Bifurcation of coisotropic invariant tori of Hamiltonian system under perturbations of symplectic structure

Ihor Parasyuk

Abstract
We study the influence of small perturbations of symplectic structure and of Hamiltonian function on the behavior of a completely integrable Hamiltonian system whose phase space is stratified by the Lagrangian invariant tori. It is shown that, in quite general case, near certain family of these tori there appears a domain which contains a Cantor set of coisotropic invariant tori of the perturbed system. The relative measure of such a set tends to one when the magnitude of the perturbations decreases to zero.

1 Introduction

Let (M, ω_0^2) be $2n$-dimensional symplectic manifold with symplectic structure ω_0^2, $\mathcal{H}_0 : M \mapsto \mathbf{R}$ be a Hamiltonian function of completely integrable Hamiltonian system, and $F := (F_1, \ldots, F_n) : M \mapsto \mathbf{R}^n$ be a mapping whose components forms a complete involutive collection of the system's first integrals. The standard KAM-theory [1, 2] deals with perturbations of Hamiltonian function of the form $\mathcal{H}_0 \mapsto \mathcal{H}_0 + \mu\mathcal{H}_1$ where μ is a small parameter.

In this report we discuss a more general situation when not only the Hamiltonian but also the symplectic structure is perturbed, i.e. $\omega_0^2 \mapsto \omega_0^2 + \mu\omega_1^2$ where ω_1^2 is a closed but *not exact* 2-form on M. Such kind of problems, for example, naturally arises when the motions of nearly integrable mechanical systems are examined in the presence of weak magnetic field with certain singularities. (The treatment of symplectic structure deformation as the influence of the magnetic field was suggested by S. P. Novikov [3]). As far as we know, until recent time the above perturbations in general case has not yet been studied within the framework of KAM-theory. Some special results in this direction were obtained in [4].

The goal of this report, which is based on the paper [5], is to show that perturbations of symplectic structure may cause transformations of some Lagrangian invariant tori of the integrable Hamiltonian system into *coisotropic* invariant tori of the perturbed one. Quasiperiodic motions on such tori represent relatively new object of study in the nonlinear oscillations theory [6, 7, 8, 9]. In our case on each coisotropic invariant torus of dimension $r > n$ one can point out n fast and $r - n$ slow angle variables.

It should be noted that in this work we use the concept of bifurcation a little conditionally because the above coisotropic invariant tori do not depend continuously on parameter μ. Besides, here we observe the bifurcation not of an individual torus, but of quite massive set of tori localized in a neighborhood of certain family of unperturbed Lagrangian tori. We can describe this "generating" family of tori in a following way. Let $c \in \mathbf{R}^n$ be such a value of F that the set $F^{-1}(c)$ has a compact connected component M_c. As is well known [1] M_c is diffeomorphic to n-dimensional torus. In some neighborhood $N(M_c)$ of M_c a free symplectic action of n-dimensional torus $T^n = \mathbf{R}^n/2\pi\mathbf{Z}^n = \{\phi = (\phi_1, \ldots, \phi_n)| \bmod 2\pi\}$ naturally arises. This action is determined by a mapping $\Phi : T^n \times N(M_c) \mapsto N(M_c)$ and leaves invariant any common level manifold of functions $F_1, \ldots, F_n$ which intersects with $N(M_c)$. For any fixed $\phi \in T^n$ we define a mapping $\Phi^\phi = \Phi(\phi, \cdot) : N(M_c) \mapsto N(M_c)$. Now in $\mathbf{R}^n$ consider the Liouville system $\dot{F} = \mu G(x, \mu) := \{F, \mathcal{H}_\prime + \mu\mathcal{H}_1\}_\mu$ which governs the evolution of the functions $F_1, \ldots, F_n$ via the flow of the perturbed system (here $\{\cdot, \cdot\}_\mu$ stands for the Poisson brackets corresponding to the perturbed symplectic structure). Usually, one starts the bifurcation analysis from studying the averaged vector field $\int_{T^n} G(\Phi^\phi(x), 0)\, d\phi := G_0(F)$. We suppose that $G_0(F)$ has a stable singular point F_*. As we shall see later, if this point satisfies certain non-degeneracy conditions then there exists a k_0-dimensional ($k_0 < n$) manifold $\mathcal{F}$ of stable singular points. It turns out that for sufficiently small μ the perturbed system has coisotropic invariant tori which form a set $\mathcal{T}_\mu$ localized near the "generating" family of Lagrangian tori $\mathcal{W} = \{x \in M : F(x) = c,\ c \in \mathcal{F}\}$. Moreover, there exists a domain $\mathcal{D}_\mu \supset \mathcal{T}_\mu$ which shrinks to $\mathcal{W}$ as $\mu \to 0$ and in which the coisotropic invariant tori occupy a set of relative Lebesgue measure close to 1, namely, $\mathrm{mes}(\mathcal{T}_\mu \bigcap \mathcal{D}_\mu)/\mathrm{mes}\, \mathcal{D}_\mu \to 1$ as $\mu \to 0$.

2 Quasistationary points

Let us put into correspondence to any $a \in \mathbf{R}^n$ the vector field $X_a = \frac{d}{dt}\big|_{t=0} \Phi^{at}$. By means of the the averaged 2-form $\bar{\omega}_1^2 = \int_{T^n} (\Phi^\phi)^* \omega_1^2 \, d\phi$ we introduce a skew-symmetric bilinear form on $\mathbf{R}^n$ as follows: $\mathcal{C}(a,b) = \bar{\omega}_1^2(X_a, X_b), \; a, b \in \mathbf{R}^n$. This form is correctly determined: it does not depend on points of M. If $\mathcal{C}$ is nontrivial, then the torus action is non-exact. This means that some X_a do not have global corresponding Hamiltonians. It is just the case we are interested in. Namely, we suppose that $k_0 := \dim \ker \mathcal{C} \neq 0$ and, moreover, that the vectors $\sigma_1, \ldots, \sigma_{k_0}$ which form a basis in $\ker \mathcal{C}$ satisfy the following number-theoretic conditions:

$\mathbf{H}_0$: $\exists \gamma_0 > 0 \; \forall m \in \mathbf{Z}^n \backslash \{0\} \; \max_{1 \le i \le k_0} |\langle m, \sigma_i \rangle| \ge \gamma_0 |m|^{-n}$.

Here $\langle \cdot, \cdot \rangle$ stands for the standard scalar product in the coordinate vector space and $|m| = \max_{1 \le i \le n} |m_i|$.

As is well known, in $N(M_c)$ there exist action-angle coordinates $p = (p_1, \ldots, p_n), \; q = (q_1, \ldots, q_n)| \bmod 2\pi$ for the unperturbed system. On the basis of the Darboux-Weinstein theorem (see e.g. [10]) and a Moser's theorem [11] one can construct a coordinate transformation $p \mapsto p + O(\mu), \; q \mapsto q + O(\mu)$ which reduces the perturbed Hamiltonian and the perturbed symplectic structure, respectively, to the form

$$H = H_0(p) + \mu H_1(p, q; \mu), \qquad \omega^2 = dp \wedge dq + \frac{\mu}{2} \sum_{i,j=1}^{n} c_{ij} dq_i \wedge dq_j$$

where c_{ij} $(i, j = 1, \ldots, n)$ are constants. Henceforth we shall assume the functions $H_0, \; H_1$ to be real analytic in $D^n_{R_0} \times \Pi^n_{R_0} \times D^1_{\mu_0}$ for some $R_0 > 0, \; \mu_0 > 0$, where we put $D^n_R = \{x \in \mathbf{C}^n : |x| < R\}, \; \Pi^n_R = \{x \in \mathbf{C}^n : |\mathrm{Im}\, x| < R\}$ for any $R > 0$.

Obviously, the Poisson brackets corresponding to ω^2 now can be written as

$$\{p_i, p_j\} = \mu c_{ij}, \quad \{q_i, p_j\} = \delta_{ij} \quad (i, j = 1, \ldots, n),$$

where δ_{ij} is the Kronecker symbol. Here and in the sequel the unwritten elements of the Poissonian structure matrix will be treated as zero.

Denote by C the linear operator in $\mathbf{R}^n$ with matrix $\{c_{ij}\}_{i,j=1}^n$ and consider the Poissonian system $\dot{p} = C \mathrm{grad}\, H_0(p)$. This system has a collection of geometric first integrals $J_i(p) = \langle \sigma_i, p \rangle$ $(i = 1, \ldots, k_0)$ which are the Casimir functions for the Poisson brackets. Now we suppose that

$\mathbf{H}_1$: the vector field $X := C\mathrm{grad}\, H_0$ has a singular point p_*.

We shall call p_* the *quasistationary point* (with respect to the perturbed system).

Let Y_{p_*} be the restriction of X to that common level manifold of functions $J_i(p)$ $(i = 1, \dots, k_0)$ which passes through p_*. Denote by DY_{p_*} the linear operator corresponding to linearization of Y_{p_*} at p_* and assume that

$\mathbf{H}_2$: the eigenvalues of DY_{p_*} are purely imaginary and pair-wise different.

We denote them as $\pm i\lambda_j(p_*)$ $(j = 1, \dots, m := (n - k_0)/2)$. Now it is easy to see that near p_* the set of quasistationary points forms a k_0-dimensional manifold $\mathcal{W}$ which locally is the image of a real analytic mapping $p_0(\cdot) : \mathcal{U} \mapsto \mathbf{R}^n$ where $\mathcal{U}$ is a neighborhood of the origin in $\mathbf{R}^{k_0}$, $p_0(0) = p_*$. Henceforth the eigenvalues $\pm i\lambda_j(p_0(\eta))$ $(j = 1, \dots, m)$ corresponding to quasistationary point $p_* = p_0(\eta)$ $(\eta \in \mathcal{U})$ will be denoted as $\pm i\lambda_j(\eta)$. Put $\Lambda_j(\eta) = \partial H_0(p_0(\eta))/\partial \eta_j$ $(j = 1, \dots, k_0)$. Now we are in position to formulate the Rüssmann's non-degeneracy conditions [12]:

$\mathbf{H}_3$: The functions $\Lambda_1(\eta), \dots, \Lambda_{k_0}(\eta), \lambda_1(\eta), \dots, \lambda_m(\eta)$ are linearly independent in $\mathcal{U}$.

3 Preliminary transformations

In this section we shall consider p_0 as an n-dimensional parameter but in the sequel we shall put $p_0 = p_0(\eta)$. Let us introduce the following "resonant" set

$$\Re^n = \Re^n(T, \gamma, \tau, \epsilon) := \bigcup_{m \in K_N} \{\omega \in \mathbf{C}^n : |\langle m, \omega \rangle| < \gamma |m|^{-\tau}\}$$

where $T,\ \gamma,\ \tau,\ \epsilon$ are positive numbers, $N = N(T, \epsilon) := \ln \epsilon^{-T}$, $K_N = \{m \in \mathbf{Z}^n : 1 \le |m| \le N\}$.

It turns out that for any fixed $R_1 < R_0$, for sufficiently large T, all sufficiently small $\mu > 0$, all $\epsilon \in (0, 1)$, and all $p_0 \in \{p \in D_{R_1} : H_0'(p) \notin \Re^n\}$ there exists a change of variables

$$p \mapsto p_0 + \mu f(p, q; p_0, \sqrt{\mu}), \quad q \mapsto q + \mu g(p, q; p_0, \sqrt{\mu}),$$

which reduces the perturbed Poisson brackets and the perturbed Hamiltonian function, respectively, to the form

$$\{p_i, p_j\} = c_{ij}, \quad \{q_i, p_j\} = \delta_{ij}, \tag{1}$$

$$H_0'(p_0)p + \mu[\frac{1}{2}H_0''(p_0)p^2 + \langle G_0(p_0), p\rangle + \sqrt{\mu}\hat{H}_2(p; p_0, \sqrt{\mu})] + O(\mu^4 + \epsilon), \quad (2)$$

with some real analytic mapping $G_0 : D_{R_1} \mapsto \mathbf{C}^n$ and function $\hat{H}_2 : D_{R_0} \times D_{R_1} \times D_{\mu_0} \mapsto \mathbf{C}$.

For any fixed $\epsilon \in (0, 1)$ the above change of variables is real analytic with respect to $p,\ q,\ p_0,\ \sqrt{\mu}$. (The rigorous formulation of this statement is somewhat cumbersome and we omit it here).

Let us give a sketch of the proof. First we carry out the scale transformation $p \mapsto p_0 + \sqrt{\mu}p$. The perturbed system becomes the Hamiltonian one with respect to Poissonian structure

$$\{p_i, p_j\} = \sqrt{\mu}c_{ij}, \quad \{q_i, p_j\} = \delta_{ij}, \quad (3)$$

and its Hamiltonian function can be represented in the form

$$H_0'(p_0)p + \sqrt{\mu}[\frac{1}{2}H_0''(p_0)p^2 + H_1(p_0, q, 0)] + \mu H_2(p, q; p_0, \sqrt{\mu}).$$

Now we can use a symplectic averaging procedure to eliminate all harmonics $\exp(i\langle m, q\rangle),\ 0 < |m| \le N(T, \epsilon)$, in function's $H_1,\ H_2$ coefficients multiplying $\sqrt{\mu},\ \mu, \ldots, \mu^4$, provided that $H_0'(p_0) \notin \Re^n$. Taking into account that $H_1(p_0, q, 0)$ does not depend on p, we are able to construct a symplectic (with respect to (3)) transformation of the type $p \mapsto p + O(\sqrt{\mu}),\ q \mapsto q + O(\mu)$, which reduces the Hamiltonian to the form

$$H_0'(p_0)p + \sqrt{\mu}[\frac{1}{2}H_0''(p_0)p^2 + \bar{H}_1(p_0)] + \mu\bar{H}_2(p; p_0, \sqrt{\mu}) + \sqrt{\mu}O(\mu^4 + \epsilon),$$

The bars over symbols of the functions mean their averaged values with respect to q. The term $\sqrt{\mu}\bar{H}_1(p_0)$ is inessential and can be omitted. After one more scaling transformation $p \mapsto \sqrt{\mu}p$ we arrive at the required result. Note that in formula (2) $G_0(p_0) = \partial\bar{H}_2(0, p_0, 0)/\partial p$.

4 Action-angle coordinates in a neighborhood of quasistationary points

Let us now restrict the parameter p_0 in Hamiltonian (2) to the set of quasistationary points. Thus we put $p_0 = p_0(\eta)$ where $p_0(\cdot) : \mathcal{U} \mapsto \mathbf{R}^n$ is the mapping introduced in the Section 2. We may assume that $|p_0(\eta)| < R_1,\ \eta \in \mathcal{U}$. It is not hard to show that there exists a linear change of variables $p \mapsto (J, u, v) = (J_1, \ldots, J_{k_0}, u_1, \ldots, u_m, v_1, \ldots, v_m)$, dependent on the k_0-dimensional parameter η, which reduces the pair (1), (2) to

$$\{v_i, u_j\} = \delta_{ij}, \quad \{q, J_j\} = \sigma_j, \quad \{q_i, q_j\} = \nu_{ij}(\eta), \tag{4}$$

and

$$\langle \Lambda(\eta), J\rangle + \mu \left[\frac{1}{2}\langle L(\eta)w, w\rangle + G_1(J;\eta) + \sqrt{\mu}G_2(J, w; \eta, \sqrt{\mu})\right] + O(\mu^4 + \epsilon), \tag{5}$$

respectively. Here

$$\begin{aligned} \Lambda(\eta) &= (\Lambda_1(\eta), \ldots, \Lambda_{k_0}(\eta)), \\ L(\eta) &= \mathrm{diag}(\lambda_1(\eta), \ldots, \lambda_m(\eta), \lambda_1(\eta), \ldots, \lambda_m(\eta)). \end{aligned}$$

and $w = (u, v) = (u_1, \ldots, u_m, v_1, \ldots, v_m)$. Now let us formulate our main preliminary result.

Proposition 1 *For any $R > 0$, $\rho \in (0, R^2/2)$, $\tau > 0$, $\gamma > 0$ there exist numbers $T = T(R, \rho) > 0$, $\mu_*(R, \rho, \tau, \gamma) > 0$, and for any $\epsilon \in (0, 1)$ there exists such a change of variables $(J, u, v) \mapsto (J, I, \phi| \bmod 2\pi)$ $(I = (I_1, \ldots, I_m),\ \phi = (\phi_1, \ldots, \phi_m))$ of the form*

$$\begin{aligned} u_i &= \sqrt{\xi_i^2 + 2I_i}\cos\phi_i + \sqrt{\mu}U_i(J, I, \phi; \eta, \xi, \sqrt{\mu}), \\ v_i &= \sqrt{\xi_i^2 + 2I_i}\sin\phi_i + \sqrt{\mu}V_i(J, I, \phi; \eta, \xi, \sqrt{\mu}), \\ q_j &= \psi_j + \sqrt{\mu}Q_i(J, I, \phi; \eta, \xi, \sqrt{\mu}) \quad (i = 1, \ldots, m,\ j = 1, \ldots, n), \end{aligned}$$

that in new coordinates the perturbed Hamiltonian function and the perturbed Poisson brackets are represented as follows:

$$H = \langle \Lambda(\eta), J\rangle + \mu\left[\langle \lambda(\eta), I\rangle + G_1(J, \eta) + \sqrt{\mu}E(J, I; \eta, \xi, \sqrt{\mu})\right] + O(\mu^4 + \epsilon), \tag{6}$$

$$\begin{gathered} \{\psi, J_j\} = \sigma_j \quad (j = 1, \ldots, k_0); \quad \{\phi_i, I_j\} = \delta_{ij} \quad (i, j = 1, \ldots, m); \\ \{\psi_i, \psi_j\} = \nu_{ij}(\eta) \quad (i, j = 1, \ldots, n). \end{gathered} \tag{7}$$

Here $\xi = (\xi_1, \ldots, \xi_m)$ are additional parameters; the functions U_i, V_i, Q_j, H, G_1, E are real analytic on the set

$$\begin{aligned} \mathcal{B} = \{\ (J, I, \phi, \eta, \xi, \sqrt{\mu}) :\ & J \in D_\rho^{k_0},\ I \in D_\rho^m,\ \phi \in \Pi_\rho^m, \\ & \eta \in \mathcal{U},\ \Omega(\eta) \notin \Re^n,\ \lambda(\eta) \notin \Re^m,\ \sqrt{2\rho} < |\xi| < R, \\ & |\sqrt{\mu}| < \sqrt{\mu_*}\ \}, \end{aligned}$$

where $\Omega(\eta) = \sum_{i=1}^{k_0} \Lambda_i(\eta)\sigma_i$, $\lambda(\eta) = (\lambda_1(\eta), \dots, \lambda_m(\eta))$; *the functions* $\Lambda(\eta)$, $\lambda(\eta)$, $\nu_{ij}(\eta)$ *are real analytic in* $\mathcal{U}$; *lastly, there exists such a number* $K = K(R, \rho, \tau, \gamma) > 0$ *that*

$$|E(J, I; \eta, \xi, \sqrt{\mu})| \leq K,$$

$$|H - \langle \Lambda(\eta), J\rangle - \mu\left[\langle \lambda(\eta), I\rangle + G_1(J, \eta) + \sqrt{\mu}E(J, I; \eta, \xi, \sqrt{\mu})\right]| \leq K(\mu^4 + \epsilon)$$

for all $(I, J, \phi, \eta, \xi, \sqrt{\mu}) \in \mathcal{B}$, $\psi \in \Pi^n_{R_1}$, $\epsilon \in (0, 1)$.

5 Main Theorem

Consider now the Hamiltonian

$$H^0 = \langle \Lambda(\eta), J\rangle + \mu\left[\langle \lambda(\eta), I\rangle + G_1(J, \eta) + \sqrt{\mu}E(J, I; \eta, \xi, \sqrt{\mu})\right].$$

The corresponding Hamiltonian system has the form

$$\begin{array}{rclrcl} \dot{I} &=& 0, & \dot{J} &=& 0, \\ \dot{\phi} &=& \mu\left[\lambda(\eta) + \sqrt{\mu}\dfrac{\partial E}{\partial I}\right], & \dot{\psi} &=& \Omega(\eta) + \mu\sum_{i=1}^{k_0}\dfrac{\partial(G_1 + \sqrt{\mu}E)}{\partial J_i}\sigma_i. \end{array} \qquad (8)$$

It generates quasiperiodic motions on coisotropic invariant tori which are the common level manifolds of functions I, J. Using the results of the previous sections, one can easily show that the torus corresponding to the values $I = 0$, $J = 0$ lies in a $O(\mu^{3/2})$-neighborhood of the torus $\mathcal{T}_0(\eta, \xi, \mu)$ which in the initial coordinates p, q is described by the equation

$$p = p_0(\eta) + \mu(A(\eta)B(\phi)\xi + p_1(q; \eta)).$$

Here $A(\eta)$ is $(n \times 2m)$-matrix of rank $2m$ whose elements are real analytic functions of $\eta \in \mathcal{U}$; $(2m \times m)$-matrix $B(\phi)$ has the form

$$B(\phi) = \begin{pmatrix} \mathrm{diag}(\cos\phi_1, \dots, \cos\phi_m) \\ \mathrm{diag}(\sin\phi_1, \dots, \sin\phi_m) \end{pmatrix};$$

the mapping $p_1 : T^n \times \mathcal{U} \mapsto \mathbf{R}^n$ is a C^∞-extension of a real analytic one defined on the direct product of T^n and the subset of $\mathcal{U}$ specified by the conditions $\Omega(\eta) \notin \Re^n$, $\lambda(\eta) \notin \Re^m$.

The system with Hamiltonian (5) can be regarded as a perturbation of that with Hamiltonian H^0. We may put $\epsilon = \mu^4$ and apply KAM-theory to establish the existence of quasiperiodic motions in the perturbed system. Note that since $\dot{\phi} = O(\mu)$ the system (8) is close to a degenerate one. For corresponding KAM-like theorems we refer to [13, 14].

Put $\Xi = \{(\eta, \xi) \in \mathbf{R}^{k_0} \times \mathbf{R}^m : \eta \in \mathcal{U},\ 0 < \xi_j < R,\ j = 1, \dots, m\}$. Now our main result can be formulated as follows.

Theorem 1 *Let the symplectic structure ω_0^2 and the Hamiltonian $\mathcal{H}_0$ of completely integrable system undergo small perturbations: $\mathcal{H}_0 \mapsto \mathcal{H}_0 + \mu\mathcal{H}_1$, $\omega_0^2 \mapsto \omega_0^2 + \mu\omega_1^2$. Suppose that the skew-symmetric bilinear form $\mathcal{C}$ corresponding to ω_1^2 and the Hamiltonian $\mathcal{H}_0$ satisfy $\mathbf{H}_0$ – $\mathbf{H}_3$. If $\mu_* > 0$ is sufficiently small then the following assertions holds true:*
1) for any $\mu \in (0, \mu_*)$ there exists a set $\Xi_\mu \subset \Xi$ of Cantor type such that if $(\eta, \xi) \in \Xi_\mu$ then $O(\mu^{3/2})$-neighborhood of each torus $\mathcal{T}_0(\eta, \xi, \mu)$ contains $(m+n)$-dimensional real analytic invariant torus $\mathcal{T}_\mu(\eta, \xi, \mu)$ of the perturbed system;
2) any motion on the torus $\mathcal{T}_\mu(\eta, \xi, \mu)$ is quasiperiodic with everywhere dense orbit;
3) $\mathrm{mes}\left(\bigcup_{(\eta,\xi)\in\Xi} \mathcal{T}_0(\eta, \xi, \mu) \setminus \bigcup_{(\eta,\xi)\in\Xi_\mu} \mathcal{T}_\mu(\eta, \xi, \mu)\right) / \mathrm{mes}\left(\bigcup_{(\eta,\xi)\in\Xi} \mathcal{T}_0(\eta, \xi, \mu)\right) \to 0$ *as* $\mu \to 0$.

Note that applying the technique developed by J. Pöshel, M.R. Herman, and M.B. Sevryuk (see [15, 9, 14]), one can show that the perturbed invariant tori from the above theorem can be included into a sufficiently smooth family of tori dependent on parameters $(\eta, \xi) \in \Xi$.

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Author

Parasyuk, Ihor
Faculty of Mechanics and Mathematics
Kyiv Taras Shevchenko University
64, Volodymyrs'ka str.
252033, Kyiv
Ukraine
pio@mechmat.univ.kiev.ua