The reduction method in the theory of Lie-algebraically integrable oscillatory Hamiltonian systems.

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0. Introduction.

As is well known [1,4] symmetry analysis of nonlinear dynamical systems on a smooth manifold M gives rise in many cases to exhibiting its many hidden but interesting properties, in particular such as being integrable by quadratures due to the Liouville-Arnold theorem [2]. In case when the manifold M can be represented as the cotangent space $T^*(K)$ to some subgroup K of a Lie group G naturally acting on it, the study of the corresponding flow can be recast via the reduction method [3] into the Hamiltonian framework due to the existence on $T^*(K)$ the canonical Poisson structure. Furthermore, if the symmetry group G naturally generalizes to the loop group $G_+(\lambda)$ over $\lambda \in D_0 \subset \mathbb{C}$, then the corresponding momentum mapping $l : T^*(K) \to \mathcal{G}_+^*(\lambda)$ provides us with a Lax type representation and related with it a complete set of commuting invariants . Such a scheme appeared to be very useful when proving the Liouville integrability of many finite-dimensional systems such as Kowalevskaya's top [3], Neumann type systems [5,6] and other. Below we study complete integrability of nonlinear oscillatory dynamical systems connected in particular both with the Cartan decomposition of a Lie algebra $\mathcal{G} = \mathcal{K} + \mathcal{P}$, where $\mathcal{K}$ is the Lie algebra of a fixed subgroup $K \subset G$ with respect to an involution $\sigma : G \to G$ on the Lie group G, and with a Poisson action of special type on a symplectic matrix manifold.

1.Integrable systems on $T^*(K)$:the general scheme.

Consider a Lie group G and an involution σ on G. If $K \subset G$ is its fixed subgroup, then the Lie algebra $\mathcal{G}$ of the Lie group G admits the Cartan decomposition $\mathcal{G} = \mathcal{K} + \mathcal{P}$ with the induced involution mapping $\sigma = id$ on $\mathcal{K}$ and $\sigma = -id$ on $\mathcal{P}$. Denote also $\mathcal{G}^* = \mathcal{K}^* + \mathcal{P}^*$ the dual decomposition of the adjoint space $\mathcal{G}^*$. The cotangent space $T^*(K) \simeq K \times \mathcal{K}^*$ by means of left translations on K. Assume now that the natural group action of G on $T^*(K)$ is extended to that of the loop group $G_+(\lambda)$, $\lambda \in D_0$, where $D_0 \subset \mathbb{C}^1$ is a disc containing zero. Let $\mathcal{G}_+(\lambda)$ be the Lie algebra of the loop group $G_+(\lambda)$ acting on the cotangent bundle $T^*(K) \simeq K \times \mathcal{K}^*$. If the action is Hamiltonian [1], one can define the corresponding momentum mapping $l : T^*(K) \to \mathcal{G}_+^*(\lambda)$. Here the adjoint space $\mathcal{G}_+^*(\lambda)$ is defined with respect to the following invariant and symmetric scalar product on $\mathcal{G}_+(\lambda)$:

$$< \xi(\lambda), \eta(\lambda) >_{-1} = res_{\lambda \in D_0} 1/\lambda < \xi(\lambda), \eta(\lambda) >_{\mathcal{G}} \tag{1.1}$$

for any $\xi(\lambda) \in \mathcal{G}_+^*(\lambda), \eta(\lambda) \in \mathcal{G}_+(\lambda)$, where $< \cdot, \cdot >_{\mathcal{G}}$ denotes the standard Killing form on $\mathcal{G}$. Any orbit passing through a point $l(u, v; \lambda) \in \mathcal{G}_+^*(\lambda)$, with $(u, v) \in T^*(K)$ being fixed, is defined naturally as

$$\underset{x(\lambda) \in \mathcal{G}_+(\lambda)}{Span} \{ \ \Pr_{\mathcal{G}_+(\lambda)} (Ad^*_{\exp(-x(\lambda))} l(u, v; \lambda))\}, \tag{1.2}$$

where $\Pr_{\mathcal{G}_+(\lambda)} : \mathcal{G}^*(\lambda, \lambda^{-1}) \to \mathcal{G}_+^*(\lambda)$ denotes the projection upon $\mathcal{G}_+^*(\lambda)$ parallelly to the subspace $\mathcal{G}_-^*(\lambda)$, $\mathcal{G}^*(\lambda, \lambda^{-1}) := \mathcal{G}_+^*(\lambda) \oplus \mathcal{G}_-^*(\lambda)$ with $\mathcal{G}_+^*(\lambda) = \mathcal{G}_-(\lambda)$ and $x(\lambda) \in \mathcal{G}_+(\lambda)$ is arbitrary element. Let $\mathcal{G}_+(\lambda) = \{\sum_{i \in \mathbb{Z}_+} x_i \lambda^i : x_i \in \mathcal{G}$, $\sigma x_i = (-1)^i x_i$ for all $i \in \mathbb{Z}_+\}$, so $\mathcal{G}_+^*(\lambda) = \{\sum_{j \in \mathbb{Z}_+} y_j \lambda^{-(j+1)} : y_j \in \mathcal{G}^*$, $\sigma y_j = (-1)^{j+1} y_j$ for all $j \in \mathbb{Z}_+\}$. Having for instance taken $\bar{l}_b(u, v; \lambda) = v(u) + \lambda^{-1} b \in \mathcal{G}_+^*(\lambda)$, one can derive that

$$l_b(u, v; \lambda) := Ad^*_{\exp(-x(\lambda))} \bar{l}_b(u, v; \lambda) = Ad^*_{u^{-1}} v(u) + \lambda^{-1} Ad^*_{u^{-1}} b = \tag{1.3}$$

$$v(e) + \lambda^{-1} Ad^*_{u^{-1}} b$$

for any $(u, v) \in K \times \mathcal{K}$ with $u := \exp x_0 \in K$ and $b \in \mathcal{G}^*$. Consider now an element $a\lambda \in \mathcal{G}^*(\lambda, \lambda^{-1})$, where $a \in \mathcal{G}^*$ is constant. Since also

$$(a\lambda, [\mathcal{G}_-(\lambda), \mathcal{G}_-(\lambda)])_{-1} = 0 = (a\lambda, \mathcal{G}_+(\lambda))_{-1} \tag{1.4}$$

for any $a \in \mathcal{G}^*$, we see that the element $a\lambda \in \mathcal{G}^*(\lambda, \lambda^{-1})$ is an infinitesimal character of the Lie subalgebra $\mathcal{G}_-(\lambda)$. Based now on the well known Adler-Kostant-Symes (AKS) theorem [10], one can formulate the following theorem.

Theorem 1.1 *All functional* $\gamma^{(a,b)}_{s,n}(u, v) := res_{\lambda \in D_0} (\lambda^s l^n_{a,b}(u, v; \lambda)\ )$, $s, n \in \mathbb{Z}$, *where*

$$l_{a,b}(u, v; \lambda) := l_b(u, v; \lambda) + a\lambda \tag{1.5}$$

are involutive on the cotangent space $T^*(K) \simeq K \times K$ *with respect to the standard Poisson bracket on* $T^*(K)$. *Since under the involution* $K \ni u :\to u^{-1} \in K$ *and* $T_e^*(K) \ni v(e) \to w(e) \in T^*(K) \simeq K^*$ *combined with the permutation* $\mathcal{G}^* \ni a \longleftrightarrow b \in \mathcal{G}^*$ *the element* $l_{a,b}(u, v; \lambda) \to l_{b,a}(u, w; \lambda)$, *making it possible to represent the flow on* $T^*(K)$ *generated by the invariant* $\gamma^{(a,b)}_{1-n,n}(u, v) \in D(T^*(K))$, $n \in Z_+$, *as the one generated by* $\gamma^{(a,b)}_{n-3,,n}(u, w)$.

In case when a Lie algebra $\mathcal{G}$ is the Lie algebra of the connected subgroup G of $SO(4, 3)$, the maximal compact subgroup $K \subset G$ with the Lie algebra $\mathcal{K}$

is isomorphic to $so(4,3)$.Thereby this pair $(\mathcal{G},\mathcal{K})$ can be used [11] for constructing integrable flows quadratic in momenta on $T^*(K)$, in particular the four -dimensional top and its generalizations.

2. Oscillatory dynamical systems on $T^*(K)$: an example.

Consider now the case when a loop group $G_-(\lambda)$ acts on $T^*(K) \simeq K \times \mathcal{K}^*$, where $\lambda \in D_\infty$ and $D_\infty \subset \mathbb{C}$ is an open disc containing the infinite point. Put $\mathcal{G}_-(\lambda)$ the Lie algebra of the group $G_-(\lambda)$ and $\mathcal{G}_-^*(\lambda)$ its adjoint space with respect to the scalar product $< \xi(\lambda), \eta(\lambda) >_0 := res_{\lambda \in D_\infty} < \xi(\lambda), \eta(\lambda) >_{\mathcal{G}}$ for any $\xi(\lambda) \in \mathcal{G}_-^*(\lambda)$ and $\eta(\lambda) \in \mathcal{G}_-(\lambda)$. As before, let $\mathcal{G}_+(\lambda) = \{\sum_{i\in\mathbb{Z}_+} x_i\lambda^i : x_i \in \mathcal{G}$, $\sigma x_i = (-1)^i x_i,\ i \in \mathbb{Z}_+\}$, $\mathcal{G}_-(\lambda) = \{\sum_{i\in\mathbb{Z}_+} y_i\lambda^{-(i+1)} : y_i \in \mathcal{G},\ \sigma y_i = (-1)^{i+1} y_i$, $i \in \mathbb{Z}_+\}$.The adjoint space $\mathcal{G}_+^*(\lambda) = \{\sum_{i\in\mathbb{Z}_+} a_i\lambda^i : a_i \in \mathcal{G}^*,\ \sigma x_i = (-1)^i x_i$, $i \in \mathbb{Z}_+\}$ contains one-parametric orbits of the Ad^*- action, which can be interpreted as some finite-dimensional integrable Hamiltonian systems on $T^*(K)$. For this to be a lot more clarified, let us consider an element $a\lambda^2 \in \mathcal{G}_-^*(\lambda)$ with $a \in \mathcal{P}$ and calculate its orbit under the action $Ad^*_{\exp(-x(\lambda))} : \mathcal{G}_-^*(\lambda) \to \mathcal{G}_-^*(\lambda)$, where $x(\lambda) \in \mathcal{G}_-(\lambda)$ is some element specified by a point $(u,v) \in T^*(K)$. We find therefore that the orbit of the element $a\lambda^2 + b \in \mathcal{G}_-^*(\lambda)$ has the form:

$$l_{a,b}(u,v;\lambda) = a\lambda^2 + \lambda[x_0,a] + [x_1,a] + 1/2[x_0,[x_0,a]] + b, \tag{2.1}$$

in which one can make identifications $[x_0,a] := q \in \mathcal{K}_a^\perp$ and $[x_1,a] = p \in \mathcal{P}_a^\perp$ with $u := (\exp x_1) \in \mathcal{K}$ and $b := [x_0,a] \in \mathcal{K}_a^* \subset \mathcal{K}^*$ due to the natural isomorphisms $ad\ a : \mathcal{K}_a^\perp \to \mathcal{P}_a^\perp$ and $ad\ a : \mathcal{P}_a^\perp \to \mathcal{K}_a^\perp \subset \mathcal{K}^*$. Similarly one can represent the forth element in (2.1) as

$$\alpha(q) := \mathrm{Pr}_{\mathcal{P}_a} 1/2[(ad\ a)^{-1}q, q], \tag{2.2}$$

where evidently $\alpha : \mathcal{K}_a^\perp \to \mathcal{P}_a$. Having assumed further that an element $a \in \mathcal{P}$ is such that $[\mathcal{G}_a^\perp, \mathcal{G}_a^\perp] \subset \mathcal{G}_a$ or equivalently $\mathcal{G}{=}\mathcal{G}_a \oplus \mathcal{G}_a^\perp$ (the symmetric expansion), one easily verifies that $[\mathcal{P}_a^\perp, \mathcal{K}_a^\perp] \subset \mathcal{P}_a$, or $\alpha(q) = 1/2[(ad\ a)^{-1}q, q]$ since $\alpha(q) \in \mathcal{P}_a$ for all $q \in \mathcal{K}_a^\perp$. In virtue of the isomorphism between $\mathcal{P}_a^\perp$ and $\mathcal{K}_a^\perp$, the orbit (2.1) evidently is diffeomorphic both to $\mathcal{K}_a^\perp \oplus \mathcal{P}_a^\perp$ and to the cotangent space $T^*(\mathcal{K}_a^\perp)$.

The space $T^*(\mathcal{K}_a^\perp)$ is endowed with the canonical Poissonian structure being equivalent to the standard Lie-Poisson structure upon the orbit (2.1):

$$\{q_i, q_j\} :=< l, [\nabla q_i(l), \nabla q_j(l) >_0 = 0, \tag{2.3}$$

$$\{q_i, p_j\} :=< l, [\nabla q_i(l), \nabla p_j(l) >_0 =< [f_j, e_i], a >_{\mathcal{G}},$$

$$\{p_i, p_j\} :=< l, [\nabla p_i(l), \nabla p_j(l) >_0 =< [f_i, f_j], q >_{\mathcal{G}},$$

for all $i,j = \overline{1,n}$ and any $(q,p) \in T^*(\mathcal{K}_a^\perp)$, where $\nabla : \mathcal{D}(T^*(\mathcal{K}_a^\perp)) \to \mathcal{K}_a$ denotes the usual gradient mapping on $\mathcal{D}(T^*(\mathcal{K}_a^\perp))$. When deriving (2.3) we made use of the following relationships: $q := \sum_{i=1}^n q_i e_i$, $p := \sum_{i=1}^n p_i f_i$, where

$\{e_j=[f_j,a]\in\mathcal{K}_a^\perp : j=\overline{1,n}\}$ and $\{f_j\in\mathcal{P}_a^\perp : j=\overline{1,n}\}$ are orthogonal bases in $\mathcal{K}_a^\perp$ and $\mathcal{P}_a^\perp$ correspondingly, that is $<e_i,e_j>_\mathcal{G}=\delta_{ij}=<f_i,f_j>_\mathcal{G}$ for all $i,j=\overline{1,n}$.

As was mentioned in [12,13] the elements $a\in\mathcal{P}$ satisfying the property $\mathcal{G}=\mathcal{G}_a\oplus\mathcal{G}_a^\perp$ can be found easily enough if one to consider a dual compact Lie algebra $\mathcal{G}=\mathcal{K}\oplus i\mathcal{P}$. Then the Hermitian symmetric expansion $\mathcal{G}=\mathcal{G}_{ia}\oplus\mathcal{G}_{ia}^\perp$ holds and the problem reduces to recounting all involutions $\sigma:\mathcal{G}\to\mathcal{G}$ in $\mathcal{G}$ commuting with the above Hermitian expansion and equal to "$-id$" upon the center of the Lie algebra $\mathcal{G}_{ia}$. The condition $\mathcal{G}=\mathcal{G}_a\oplus\mathcal{G}_a^\perp$ involved above on an element $a\in\mathcal{P}$ implies obviously that $\mathcal{G}_a^\perp=ad\ a(\mathcal{G})=ad\ a(\mathcal{G}_a^\perp)$, since by definition $ad\ a(\mathcal{G}_a)=0$. Thus the element $a\in\mathcal{P}$ defines the projection operator $P_a:\mathcal{G}\to\mathcal{G}$ on $\mathcal{G}$ compatible with the involution $\sigma:\mathcal{G}\to\mathcal{G}$, that is $P_a\sigma=\sigma P_a$, where $P_a^2=P_a$. The latter condition appears to be useful for practical calculations on which we shall not dwell here. To end this section, let us write down the corresponding Hamiltonian flows on $T^*(\mathcal{K}_a^\perp)$ in the componentwise form. The vector $(q,p)\in T^*(\mathcal{K}_a^\perp)$ is a set of canonical coordinates on the orbit (2.1) since due to the imbedding $[\mathcal{P}_a^\perp,\mathcal{P}_a^\perp]\subset\mathcal{K}_a$, the bracket $\{p_i,p_j\}=0$ for all $i,j=\overline{1,n}$. As a result one obtains the following expression for the orbit point (2.1) :

$$l_{a,b}(q,p;\lambda)=a\lambda^2+\lambda\sum_{i=1}^n q_ie_i+(\sum_{i=1}^n p_if_i+1/2\sum_{i,j=1}^n q_iq_j[e_i,f_j]+b\lambda, \qquad (2.4)$$

where in virtue of (2.3)

$$\{q_i,q_j\}=0=\{p_i,p_j\},\{p_i,q_j\}=<f_j,f_i>_\mathcal{G} \qquad (2.5)$$

for all $i,j=\overline{1,n}$. Evaluating the functional $H=1/2res_{\lambda\in D_\infty}\lambda^{-1}\langle l_{a,b}(q,p;\lambda),l_{a,b}(q,p;\lambda)\rangle_\mathcal{G}$ on the orbit space $T^*(\mathcal{K}_a^\perp)$ at $b\in\ \mathcal{P}_a$, one gets the Hamiltonian function

$$H(q,p)=1/2\sum_{j=1}^n p_j^2+1/2\sum_{i,j=1}^n q_iq_j\langle[e_i,f_j],b\rangle_\mathcal{G}+$$

$$1/8\sum_{i,j=1}^n\sum_{s,l=1}^n q_iq_s<[e_i,f_j],[e_s,f_l]>_\mathcal{G}\ q_jq_l, \qquad (2.6)$$

describing an unharmonic oscillatory dynamical system of particles on the axis $\mathbb{R}\ni q^j$, $j=\overline{1,n}$, interacting with each other by means of a forth order potential. Based on theorem 1.1. one can formulate the following theorem.

Theorem 2.1. *The unharmonic oscillatory dynamical system (2.6) on the orbit space $T^*(\mathcal{K}_a^\perp)$ with the Poisson brackets (2.5) is a completely Liouville-Arnold integrable [1,2,18] Hamiltonian system.*

Choosing different semisimple Lie algebras $\mathcal{G}$ admitting the Hermitian symmetric expansion $\mathcal{G}_a\oplus\mathcal{G}_a^\perp=\mathcal{G}$ for some element $a\in\mathcal{P}$, where $\mathcal{G}=\mathcal{K}\oplus\mathcal{P}$ is the Cartan decomposition, one can build all of fourth order potential canonical Hamiltonian systems on $T^*(\mathcal{K}_a^{\perp}) \simeq T^*(\mathbb{R}^n)$ from [1].

3. Unharmonic oscillatory Hamiltonian systems and their Lie-algebraic integrability.

Consider now a dual matrix manifold $M := M_{n,2} \times M_{n,2}$ of dimension $(n\times 2)$, $n \in \mathbb{Z}_+$, endowed with the following natural symplectic structure

$$\omega^{(2)} = Sp(dQ^{\intercal} \wedge dF), \tag{3.1}$$

where $(F,Q) \in M$ and "Sp" means the standard trace operation. Let $A_+(\lambda)$ mean an analytical inside an open ring $D_0 \ni 0$ loop group acting on the manifold M as follows: for any $(F,Q) \in M$ and $g(\lambda) \in A_+(\lambda)$

$$F \overset{g(\lambda)}{:\rightarrow} F_{g(\lambda)} := res_{\lambda\in D_0}\frac{1}{\lambda-\Omega}Fg^{-1}(\lambda),$$

$$Q^{\intercal} \overset{g(\lambda)}{:\rightarrow} Q^{\intercal}_{g(\lambda)} := res_{\lambda\in D_0}\ g(\lambda)Q^{\intercal}\frac{1}{\lambda-\Omega}, \tag{3.2}$$

where $\Omega \in M_{n,n}$ is some matrix whose spectrum $\sigma(\Omega) \subset D_0$. Denote $\mathcal{A}_+(\lambda)$ the Lie algebra of the Lie group $A_+(\lambda)$, and put

$$\mathcal{A}_+(\lambda) = \{\sum_{j\in\mathbb{Z}_+} a_j\lambda^j : a_j \in sl(2;\mathbb{R}),\ j \in \mathbb{Z}_+\}. \tag{3.3}$$

The group action (3.2) as one can easily verify is Poissonian, leaving the symplectic structure (3.1) invariant. Thus if a one parametric subgroup $\{\exp(a(\lambda)t) : a(\lambda) \in \mathcal{A}_+(\lambda),\ t \in \mathbb{R}\}$ acts on M,the corresponding Hamiltonian function comes as follows:

$$H_a = -res_{\lambda\in D_0} Sp(Q^{\intercal}\frac{1}{\lambda-\Omega}Fa(\lambda)) := -2 < l(F,Q;\lambda), a(\lambda) >_0, \tag{3.4}$$

where

$$l(F,Q;\lambda) := \frac{1}{2}Q^{\intercal}\frac{1}{\lambda-\Omega}F \tag{3.5}$$

is the momentum mapping [1,2] and $< \cdot,\cdot >_r$ $r \in \mathbb{Z}$, is a scalar product on $\mathcal{A}(\lambda,\lambda^{-1})$ defined by the expression:

$$< l(\lambda), a(\lambda) >_r := res_{\lambda\in D_0}\lambda^{-r}Sp(l(\lambda)a(\lambda)). \tag{3.6}$$

It is easy to verify that the momentum mapping $l : M \rightarrow \mathcal{A}_+^*(\lambda)$ defined by (3.5) is equivariant [1], that is the diagram

$$\begin{array}{ccc} M & \overset{l}{\rightarrow} & \mathcal{A}_+^*(\lambda) \\ g(\lambda)\downarrow & & \downarrow Ad^*_{g^{-1}(\lambda)} \\ M & \overset{l}{\rightarrow} & \mathcal{A}_+^*(\lambda) \end{array} \tag{3.7}$$

is commutative for all $g(\lambda) \in \mathcal{A}_+^*(\lambda)$, meaning [1] that the loop group $\mathcal{A}_+(\lambda)$ action on M is Hamiltonian.

Define now a Lie algebras homomorphism

$$\alpha : \mathcal{A}_+(\lambda) \to \mathcal{G}_+(\lambda) \subset \lambda^2 \mathcal{A}_+(\lambda) \oplus \sigma_+ \mathbb{R}, \tag{3.8}$$

where for any $a(\lambda) \in \mathcal{A}_+(\lambda)$

$$\alpha(a)(\lambda) := \lambda^2 a(\lambda) \oplus a_{21}^{(0)} \sigma_+ \tag{3.9}$$

with $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ being a $sl(2;\mathbb{R})$ matrix basis.. It is verified that the mapping (3.8) is a homomorphism and the image $\alpha\mathcal{A}_+(\lambda) := \mathcal{G}_+(\lambda)$ constitutes a Lie algebra over $\mathbb{R}$. Thus there exists a loop group $\mathcal{G}_+(\lambda)$ whose Lie algebra coincides with this Lie algebra $\mathcal{G}_+(\lambda)$. Thereby one can define now another loop group $G_+(\lambda)-$action on M defined by the formulas (3.2) but with an element $g(\lambda) \in A_+(\lambda)$ replaced by an element $\alpha g(\lambda) \in G_+(\lambda)$, where $\alpha : A_+(\lambda) \to G_+(\lambda)$ is the corresponding to the mapping (3.8) loop groups homomorphism. Therefore, similarly to (3.4) one finds a momentum mapping $l_\alpha : M \to \mathcal{G}_+^*(\lambda)$ with respect to the modified loop group action $G_+(\lambda) \times M \overset{\alpha}{\to} M$ equivalent to that of $A_+(\lambda) \times M \to M$. A simple calculation yields

$$l_\alpha(F, Q; \lambda) = l(F, Q; \lambda) + \lambda l_{12}^{(0)} \sigma^+, \tag{3.10}$$

where by definition, $l := \sum_{j \in \mathbb{Z}_+} l^{(j)} \lambda^{-(j+1)}$ .When deriving (3.10) we based on the Hamiltonian function expression

$$H_a^\alpha = -2 < l_\alpha(F, Q; \lambda), \alpha(a)(\lambda) >_{-2} \tag{3.11}$$

generated by a one parametric subgroup $\{\exp(\alpha a(\lambda)t) \in G_+(\lambda) : a(\lambda) \in \mathcal{A}_+(\lambda), t \in \mathbb{R}\ \}$ and made use of the properties $Sp(\sigma_\pm \sigma^\pm) = 1$, $Sp(\sigma_- \sigma^+) = 0 = Sp(\sigma_+ \sigma^-)$ for the dual bi-orthogonal basis $\{\sigma^\pm,\ \sigma^0\} \in sl^*(2;\mathbb{R})$. Notice now that the element $\eta := \lambda^2 \sigma^+ - 2\sigma^- \in \mathcal{G}^*(\lambda, \lambda^{-1})$ is an infinitesimal character of the Lie subalgebra $\mathcal{G}_+(\lambda)$, where by definition $\mathcal{G}(\lambda, \lambda^{-1}) := \mathcal{G}_+(\lambda) \oplus \mathcal{G}_-(\lambda)$ and

$$< \eta, [\mathcal{G}_+(\lambda), \mathcal{G}_+(\lambda)] >_{-2} = 0 = < \eta, \mathcal{G}_-(\lambda) >_{-2} . \tag{3.12}$$

Owing to the property (3.12) and AKS-theorem [7-9], the extended momentum mapping

$$S(F, Q; \lambda) := \lambda^2 \sigma^+ - 2\sigma^- + l_\alpha(F, Q; \lambda) \tag{3.13}$$

generates on the manifold M an involutive with respect to (3.1) invariants $\gamma_j \in \mathcal{D}(M)$, $j = \overline{-1, n}$, via the expression:

$$\det S(F, Q; \lambda) = -\lambda^2 + \lambda\gamma_{-1} + \gamma_0 + \sum_{j=1}^{n} \frac{\gamma_j}{\lambda - \Omega_j}, \tag{3.14}$$

where we have put for definiteness $\Omega := diag\{\Omega_j \in \mathbb{R}/\{0\} : j = \overline{1,n}\}$, $Q := Fh$, $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $F := \begin{pmatrix} q_1, & q_2, & ..., & q_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix}^\tau \in M_{n,2}$. As a result of simple calculation one finds from (3.14) that

$$\gamma_j = -\frac{1}{2}p_j^2 + \frac{1}{4} < q, \Omega q > q_j^2 - < q, q > \Omega_j^2 q_j^2 + \tag{3.15}$$

$$\frac{1}{4}\sum_{k \neq j=1}^{n} (p_j q_k - p_k q_j)^2/(\Omega_j - \Omega_k),$$

where $j = \overline{1,n}$, and $< \cdot, \cdot >$ is the usual scalar product in $\mathbb{R}^n$. The corresponding symplectic structure (3.1) turns into the following canonical one: $\omega^{(2)}(F,Q) = 2\omega^{(2)}(q,p)$, where $\omega^{(2)}(q,p) := \sum_{j=1}^n dp_j \wedge dq_j$. Thus all Hamiltonian flows generated by invariants (3.15) on the space $M \simeq T^*(\mathbb{R}^n)$ are Liouville-Arnold integrable by quadratures since $\{\gamma_j, \gamma_k\} = 0$ for all $j, k = \overline{-1, n}$. In particular for the Hamiltonian function $H := \sum_{j=1}^n \Omega_j \gamma_j$ the corresponding dynamical system on $T^*(\mathbb{R}^n)$ is given as follows:

$$dq_j/dx = p_j, \quad dp_j/dx + \Omega_j^2 q_j - \Omega_j q_j < q, q >= \tag{3.16}$$

$$q_j(< q, \Omega q > -3/4 < q, q >^3),$$

where $j = \overline{1,n}$. Similar to (3.16) oscillatory equations constrained to live on the cotangent space $T^*(\mathbb{S}^{n-1})$ to the unit sphere $\mathbb{S}^{n-1} = \{q \in \mathbb{R}^n :< q, q >= 1\}$ were for the first time derived and studied in detail in [5,14], having based exclusively on the algebraic-geometric techniques [15]. Later on these results were rederived in [5,16] from the Lie-algebraic viewpoint [6]. As was shown in [17] by means of direct calculation, the extended momentum mapping (3.13) satisfies the following dynamical $r-$matrix identity:

$$\{S(q,p;\lambda), \oplus S(q,p;\mu)\} = [r_{12}(\lambda,\mu), S(q,p;\lambda) \otimes \mathbb{I}] - [r_{21}(\lambda,\mu), \mathbb{I} \otimes S(q,p;\mu)], \tag{3.17}$$

where $r_{21}(\lambda, \mu) := r_{12}(\mu, \lambda)$ and

$$r_{12}(\lambda,\mu) = P/(\lambda - \mu) - (< q, q > -\lambda - \mu)\sigma_- \otimes \sigma^+, \tag{3.18}$$

$Px \otimes y := y \otimes x$ for any $x, y \in \mathbb{R}^2$ and all $\lambda \neq \mu \in \mathbb{C}$. There is an important problem of deriving this $r-$matrix (3.18) from the pure Lie-algebraic viewpoint as it was done in [17] subject to the Calogero type models.

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