Non-Linear Oscillations in a Time-Optimal Feedback System

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Abstract

There exists a very broad class of industrial devices which need to change their position in a minimum time. Dynamics of the above devices, called *position mechanisms,* depends essentially on the motion resistance and may be defined by differential equation:

$$\dot{x} = y,\ x(0) = x_0; \quad \dot{y} = f(y) + u,\ y(0) = y_0, \qquad (*)$$

where *x, y* is position and velocity of the mechanism respectively, *f* is a non-linear and discontinuous function of motion resistance, *u* is a measurable function of control. Time-optimal problem of the system (∗) will be understood as a transfer of any initial state $\mathbf{z}_0 = (x_0, y_0) \in R^2$ to the target state $\mathbf{z}_1 = (x_1, y_1) \in R^2$ in a minimum time $t^* < \infty$. Time-optimality of the controlled processes of the object (∗) may be ensured only in a closed-loop system which attributes to each of the state $\mathbf{z} = (x, y)$ a time optimal value $u*$ of the control function u. Thus, the open controlled system (∗) should be replaced by a feedback system $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, v(\mathbf{z}))$, where $\mathbf{f}: R^2 \times R \to R^2$ and control function $\mathbf{v}$: $R^2 \to \mathbf{U} \subset R^1$.

Time optimal feedback system synthesis leads to the following structures creation:

$$\dot{x} = y; \quad \dot{y} = \begin{cases} F^+(y), & (x, y) \in \mathbf{T}^+ \cup \mathbf{R}^+ \\ F^-(y), & (x, y) \in \mathbf{T}^- \cup \mathbf{R}^- \end{cases} \qquad (**)$$

where F^+, F^- define the right-hand side of (∗) by control function $u*$ that induces the time-optimal controlled process, $\mathbf{T}^+$, $\mathbf{T}^-$ are the trajectories of the system (∗) solutions generated by $u* = \pm 1$ respectively, those reach the target state $\mathbf{z}_1 = (x_1, y_1)$. These trajectories may be defined as $\mathbf{T}^+ = \{(x, y) : y = h(x), x \geq x_1\}$, $\mathbf{T}^- = \{(x, y) : y = h(x), x \leq x_1\}$ where h is of $C(R)$ class. In real closed-loop system such as (∗∗) both the internal uncertainty and external perturbations may appear. So, the created feedback system becomes non-time-optimal one. For this case we create the factors $p = \sup Q\{h(x)\ x \in [0, \infty),\ y \in (-\infty, 0)\}$ and $r = \inf Q\{h(x),\ x \in [0, \infty), y \in (-\infty, 0]$. There has been shown that if the function h differs from the time optimal one and $p \in [0,1)$ then the system (∗∗) induces the convergent non-linear oscillations, i.e. state trajectory goes round the target $\mathbf{z}_1$ and reaches it in finite time after performing undefined number of encirclements. Instead, if the factor $r > 1$ then the

system (∗∗) induces the divergent non-linear oscillations, i.e. state trajectory goes round the target $\mathbf{z}_1$ divergently and tends to form the limit curve with the target state $\mathbf{z}_1$ in its interior.

1. Introduction

Industrial devices, such as saddles of machine tools, tracer machines, industrial manipulators, several parts of industrial robots, or the position mechanisms of industrial automata, need to change their position in a minimum time, particularly when it is necessary to move the mechanism before another technological operation can proceed. Synthesis of a time-optimal control structure becomes therefore an important, economical problem.

Dynamics of the above devices, called *position mechanisms*, depends essentially on the motion resistance and may be defined by the following differential equation:

$$\dot{x} = y,\ \ x(0) = x_0; \quad \dot{y} = f(y) + u,\ \ y(0) = y_0 \tag{1.1}$$

where *x*, *y* is position and velocity of the mechanism respectively, *f* is a function of motion resistance, *u* is a measurable function of control. Typically, *as a time-optimal problem of the system* (1.1) *will be understood a transfer of any initial state* $\mathbf{z}_0 = (x_0, y_0) \in R^2$ *to any target state* $\mathbf{z}_1 = (x_1, y_1) \in R^2$ *in a minimum time* $t^* < \infty$.

Knowledge of the time-optimal solution plays an essential role in practical applications. Usually, there is created a closed-loop system which attributes to each of the state a time-optimal value of the control function *u*. The most useful concept of feedback system synthesis is given in the definition as below.

DEFINITION 1.1 (*Regular synthesis of feedback system*)

Given a controlled dynamic system:

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{u}), \quad \mathbf{z} \in R^n, \quad \mathbf{u} \in \mathbf{U} \subset R^m, \quad \mathbf{f} : R^n \times R^m \to R^n, \tag{1.2}$$

The concept of regular synthesis [2], [4] of a time-optimal control structure for such a system leads to a feedback control function $\mathbf{v} : R^n \to \mathbf{U}$ creation satisfying the following properties:

a) each time-optimal solution of (1.2) is a standard solution (Caratheodory or *C*-solution) of the following closed-loop system:

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{v}(\mathbf{z})), \quad \mathbf{v} : R^n \to R^m \tag{1.3}$$

b) each standard solution of (1.3) is a time-optimal solution of (1.2). ■

Obviously, (1.3) describes the dynamic behaviour of a closed-loop system displacing (1.2) from any initial state $\mathbf{z}_0 \in R^n$ to any target state $\mathbf{z}_1 \in R^n$.

To justify the construction of such a controller, the following reasons are generally given:

1) *There is no need to compute the optimal control for every new initial state separately* and

2) *The controller acting upon* (1.3) *is sensitive to instantaneous perturbations, i.e. if at any instant of the process the system is deviated from its optimal trajectory, the remaining portion of the process will again lead to the desired final state (target) and will be optimal with respect to this new initial state.*

Subsequent paragraphs deal with the time-optimal problem of the dynamic controlled object (1.2) by assumption that the motion resistance functions is piecewise continuous with finite number of discontinuity. Under that assumption the function $f(y)$ describe as broad as possible class of motion resistance in particular all types of friction. The assumed discontinuity of differential equations (1.2) right-hand side makes it impossible to apply the classical theory of optimisation under minimum time criterion. Time-optimal problem has been solved therefore using special mathematical methods, among other *Theory of Differential Inequalities.* Knowing the time-optimal solutions of controlled object (1.2) we will create closed-loop systems using principles of the *Regular Synthesis* method. Engineering systems constructed in such a way, acting under real technological circumstances, may induce singular phenomena such as *limit cycles, convergent and divergent non-linear oscillations*

2. Preliminaries

GENERAL NOTATIONS 2.1.

a) Any solution of the systems (1.1) by $u \in [-1,+1]$ starting from the initial state $\mathbf{z}_0 = (x_0, y_0) \in R^2$ in increasing time will be denoted $\mathbf{q}(t;\mathbf{z}_0) = (x(t,\mathbf{z}_0), y(t, y_0))$.

b) The solutions of the system (1.1) generated by the control function $u \equiv +1$ and $u \equiv -1$ starting from any point $\mathbf{z}_i = (x_i, y_i) \in R^2$ in increasing time will be noted $\mathbf{q}_+(t;\mathbf{z}_i) = (x_+(t,\mathbf{z}_i), y_+(t, y_i))$, and $\mathbf{q}_-(t;\mathbf{z}_i) = (x_-(t,\mathbf{z}_i), y_-(t, y_i))$ respectively or shortly (in particular in the figures) $\mathbf{q}_+$ and $\mathbf{q}_-$.

c) The solutions of the system (1.1) generated by the control functions $u \equiv +1$ and $u \equiv -1$ starting from any point $\mathbf{z}_i = (x_i, y_i) \in R^2$ in *decreasing time* (backwards, i.e. $t \le 0$) will be denoted $\mathbf{q}_+(-t;\mathbf{z}_i) = (x_+(-t;\mathbf{z}_i), y_+(-t;y_i))$ and $\mathbf{q}_-(-t;\mathbf{z}_i) = x_-(-t;\mathbf{z}_i), y_-(-t;y_i))$ respectively.

d) Trajectories of the solutions $\mathbf{q}_+(t;\mathbf{z}_0)$ and $\mathbf{q}_-(t;\mathbf{z}_0)$, starting from any point $\mathbf{z}_0 \in R^2$ and reaching the target state $\mathbf{z}_1 \in R^2$ will play an essential role. They will be called *Terminal Trajectories*, will be denoted $\mathbf{T}^+$ and $\mathbf{T}^-$ respectively and will be constitute by the trajectories of the solutions $\mathbf{q}_+(-t;\mathbf{z}_1)$ and $\mathbf{q}_-(-t;\mathbf{z}_1)$ respectively. Thus,

$$\mathbf{T}^+ = \{ \mathbf{z} = (x, y) : \mathbf{q}_+(-t;\mathbf{z}_1) \}; \quad \mathbf{T}^- = \{ \mathbf{z} = (x, y) : \mathbf{q}_-(-t;\mathbf{z}_1) \} \tag{2.1}$$

e) In the text and in particular in mathematical formulas the following abbreviations are of use: *a.e.* means *almost everywhere*, *PC* means *piecewise continuous*, **CLS** means *closed-loop system*. ■

The right-hand side of the **CLS** investigated in this work is discontinuous in terms of its argument. The standard solutions of the system such as (1.3) thus become inappropriate [2], [4]. Namely, they cannot characterise all the motions that can occur in the system modelled by (1.3) and, conversely, not all standard solutions admit physical interpretation. For discontinuous differential equations, apparently the most complete definition of solution is that of Filippov [3]. In this paper both the standard and Filippov class of the solutions will be used.

DEFINITION 2.2. (*Caratheodory and Filippov solutions*)

Let $\mathbf{x}:\mathbf{I}\rightarrow R^n$ (**I** is an interval in R^1) be an absolutely continuous function on each compact subinterval of **I**. Then **x** is called:

a) a standard (or Caratheodory or *C*) solution of a differential equation

$$\dot{\mathbf{x}}(t)=\mathbf{g}(t,\mathbf{x}(t)),\quad \mathbf{g}:R^1\times R^n\rightarrow R^n, \tag{2.2}$$

iff **x** satisfies (2.2) almost everywhere on **I**;

b) a Filippov (or *F*) solution of (2.2) iff

$$\dot{\mathbf{x}}(t)\in F(\mathbf{g}(t,\mathbf{x}(t))\quad \text{a.e. on } \mathbf{I}$$

where operator *F* is given by the formula:

$$F(\mathbf{g}(t,\mathbf{x}))=\bigcap_{\varepsilon>0}\ \bigcap_{\mu(\mathbf{z})=0}\overline{cvx}\,\mathbf{g}(t,(\mathbf{x}+\varepsilon\mathbf{B})\setminus\mathbf{Z}),$$

B is the open unit ball in R^n, $\mu\,(\mathbf{Z})$ is Lebesgue measure of the set **Z**,

$\overline{cvx}\ \mathbf{M}$ denotes closure of the convex hull of $\mathbf{M}\subset R^n$ [3]. ■

The above definition of the *F*-solution is not easy to deal with. However, in this work, we shall need it in special local situations.

3. Time-Optimal Problem

In engineering systems controlling the object (1.1) we distinguish two types of the target state to which the above object should be brought, namely:

a) a motionless (stationary) target, i.e. $\mathbf{z}_1=(x_1,y_1)$ lies on the *x*-axis ($x_1\in R^1$, $y_1=0$)

b) a moving (non-stationary) target, i.e. $\mathbf{z}_1=(x_1,y_1)$ does not lay on the *x*-axis ($x_1\in R^1$, $y_1\neq 0$).

In what follows we will see that in created closed-loop time-optimal structure an essential role there plays so called *Switching Curve*. The practical identification of real motion resistances (i.e. determination of the function f) as well as the object parameter values, is beset with difficulties. Consequently, the model of the dynamic controlled object taken for the closed-loop system synthesis is an inaccurate mapping of this model. Thus, in the designed control structure, the switching curve is not time-optimal one. Moreover, the real time-optimal controller creates its own control function which may differ considerable from that required, due to technical reasons [2], [6]. Therefore, one may state that the switching curve (and thereby control function) induced in real feedback systems is a different from the time optimal one [9], [10]. In what follows the switching curve different from the time-optimal one will be called *Inaccurate Switching Curve*.

In this part we investigate the influence of switching curve inaccuracy, towards time-optimal one, upon the dynamic behaviour of the closed-loop time-optimal system controlling the dynamic object (1.1). Analysis will be performed separately for the motionless target and the moving one.

3.1. Time-Optimal Solution

Given dynamic controlled object (1.1). In order to describe the largest possible class of motion resistance, in particular all types of friction, we assume that the function $f(y)$ is piecewise continuous with finite number of first kind discontinuity. From technical point of view the following constraints should be taken into account:

$$|u| \le 1, \quad |y| \le y_m, \quad |\dot{y}| \le \dot{y}_m$$

Global controllability of the system (1.1) is ensured if almost everywhere:

$$\left| f(y) \right| \le 1 - \varepsilon\,, \quad 0 < \varepsilon < 1, \quad y \in R^1$$

In what follows the system (1.1) will be replaced by the following generalised form:

$$\left.\begin{aligned} &a)\ \dot{x} = y, \quad x(0) = x_0 \\ &b)\ \dot{y} \in F(y), \quad y(0) = y_0 \end{aligned}\right\} \tag{3.1}$$

where

$$F(y) = [\, f^-(y),\, f^+(y)\,] \tag{3.2}$$

The functions $f^+(y)$ and $f^-(y)$ are piecewise continuous with finite number of discontinuity. Assume, there exist the constants $a < 0$, $0 < b$, $0 < m < M < \infty$, such that the following conditions are fulfilled:

$$\left.\begin{aligned} &f^-: (a,b] \to (-M,-m), \quad f^-(a) = 0 \\ &f^+: [a,b) \to (m, M), \quad f^+(b) = 0 \end{aligned}\right\} \tag{3.3}$$

The model (3.1) includes the following cases which have essential meaning in engineering applications [4], [5]:

i) Velocity y and acceleration $\dot{y}$ are not bounded (i.e. $y_m = \dot{y}_m = \infty$).
ii) Velocity y is bounded, acceleration $\dot{y}$ is not (i.e. $y_m < \infty, \quad \dot{y}_m = \infty$).
iii) Acceleration $\dot{y}$ is bounded, velocity y is not (i.e. $y_m = \infty, \quad \dot{y}_m < \infty$).
iv) Both velocity y and acceleration $\dot{y}$ are bounded (i.e. $y_m < \infty, \quad \dot{y}_m < \infty$).

By a solution of the inclusion (3.1b) we will understand the standard (Caratheodory or C) solution, i.e. a function $y : \mathbf{J} \to R^1$ ($\mathbf{J}$ is an interval in R^1, containing more than one point) is the solution of (3.1b) iff it is absolutely continuous function on each compact subinterval of **J**, fulfilling (3.1b) almost everywhere on **J**.

In the sequel the solutions of equation $\dot{y} = f^-(y)$ and $\dot{y} = f^+(y)$ play an essential role.

LEMMA 3.1.

Let the functions $f^+(y)$ and $f^-(y)$ be piecewise continuous and satisfy assumptions (3.3). Then, for each initial value $y_0 \in R^1$ differential equations

$$\text{a) } \dot{y} = f^+(y), \quad y(0) = y_0; \qquad \text{b) } \dot{y} = f^-(y), \quad y(0) = y_0 \tag{3.4}$$

have the unique solutions defined on interval $t \in [0, \infty)$.

PROOF: The proof follows the same pattern as that used in the proofs of Theorems 4.1 and 5.3 in the paper [1]. ■

REMARK 3.2

Any solution of inclusion (3.1b) will be denoted $y(t, y_0)$, instead by $y_+(t, y_0)$ and $y_-(t, y_0)$ will be denoted the unique C-solutions of differential equations (3.4a) and (3.4b) respectively. It is obvious that the solutions y_+ and y_- are increasing and decreasing functions respectively. ■

LEMMA 3.3

Let the functions $f^+(y)$ and $f^-(y)$ be defined by (3.3). Then, for each initial state $\mathbf{z}_0 \in R^2$, differential equations

$$\left.\begin{aligned} &\dot{x} = y, \quad x(0) = x_0, \\ &\dot{y} = f^+(y), \quad y(0) = y_0 \end{aligned}\right\} \tag{3.5a}$$

$$\left.\begin{aligned} &\dot{x} = y, \quad x(0) = x_0, \\ &\dot{y} = f^-(y), \quad y(0) = y_0 \end{aligned}\right\} \tag{3.5b}$$

have the unique *C*-solutions $\mathbf{q}_+(t;\mathbf{z}_0)$ and $\mathbf{q}_-(t;\mathbf{z}_0)$ respectively, defined in the time interval $t \in [0, \infty)$.

PROOF: The proof results directly from Lemma 3.1. ■

REMARK 3.4. (*Properties of the solutions* $\mathbf{q}_+$ *and* $\mathbf{q}_-$).

a) The co-ordinates of the $\mathbf{q}_+(t;\mathbf{z}_0)$ solution have got the following properties. Let $y_0 < 0$. Then there exists a finite time $0 < t_1$ such that $y_+(t,y_0) < 0$ on $[0,t_1)$, $y_+(t_1,y_0) = 0$ and $y_+(t,y_0) > 0$ on (t_1,∞). Co-ordinate $y_+(t,y_0)$ is increasing function on $[0,\infty)$ but $x_+(t,\mathbf{z}_0)$ is decreasing on $[0,t_1]$ and increasing one on $[t_1,\infty)$. If $y_0 \geq 0$ then co-ordinates $x_+(\cdot,\mathbf{z}_0)$ and $y_+(\cdot,y_0)$ hold the same properties as above in the interval $[t_1,\infty)$.

b) The co-ordinates of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution have got the following properties. Let $y_0 > 0$. Then there exists a finite time $0 < t_1$ such that $y_-(t,y_0) > 0$ on $[0,t_1)$, $y_-(t_1;y_0) = 0$, $y_-(t,y_0) < 0$, on (t_1,∞). Furthermore, $y_-(t,y_0)$ is decreasing function on $[0,\infty)$, but $x_-(t,\mathbf{z}_0)$ is increasing on $[0,t_1]$ and decreasing one on $[t_1,\infty)$. If $y_0 \leq 0$ then as in a) $x_-(\cdot,\mathbf{z}_0)$ and $y_-(\cdot,y_0)$ hold the same properties as defined above in the interval $[t_1,\infty)$. ■

The *Terminal Trajectories* $\mathbf{T}^+$ and $\mathbf{T}^-$ forms in the state-plane a curve $\mathbf{T} = \mathbf{T}^- \cup \mathbf{T}^+$ that will be called the *Switching Curve*. From the fact that the solutions $y_+(t,y_0)$ and $y_-(t,y_0)$ are increasing and decreasing functions respectively it follows immediately that this switching curve $\mathbf{T}$ divides the state-plane into two sets of states

$$\left.\begin{aligned} \mathbf{R}^+ &= \{(x,y) : (x',y) \in \mathbf{T} \Rightarrow x < x'\} \\ \mathbf{R}^- &= \{(x,y) : (x',y) \in \mathbf{T} \Rightarrow x > x'\} \end{aligned}\right\} \tag{3.6}$$

THEOREM 3.5

Given dynamic object (3.1). Let the functions $f^+(y)$ and $f^-(y)$ be piecewise continuous and satisfy assumptions (3.3). Then, from each initial state $\mathbf{z}_0 = (x_0, y_0)$, $x_0 \in R^1$, $y_0 \in [a, b]$ there starts the trajectory of the time-optimal solution $\mathbf{q}^*(t;\mathbf{z}_0)$ displacing the object (3.1) to any target state $\mathbf{z}_1 = (x_1,y_1)$, $x_1 \in R^1$, $y_1 \in [a, b]$ in a minimum time $t^* < \infty$.

Thesis a) Let $\mathbf{z}_0 \in \mathbf{R}^+$. Then there exists a finite time $t_1 > 0$ such that time-optimal solution:

$$\mathbf{q}^*(t;\mathbf{z}_0)=\left|\begin{array}{ll}\mathbf{q}_+(t;\mathbf{z}_0), & t\in[0,t_1]\\ \mathbf{q}_-(t-t_1;\mathbf{q}_+(t_1;\mathbf{z}_0)), & t\in[t_1,t^*]\end{array}\right. \tag{3.7a}$$

Thesis b) Let $\mathbf{z}_0\in\mathbf{R}^-$. Then there exists a finite time $t_1>0$ such that time-optimal solution:

$$\mathbf{q}^*(t;\mathbf{z}_0)=\left|\begin{array}{ll}\mathbf{q}_-(t;\mathbf{z}_0), & t\in[0,t_1]\\ \mathbf{q}_+(t-t_1;\mathbf{q}_+(t_1;\mathbf{z}_0)), & t\in[t_1,t^*]\end{array}\right. \tag{3.7b}$$

PROOF: The proof of theorem is given in [9]. ■

3.2. Closed-Loop System

3.2.1. Time-optimal closed-loop system

In the previous chapter 3.1 there has been proved that if only (3.3) is fulfilled then there exists the unique time-optimal solution of the differential inclusion (3.1) that is a mapping of dynamic behaviour of the controlled dynamic object. That solution together with the method of *Regular Synthesis* (Definition 1.1) leads to the following closed-loop time-optimal system creation:

$$\left.\begin{array}{l}\dot{x}=y, x(0)=x_0\\ \dot{y}=\left|\begin{array}{ll}f^+(y),\ (x,y)\in\mathbf{T}_*^+\cup\mathbf{R}_*^+, & y(0)=y_0\\ f^-(y),\ (x,y)\in\mathbf{T}_*^-\cup\mathbf{R}_*^-, & y(0)=y_0\end{array}\right.\end{array}\right\} \tag{3.8}$$

where in accordance with (2.1) and adequately (3.6)

$$\mathbf{T}_*^+=\{\ \mathbf{q}_+(t;\mathbf{z}_1),\ \ t\le 0\ \};\qquad \mathbf{T}_*^-=\{\ \mathbf{q}_-(t;\mathbf{z}_1),\ \ t\le 0\ \} \tag{3.9}$$

$$\left.\begin{array}{l}\mathbf{R}_*^+=\{(x,y):(x',y)\in\mathbf{T}^*\Rightarrow x<x'\}\\ \mathbf{R}_*^-=\{(x,y):(x',y)\in\mathbf{T}^*\Rightarrow x>x'\}\end{array}\right\} \tag{3.10}$$

In (3.10) $\mathbf{T}^*$ denotes in standard way the time optimal switching curve given as usually by:

$$\mathbf{T}^*=\mathbf{T}_*^+\cup\mathbf{T}_*^- \tag{3.11}$$

It should be emphasised that the system (3.8) induces the solutions $\mathbf{q}_+$ and $\mathbf{q}_-$ only.

From uniqueness of the $\mathbf{q}_+$ and $\mathbf{q}_-$ solutions and their properties (shown in Remark 3.4) it follows that the branches $\mathbf{T}_*^+$ and $\mathbf{T}_*^-$ may be defined equivalently as:

$$\mathbf{T}_*^+ = \{(x,y) : x = h^*(y), \quad y \in [a, y_1]\}$$ (3.12)

$$\mathbf{T}_*^- = \{(x,y) : x = h^*(y), \quad y \in [y_1, b]\}$$ (3.13)

$$\mathbf{T}^* = \{(x,y) : x = h^*(y), \quad y \in [a, b]\}$$ (3.14)

where the function h^* is piecewise $C^1(R^1)$, $h^*(y_1) = x_1$ and it results from differential equations

$$[h^*(y)]' = \frac{\mathrm{d}h^*(y)}{\mathrm{d}y} = \left| \begin{array}{l} \dfrac{y}{f^+(y)} = F^+(y), \quad \text{a.e. on } [a, y_1] \\ \dfrac{y}{f^-(y)} = F^-(y) \quad \text{,a.e. on } [y_1, b] \end{array} \right. \tag{3.15}$$

obtained immediately from differential equations (3.12) and (3.13) (after having regard the properties of the $\mathbf{q}_+$ and $\mathbf{q}\_$ solutions shown in Remark 3.4).

The uniqueness of the solutions $\mathbf{q}_+(t;\mathbf{z}_0)$ and $\mathbf{q}\_(t;\mathbf{z}_0)$ implies that the trajectories of those solutions starting from any initial state $\mathbf{z}_0 \in R^1 \times$[a,b] may be described in the state-plane by the functions of piecewise $C^1(R^1)$ class, $x = g_+(y)$ and $x = g_-(y)$ respectively, similarly as it was done when defining the switching curve **T** by the function $x = h(y)$. Using the same manner as that adopted in the formula (3.15) we have:

$$\left. \begin{array}{l} g'_+(y) = \dfrac{\mathrm{d}g_+(y)}{\mathrm{d}y} = \dfrac{y}{f^+(y)} = F^+(y), \quad \text{a.e. on } [y_0, \infty) \\ g'_-(y) = \dfrac{\mathrm{d}g_-(y)}{\mathrm{d}y} = \dfrac{y}{f^-(y)} = F^-(y), \quad \text{a.e. on } (-\infty, y_0] \end{array} \right\} \tag{3.16}$$

3.2.2. Closed-loop system with inaccurate switching curve

We have already mentioned that the controller of engineering system induces its own switching curve because of technical reasons shown in previous paragraph. This curve, generally speaking, is not time-optimal one. Such a switching curve has been called *inaccurate switching curve*. It may be defined by a function $x = h(y)$, where h is piecewise of $C^1(R^1)$ class and it will be noted here **T**. Thus, a formula describing that curve results from (3.12),(3.13) and (3.14) after setting $\mathbf{T}^+ \to \mathbf{T}_*^+$, $\mathbf{T}^- \to \mathbf{T}_*^-$, $h \to h^*$, $\mathbf{T} \to \mathbf{T}^*$. We have:

$$\mathbf{T}^+ = \{ (x,y) : x = h(y), \quad y \in [a, y_1] \}$$ (3.17)

$$\mathbf{T}^- = \{ (x,y) : x = h(y), \quad y \in [y_1, b] \}$$ (3.18)

$$\mathbf{T} = \{ (x,y) : x = h(y), \quad y \in [a, b] \}$$ (3.19)

where $h(y_1) = x_1$.

Similarly as the time-optimal switching curve, the inaccurate one divides the state-plane into two regions, defined after putting into (3.10) $\mathbf{R}^+ \to \mathbf{R}_*^+$ and $\mathbf{R}^- \to \mathbf{R}_*^-$. We have:

$$\left.\begin{aligned} \mathbf{R}^+ &= \{(x,y) : (x',y) \in \mathbf{T} \Rightarrow x < x'\} \\ \mathbf{R}^- &= \{(x,y) : (x',y) \in \mathbf{T} \Rightarrow x > x'\} \end{aligned}\right\} \qquad (3.20)$$

The real closed-loop system will be defined therefore by (3.8) after putting $\mathbf{T}^+ \to \mathbf{T}_*^+$, $\mathbf{R}^+ \to \mathbf{R}_*^+$, $\mathbf{R}^- \to \mathbf{R}_*^-$. Thus, we have:

$$\left.\begin{aligned} &\dot{x} = y, \quad x(0) = x_0 \\ &\dot{y} = F(x,y) = \left| \begin{aligned} &f^+(y),(x,y) \in \mathbf{T}^+ \cup \mathbf{R}^+, \quad y(0) = y_0 \\ &f^-(y),(x,y) \in \mathbf{T}^- \cup \mathbf{R}^-, \quad y(0) = y_0 \end{aligned} \right. \end{aligned}\right\} \qquad (3.21)$$

From here, as a solution of (3.21) we will mean the solution in increasing time t only. For the sake of convenience we denote:

$$\frac{\mathrm{d}\, h(y)}{d\ y} = h'(y) \quad \text{and} \quad \mathbf{B}_\alpha^\beta = \{\ (x,y) \in R^1 \times [\alpha, \beta],\ \ a \le \alpha \le \beta \le b\ \}$$

3.2.3. Dynamic behaviour of the CLS with inaccurate switching curve, operating by motionless target

When talking in this paragraph about the target state we understand it as a motionless one, i.e. $\mathbf{z}_1 = (x_1, 0)$, $x_1 \in R^1$. For this case the formulas (3.17), (3.18) take the following form:

$$\mathbf{T}^+ = \{\ (x,y)\ : x = h(y), \quad y \in [a, 0]\ \} \qquad (3.22)$$

$$\mathbf{T}^- = \{\ (x,y)\ : x = h(y), \quad y \in [0, b\ ]\ \} \qquad (3.23)$$

Denote as f_0^+ and f_0^- the practical estimation of the functions f^+ and f^- taken for the function $h(y)$ formation. After setting $f_0^+ \to f^+$ and $f_0^- \to f^-$ into (3.4) and (3.5) respectively we get

$$h'(y) = \frac{\mathrm{d}h(y)}{\mathrm{d}y} = \frac{y}{f_0^+(y)}, \quad \text{a.e. on [a, 0]}$$

$$h'(y) = \frac{\mathrm{d}h(y)}{\mathrm{d}y} = \frac{y}{f_0^-(y)}, \quad \text{a.e. on } [0, \mathrm{b}]$$

Because the functions f_0^+ and f_0^- should fulfil the same assumption (3.3) as the functions f^+ and f^-, the above together with (3.3) imply

$$h' : [\mathrm{a}, \mathrm{b}] \to (-\infty, 0) \tag{3.24}$$

LEMMA 3.6

Assume, the functions f^+ and f^- are piecewise continuous on [a, b] and satisfy (3.3), the curves $\mathbf{T}^+$ and $\mathbf{T}^-$ are defined by (3.22) and (3.23) respectively.

Thesis 1. If inequality

$$\frac{y}{f^+(y)} = F^+(y) \le h'(y) \quad \text{a.e. on } [\mathrm{a}, 0] \tag{3.25}$$

is fulfilled then the system (3.21) has the unique *C*-solution in the set $\mathbf{R}^+ \cup \mathbf{T}^+$.

Thesis 2. If inequality

$$\frac{y}{f^-(y)} = F^-(y) \le h'(y) \quad \text{a.e. on } [0, \mathrm{b}] \tag{3.26}$$

is fulfilled then the system (3.21) has the unique *C*-solution in the set $\mathbf{R}^- \cup \mathbf{T}^-$.

PROOF: The proof is given in [10]. ■

Assume the following estimations of the functions $f^+(y)$, $f^-(y)$ and $h(y)$:

$$m \le K_1^+(y) \le f^+(y) \le K_2^+(y) \le M \quad \text{on } [0, b),$$
$$K_1^+(b) = f^+(b) = K_2^+(b) = 0 \tag{3.27}$$

$$-M \le -K_2^-(y) \le f^-(y) \le -K_1^-(y) \le -m \quad \text{on } (0, b] \tag{3.28}$$

$$-M \le -k_2^-(y) \le f^-(y) \le -k_1^-(y) \le -m \quad \text{on } (a, 0],$$
$$k_1^-(a) = f^-(a) = k_2^-(a) = 0 \tag{3.29}$$

$$m \le k_1^+(y) \le f^+(y) \le k_2^+(y) \le M \quad \text{on } [a, 0) \tag{3.30}$$

Figure 1 illustrates the above estimations.

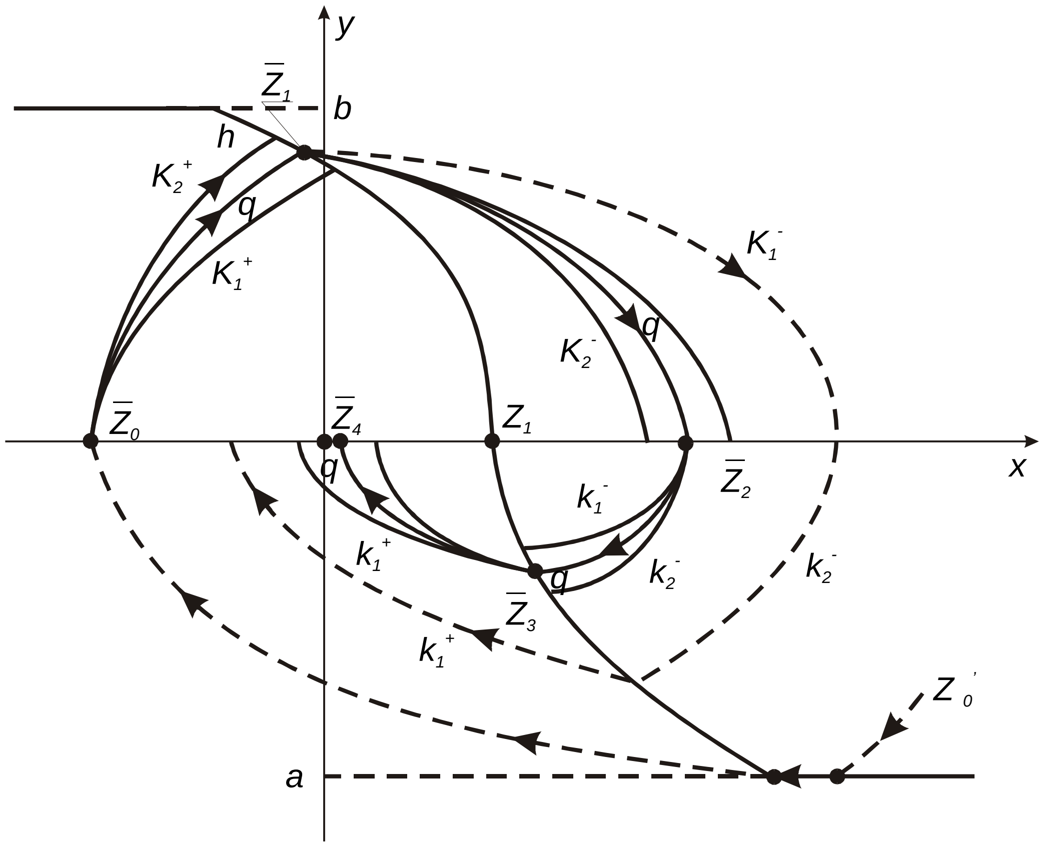

Fig.1. Estimating functions

The inaccurate switching curve **T** induced by a real controller will be estimated in the following way:

$$-\alpha\,|\,y\,|^{\beta} \le \frac{y}{h_1(y)} \le h'(y) \le \frac{y}{h_2(y)}\ ,\quad \alpha > 0,\ \beta > 0,\ \text{on } [a, 0] \tag{3.31}$$

$$-\alpha\,|\,y\,|^{\beta} \le \frac{y}{H_1(y)} \le h'(y) \le \frac{y}{H_2(y)}\ ,\quad \alpha > 0,\ \beta > 0,\ \text{on } [0, \text{b}] \tag{3.32}$$

Figyre 2 visualises the above estimations of the inaccurate switching curve. The functions $k_i^+, k_i^-, K_i^+, K_i^-, h_i, H_i,\ i = 0, 1$ appeared in definitions (3.27) -(3.32) will be called *estimating functions*.

Notice, for $y \ge 0$ they are denoted in capital letters and those for $y \le 0$ in small letters.

A subject of this chapter is to formulate assumptions dealing *estimating functions* for two following cases:

a) the system (3.21) starting from any point $\mathbf{z}_0 \in \mathbf{B}_a^b \backslash \mathbf{z}_1$ is brought to the target state $\mathbf{z}_1$ in a finite time;

b) for any starting point $\mathbf{z}_0 \in \mathbf{B}_a^b \backslash \mathbf{z}_1$ there exists the unique solution $\mathbf{q}(t;\mathbf{z}_0)$, $t \in [0,\infty)$, moreover, there exists a ball $\mathbf{B}(\mathbf{z}_1, \delta)$ and a time $t = t_\delta$ such that

$$\mathbf{q}(t;\mathbf{z}_0) \neq \mathbf{z}_1, \quad t \in [0,\infty) \quad \text{and} \quad \mathbf{q}(t;\mathbf{z}_0) \notin \mathbf{B}(\mathbf{z}_1, \delta), \quad t \geq t_\delta ;$$

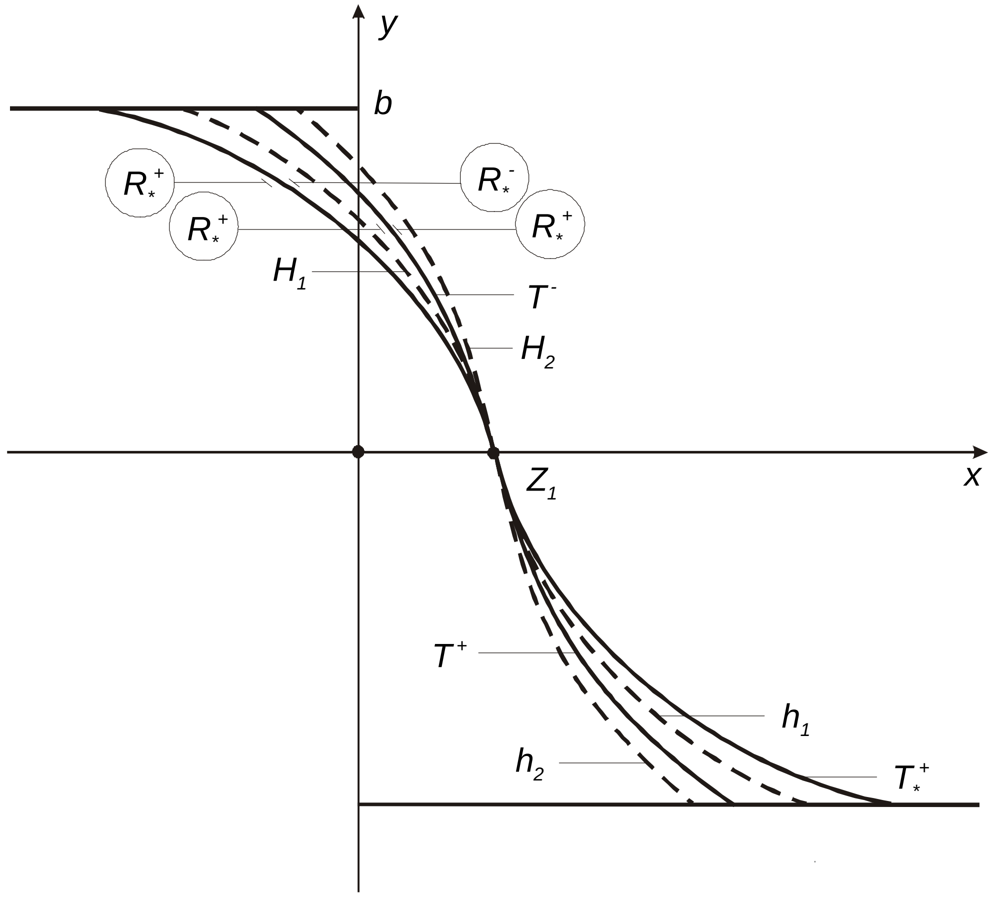

Fig.2. Estimating functions of the *inaccurate switching curve*

In what follows the following notations will be applied:

$$p = \sup_{\substack{y \in [0,b] \\ y \in [a,0]}} \frac{\int_0^y s\left(\frac{1}{K_1^-(s)} - \frac{1}{H_2(s)}\right)ds}{\int_0^y s\left(\frac{1}{K_2^+(s)} + \frac{1}{H_2(s)}\right)ds} \cdot \frac{\int_0^y s\left(\frac{1}{k_1^+(s)} - \frac{1}{h_2(s)}\right)ds}{\int_0^y s\left(\frac{1}{k_2^-(s)} + \frac{1}{h_2(s)}\right)ds}, \tag{3.33}$$

$$r = \inf_{\substack{y \in [0,b] \\ y \in [a,0]}} \frac{\int_0^y s\left(\frac{1}{K_2^-(s)} - \frac{1}{H_1(s)}\right)ds}{\int_0^y s\left(\frac{1}{K_1^+(s)} + \frac{1}{H_1(s)}\right)ds} \cdot \frac{\int_0^y s\left(\frac{1}{k_2^+(s)} - \frac{1}{h_1(s)}\right)ds}{\int_0^y s\left(\frac{1}{k_1^-(s)} + \frac{1}{h_1(s)}\right)ds} \tag{3.34}$$

THEOREM 3.7

Given the closed-loop system (3.21). Let the following assumptions be fulfilled:

a) The right-hand side of the system (3.21) satisfies inequalities (3.27) - (3.30).

b) The switching curve **T** is defined by (3.22),(3.23) and satisfies inequalities (3.31),(3.32).

c) The functions appearing in estimating inequalities (3.27) - (3.32) are piecewise continuous on their domains.

d)

$$h_1(y) \geq k_2^+(y) \quad \text{on} \quad [a, 0], \tag{3.35}$$

$$H_1(y) \geq K_2^-(y) \quad \text{on} \quad [0, b]. \tag{3.36}$$

Thesis 1. If the factor defined by (3.33) $p \in [0,\ 1)$ then for each starting point $\mathbf{z}_0 \in \mathbf{B}_a^b \backslash \mathbf{z}_1$ there exists $\bar{t} < \infty$ such that the C-solution $\mathbf{q}(t;\mathbf{z}_0)$ of the system (3.39) satisfies relations:

$$\mathbf{q}(t;\mathbf{z}_0) \neq \mathbf{z}_1, \quad t \in [0,\ \bar{t}\,), \quad \mathbf{q}(\bar{t}\,;\mathbf{z}_0) = \mathbf{z}_1$$

Thesis 2. If the factor defined by (3.34) $r > 1$ then for each starting state $\mathbf{z}_0 \in \mathbf{B}_a^b \backslash \mathbf{z}_1$ the closed-loop system (3.21) has the unique C-solution $\mathbf{q}(t;\mathbf{z}_0)$, $t \in [0, \infty)$ and there exists a ball $\mathbf{B}(\mathbf{z}_1, \delta)$, $\delta > 0$ and a time $t_\delta < \infty$ such that:

$$\mathbf{q}(t;\mathbf{z}_0) \neq \mathbf{z}_1, \quad t \in [0,\ \infty) \quad \text{and} \quad \left\| \mathbf{q}(t;\mathbf{z}_0) - \mathbf{z}_1 \right\| \geq \delta, \quad t \geq t_\delta$$

PROOF: The proof is given in [10]. ■

Now, we are going to interpret the results of Theorem 3.7. Physical meaning of those results are contained in the below Remark.

REMARK 3.8

Let the switching curve **T** induced by engineering controller be less steep than the time optimal one $\mathbf{T}^*$, i.e. inequalities (3.25) and (3.26) occur. If only $p \in (0,\ 1)$ (the factor p is defined by (3.33)) then the trajectory of the C-solution generated by the **CLS** (3.21) oscillates and encircles the target state $\mathbf{z}_1$, indefinitely many number times, while converging. Instead, if $r > 1$ (the factor r is defined by (3.34)) then the trajectory of the **CLS** (3.21) C-solution is oscillatory one and spirals round the target $\mathbf{z}_1$ while diverging and after performing a number of encirclements it reaches the straight-line $R^1 \times a$ or $R^1 \times b$. The next portion of that trajectory tends to the limit-cycle formation with amplitude constrained by the border values a and b.

Obviously, if $p \geq 1$ and $r \leq 1$, then the oscillatory process tends to a limit cycle formation too. ■

THEOREM 3.9

Given closed-loop system (3.21). Assume: the functions f^+ and f^- are piecewise continuous on [a, b] and satisfy (3.3), the curves $\mathbf{T}^+$ and $\mathbf{T}^-$ are defined by (3.22) and (3.23) respectively.

Thesis 1. Let there exist the constants α > 0 and β > 0 such that

$$-\alpha|y|^{\beta} \le h'(y) < \frac{y}{f^+(y)} \quad \text{on} \quad [a, 0) \tag{3.37}$$

then from each $\mathbf{z}_0 \in \mathbf{T}^+$ there starts the trajectory of the unique *F*-solution that lying totally on the curve $\mathbf{T}^+$ reaches the target $\mathbf{z}_1$ in a finite time. No *C*-solution starts from $\mathbf{z}_0 \in \mathbf{T}^+$ and none lies on it.

Thesis 2. Let there exist the constants α > 0 and β > 0 such that

$$-\alpha|y|^{\beta} \le h'(y) < \frac{y}{f^-(y)} \quad \text{on} \quad (0, b], \tag{3.38}$$

then from each $\mathbf{z}_0 \in \mathbf{T}^-$ there starts the trajectory of the unique *F*-solution that lying totally on the curve $\mathbf{T}^-$ reaches the target $\mathbf{z}_1$ in finite time. No *C*-solution starts from $\mathbf{z}_0 \in \mathbf{T}^-$ and none lies on it.

PROOF: The proof is given in [10]. ■

The trajectory of the *F*-solution may be interpreted from physical point of view. That interpretation has been contained in the below Remark.

REMARK 3.10

Technical interpretation of the *F*-solution will be done for $y_1 > 0$ only, the case $y_1 < 0$ being analogous. If the closed-loop system (3.21) operates with inaccuracy (3.37) and (3.38) then from each $\mathbf{z}_0 \in \mathbf{T}^-$ there starts a trajectory of the unique *F*-solution that coincides totally with the switching curve $\mathbf{T}^-$. On the other hand, (3.21) implies that from each $\mathbf{z}_0 \in \mathbf{T}^-$ there should start the trajectory of the *C*-solution $\mathbf{q}_-$. After comparing an inclination of the trajectory of the $\mathbf{q}_-$ solution defined by (3.16) with an inclination of the switching curve **T** in the point $\mathbf{z}_0 \in \mathbf{T}^-$ one states that trajectory of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution tends to penetrate into $\mathbf{R}^+$region. The low of control given by (3.21) implies that none *C*-solution $\mathbf{q}_-(t;\mathbf{z}_0)$ exists in $\mathbf{R}^+$. Practically, the trajectory generated by the **CLS** (3.21) on leaving $\mathbf{z}_0 \in \mathbf{T}^-$ penetrates into $\mathbf{R}^+$ where it is immediately forced to re-penetrate the switching curve $\mathbf{T}^-$ (see Figure 3). The real **CLS** (3.21) with switching curve **T** inaccuracy given by (3.37) and

(3.38) generates a trajectory which starts to oscillate around $\mathbf{T}^{-}$ with some frequency and amplitude depending on the delay time inherent in the switching operation, which evidently exists in every real structure. This trajectory of the F-solution is therefore *a limit of the real oscillatory process (sliding, chattering) when the delay time tends to zero, i.e. when the frequency tends to infinity*.

The same interpretation may be offered for F-solutions starting from $\mathbf{z}_0 \in \mathbf{T}^{+}$. ■

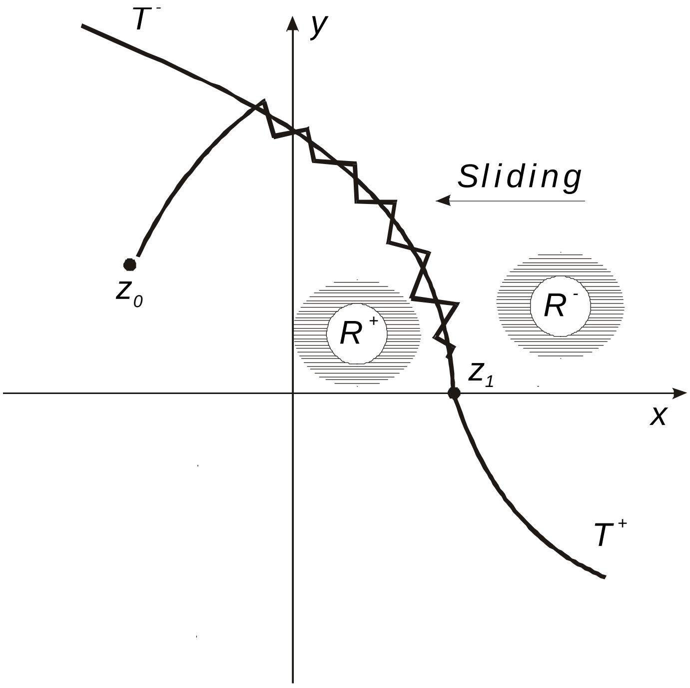

Fig.3. Sliding process

The F-solution becomes therefore a generalised description of the oscillatory sliding process which has an essential practical meaning. The F-solution is independent of whatever variation of $F_{-}(y)$ and $F_{+}(y)$ if only (3.37) and (3.38) is fulfilled.

3.2.4. Dynamic behaviour of the CLS with inaccurate switching curve, operating by the moving target

When talking in this paragraph about the target state it will be understood as a moving one, i.e. $\mathbf{z}_1 = (x_1,\ y_1),\ x_1 \in R^1,\ y_1 \neq 0$. For this case the switching curve, state-plane partitioning, mapping of closed-loop system dynamics will be defined exactly by the formulas (3.17) - (3.21)

Analogously as it was done in the case of motionless target, the switching curve inaccuracy will be defined by the following inequalities:

$$\left.\begin{aligned} \frac{y}{f^{-}(y)} &= F^{-}(y) < h'(y) \le 0, \quad \text{a.e. on} \quad [y_1, b], \\ \frac{y}{f^{+}(y)} &= F^{+}(y) < h'(y) \le 0, \quad \text{a.e. on} \quad [a, y_1] \end{aligned}\right\} \tag{3.39}$$

and

$$\left.\begin{aligned} \frac{y}{f^{-}(y)} &= F^{-}(y) > h'(y) \geq -\alpha\, y, \quad \text{a..e. on} \quad [y_1, b], \\ \frac{y}{f^{+}(y)} &= F^{+}(y) > h'(y) \geq \alpha\, y, \quad \text{a.e. on} \quad [a, y_1] \end{aligned}\right\} \tag{3.40}$$

where $\alpha > 0$.

In this case the target state $\mathbf{z}_1$ lies out of the x-axis, i.e. $x_1 \in R^1$, $y_1 \neq 0$. Only the case $y_1 > 0$ will be considered, the case $y_1 < 0$ being analogous.

Now, we are going to analyse the control process of the system (3.21) by the assumptions (3.39).

Let us denote:

$$\mathbf{S} = \left\{ (x, y) : (x', y) \in \mathbf{T}^{+}, (x'', y) \in \mathbf{P} \Rightarrow x' < x \leq x'' \right\}$$

where $\mathbf{P}$ is the set of the states lying on the trajectory of the $\mathbf{q}_{\_}$ solution, starting from the target state $\mathbf{z}_1$,

$$\mathbf{P} = \{\ (x, y)\ :\ \mathbf{q}_{\_}(t; \mathbf{z}_1),\quad t \geq 0\ \}$$

In the Figure 4 there have been shown both the set **S** and **P**.

Following the same way of argument as that used in analysis of the case of motionless target we are able to proof theses contained in the corollary as given below.

COROLLARY 3.11

Let assumptions (3.39) occur. The trajectory of the *C*-solution generated by the **CLS** (3.21) oscillates round the set **S** and tends to limit curve formation. The limit curve can take on two different forms:

a) co-ordinates $y(t)$ of the limit curve reach the limit values a or b. Then each trajectory of the *C*-solution is divergent in relation to the set **S**;

b) co-ordinates $y(t)$ of the limit curve do not reach the limit values a, b. Then each trajectory starting from the inside this limit curve tends to it, while diverging, whereas any trajectory starting from the outside of the limit curve tends towards it, while converging.

We are now going to analyse the control process of the closed-loop system (3.21) for assumption (3.40). Let us assume that the target state lies over the x-axis i.e. $\mathbf{z}_1 = (x_1, y_1)$, $y_1 > 0$. Let us define the following subsets:

$$\left.\begin{aligned} \mathbf{R}_1^{+} &= \left\{ (x, y) : \in \mathbf{R}^{+} : y \geq y_1 \right\} \\ \mathbf{R}_1^{-} &= \left\{ (x, y) : \in \mathbf{R}^{-} : y \geq y_1 \right\} \end{aligned}\right\} \tag{3.41}$$

$$\left.\begin{array}{l}\mathbf{T}_1^+ = \left\{ (x,y) \in \mathbf{T}^+ : y \in [a,0) \right\} \\ \mathbf{T}_2^+ = \left\{ (x,y) \in \mathbf{T}^+ : y \in (0,y_1) \right\}\end{array}\right\} \tag{3.42}$$

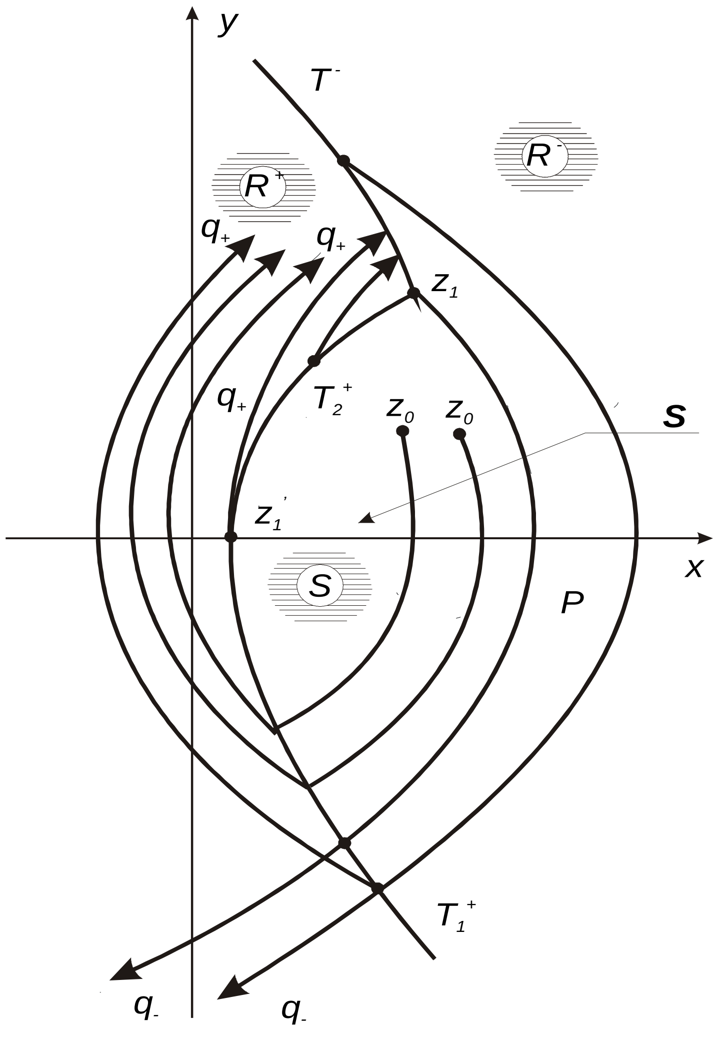

Fig.4. *Inaccurate switching curve* by moving target

Obviously, the switching curve $\mathbf{T}^+$ may be now defined by relation:

$$\mathbf{T}^+ = \mathbf{T}_1^+ \cup \mathbf{z}_1' \cup \mathbf{T}_2^+ \tag{3.43}$$

where $\mathbf{z}_1'$ is a point in which the switching curve $\mathbf{T}^+$ penetrates the *x*-axis.

Theses contained in the corollary given below can be proved in the same way as was done in the analysis of the case of motionless target.

COROLLARY 3.12

Given the closed-loop system (3.21), operating by moving target $\mathbf{z}_1 = (x_1, y_1)$, $y_1 > 0$. Let assumption (3.40) be true.

a) If $\mathbf{z}_0 \in \mathbf{T}^-$, then the closed-loop system (3.21) has no *C*-solution, whereas there exists the unique *F*-solution, the trajectory of which lying totally on the curve $\mathbf{T}^-$ reaches the target state $\mathbf{z}_1$ in finite time.

b) If $\mathbf{z}_0 \in \mathbf{T}_1^+$, then the closed-loop system (3.21) has no *C*-solution, whereas there exists the unique *F*-solution, the trajectory of which lies totally on the switching curve $\mathbf{T}_1^+$ and reaches the point $\mathbf{z}_1' = (h(0), 0)$ (i.e. the point of intersection the *x*-axis with the switching curve $\mathbf{T}^+$ in finite time.

c) From the state $\mathbf{z}_1'$ there starts the unique *F*-solution $\mathbf{q}(t;\mathbf{z}_1') \equiv (h(0), 0)$, $t\in [0, \infty)$. None solution of the closed-loop system (3.21) can leave the point $\mathbf{z}_1'$. The closed-loop system (3.21) is stepped continuously in the point $\mathbf{z}_1'$. This state will becomes *solution end-point.*

The solutions defined in the point a) and b) may be interpreted from physical point of view in the same manner as it was done in the *REMARK 3.10.* as oscillations performed around the curve $\mathbf{T}^-$ and $\mathbf{T}_1^+$ respectively.

F-solution $\mathbf{q}(t;\mathbf{z}_1') \equiv (h(0), 0)$, $t\in [0, \infty)$ may be interpreted in the following physical way:

The control low (3.21) implies that from the point $\mathbf{z}_1'$ there starts the *C*-solution $\mathbf{q}_+(t;\ \mathbf{z}_1')$ that penetrates into $\mathbf{R}^-$ set where it is immediately forced to intersect the curve $\mathbf{T}_1^+$. In this new point of $\mathbf{T}_1^+$ intersection there starts the trajectory of $\mathbf{q}_+$ which along $\mathbf{T}_1^+$ reaches ones again the point $\mathbf{z}_1'$. This cycle of process is repeated infinite number of times. In such a way the system starts to oscillate in a certain neighbourhood of the state $\mathbf{z}_1'$.

The same conclusions may be offered for the case of moving target $\mathbf{z}_1 = (x_1,\ y_1)$, $y_1 < 0$. ■

Now, we are going to show that if only the moving target is a case, then *phenomenon* of non-linear oscillations may appear in the dynamic behaviour of the **CLS**, even as the switching curve **T** *satisfies the conditions of time optimality*.

We consider the case $y_1 > 0$ only, the case $y_1 < 0$ being analogous.

If the switching curve is time-optimal one then

$$\left.\begin{aligned} &a)\ h'(y) = \frac{y}{f^+(y)} = F^+(y), \quad y \in [a, y_1], \\ &b)\ h'(y) = \frac{y}{f^-(y)} = F^-(y), \quad y \in [y_1, b] \end{aligned}\right\} \tag{3.44}$$

and an engineering **CLS** (3.21) becomes the time-optimal one. In what follows we will use the notations as shown in the Figure 5.

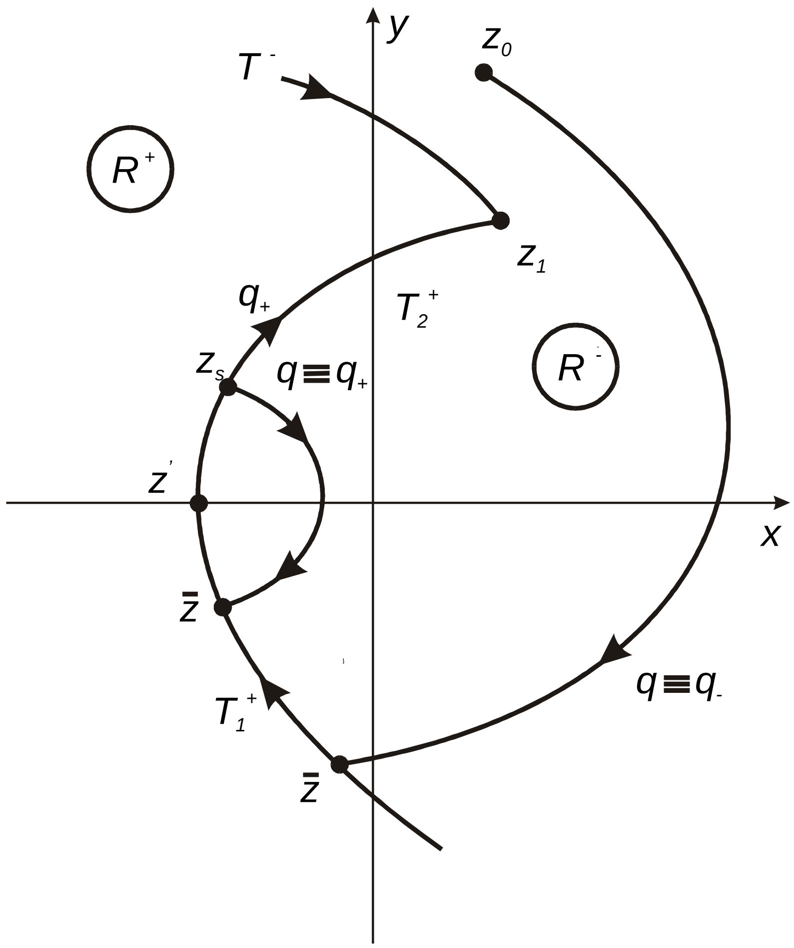

Fig.5. Limit cycle

Let us take into consideration the state $\mathbf{z}_s \in \mathbf{T}_2^+ \backslash \mathbf{z}_1$. It is obvious that from that point there starts the trajectory of the time-optimal *C*-solution that transfers the **CLS** (3.21) along the trajectory $\mathbf{T}_2^+$ to the target state $\mathbf{z}_1$ at finite time.

From (3.21) it follows also, that from each state $\mathbf{z}_s \in \mathbf{T}_2^+ \backslash \mathbf{z}_1$ there starts the trajectory of the *C*-solution $\mathbf{q}_-$ of the **CLS** (3.21) defined on an open interval $t \in (0,\ t')$, $t' > 0$, i.e.

$$\mathbf{q}_-(t;\mathbf{z}_s) \in \mathbf{R}^-,\ t \in (0,\ t'),\ \ t' > 0 \tag{3.45}$$

Obviously, there exists a time $\bar{t} > t'$ such that $\mathbf{q}_-(\bar{t};\mathbf{z}_s) = \bar{\mathbf{z}} \in \mathbf{T}_1^+$ (see Figure 5). Extending the above solution up to the bound of the domain where it exists we get:

$$\mathbf{q}_-(0;\ \mathbf{z}_s) \in \mathbf{T}_2^+ \backslash \mathbf{z}_1;\quad \mathbf{q}_-(t;\ \mathbf{z}_s) \in \mathbf{R}^-,\quad t \in (0,\ \bar{t});\quad \mathbf{q}_-(\bar{t};\mathbf{z}_s) = \bar{\mathbf{z}} \in \mathbf{T}_1^+$$

Perceive, that from the above mentioned point $\bar{\mathbf{z}} \in \mathbf{T}_1^+$ there starts the trajectory of the *C*-solution which, along the curve $\mathbf{T}_1^+$, tends to reach once again the same point $\mathbf{z}_s \in \mathbf{T}_2^+$ (see Figure 5).

From the above considerations there appear conclusions contained in the below corollary.

COROLLARY 3.13

Given the closed-loop system (3.21) with the time-optimal switching curve and operating by moving target $y_1 > 0$.

From each state $\mathbf{z}_s \in \mathbf{T}_2^+ \backslash \mathbf{z}_1$ there start two non-unique solutions. The first one transfers the **CLS** to the target $\mathbf{z}_1$ along the curve $\mathbf{T}_2^+$ in minimum time. The second one on leaving curve $\mathbf{T}_2^+$ penetrates into the set $\mathbf{R}^-$ and next intersects the switching curve $\mathbf{T}_1^+$. From this point of intersection there starts the trajectory of the *C*-solution, lying totally on the curve $\mathbf{T}_1^+$, that brings the **CLS** once again to the starting point $\mathbf{z}_s$. Because of technical reasons, the engineering **CLS** may identify the point $\mathbf{z}_s \in \mathbf{T}_2^+ \backslash \mathbf{z}_1$ with even the least, but systematic error, as belonging to the $\mathbf{R}^-$. the **CLS** generates the oscillations the amplitude of which tends to limit curve formation, that does not contain in its interior the target state $\mathbf{z}_1$. (see Figure 5).

The same interpretation may be offered for the case $y_1 < 0$. ■

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