulse effect (2) is fulfilled;

4) the vector-function $x(t)$ satisfies the boundary conditions (3), (4).

The condition (5) provides the solution of problems of the type

$$(P(t)x'(t))' - Q(t)x(t) = 0,\ t \in J_0,$$

$$x(t_i) = a_{i0}, P(t_i)x'(t_i) = a_{i1},\ i = 1, \dots, r,$$

for arbitrary a_{i0}, $a_{i1}, i = 1, \dots, r$. The last statement follows from Theorem 6.1. [1, p. 45].

Definition 2. A continuous from the left $n \times n$ matrix-valued function $G(t, s)$, where $G : [0, l]^2 \to M_n(R)$, $M_n(R)$ is a set of real $n \times n$-matrices, is called the Green function of two-point problem (3), (4) for linear impulse system (1), (2), if the following conditions are fulfilled:

1) at every fixed $s \in J_0$ row of the matrix $G(t, s)$ are solutions of the homogeneous boundary problem

$$(P(t)x'(t))' - Q(t)x(t) = 0,\ t \in J_0 = [0, l] \setminus \cup_{i=1}^{r} \{t_i\} \tag{6}$$

with impulse effect

$$P(t_i + 0)x'(t_i + 0) - P(t_i)x'(t_i) + \alpha_i x(t_i) + \beta_i x'(t_i) = 0,\ i = 1, \dots, r, \tag{7}$$

and the boundary conditions (3), (4);

2) at every interval $(t_i, t_{i+1}), i = 0, 1, \dots, r$ of the partition of the intervals $[0, s)$ and $(s, l]$, where $s \in J_0$, the function $G(t, s)$ is continuously differentiated and satisfies the following conditions:

$$G(s + 0, s) - G(s - 0, s) = 0, \tag{8}$$

$$G'_t(s + 0, s) - G'_t(s - 0, s) = P^{-1}(s) \tag{9}$$

The given definition of the Green function for the impulse system is based on the definition of the Green function [1, p. 97] for the linear differential operator of the m order with singular matrix coefficient in the vicinity of the highest derivative which is defined on the set of m-continuous differentiated functions.

Theorem 1. Let the singular boundary valued problem with the impulse effect (6), (7) and with the boundary conditions (3), (4) has only a trivial solution. Then the Green function of problem (1) - (4) exists and is the is unique.

Proof. The Green function $G : [0, l]^2 \to M_n(R)$ of the problem (1) - (4) may be written as

$$G(t,s) = \begin{cases} X(t)C^-(s), & t < s, \\ X(t)C^+(s), & t > s, \end{cases} \tag{10}$$

where $X(t) = [x_1(t), \ldots, x_{2n}(t)]$ is $n \times 2n$ matrix constructed with the help of linear independent solutions $x_k = x_k(t),\ k = 1, \ldots, 2n$ of the system of homogeneous linear differential equations (6) on the set J_0. In the formula (10) $C^-(s)$, $C^+(s)$ are some matrix the elements of which are unknown to be sectionally-continuous functions on the set J_0. Really, every row $G_j(t, s), j = 1, \ldots, n$, of the Green matrix $G(t, s)$ for a fixed $s \in J_0$ is a solution of the homogeneous boundary problem (6), (7). Thus,

$$G_j(t,s) = \begin{cases} \sum_{v=1}^{2n} c_{vj}^-(s)x_v(t), & t < s, \\ \sum_{v=1}^{2n} c_{vj}^+(s)x_v(t), & t > s, \end{cases} \tag{11}$$

where $c_{vj}^-(s), c_{vj}^+(s),\ v = 1, \ldots, 2n; j = 1, \ldots, n$ are some scalar functions which are elements of $2n \times n$ matrices $C^-(s)$, $C^+(s)$. The equality (11) is equivalent to relation (10).

Taking into consideration condition (8) for the continuous Green function $G = G(t, s)$ at the point of a diagonal square $[0, l] \times [0, l]$ and the condition (9) of the jump of its derivative, from the function (11) we obtain:

$$X(t)C(s) = 0, \tag{12}$$

$$P(s)X'(t)C(s) = I_n, \tag{13}$$

where $C(s) := C^+(s) - C^-(s)$.

The relation (12), (13) is the system of $2n^2$ non-homogeneous linear equations the matrix of the coefficients of which is not degenerated for every $t \in J_0$. The last statement follows from the fact that a block matrix

$$\begin{pmatrix} X \\ PX' \end{pmatrix}$$

formed by linear independent solutions of the system of linear differential equations of the first order of the type

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & P^{-1} \\ Q & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which is equivalent to the system (6), (7) under the given conditions. The system (12), (13) has the unique solution. By reasoning similar to given one in [1, p. 31] taking into consideration the assumption about the non-existence of non-trivial solutions of the problem (6),(7),(3),(4), we found that matrix- functions $C^{-}(s)$ and $C^{+}(s)$ in the formula (10) are uniquely determined by matrix $C(s) = C^{+}(s) - C^{-}(s)$.

Thus, we establish that the Green function $G(t, s)$ of the problem (1)-(4) is given by the formula of the type (10), where coefficients $C^{-}(s)$ and $C^{+}(s)$ are uniquely determined by the matrix $C = C(s)$, which is the unique solution of a boundary problem uniquely connected with the problem (1) - (4). The theorem is proved.

By the determined Green function $G(t, s)$ the formula for the solution of two-point boundary problem with an impulse effect (1) - (4) may be written. The following theorem holds true.

Theorem 2. Let the conditions be satisfied:

1) the boundary value problem (6), (7), (3), (4) has only a trivial solution;

2) vector-function $f(t)$ is continuous on every interval $(t_i, t_{i+1}], i = 0, 1, \dots, r$;

3) $\gamma_i, i = 0, 1, \dots, r$ are constant n-dimensional vectors.

Then problem (1) - (4) has unique solution $x(t)$ of the form

$$x(t) = \int_0^l G(t, s) f(s) ds + \sum_{v=1}^{r} G(t, t_v + 0)\gamma_v, \ t \in [0, l]. \tag{14}$$

The above theorem is proved by a substitution of the expression (14) to the system of linear differential equations of the second order (1), to the impulse effect (2) and to the two-point boundary conditions (3), (4).

References

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