RHEONOMIC COORDINATE METHOD APPLIED TO NONLINEAR VIBRATION SYSTEMS WITH HEREDITARY ELEMENTS

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Results pointed out in this paper, are inspired by papers O. A. Goroshko and N. P. Puchko (see Ref. [13] and [14]), about Lagrange's equations for the multybodies hereditary systems, and a rheological models of the bodes properties presented in the monography written by G.M. Savin and Ya.Ya. Ruschitsky (see Ref. [24]), as well as a monography on rheonimic dynamics written by V.A. Vuji~i} (see [6]). By using rhelogical body models for designing deformable rheological hereditary elements with hybrid rheological elastoviscosic and/or viscoelastic properties (see Ref. [23], [24] and [18]), discrete oscillatory systems with hereditary elements as a constraints, are designed, as systems with one degree of freedom as well as with many degrees of freedom. For these oscillatory hereditary systems, the integro-differential equations second and/or third kind are composed. The solutions of these integrodifferential equations are studied. Energy transfer from one kind to other is studied for the multy-frequency forced vibrations of the hereditary systems. For example, the rheological pendulum on the wool's thread with changeable length is modeled by rheonomic coordinate as well as by Burgers rheological hereditary element. By using defined rheological pendulum basic properties of the rheonomic coordinate in the sense of the Vuji~i}'s rheonomic coordinate are introduced. The force, as well as the power of the rate of rheological and rheonomic constraints change are determined.

For the designed discrete hereditary systems with corresponding rheological and relaxational hereditary elements the integrodifferential equations second and/or third kind are composed. On the basis of the analysis of the discrete hereditary oscillatory systems the Goroshko's definition on dynamically determinated or indeterminated discrete hereditary systems was confirmed.

Keywords: oscillatory hereditary systems. rheological elements, rheonomic coordinate, rheological pendulum, rheological and relaxational kernels.

I. Introduction

The paper of academician Goroshko (see Ref. [13] and [14]) was inspiration for research in area of hereditary discrete systems, as well as a mutual work on the monography: *Analytical Dynamics of the Discrete Hereditary Systems* which is in preparation for publishing in Serbian as well as in English. Some examples were consider in the following papers: [27], [28], [32], [33] and [34].

For active constructions we can use various types of control and regulation of dynamical system parameters. In the modeling of an active construction, different kinds of active element can be used. Some of these elements are active hereditary elements with different kinds of viscoelasticity or hereditary elasticity, with different time relaxation, as well as time retardation (see Ref. [23], [24], [25] and [18]. See also [29], and [30].). Active properties of construction can arise by active force or external excitations, active temperature fields, active electrical or optical fields, or by changeable distances between bearings, as a rheonomic coordinate, as well as by changeable rigidity. Active construction can be realized by subsystem as an active element with external excitation.

As in the active constructions is not possible control without sensors as well sensors work by modulations of amplitude, phase as well as by modulations of frequency it is necessary to introduce the sensors parameters as a active excitations into active elements as well into active construction. Par example, optical sensors work by modulations of amplitude, of phase as well as by modulations of frequency of light waves, which arises with changeable optical parameters of material in the changes of stress and strain state in the material of construction during the way of the light waves.

Active elements are elements by the use of which we can observe and control stress and strain states in construction, as well as a temperature field state, by the use of sensor observed active parameters of the dynamical state of construction.

Active elements can be designed by the use of properties of dynamical adaptations, as an electromechanical, termomechanical or mechanical. In mechanical way the construction rigidity of the defined sections can be made changeable.

In this paper we would like to investigate equations of dynamics of active discrete hereditary elements as well as systems by introducing rheonomic coordinate in the standard hereditary element.

We will consider [10, [11], [12], [16], [17], [4] and [15] as well as [20], [21], [22] and [31] to be our basic literature.

II. EQUATIONS OF DYNAMICS OF A DISCRETE SYSTEM WITH FINITE CONSTRAINTS AND STANDARD HEREDITARY ELEMENTS

We investigate dynamical system (see Figure No.1) of N material particles with masses m_ν, $\nu = 1,2,3,...,N$, the vector positions of which are $\vec{r}_\nu = y_\nu^i \vec{e}_i, i = 1,2,3;\ \nu = 1,2,3,...,N$. Material particles are constrained by S finite constraints (see [1], [2], [3], [26], [16], [17], [11], [10] and [12]):

$$\bar{f}_\mu(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) = f_\mu\left(y^1, y^2, ..., y^{3N}\right) \quad \mu = 1,2,3,...,S \tag{1}$$

and where we introduce the following notations: $y_\nu^k =: y^{3\nu-(3-k)}, k = 1,2,3;\quad m_{3\nu-k} = m_{3\nu}, k = 1,2,3,$ $\nu = 1,2,3,...,N$; as well as by K standard hereditary elements neglected mass and material properties parameters of which are: $n_{(\nu,\nu+1)k}$, $k = 1,2,3,....,K$, are times of relaxation, and $c_{(\nu,\nu+1)k}$ and $\tilde{c}_{(\nu,\nu+1)k}$. are an instantaneous rigid stifness modulus as prolonged ones.

Relations between reactions and deformations of the hereditary element in the discrete system can be defined in one of the following ways:

* in the relaxational forms by using integral stress strain state relations:

$$P_{(\nu,\nu+1)k} = c_{(\nu,\nu+1)k}\left[\bar{\rho}_{(\nu,\nu+1)k}(t) - \int_0^t \mathsf{R}_{(\nu,\nu+1)k}(t-\tau)\bar{\rho}_{(\nu,\nu+1)k}(\tau)d\tau\right], \tag{2}$$

$$\nu = 1,2,3,...,N,\ k = 1,2,3,....,K_\nu$$

where are

$$\mathsf{R}_{(\nu,\nu+1)k}(t-\tau) = \frac{c_{(\nu,\nu+1)k} - \tilde{c}_{(\nu,\nu+1)k}}{n_{(\nu,\nu+1)k}c_{(\nu,\nu+1)k}} e^{-\frac{t-\tau}{n_{(\nu,\nu+1)k}}},\ \nu = 1,2,3,...,N,\ k = 1,2,3,....,K_\nu \tag{3}$$

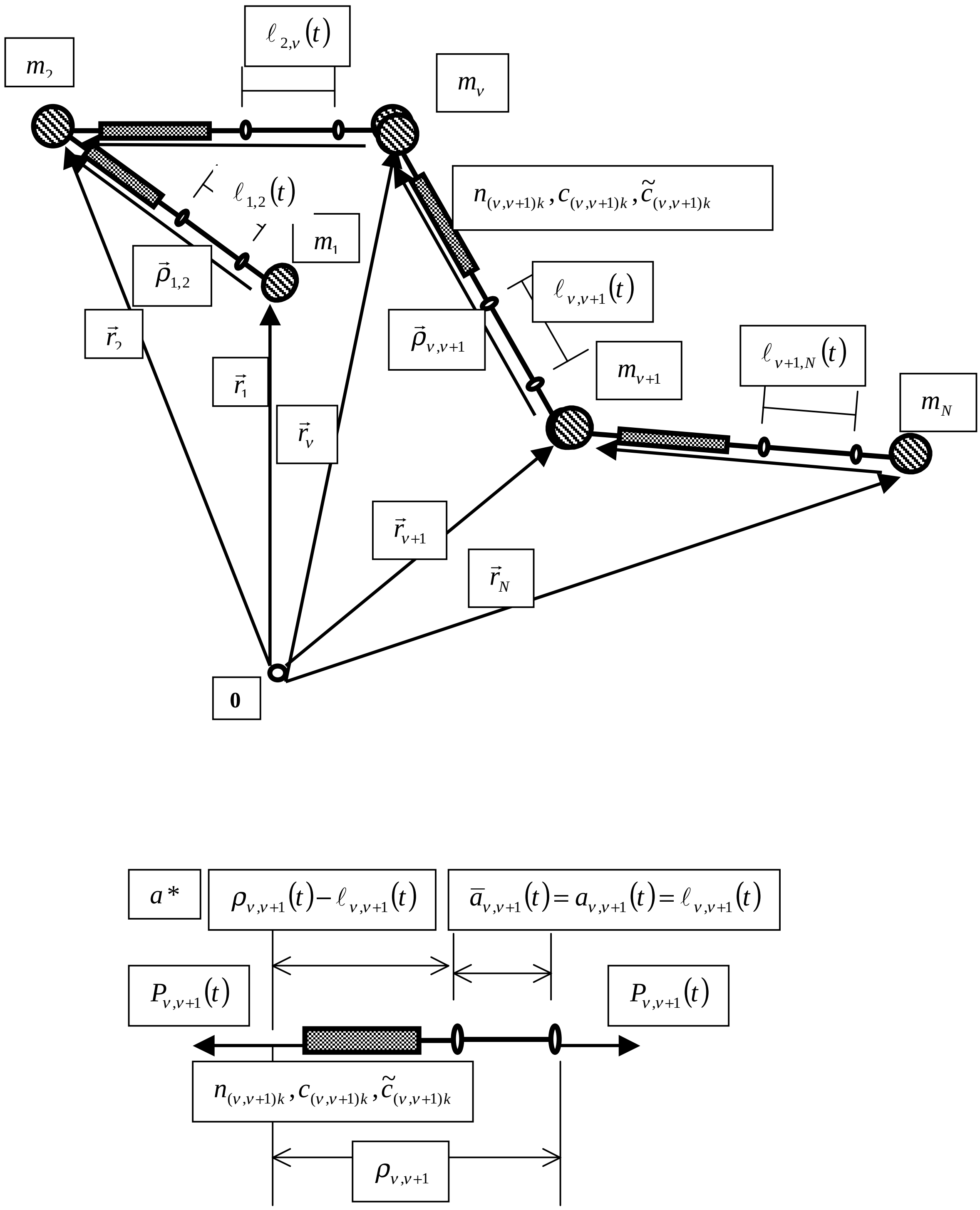


Figure No. 1. Model of discrete hereditary system with rheonomic constraints and with N material particles
$a*$ Hereditary and rheonomic elements in series

kernels of relaxation (see Ref. [15] and [13], [14]), and

$$\rho_{(\nu,\nu+1)k} = \left|\vec{\rho}_{(\nu,\nu+1)k}\right| = \left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right| \qquad \text{i}$$

$$\bar{\rho}_{(\nu,\nu+1)k} = \left|\vec{\rho}_{(\nu,\nu+1)k}\right| - \rho_{(\nu,\nu+1)k0} = \left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right| - \rho_{(\nu,\nu+1)k0}\,. \qquad \text{(a)}$$

and $\rho_{(\nu,\nu+1)k0}$ is natural length of a hereditary element in natural stress-strain state, when the strain and stress in the element are equal to zero.

* in the retardation forms by using integral stress strain state relations:

$$\overline{\rho}_{(\nu,\nu+1)k} = \frac{1}{c_{(\nu,\nu+1)k}}\left[P_{(\nu,\nu+1)k}(t) + \int_0^t \mathrm{K}_{(\nu,\nu+1)k}(t-\tau)P_{(\nu,\nu+1)k}(\tau)d\tau\right], \quad (4)$$

$$\nu = 1,2,3,...,N,\ k = 1,2,3,....,K_\nu$$

where are

$$\mathrm{K}_{(\nu,\nu+1)k}(t-\tau) = \frac{c_{(\nu,\nu+1)k} - \widetilde{c}_{(\nu,\nu+1)k}}{n_{(\nu,\nu+1)k}c_{(\nu,\nu+1)k}} e^{-\frac{(t-\tau)\widetilde{c}_{(\nu,\nu+1)k}}{n_{(\nu,\nu+1)k}c_{(\nu,\nu+1)k}}},\ \nu = 1,2,3,...,N,\ k = 1,2,3,....,K_\nu \quad (5)$$

is a kernel of rheology, and $\rho_{(\nu,\nu+1)k} = \left|\vec{\rho}_{(\nu,\nu+1)k}\right| = \left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|$.

* in differential form:

$$n_{(\nu,\nu+1)k}\dot{P}_{(\nu,\nu+1)k}(t) + P_{(\nu,\nu+1)k}(t) = n_{(\nu,\nu+1)k}c_{(\nu,\nu+1)k}\dot{\overline{\rho}}_{(\nu,\nu+1)k} + \widetilde{c}_{(\nu,\nu+1)k}\overline{\rho}_{(\nu,\nu+1)k} \quad (6)$$

$$\nu = 1,2,3,...,N,\ k = 1,2,3,....,K_\nu$$

Finite constraints (1) must satisfy the following velocity condition:

$$\dot{f}_\mu = \sum_{\alpha=1}^{\alpha=3N}\frac{\partial f_\mu}{\partial y^\alpha}\dot{y}^\alpha = 0,\ \alpha = 1,2,3,....,3N,\ \mu = 1,2,3,....,S \quad (7)$$

as well as the acceleration conditions:

$$\ddot{f}_\mu = \sum_{\alpha=1}^{\alpha=3N}\frac{\partial f_\mu}{\partial y^\alpha}\ddot{y}^\alpha + \sum_{\alpha=1}^{\alpha=3N}\sum_{\beta=1}^{\beta=3N}\frac{\partial^2 f_\mu}{\partial y^\alpha \partial y^\beta}\dot{y}^\alpha\dot{y}^\beta = 0 \quad (8)$$

As this finite constraints are independent the differential determinate of matrix is different them zero:

$$\Delta = \left|\Delta_{\mu\alpha}\right| = \left|\frac{\partial f_\mu}{\partial y^\alpha}\right| \neq 0\ ,\ \alpha = 1,2,3,....,3N,\ \mu = 1,2,3,....,S \quad (9)$$

By using previous velocity conditions we can write ortogonality conditions $\left(grad_\nu f_\mu, \vec{v}^\nu\right) = 0$, $\nu = 1,2,3,...,N$, $\mu = 1,2,3,....,S$ between mass particles and gradients of the finite constraints , for ideal constraints reactions we can write the following:

$$\vec{\mathbf{R}}_\nu = \sum_{\mu=1}^{\mu=S}\lambda_\mu grad_\nu f_\mu(\vec{r}_1,...,\vec{r}_N),\ \nu = 1,2,3,...,N \quad (10)$$

in which the λ_μ are Lagrange's multiplikators of the finite constraints, as well as

$$grad_\nu f_\mu = \vec{i}\frac{\partial f_\mu}{\partial y_\nu^{3\nu-3}} + \vec{j}\frac{\partial f_\mu}{\partial y_\nu^{3\nu-2}} + \vec{k}\frac{\partial f_\mu}{\partial y_\nu^{3\nu-1}} \quad (10^*).$$

The resulting reactions of the K standard hereditary elements into ν-rd ($\nu+1$ with opposite direction) mass material particle is:

$$P_{\nu,\nu+1}(t) = \sum_{k=1}^{k=K}P_{(\nu,\nu+1)k}(t)\frac{\vec{\rho}_{(\nu,\nu+1)k}}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|} = \sum_{k=1}^{k=K}P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|} \quad (11)$$

Resulting reaction forces of finite constraints and hereditary elements in the observed system are:

$$\vec{\mathbf{R}}_\nu = \sum_{\mu=1}^{\mu=S}\lambda_\mu grad_\nu f_\mu(\vec{r}_1,...,\vec{r}_N) + P_{\nu,\nu+1}(t) + \vec{\mathbf{R}}_{\nu T} \quad (12^*)$$

$$\vec{\mathbf{R}}_\nu = \sum_{\mu=1}^{\mu=S}\lambda_\mu grad_\nu f_\mu(\vec{r}_1,\vec{r}_2,...,\vec{r}_N) + \sum_{k=1}^{k=K_\nu}P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|} + \vec{\mathbf{R}}_{\nu\mathbf{T}} \quad (12)$$

From a principle of the work on the virtual system displacements can be written in the following form:

$$\sum_{\nu=1}^{\nu=N}\left\{\vec{\mathrm{I}}_{\nu}+\vec{\mathrm{F}}_{\nu}+\vec{\mathrm{R}}_{\nu}+\vec{\mathrm{P}}_{\nu}+\vec{\mathrm{R}}_{\nu\mathrm{T}}\right\}\delta\vec{r}_{\nu}=0 \qquad (13^{*})$$

or:

$$\sum_{\nu=1}^{\nu=N}\left\{m_{\nu}\ddot{\vec{r}}_{\nu}-\vec{\mathrm{F}}_{\nu}(t)-\sum_{\mu=1}^{\mu=S}\lambda_{\mu}grad_{\nu}\,f_{\mu}(\vec{r}_{1},\dots,\vec{r}_{N})-\sum_{k=1}^{k=K_{\nu}}P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k}\right|}-\vec{\mathrm{R}}_{\nu\mathrm{T}}\right\}\delta\vec{r}_{\nu}=0 \qquad (13)$$

Dynamical Lagrange's equations first kind arise from previous equation in the following form

$$m_{\nu}\ddot{\vec{r}}_{\nu}=\vec{\mathrm{F}}_{\nu}(t)+\sum_{\mu=1}^{\mu=S}\lambda_{\mu}grad_{\nu}\,f_{\mu}(\vec{r}_{1},\dots,\vec{r}_{N})+\sum_{k=1}^{k=K_{\nu}}P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k}\right|}+\vec{\mathrm{R}}_{\nu\mathrm{T}} \qquad (14)$$

$$\nu=1,2,3,\dots,N$$

Lets define some relations by which the investigation and description of the dynamics of this problem is simpler. For this reason the equations (14) are rewritten for ν and $\nu+1$ in the forms:

$$\ddot{\vec{r}}_{\nu}=\frac{1}{m_{\nu}}\left\{\vec{\mathrm{F}}_{\nu}(t)+\sum_{\mu=1}^{\mu=S}\lambda_{\mu}grad_{\nu}\,f_{\mu}(\vec{r}_{1},\dots,\vec{r}_{N})+\sum_{k=1}^{k=K_{\nu}}P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k}\right|}+\vec{\mathrm{R}}_{\nu\mathrm{T}}\right\} \qquad (15)$$

$$\ddot{\vec{r}}_{\nu+1}=\frac{1}{m_{\nu+1}}\left\{\vec{\mathrm{F}}_{\nu+1}(t)+\sum_{\mu=1}^{\mu=S}\lambda_{\mu}grad_{\nu+1}\,f_{\mu}(\vec{r}_{1},\dots,\vec{r}_{N})+\sum_{k=1}^{k=K_{\nu}}P_{(\nu+1,\nu+2)k}(t)\frac{\vec{r}_{(\nu+2)k}-\vec{r}_{(\nu+1)k}}{\left|\vec{r}_{(\nu+2)k}-\vec{r}_{(\nu+1)k}\right|}+\vec{\mathrm{R}}_{\nu+1\mathrm{T}}\right\}$$

$$\nu=1,2,3,\dots,N$$

By subtraction of the previous equations (15) result becomes:

$$\ddot{\vec{r}}_{\nu+1}-\ddot{\vec{r}}_{\nu}=\frac{1}{m_{\nu+1}}\vec{\mathrm{F}}_{\nu+1}(t)-\frac{1}{m_{\nu}}\vec{\mathrm{F}}_{\nu}(t)+\frac{1}{m_{\nu+1}}\sum_{\mu=1}^{\mu=S}\lambda_{\mu}grad_{\nu+1}\,f_{\mu}(\vec{r}_{1},\dots,\vec{r}_{N})-\frac{1}{m_{\nu}}\sum_{\mu=1}^{\mu=S}\lambda_{\mu}grad_{\nu}\,f_{\mu}(\vec{r}_{1},\dots,\vec{r}_{N})+$$

$$+\frac{1}{m_{\nu+1}}\sum_{k=1}^{k=K_{\nu}}P_{(\nu+1,\nu+2)k}(t)\frac{\vec{r}_{\nu+2}-\vec{r}_{\nu+1}}{\left|\vec{r}_{\nu+2}-\vec{r}_{\nu+1}\right|}+\frac{1}{m_{\nu}}\sum_{k=1}^{k=K_{\nu}}P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{\nu+1}-\vec{r}_{\nu}}{\left|\vec{r}_{\nu+1}-\vec{r}_{\nu}\right|}+\frac{1}{m_{\nu+1}}\vec{\mathrm{R}}_{\nu+1\mathrm{T}}-\frac{1}{m_{\nu}}\vec{\mathrm{R}}_{\nu\mathrm{T}} \qquad (16)$$

$$\nu=1,2,3,\dots,N$$

The last relation may be written as:

$$\ddot{\vec{r}}_{\nu+1}-\ddot{\vec{r}}_{\nu}=\frac{1}{m_{\nu+1}}\vec{\mathrm{F}}_{\nu+1}(t)-\frac{1}{m_{\nu}}\vec{\mathrm{F}}_{\nu}(t)+\sum_{\mu=1}^{\mu=S}\lambda_{\mu}\left[\frac{1}{m_{\nu+1}}grad_{\nu+1}\,f_{\mu}(\vec{r}_{1},\dots,\vec{r}_{N})-\frac{1}{m_{\nu}}grad_{\nu}\,f_{\mu}(\vec{r}_{1},\dots,\vec{r}_{N})\right]+$$

$$+\sum_{k=1}^{k=K_{\nu}}\left\{\frac{1}{m_{\nu+1}}P_{(\nu+1,\nu+2)k}(t)\frac{\vec{\rho}_{(\nu+1,\nu+2)k}}{\rho_{(\nu+1,\nu+2)k}}-\frac{1}{m_{\nu}}P_{(\nu,\nu+1)k}(t)\frac{\vec{\rho}_{(\nu,\nu+1)k}}{\rho_{(\nu,\nu+1)k}}\right\}+\frac{1}{m_{\nu+1}}\vec{\mathrm{R}}_{\nu+1\mathrm{T}}-\frac{1}{m_{\nu}}\vec{\mathrm{R}}_{\nu\mathrm{T}} \qquad (16^{*})$$

$$\nu=1,2,3,\dots,N$$

By using the fact that distance, between any two material particles, from the system, is changeable, a constraint is expressed by equation of the form $\vec{\rho}_{\nu,\nu+1}^{\,2}=(\vec{r}_{\nu+1}-\vec{r}_{\nu})^{2}$. By two time derivatives of this constraint relation and knowing that $\vec{v}_{rel(\nu+1,\nu)}=\vec{v}_{\nu+1,\nu}=\dot{\vec{r}}_{\nu+1}-\dot{\vec{r}}_{\nu}$ is relative velocity of $\nu+1$st material mass particle around ν-st material mass particle in the relative rotation, we can write the following relation (see Ref. [8]):

$$\left(\vec{r}_{\nu+1}-\vec{r}_{\nu},\ddot{\vec{r}}_{\nu+1}-\ddot{\vec{r}}_{\nu}\right)=\dot{\rho}_{\nu+1,\nu}^{2}+\rho_{\nu+1,\nu}\ddot{\rho}_{\nu+1,\nu}-v_{\nu+1,\nu}^{2}\quad,\quad \nu=1,2,3,\dots,N \qquad (17)$$

Having in mind the previous relations (16*) the previous relation (17) becomes:

$$\sum_{k=1}^{k=K_{\nu}}\left\{\frac{1}{m_{\nu+1}}P_{(\nu+1,\nu+2)k}(t)\frac{\left(\vec{\rho}_{(\nu+1,\nu+2)k},\vec{\rho}_{(\nu,\nu+1)k}\right)}{\rho_{(\nu+1,\nu+2)k}}-\frac{1}{m_{\nu}}P_{(\nu,\nu+1)k}(t)\rho_{(\nu,\nu+1)k}\right\}=\dot{\rho}_{\nu+1,\nu}^{2}+\rho_{\nu+1,\nu}\ddot{\rho}_{\nu+1,\nu}-v_{\nu+1,\nu}^{2}-$$

$$-\left[\frac{1}{m_{\nu+1}}\left(\vec{\mathrm{F}}_{\nu+1}(t),\vec{\rho}_{\nu,\nu+1}\right)-\frac{1}{m_{\nu}}\left(\vec{\mathrm{F}}_{\nu}(t),\vec{\rho}_{\nu,\nu+1}\right)\right]-\left[\frac{1}{m_{\nu+1}}\left(\vec{\mathrm{R}}_{\nu+1\mathrm{T}},\vec{\rho}_{\nu,\nu+1}\right)-\frac{1}{m_{\nu}}\left(\vec{\mathrm{R}}_{\nu\mathrm{T}},\vec{\rho}_{\nu,\nu+1}\right)\right]-$$

$$-\sum_{\mu=1}^{\mu=S}\lambda_{\mu}\left[\frac{1}{m_{\nu+1}}\left(grad_{\nu+1}f_{\mu}(\vec{r}_1,...,\vec{r}_N),\vec{\rho}_{\nu,\nu+1}\right)-\frac{1}{m_{\nu}}\left(grad_{\nu}f_{\mu}(\vec{r}_1,...,\vec{r}_N),\vec{\rho}_{\nu,\nu+1}\right)\right] \quad (18)$$

$$\nu=1,2,3,...,N$$

Into previous equation (18) , the expression of the reaction-force of the standard hereditary element by using relations (2) of the stress-strain state or coordinates of deformation, or by using relations (4) should be introduced, or the corresponding equations (6) should join these equations (18) . The resulting system of equations can be solved as an explicit independent system of equations in relation to the coordinates of vector positions of mass particles. This is for the reason that we separate only equations with "internal" system coordinate by the use of which the internal relative positions between material mass particles are defined. We must have in mind that stress-strain relations use coordinates of hereditary element deformations $\bar{\rho}_{\nu,\nu+1}=\rho_{\nu,\nu+1}-\rho_{(\nu,\nu+1)0}$ which differ between relative coordinate $\rho_{\nu,\nu+1}$ - distance between two material particles and their distance $\rho_{(\nu,\nu+1)0}$, in hereditary element natural state if it was in such state at the initial moment.

We accept that, at initial moment, the hereditary element was in natural state without deformation, without stress as well as without strain. If we suppose that hereditary elements have history than into stress-strain relations boundaries of integral are different: boundaries: from zero to t changes into boundaries: from $-\infty$ to t .

$$P_{(\nu,\nu+1)k}=c_{(\nu,\nu+1)k}\left\{\left[\rho_{(\nu,\nu+1)k}(t)-\rho_{(\nu,\nu+1)k0}\right]-\int_0^t \mathbf{R}_{(\nu,\nu+1)k}(t-\tau)\left[\rho_{(\nu,\nu+1)k}(\tau)-\rho_{(\nu,\nu+1)k0}\right]d\tau\right\} \quad (19)$$

$$\nu=1,2,3,...,N\ ,\ k=1,2,3,....,K_{\nu}$$

When we observe system in space, than it is useful, that square $v^2_{\nu+1,\nu}$ of relative velocity $\vec{v}_{rel(\nu+1,\nu)}=\vec{v}_{\nu+1,\nu}=\dot{\vec{r}}_{\nu+1}-\dot{\vec{r}}_{\nu}$ of relative motion $\nu+1$st material particle around ν-th material particle be expressed by sphere coordinate in relation to the relative pole. Than the radius $\rho_{\nu,\nu+1}$, and circular and meridional angles: $\varphi_{\nu+1,\nu}$ and $\psi_{\nu+1,\nu}$ are used. For next coordinates of the material mass particle in relation to the previous mass particle coordinate we can write:

$$y^k_{\nu}=:y^{3\nu-(3-k)},k=1,2,3;\quad m_{3\nu-k}=m_{3\nu},k=1,2,3,\ \nu=1,2,3,...,N$$

$$y^1_{\nu+1}=y^1_{\nu}+\rho_{\nu,\nu+1}\cos\psi_{\nu+1,\nu}\cos\varphi_{\nu+1,\nu}$$

$$y^2_{\nu+1}=y^2_{\nu}+\rho_{\nu,\nu+1}\cos\psi_{\nu+1,\nu}\sin\varphi_{\nu+1,\nu} \quad (20)$$

$$y^3_{\nu+1}=y^3_{\nu}+\rho_{\nu,\nu+1}\sin\psi_{\nu+1,\nu}$$

If we observe νth set of coordinates of the νth mass particle in relation to the set coordinates of first mass particle we can write:

$$y^1_{\nu+1}=y^1_1+\sum_{k=1}^{k=\nu}\rho_{k,k+1}\cos\psi_{k+1,k}\cos\varphi_{k+1,k}$$

$$y^2_{\nu+1}=y^2_1+\sum_{k=1}^{k=\nu}\rho_{k,k+1}\cos\psi_{k+1,k}\sin\varphi_{k+1,k} \quad (21)$$

$$y^3_{\nu+1}=y^3_1+\sum_{k=1}^{k=\nu}\rho_{k,k+1}\sin\psi_{k+1,k}$$

$$\ddot{\vec{r}}_{\nu+1} = \ddot{\vec{r}}_{\nu} + \frac{d^2}{dt^2}\left\{\rho_{\nu,\nu+1}\left[\vec{i}\cos\psi_{\nu+1,\nu}\cos\varphi_{\nu+1,\nu} + \vec{j}\cos\psi_{\nu+1,\nu}\sin\varphi_{\nu+1,\nu} + \vec{k}\sin\psi_{\nu+1,\nu}\right]\right\} \quad (22)$$

Now in the pair from the system (15) and by using system (21) we can write:

$$\ddot{\vec{r}}_{\nu} = \frac{1}{m_\nu}\left\{\vec{F}_\nu(t) + \sum_{\mu=1}^{\mu=S}\lambda_\mu grad_\nu f_\mu(\vec{r}_1,...,\vec{r}_N) + \sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+1)k}(t)\frac{\vec{\rho}_{(\nu,\nu+1)k}}{\rho_{(\nu,\nu+1)k}} + \vec{R}_{\nu T}\right\} \quad (15^*)$$

$$\ddot{\vec{r}}_{\nu+1} = \frac{1}{m_{\nu+1}}\left\{\vec{F}_{\nu+1}(t) + \sum_{\mu=1}^{\mu=S}\lambda_\mu grad_{\nu+1} f_\mu(\vec{r}_1,...,\vec{r}_N) + \sum_{k=1}^{k=K_\nu} P_{(\nu+1,\nu+2)k}(t)\frac{\vec{\rho}_{(\nu+1,\nu+2)k}}{\rho_{(\nu+1,\nu+2)k}} + \vec{R}_{\nu+1T}\right\}$$

$$\nu = 1,2,3,...,N$$

We can see that a system can be obtained from which the Descartes coordinate $y_\nu^k =: y^{3\nu-(3-k)}, k = 1,2,3;$ by the use of which absolute positions of mass particles of the system are defined, are eliminated. These equations contain system of generalized coordinates without absolute coordinates of first mass particle. These chosen coordinates are internal coordinates of the system by the use of which the internal relative positions between mass particles of the system are determined.

For example, by using (16) or (16*) and the last relation (22) we can obtain:

$$\ddot{\vec{r}}_{\nu+1} - \ddot{\vec{r}}_{\nu} = \frac{d^2}{dt^2}\left\{\rho_{\nu,\nu+1}\vec{i}\left[\vec{i}\cos\psi_{\nu+1,\nu}\cos\varphi_{\nu+1,\nu} + \vec{j}\cos\psi_{\nu+1,\nu}\sin\varphi_{\nu+1,\nu} + \vec{k}\sin\psi_{\nu+1,\nu}\right]\right\} =$$

$$= \frac{1}{m_{\nu+1}}\vec{F}_{\nu+1}(t) - \frac{1}{m_\nu}\vec{F}_\nu(t) + \sum_{\mu=1}^{\mu=S}\lambda_\mu\left[\frac{1}{m_{\nu+1}}grad_{\nu+1} f_\mu(\vec{r}_1,...,\vec{r}_N) - \frac{1}{m_\nu}grad_\nu f_\mu(\vec{r}_1,...,\vec{r}_N)\right] +$$

$$+ \sum_{k=1}^{k=K_\nu}\left\{\frac{1}{m_{\nu+1}}P_{(\nu+1,\nu+2)k}(t)\frac{\vec{\rho}_{(\nu+1,\nu+2)k}}{\rho_{(\nu+1,\nu+2)k}} - \frac{1}{m_\nu}P_{(\nu,\nu+1)k}(t)\frac{\vec{\rho}_{(\nu,\nu+1)k}}{\rho_{(\nu,\nu+1)k}}\right\} + \frac{1}{m_{\nu+1}}\vec{R}_{\nu+1T} - \frac{1}{m_\nu}\vec{R}_{\nu T} \quad (16^{**})$$

$$\nu = 1,2,3,...,N$$

in which only internal coordinates of the system are contained: mutual distances of mass particles $\rho_{\nu,\nu+1}$,, and circular and meridional angles: $\varphi_{\nu+1,\nu}$ i $\psi_{\nu+1,\nu}$, relative rotation motion $\nu+1$ st material particle around ν-th mass particle. Square of the relative velocity of the relative rotation motion $\nu+1$ st material particle around ν-th mass particle is:

$$v_{\nu+1,\nu}^2 = \dot{\rho}_{\nu,\nu+1}^2 + \left[\rho_{\nu,\nu+1}\dot{\varphi}_{\nu+1,\nu}\cos\psi_{\nu+1,\nu}\right]^2 + \rho_{\nu,\nu+1}^2\dot{\psi}_{\nu,\nu+1}^2 \quad (23)$$

$$\nu = 1,2,3,...,N$$

For the case that we have plane discrete system motion previous relation becomes:

$$y_{\nu+1}^1 = y_1^1 + \sum_{k=1}^{k=\nu}\rho_{k,k+1}\cos\varphi_{k+1,k}$$

$$y_{\nu+1}^2 = y_1^2 + \sum_{k=1}^{k=\nu}\rho_{k,k+1}\sin\varphi_{k+1,k} \quad (24)$$

$$\ddot{\vec{r}}_{\nu+1} - \ddot{\vec{r}}_{\nu} = \frac{d^2}{dt^2}\left\{\rho_{\nu,\nu+1}\left[\vec{i}\cos\varphi_{\nu+1,\nu} + \vec{j}\sin\varphi_{\nu+1,\nu}\right]\right\} \quad (25)$$

$$\nu = 1,2,3,...,N$$

In similar way as in the case of the space system motion the relation became:

$$\ddot{\vec{r}}_{\nu+1} - \ddot{\vec{r}}_{\nu} = \frac{d^2}{dt^2}\left\{\rho_{\nu,\nu+1}\left[\vec{i}\cos\varphi_{\nu+1,\nu} + \vec{j}\sin\varphi_{\nu+1,\nu}\right]\right\} =$$

$$= \frac{1}{m_{\nu+1}}\vec{F}_{\nu+1}(t) - \frac{1}{m_\nu}\vec{F}_\nu(t) + \sum_{\mu=1}^{\mu=S}\lambda_\mu\left[\frac{1}{m_{\nu+1}}grad_{\nu+1} f_\mu(\vec{r}_1,...,\vec{r}_N) - \frac{1}{m_\nu}grad_\nu f_\mu(\vec{r}_1,...,\vec{r}_N)\right] +$$

$$+\sum_{k=1}^{k=K_v}\left\{\frac{1}{m_{v+1}}P_{(v+1,v+2)k}(t)\frac{\vec{\rho}_{(v+1,v+2)k}}{\rho_{(v+1,v+2)k}}-\frac{1}{m_v}P_{(v,v+1)k}(t)\frac{\vec{\rho}_{(v,v+1)k}}{\rho_{(v,v+1)k}}\right\}+\frac{1}{m_{v+1}}\vec{\mathrm{R}}_{v+1\mathrm{T}}-\frac{1}{m_v}\vec{\mathrm{R}}_{v\mathrm{T}} \quad (26)$$

$v=1,2,3,...,N$

For that case we suppose that all active forces are in the same motion plane of the discrete mass particle system.

The previous relations, now, are explicite function only of the internal coordinates: mutual distance $\rho_{v,v+1}$, between two mass particles and circular angles $\varphi_{v+1,v}$ of the relative rotation motion $v+1$st material particle around ν-th mass particle in the motion plane. Square of the relative velocity of the relative rotation motion $v+1$st material particle around ν-th mass particle is:

$$v_{v+1,v}^2=\dot{\rho}_{v,v+1}^2+\left[\rho_{v,v+1}\dot{\varphi}_{v+1,v}\right]^2 \quad , \quad v=1,2,3,...,N \qquad (27)$$

System of N relations (26) is expressed in vectorial forms, and obtains the form of $2N$ scalar relation in which apear $S\le N$ reactions $\mathrm{P}_{v,v+1}(t)$ of the stressed hereditary elements and $(N-2)$ internal system coordinattes: mutual distance between mass particles $\rho_{v,v+1}$, and circular angles $\varphi_{v+1,v}$ of the relative rotation motion $v+1$st material particle around ν-th mass particle in the motion plane. Square of the relative velocity of the relative rotation motion $v+1$st material particle around ν-th mass particle in the motion plane is same as (27).

If we have a dynamicaly determined system it is possible to eliminate reactions $\mathrm{P}_{v,v+1}(t)$ of the stressed hereditary elements and obtain independent system of the internal system coordinates. By these equations we obtain functional relations between internal coordinates: angle rotation motion $\varphi_{v+1,v}$ and distance $\rho_{v,v+1}$, between mass particles.

EXAMPLE 1. For the system of two mass particles (see Figure No.2) constrained by one hereditary element in the space from system of the relations (16**) by elimitation reaction of the stressed hereditary element we can obtain the following relations between internal system coordinate ρ, φ and ψ in the following forms:

$$\frac{1}{\cos\psi\sin\varphi}\left\{\frac{d^2}{dt^2}\{\rho\cos\psi\sin\varphi\}+\frac{1}{m_1}F_{01y}\cos\Omega t\right\}=\frac{1}{\cos\psi\cos\varphi}\left\{\frac{d^2}{dt^2}\{\rho\cos\psi\cos\varphi\}+\frac{1}{m_1}F_{01x}\cos\Omega t\right\}$$

(b)

$$\frac{1}{\cos\psi\sin\varphi}\left\{\frac{d^2}{dt^2}\{\rho\cos\psi\sin\varphi\}+\frac{1}{m_1}F_{01y}\cos\Omega t\right\}=\frac{1}{\sin\psi}\left\{\frac{d^2}{dt^2}\{\rho\sin\psi\}+\frac{1}{m_1}F_{01z}\cos\Omega t\right\}$$

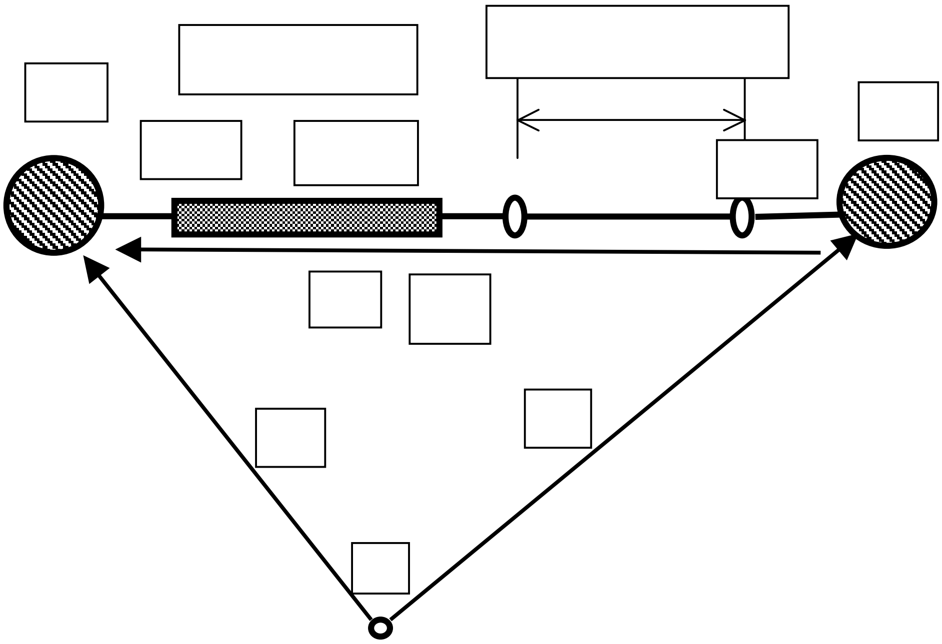
from which we obtain:

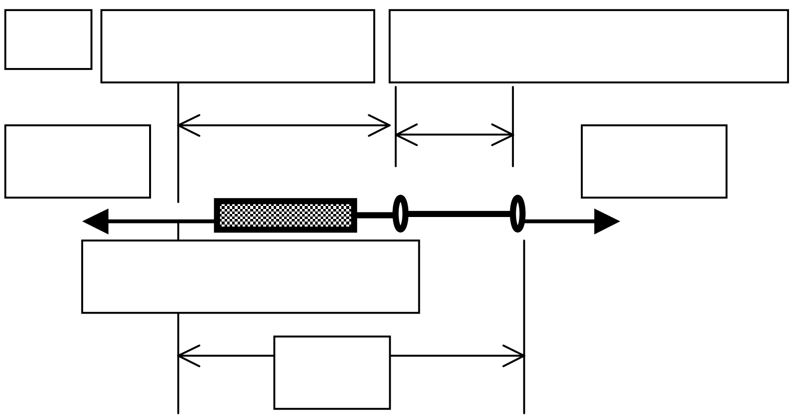
$$[\rho(t)\ddot{\varphi}(t)+2\dot{\rho}(t)\dot{\varphi}(t)]\cos\psi-2\rho(t)\dot{\varphi}(t)\dot{\psi}(t)\sin\psi=\frac{1}{m_1}\left[F_{01x}\sin\varphi-F_{01y}\cos\varphi\right]\cos\Omega t$$

(c)

$$[\rho(t)\ddot{\psi}(t)+2\dot{\rho}(t)\dot{\psi}(t)]+\rho(t)[\dot{\varphi}(t)]^2\cos\psi\sin\psi=\frac{1}{m_1\sin\varphi}\left[F_{01z}\cos\psi\sin\varphi-F_{01y}\sin\psi\right]\cos\Omega t$$

The last system equations yeld relation between angles velocities components $\dot{\psi}$ meridional and $\dot{\varphi}$ circular of the relative rotation motion of the second material particle around the first mass particle in the space of the dynamic of this discrete mass particles system. We can see that these angle velocity components of the relative rotation motion of the second material particle around first mass particle in the space are coupled as functions of distance ρ between these mass particles.





Square of relative motion velocity of the relative rotation motion second material particle around first mass particle in the space is:

$$v_r^2 = \dot{\rho}^2 + [\rho\dot{\varphi}\cos\psi]^2 + \rho^2\dot{\psi}^2 \tag{e}$$

By introducing expression (e) of the square relative velocity motion into expression (18) for the reological reaction $\mathrm{P}_{1,2}(t) = P(t)$ as a force of the internal influence between mass particles, we can obtain the following expression:

$$P(t) = -\frac{m_1 m_2}{m_1 + m_2}\left\{\frac{\dot{\vec{\rho}}^2 + (\vec{\rho}, \ddot{\vec{\rho}}) - \vec{v}_r^2}{\rho} - \frac{1}{m_1\rho}(\vec{\rho}, \vec{F}_1)\right\} = -\frac{m_1 m_2}{m_1 + m_2}\left\{\ddot{\rho} - \rho\dot{\varphi}^2\cos^2\psi - \rho\dot{\psi}^2 - \frac{1}{m_1\rho}(\vec{\rho}, \vec{F}_1)\right\} \tag{f}$$

By introducing previous expression (f) of the mutual influence force into integral relation (2) rheological-hereditary relation for the case of the forced motion of the system in space rheological relation becomes the following integro-diferential form:

$$\rho(t) - \rho_0 - \int_0^t \mathrm{R}(t-\tau)[\rho(\tau) - \rho_0]d\tau + \frac{m_1 m_2}{c(m_1 + m_2)}\left\{\ddot{\rho}(t) - \rho(t)\dot{\varphi}^2\cos^2\psi - \rho(t)\dot{\psi}^2\right\} =$$
$$= \frac{m_2}{(m_1 + m_2)}\frac{1}{c}\left[F_{o1x}\cos\psi\cos\varphi + F_{01y}\cos\psi\sin\varphi + F_{01z}\sin\psi\right]\sin\Omega t \tag{g}$$

System of coupled differenttial and integrodiferential equations can be transformed into the following system equation the first kind:

$$\frac{d\rho(t)}{dt} = u(t) \qquad \frac{d\varphi(t)}{dt} = v(t) \qquad \frac{d\psi(t)}{dt} = w(t)$$

$$\frac{du(t)}{dt} = \frac{c(m_1 + m_2)}{m_1 m_2}\left\{-\rho(t) + \rho_0 + \int_0^t \mathrm{R}(t-\tau)[\rho(\tau) - \rho_0]d\tau\right\} + c\left\{\rho(t)[v(t)]^2\cos^2\psi(t) + \rho(t)[w(t)]^2\right\} +$$
$$+ \frac{1}{m_1}\left[F_{o1x}\cos\psi(t)\cos\varphi(t) + F_{01y}\cos\psi(t)\sin\varphi(t) + F_{01z}\sin\psi(t)\right]\sin\Omega t$$

$$\frac{dv(t)}{dt} = \frac{1}{\rho(t)\cos\psi(t)}\left\{2\rho(t)v(t)w(t)\sin\psi(t) - 2u(t)v(t)\cos\psi(t) + \frac{1}{m_1}\left[F_{01x}\sin\varphi(t) - F_{01y}\cos\varphi(t)\right]\cos\Omega t\right\} \tag{l}$$

$$\frac{dw(t)}{dt} = \frac{1}{\rho(t)}\left\{-2u(t)w(t) - \rho(t)[v(t)]^2\cos\psi(t)\sin\psi(t) + \frac{1}{m_1\sin\varphi(t)}\left[F_{01z}\cos\psi(t)\sin\varphi(t) - F_{01y}\sin\psi(t)\right]\cos\Omega t\right\}$$

EXAMPLE 2. For the system of a mass material particles couple by one hereditary element in a plane (see Figure No. 3), by using relation (16**) and by eliminating the reaction of the hereditary element we can obtain the following relation between internal system coordinates ρ and φ:

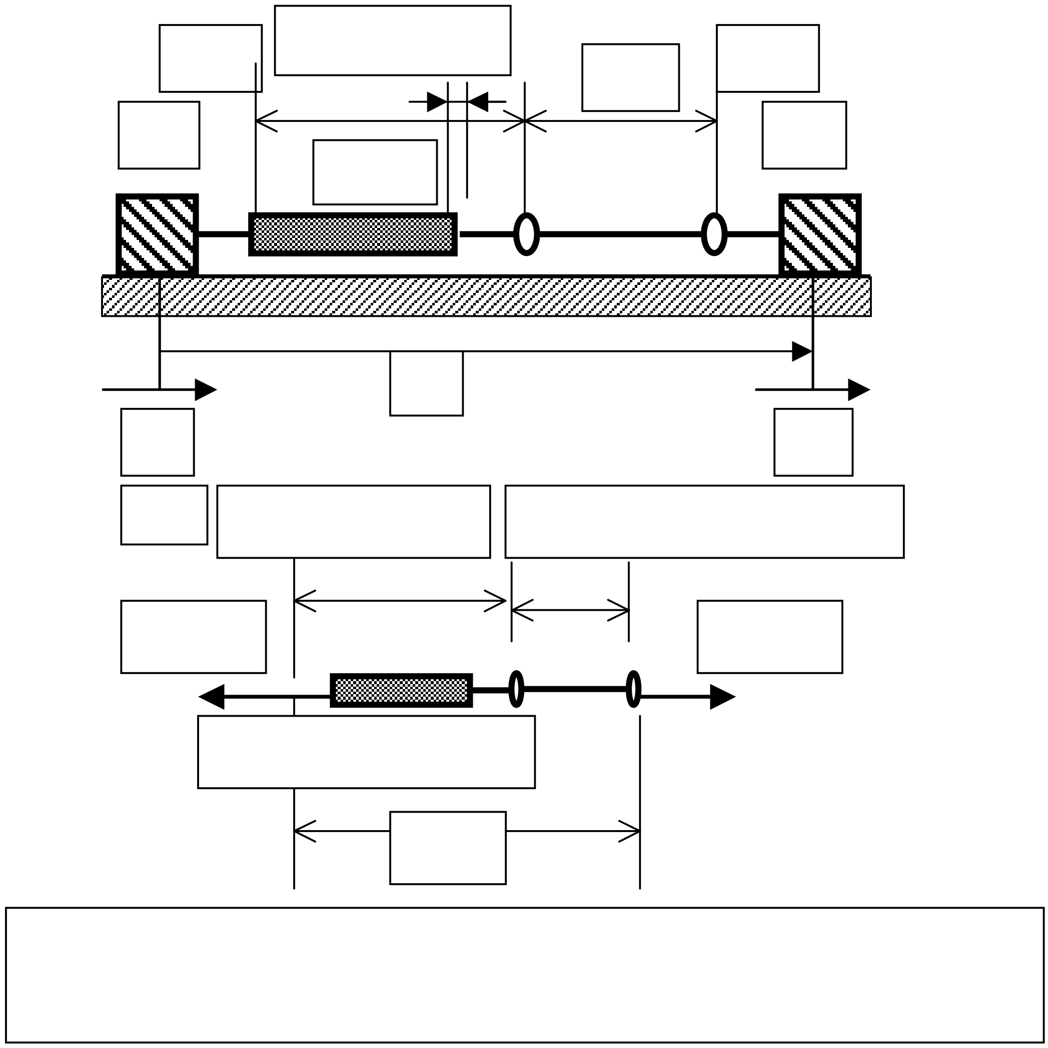
$$\cos\varphi\frac{d^2}{dt^2}\{\rho\sin\varphi\} = \sin\varphi\frac{d^2}{dt^2}\{\rho\cos\varphi\} - \frac{m_2}{m_1}F_{01}\sin\beta\cos\Omega_1 t \tag{a}$$

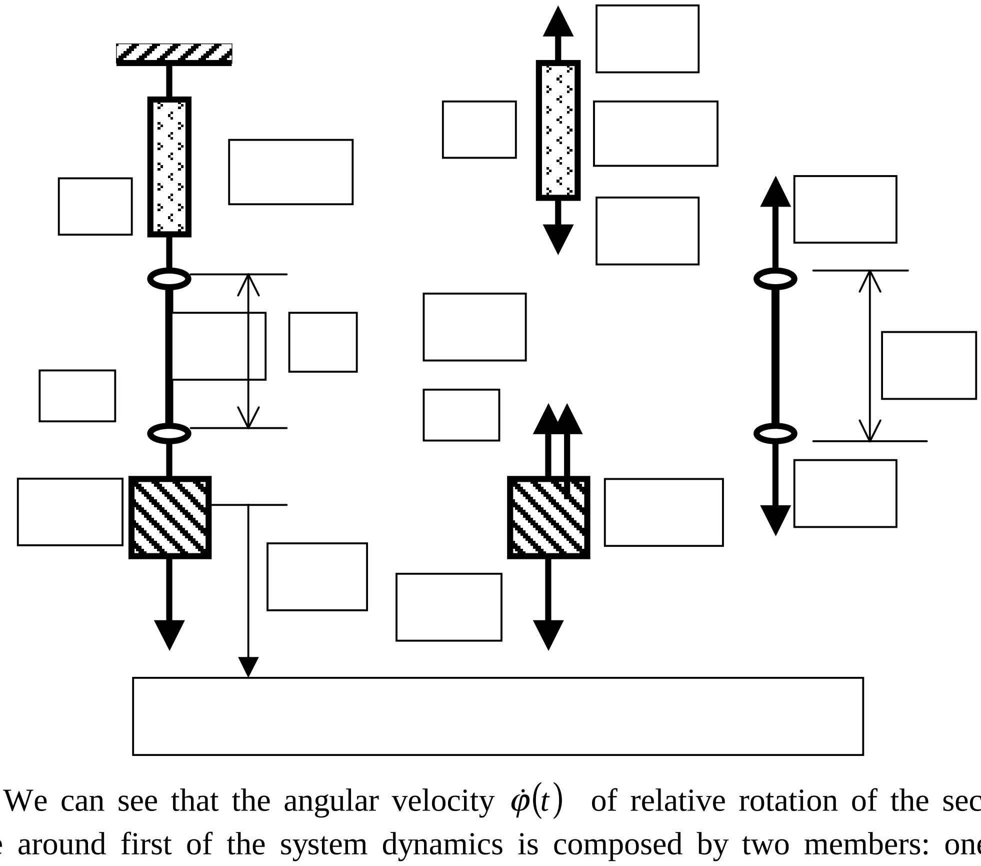
$$[2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}] = -\frac{m_2}{m_1}F_{01}\sin\beta\cos\Omega_1 t \tag{b}$$

Solution of this previous equation (b) , which has the form of differential equation of the first kind of $\dot{\varphi}(t)$ depending of the internal coordinate $\rho(t)$, takes following form:

$$\dot{\varphi}(t) = \dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^2} - \frac{m_2}{m_1}F_{01}\sin\beta\left[\frac{1}{\rho(t)}\int_0^t \rho(\tau)\cos\Omega_1\tau d\tau\right] \tag{c}$$

Last differential equation (c) gives a relation between angular velocity $\dot{\varphi}(t)$ of the relative rotation second material particle around first of the system dynamics of these mass particles and internal coordinate $\rho(t)$-distance between mass particles, in the motion plane.





We can see that the angular velocity $\dot{\varphi}(t)$ of relative rotation of the second material particle around first of the system dynamics is composed by two members: one is opposite proportional to the square of the distance ρ , and second member is in the integral form and is dependent on the external excitation force. First componet of velocity that corresponds to proper free motion, arises as the result of the initial conditions perturbation of the equilibrium position.

The square relative velocity of the relative rotation of the second mass particle around first is:

$$v_r^2 = \dot{\rho}^2 + \rho^2\omega^2 = [\dot{\rho}(t)]^2 + [\rho(t)]^2 \left\{ \dot{\varphi}_0 \frac{\rho_0^2}{[\rho(t)]^2} - \frac{m_2}{m_1} F_{01} \sin\beta \left[\frac{1}{\rho(t)} \int_0^t \rho(\tau)\cos\Omega_1\tau d\tau \right] \right\}^2 \quad \text{(d)}$$

By introducind expression (d) of the square relative velocity of the relative rotation of the second mass particle around first into expression (18) for the force $\mathrm{P}_{1,2}(t) = P(t)$ of the mutual influence between mass particles, and after this result has been introduced in relation (2) of the rheological-hereditary relation for the case of the forced system motion in the plane, and the rheological relation becomes:

$$\rho(t) - \rho_0 - \int_0^t \mathrm{R}(t-\tau)[\rho(\tau) - \rho_0] d\tau = -\frac{m_1 m_2}{c(m_1+m_2)} \left\{ \ddot{\rho}(t) - [\rho(t)] \left\{ \dot{\varphi}_0 \frac{\rho_0^2}{[\rho(t)]^2} - \frac{m_2}{m_1} F_{01} \sin\beta \left[\frac{1}{\rho(t)} \int_0^t \rho(\tau)\cos\Omega_1\tau d\tau \right] \right\}^2 \right\} \quad \text{(e)}$$

This rheological relation (e) is integro-differntial equation from which we can determine relative distance $\rho(t)$ between mass particles as a time function and as a soluttion of this equation. After solving previous equation we have main part of defined problem solution.

EXAMPLE 3. For the case of the three mass particles connected by the three hereditary elements in the plane, for the relations between internal system coordinate we obtain the following:

$$\begin{aligned}
&\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\left\{\sin(\varphi_{13} - \varphi_{23})\left[\sin\varphi_{12} \frac{d^2}{dt^2}(\rho_{12}\cos\varphi_{12}) - \cos\varphi_{12} \frac{d^2}{dt^2}(\rho_{12}\sin\varphi_{12})\right]\right\} + \\
&+\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\left\{\sin(\varphi_{12} - \varphi_{23})\left[\sin\varphi_{13} \frac{d^2}{dt^2}(\rho_{13}\cos\varphi_{13}) - \cos\varphi_{13} \frac{d^2}{dt^2}(\rho_{13}\sin\varphi_{13})\right]\right\} = \\
&= \frac{1}{m_1}\left\{\sin(\varphi_{13} - \varphi_{23})\left[\sin\varphi_{12} \frac{d^2}{dt^2}(\rho_{13}\cos\varphi_{13}) - \cos\varphi_{12} \frac{d^2}{dt^2}(\rho_{13}\sin\varphi_{13})\right]\right\} + \\
&+ \frac{1}{m_1}\left\{\sin(\varphi_{12} - \varphi_{23})\left[\sin\varphi_{13} \frac{d^2}{dt^2}(\rho_{12}\cos\varphi_{12}) - \cos\varphi_{13} \frac{d^2}{dt^2}(\rho_{12}\sin\varphi_{12})\right]\right\}
\end{aligned} \quad \text{(f)}$$

$$\begin{aligned}
&\left(\frac{1}{m_1} + \frac{1}{m_3}\right)\left\{\sin(\varphi_{12} - \varphi_{23})\left[\sin\varphi_{13} \frac{d^2}{dt^2}(\rho_{13}\cos\varphi_{13}) - \cos\varphi_{13} \frac{d^2}{dt^2}(\rho_{13}\sin\varphi_{13})\right]\right\} + \\
&+\left(\frac{1}{m_1} + \frac{1}{m_3}\right)\left\{\sin(\varphi_{13} - \varphi_{23})\left[\sin\varphi_{12} \frac{d^2}{dt^2}(\rho_{12}\cos\varphi_{12}) - \cos\varphi_{12} \frac{d^2}{dt^2}(\rho_{12}\sin\varphi_{12})\right]\right\} = \\
&= \frac{1}{m_1}\left\{\sin(\varphi_{12} - \varphi_{23})\left[\sin\varphi_{13} \frac{d^2}{dt^2}(\rho_{12}\cos\varphi_{12}) - \cos\varphi_{13} \frac{d^2}{dt^2}(\rho_{12}\sin\varphi_{12})\right]\right\} + \\
&+ \frac{1}{m_1}\left\{\sin(\varphi_{13} - \varphi_{23})\left[\sin\varphi_{12} \frac{d^2}{dt^2}(\rho_{13}\cos\varphi_{13}) - \cos\varphi_{12} \frac{d^2}{dt^2}(\rho_{13}\sin\varphi_{13})\right]\right\}
\end{aligned} \quad \text{(g)}$$

The third relation is similar to the previous, for permuted index. The following relations is obtained as well as:

$$tg\varphi_{23} = \frac{\rho_{12}\sin\varphi_{12} - \rho_{13}\sin\varphi_{13}}{\rho_{13}\cos\varphi_{13} - \rho_{12}\cos\varphi_{12}} \qquad \rho_{23}^2 = \rho_{12}^2 + \rho_{13}^2 - 2\rho_{12}\rho_{13}\cos(\varphi_{12} - \varphi_{13}) \quad \text{(f)}$$

These equations must be solved as a system of coupled equations expressed by internal coordinates and must be solved together with corresponding of (2) (or (4) or (6)) and (18). This is a system with complete equations with coresponding number tto the unknown coordinates.

III. COVARIANT DIFFERENTIAL EQUATIONS OF THE MOTION OF THE DISCRETE HEREDITARY SYSTEM

By using principle of the work on the virtual displacemet we can write (see Ref. [5]):

$$\sum_{\nu=1}^{\nu=N}\left\{\vec{\mathrm{I}}_\nu + \vec{\mathrm{F}}_\nu + \vec{\mathrm{R}}_\nu + \vec{\mathrm{P}}_\nu + \vec{\mathrm{R}}_{\nu\mathrm{T}}\right\}\delta\vec{r}_\nu = 0 \qquad (13^*)$$

or:

$$\sum_{\nu=1}^{\nu=N}\left\{m_\nu\ddot{\vec{r}}_\nu - \vec{\mathrm{F}}_\nu(t) - \sum_{\mu=1}^{\mu=S}\lambda_\mu\, grad_\nu\, f_\mu(\vec{r}_1,\ldots,\vec{r}_N) - \sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|} - \vec{\mathrm{R}}_{\nu\mathrm{T}}\right\}\delta\vec{r}_\nu = 0 \quad (13)$$

Now, the virtual displacement can be expressed by using generalized coordinates in the form: $\delta\vec{r}_\nu = \sum_{\alpha=1}^{\alpha=n}\frac{\partial\vec{r}_\nu}{\partial q^\alpha}\delta q^\alpha$ and introduced into the previous equation (13) for the work of the active and reactive forces on the virtual displacements, and we obtaine the following:

$$\sum_{\nu=1}^{\nu=N}\left\{m_\nu\ddot{\vec{r}}_\nu - \vec{\mathrm{F}}_\nu(t) - \sum_{\mu=1}^{\mu=S}\lambda_\mu\, grad_\nu\, f_\mu(\vec{r}_1,\ldots,\vec{r}_N) - \sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|} - \vec{\mathrm{R}}_{\nu\mathrm{T}}\right\}\sum_{\alpha=1}^{\alpha=n}\frac{\partial\vec{r}_\nu}{\partial q^\alpha}\delta q^\alpha = 0 \qquad (27)$$

Now, by changing order of sumarizing we can obtain:

$$\sum_{\alpha=1}^{\alpha=n}\delta q^\alpha\sum_{\nu=1}^{\nu=N}\left\{m_\nu\left(\ddot{\vec{r}}_\nu, \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) - \left(\vec{\mathrm{F}}_\nu(t), \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) - \sum_{\mu=1}^{\mu=S}\lambda_\mu\left(grad_\nu\, f_\mu(\vec{r}_1,\ldots,\vec{r}_N), \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) - \sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+1)k}(t)\frac{\left(\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}, \frac{\partial\vec{r}_{(\nu)k}}{\partial q^\alpha}\right)}{\left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|} - \left(\vec{\mathrm{R}}_{\nu\mathrm{T}}, \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right)\right\} = 0 \qquad (27^*)$$

By analysing the member from previous expression we have the following fictive, active and reactive forces:

$$\mathrm{I}_\alpha = -\sum_{\nu=1}^{\nu=N} m_\nu\left(\ddot{\vec{r}}_\nu, \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) = -\sum_{\nu=1}^{\nu=N} m_\nu\left(\frac{d\vec{v}_\nu}{dt}, \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) = -\sum_{\nu=1}^{\nu=N} m_\nu\left(\frac{d}{dt}\sum_{\beta=0}^{\beta=n}\frac{\partial\vec{r}_\nu}{\partial q^\beta}\dot{q}^\beta, \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) =$$

$$= -\sum_{\nu=1}^{\nu=N} m_\nu\left(\sum_{\gamma=1}^{\gamma=n}\sum_{\beta=1}^{\beta=n}\frac{\partial^2\vec{r}_\nu}{\partial q^\beta\partial q^\gamma}\dot{q}^\beta\dot{q}^\gamma + \sum_{\beta=1}^{\beta=n}\frac{\partial\vec{r}_\nu}{\partial q^\beta}\ddot{q}^\beta, \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) = -\left[a_{\alpha\beta}\left(\ddot{q}^\beta + \Gamma^\alpha_{\gamma\delta}\dot{q}^\gamma\dot{q}^\delta\right)\right] = -a_{\alpha\beta}\frac{D\dot{q}^\beta}{dt}$$

$$\alpha = 1,2,3,\ldots,n; \quad n = 3N - S \qquad (28)$$

$$\mathrm{Q}_\alpha = \sum_{\nu=1}^{\nu=N}\left(\vec{\mathrm{F}}_\nu(t), \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) \qquad (29)$$

$$\mathrm{Q}_\alpha^f = \sum_{\nu=1}^{\nu=N}\sum_{\mu=1}^{\mu=S}\lambda_\mu\left(grad_\nu\, f_\mu(\vec{r}_1,\ldots,\vec{r}_N), \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) = 0 \qquad (30)$$

$$\mathrm{P}_\alpha = \sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+1)k}(t)\frac{\left(\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}, \frac{\partial\vec{r}_{(\nu)k}}{\partial q^\alpha}\right)}{\left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|} = \sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+1)k}(t)\frac{\left(\vec{\rho}_{(\nu,\nu+1)k}, \frac{\partial\vec{r}_{(\nu)k}}{\partial q^\alpha}\right)}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|} \qquad (31)$$

$$\mathrm{Q}_{\alpha}^{*} = \sum_{\nu=1}^{\nu=N}\left(\vec{\mathrm{R}}_{\nu T}(t), \frac{\partial \vec{r}_{\nu}}{\partial q^{\alpha}}\right) \tag{32}$$

$$\sum_{\alpha=1}^{\alpha=n}\left[\mathrm{I}_{\alpha} + \mathrm{Q}_{\alpha} + \mathrm{P}_{\alpha} + \mathrm{Q}_{\alpha}^{*}\right]\delta q^{\alpha} = 0 \tag{33}$$

Dynamic equations system in the covariant coordinates can written in following form:

$$\mathrm{I}_{\alpha} + \mathrm{Q}_{\alpha} + \mathrm{P}_{\alpha} + \mathrm{Q}_{\alpha}^{*} = 0 \qquad \alpha = 1,2,3,\dots,n; \quad n = 3N - S \tag{34}$$

or:

$$a_{\alpha\beta}\frac{D\dot{q}^{\beta}}{dt} = \mathrm{Q}_{\alpha} + \mathrm{Q}_{\alpha}^{*} + \mathrm{P}_{\alpha} \qquad \alpha = 1,2,3,\dots,n; \quad n = 3N - S \tag{34*}$$

IV. THE RHEONOMIC COORDINATE METHOD APPLIED TO DISCRETE HEREDITARY SYSTEMS.

MODIFIED SYSTEM OF THE COVATIANT DIFFERENTIAL EQUATIONS OF MOTION OF A DISCRETE HEREDITARY SYSTEM WITH RHEONOMIC CONSTRAINTS

Lets cosider K, $K = \sum_{\nu=1}^{\nu=N} K_{\nu}$ standard hereditary elements of neglected mass and rheological properties defined by material parameters: $n_{(\nu,\nu+1)k}$, $k = 1,2,3,\dots,K_{\nu}$ times of relaxations; coefficients of rigidity $c_{(\nu,\nu+1)k}$ and $\tilde{c}_{(\nu,\nu+1)k}$ are an instanteneous rigidity and a prolonged one. Relations between reaction and deformation of the stressed and strained hereditary element in the discrete system can be expressed by the following different forms:

* in the relaxational forms by using integral relation:

$$P_{(\nu,\nu+1)k} = c_{(\nu,\nu+1)k}\left[\bar{\rho}_{(\nu,\nu+1)k}(t) - \int_{0}^{t} \mathrm{R}_{(\nu,\nu+1)k}(t-\tau)\bar{\rho}_{(\nu,\nu+1)k}(\tau)d\tau\right], \tag{35}$$

$$\nu = 1,2,3,\dots,N,\ k = 1,2,3,\dots,K_{\nu}$$

where are:

$$\boldsymbol{R}_{(\nu,\nu+1)k}(t-\tau) = \frac{c_{(\nu,\nu+1)k} - \tilde{c}_{(\nu,\nu+1)k}}{n_{(\nu,\nu+1)k}c_{(\nu,\nu+1)k}} e^{-\frac{t-\tau}{n_{(\nu,\nu+1)k}}}, \quad \nu = 1,2,3,\dots,N,\ k = 1,2,3,\dots,K_{\nu} \tag{36}$$

a kernel of relaxation, and

$$\rho_{(\nu,\nu+1)k} = \left|\vec{\rho}_{(\nu,\nu+1)k}\right| = \left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right| \quad \bar{\rho}_{(\nu,\nu+1)k} = \left|\vec{\rho}_{(\nu,\nu+1)k}\right| - \rho_{(\nu,\nu+1)k0} = \left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right| - \rho_{(\nu,\nu+1)k0}. \tag{a}$$

$\rho_{(\nu,\nu+1)0}$ is the natural state length of the hereditary element without stress and strain, and without history of stresed and strained states.

If now, between of the hereditary elements and one of the two material particles of the end of hereditary element we put a rheonomic constarint in the form of the exactly defined length segment as a function of time in the form $\ell_{\nu,\nu+1}(t) = a_{\nu,\nu+1}(\Omega t) = a_{\nu,\nu+1}(q^{0})$ we defined a hereditary discrete system with rheonomic constraints. In this case we made a new hereditary element with rheonomic modifications. Hereditary element and rheonomic element are connected in series.

This rheonomic modification we can introduce into hereditary element simple in paralel, or serial conection, as well as in other ways, as an element introduced into the complecs system of the conected hereditary elements.

For that reason , into reserch of the system defined in that way, we introduce a time function as a rheonomic coordinate in the sence of V. Vuji~i}. As a rheonomic coordinate we can choose for example the following: $q^0 = \Omega t$.

Follows the idea of V. Vuji~i} (see Ref. [6], [7], for the description of system dynamics we choose the generalized coordinate, of which number $n = 3N - S$ is a diference between number of $3N$ positions coordinates and number S of the finite constraints, and we join a rheonomic coordinate q^0 as a more them $n = 3N - S$. This rheonomic coordinate q^0 must be chosen depending on functions introduced by rheonomic segment of lenght, as it is possible to see on the figure.

In this case we can choose as a rheonomic coordinate q^0 : time, or lenght, or for example $q^0 = \Omega t$ where Ω is parameter as a frequency. This rheonomic coordinate can be chosen as dimensionless, or lenth, or an angle in radinas. This depends on the concrete introduced rheonomic constraints into hereditary system. Next research is for different kinds of rheonomic coordinate and we have not defined the type of chosen rheonomic coordinate.

In acordance with introduced rheonomic lenth in series conected with hereditary element we must correct some relations for the stress-strain state of the hereditary element. For that reason we composed following relations:

$$\rho_{\nu,\nu+1} = \left|\vec{\rho}_{\nu,\nu+1}\right| = \left|\vec{r}_{\nu+1} - \vec{r}_\nu\right| \quad \text{i}$$

$$\overline{\rho}_{\nu,\nu+1} = \left|\vec{\rho}_{\nu,\nu+1}\right| - \rho_{(\nu,\nu+1)0} - \ell_{\nu,\nu+1}(\Omega t) = \left|\vec{r}_{\nu+1} - \vec{r}_\nu\right| - \rho_{(\nu,\nu+1)0} - a_{\nu,\nu+1}\left(q^0\right). \qquad \text{(a*)}$$

For the compositon of dynamical equations of the discrete hereditary system with rheonomic constraints -lenght as it is defined, we can use the equation of virtual work in the form:

$$\sum_{\nu=1}^{\nu=N}\left\{\vec{\mathrm{I}}_\nu + \vec{\mathrm{F}}_\nu + \vec{\mathrm{R}}_\nu + \vec{\mathrm{P}}_\nu + \vec{\mathrm{R}}_{\nu\mathrm{T}}\right\}\delta\vec{r}_\nu = 0 \qquad (13^*)$$

or:

$$\sum_{\nu=1}^{\nu=N}\left\{m_\nu\ddot{\vec{r}}_\nu - \vec{\mathrm{F}}_\nu(t) - \sum_{\mu=1}^{\mu=S}\lambda_\mu grad_\nu\, f_\mu(\vec{r}_1,...,\vec{r}_N) - \sum_{k=1}^{k=K_\nu}P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|} - \vec{\mathrm{R}}_{\nu\mathrm{T}}\right\}\delta\vec{r}_\nu = 0 \quad (13)$$

Now, the virtual displacement must be expressed by n generalized coordinates with joined rheonomic coordinate q^0, as a system of $n+1$ coordinates $q^\alpha, \alpha = 0,1,2,3,....,n$, $n = 3N - S$.

Now we have a modified - extended system of $n+1$ coordinates, and virtual displacement is: $\delta\vec{r}_\nu = \sum_{\alpha=0}^{\alpha=n}\frac{\partial\vec{r}_\nu}{\partial q^\alpha}\delta q^\alpha$. By introduce this expression into initial vector equation (13), this equation takes the following form:

$$\sum_{\nu=1}^{\nu=N}\left\{m_\nu\ddot{\vec{r}}_\nu - \vec{\mathrm{F}}_\nu(t) - \sum_{\mu=1}^{\mu=S}\lambda_\mu grad_\nu\, f_\mu(\vec{r}_1,...,\vec{r}_N,t) - \sum_{k=1}^{k=K_\nu}P_{(\nu,\nu+1)k}(t)\frac{\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|} - \vec{\mathrm{R}}_{\nu\mathrm{T}}\right\}\sum_{\alpha=0}^{\alpha=n}\frac{\partial\vec{r}_\nu}{\partial q^\alpha}\delta q^\alpha = 0 \qquad (38)$$

Now by changing the order of sumarizing we obtain the following:

$$\sum_{\alpha=0}^{\alpha=n}\delta q^{\alpha}\sum_{\nu=1}^{\nu=N}\left\{m_{\nu}\left(\ddot{\vec{r}}_{\nu},\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right)-\left(\vec{\mathrm{F}}_{\nu}(t),\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right)-\sum_{\mu=1}^{\mu=S}\lambda_{\mu}\left(grad_{\nu}\, f_{\mu}(\vec{r}_1,\ldots,\vec{r}_N,t),\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right)-\sum_{k=1}^{k=K_{\nu}}P_{(\nu,\nu+1)k}(t)\frac{\left(\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k},\frac{\partial\vec{r}_{(\nu)k}}{\partial q^{\alpha}}\right)}{\left|\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k}\right|}-\left(\vec{\mathrm{R}}_{\nu\mathrm{T}},\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right)\right\}=0$$

(39)

For the generalised active and reactive forces in the extended forms we must vrite:

$$\mathrm{I}_{\alpha}=-\sum_{\nu=1}^{\nu=N}m_{\nu}\left(\ddot{\vec{r}}_{\nu},\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right)=-\sum_{\nu=1}^{\nu=N}m_{\nu}\left(\frac{d\vec{v}_{\nu}}{dt},\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right)=-\sum_{\nu=1}^{\nu=N}m_{\nu}\left(\frac{d}{dt}\sum_{\beta=0}^{\beta=n}\frac{\partial\vec{r}_{\nu}}{\partial q^{\beta}}\dot{q}^{\beta},\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right)=$$

$$=-\sum_{\nu=1}^{\nu=N}m_{\nu}\left(\sum_{\gamma=0}^{\gamma=n}\sum_{\beta=0}^{\beta=n}\frac{\partial^2\vec{r}_{\nu}}{\partial q^{\beta}\partial q^{\gamma}}\dot{q}^{\beta}\dot{q}^{\gamma}+\sum_{\beta=0}^{\beta=n}\frac{\partial\vec{r}_{\nu}}{\partial q^{\beta}}\ddot{q}^{\beta},\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right)=-\left[a_{\alpha\beta}\left(\ddot{q}^{\beta}+\Gamma^{\alpha}_{\gamma\delta}\dot{q}^{\gamma}\dot{q}^{\delta}\right)\right]=-a_{\alpha\beta}\frac{D\dot{q}^{\beta}}{dt}$$

$$\alpha=0,1,2,3,\ldots,n;\quad n=3N-S \qquad (40)$$

$$\mathrm{Q}_{\alpha}=\sum_{\nu=1}^{\nu=N}\left(\vec{\mathrm{F}}_{\nu}(t),\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right) \qquad (41)$$

$$\mathrm{Q}^{f}_{\alpha}=\sum_{\nu=1}^{\nu=N}\sum_{\mu=1}^{\mu=S}\lambda_{\mu}\left(grad_{\nu}\, f_{\mu}(\vec{r}_1,\ldots,\vec{r}_N,t),\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right) \qquad (42)$$

$$\mathrm{P}_{\alpha}=\sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_{\nu}}P_{(\nu,\nu+1)k}(t)\frac{\left(\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k},\frac{\partial\vec{r}_{(\nu)k}}{\partial q^{\alpha}}\right)}{\left|\vec{r}_{(\nu+1)k}-\vec{r}_{(\nu)k}\right|}=\sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_{\nu}}P_{(\nu,\nu+1)k}(t)\frac{\left(\vec{\rho}_{(\nu,\nu+1)k},\frac{\partial\vec{r}_{(\nu)k}}{\partial q^{\alpha}}\right)}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|} \qquad (43)$$

$$\mathrm{Q}^{*}_{\alpha}=\sum_{\nu=1}^{\nu=N}\left(\vec{\mathrm{R}}_{\nu T}(t),\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right) \qquad (44)$$

$$\sum_{\alpha=0}^{\alpha=n}\left[\mathrm{I}_{\alpha}+\mathrm{Q}_{\alpha}+\mathrm{Q}^{f}_{\varepsilon}+\mathrm{P}_{\alpha}+\mathrm{Q}^{*}_{\alpha}\right]\delta q^{\alpha}=0 \qquad (44)$$

Equations of system dynamics into covarint coordintes are written in the following form:

$$\mathrm{I}_{\alpha}+\mathrm{Q}_{\alpha}+\mathrm{Q}^{f}_{\varepsilon}+\mathrm{P}_{\alpha}+\mathrm{Q}^{*}_{\alpha}=0 \qquad \alpha=0,1,2,3,\ldots,n; n=3N-S \qquad (45)$$

or,

$$a_{\alpha\beta}\frac{D\dot{q}^{\beta}}{dt}=\mathrm{Q}_{\alpha}+\mathrm{Q}^{f}_{\alpha}+\mathrm{Q}^{*}_{\alpha}+\mathrm{P}_{\alpha} \qquad \alpha=0,1,2,3,\ldots,n; n=3N-S \qquad (45^*)$$

The generalized inertia force is

$$\mathrm{I}_{\alpha}=-\sum_{\nu=1}^{\nu=N}m_{\nu}\left(\ddot{\vec{r}}_{\nu},\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\right)=-\sum_{\nu=1}^{\nu=N}m_{\nu}\left(\frac{d\vec{v}_{\nu}}{dt},\frac{\partial\dot{\vec{r}}_{\nu}}{\partial\dot{q}^{\alpha}}\right)=-\left\{\frac{d}{dt}\frac{\partial E_k}{\partial\dot{q}^{\alpha}}-\frac{\partial E_k}{\partial q^{\alpha}}\right\} \qquad (46)$$

and kinetic energy is

$$E_k=\frac{1}{2}\sum_{\nu=1}^{\nu=N}m_{\nu}\vec{v}_{\nu}^{\,2}=\frac{1}{2}\sum_{\nu=1}^{\nu=N}m_{\nu}\left[\sum_{\alpha=0}^{\alpha=n}\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}}\dot{q}^{\alpha}\right]^2=\frac{1}{2}\sum_{\alpha=0}^{\alpha=n}\sum_{\beta=0}^{\beta=n}\dot{q}^{\alpha}\dot{q}^{\beta}\sum_{\nu=1}^{\nu=N}m_{\nu}\left(\frac{\partial\vec{r}_{\nu}}{\partial q^{\alpha}},\frac{\partial\vec{r}_{\nu}}{\partial q^{\beta}}\right) \qquad (47)$$

The basic initial equation of the motion is in the form:

$$\sum_{\alpha=0}^{\alpha=n}\left[\frac{d}{dt}\frac{\partial E_k}{\partial\dot{q}^{\alpha}}-\frac{\partial E_k}{\partial q^{\alpha}}-\mathrm{Q}_{\alpha}-\mathrm{Q}^{f}_{\varepsilon}-\mathrm{P}_{\alpha}-\mathrm{Q}^{*}_{\alpha}\right]\delta q^{\alpha}=0 \qquad (48)$$

From last equation we can write extended and modified system of Lagrange's differential equations first kind with $n+1$ equations. These equation contain generalized rheological reaction of hereditary elements, rheonomic constraints reactions, and other usually active and reactive forces.

By using extended coordinate system which consists of the $\vec{n}$ generalized coordinates extended with rheonomic coordinate, the extended and modified system of Lagrange's differential equations first kind with $n+1$ equations. must be named as a Lagrage-Vuji~i}-Goroshko system of differential equations for the discrete hereditary system with rheonomic constraints. These equations are in the form:

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^\alpha} - \frac{\partial E_k}{\partial q^\alpha} - \mathrm{Q}_\alpha - \mathrm{Q}_\varepsilon^f - \mathrm{P}_\alpha - \mathrm{Q}_\alpha^* = 0 \ , \quad \alpha = 0,1,2,3,\dots,n; \quad n = 3N - S \qquad (49)$$

Now, we are focused on the analysis of generalized forces work. Generalized reactions work of the rheonomic constraints (42) is nor equal to zero, as in the case of the scleronomic finite constraints. By using velocity conditions we can write:

$$\sum_{\nu=1}^{\nu=N}\left(grad_\nu\, f_\mu(\vec{r}_1,\vec{r}_2,....,\vec{r}_N,t), \vec{v}_\nu\right) + \frac{\partial f_\mu(\vec{r}_1,\vec{r}_2,....,\vec{r}_N,t)}{\partial t} = 0 \qquad (50)$$

where is:

$$\vec{v}_\nu = \sum_{\beta=0}^{\beta=n}\frac{\partial \vec{r}_\nu}{\partial q^\beta}\dot{q}^\beta \qquad (51)$$

and

$$\sum_{\nu=1}^{\nu=N}\left(grad_\nu\, f_\mu(\vec{r}_1,\vec{r}_2,....,\vec{r}_N,q^0), \sum_{\beta=0}^{\beta=n}\frac{\partial \vec{r}_\nu}{\partial q^\beta}\dot{q}^\beta\right) + \frac{\partial f_\mu(\vec{r}_1,\vec{r}_2,....,\vec{r}_N,t)}{\partial q^0}\dot{q}^0 = 0 \qquad (52)$$

We can see that generalized rheonomic reaction work diferent them is zero, and that is the function of rheonomic coordinates in the form:

$$\mathrm{Q}_\alpha^f = \sum_{\nu=1}^{\nu=N}\sum_{\mu=1}^{\mu=S}\lambda_\mu\left(grad_\nu\, f_\mu(\vec{r}_1,...,\vec{r}_N,t), \frac{\partial \vec{r}_\nu}{\partial q^\alpha}\right) \neq 0 \qquad (53)$$

The generalized rheological constrint reaction can be expressed by using (43) and rheological reaction of the stress-strain state of the hereditary element (35):

$$\mathrm{P}_\alpha = \sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_\nu} c_{(\nu,\nu+1)k}\left\{\rho_{(\nu,\nu+1)k}(t) - \rho_{(\nu,\nu+1)k0} - a_{(\nu,\nu+1)k}(q^0) - \int_0^t \mathbf{R}_{(\nu,\nu+1)k}(t-\tau)\left[\rho_{(\nu,\nu+1)k}(\tau) - \rho_{(\nu,\nu+1)k0} - a_{(\nu,\nu+1)k}(q^0(\tau))\right]d\tau\right\}\frac{\left(\vec{\rho}_{(\nu,\nu+1)k}, \frac{\partial \vec{r}_{(\nu)k}}{\partial q^\alpha}\right)}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|}$$

(54)

The generalized rheological constraint reaction can be expressed by two componets: one as an elastic properties reaction, and one as a rheological properties reaction of the hereditary element:

$$\mathrm{P}_\alpha^c = \sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_\nu} c_{(\nu,\nu+1)k}\left\{\rho_{(\nu,\nu+1)k}(t) - \rho_{(\nu,\nu+1)k0} - a_{(\nu,\nu+1)k}(q^0)\right\}\frac{\left(\vec{\rho}_{(\nu,\nu+1)k}, \frac{\partial \vec{r}_{(\nu)k}}{\partial q^\alpha}\right)}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|} \qquad (55)$$

$$\mathrm{P}_\alpha^{\bar{c}} = -\sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_\nu} c_{(\nu,\nu+1)k}\left\{\int_0^t \mathbf{R}_{(\nu,\nu+1)k}(t-\tau)\left[\rho_{(\nu,\nu+1)k}(\tau) - \rho_{(\nu,\nu+1)k0} - a_{(\nu,\nu+1)k}(q^0(\tau))\right]d\tau\right\}\frac{\left(\vec{\rho}_{(\nu,\nu+1)k}, \frac{\partial \vec{r}_{(\nu)k}}{\partial q^\alpha}\right)}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|}$$

(56)

By multiplying with dq^α and by sumarizing by index α , an after integrations by $d\vec{r}_\nu$, we can write following expressions::

$$\sum_{\alpha=0}^{\alpha=n} \mathrm{P}_{\alpha}^{c}\, dq^{\alpha} = \sum_{\alpha=0}^{\alpha=n}\sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_\nu} c_{(\nu,\nu+1)k}\left\{\rho_{(\nu,\nu+1)k}(t)-\rho_{(\nu,\nu+1)k0}-a_{(\nu,\nu+1)k}\left(q^{0}\right)\right\}\frac{\left(\vec{\rho}_{(\nu,\nu+1)k}, \frac{\partial \vec{r}_{(\nu)k}}{\partial q^{\alpha}}\right)}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|} dq^{\alpha} \quad (57)$$

$$\sum_{\alpha=0}^{\alpha=n} \mathrm{P}_{\alpha}^{c}\, dq^{\alpha} = \sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_\nu} c_{(\nu,\nu+1)k}\left\{\rho_{(\nu,\nu+1)k}(t)-\rho_{(\nu,\nu+1)k0}-a_{(\nu,\nu+1)k}\left(q^{0}\right)\right\}\frac{\left(\vec{\rho}_{(\nu,\nu+1)k}, \sum_{\alpha=0}^{\alpha=n} dq^{\alpha}\frac{\partial \vec{r}_{(\nu)k}}{\partial q^{\alpha}}\right)}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|} \quad (58)$$

Now we introduce a function as a rheological potential which corresponds to elastic properties of the hereditary element:

$$\Pi^{c} = \sum_{\alpha=0}^{\alpha=n}\int_{0}^{q^{\alpha}} \mathrm{P}_{\alpha}^{c}\, dq^{\alpha} = \sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_\nu} c_{(\nu,\nu+1)k}\int_{0}^{\vec{r}_{\nu}}\left\{\rho_{(\nu,\nu+1)k}(t)-\rho_{(\nu,\nu+1)k0}-a_{(\nu,\nu+1)k}\left(q^{0}\right)\right\}\frac{\left(\vec{\rho}_{(\nu,\nu+1)k}, d\vec{r}_{(\nu)k}\right)}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|} \quad (59)$$

Also, we introduce a function as a rheological potential which corresponds to the rheological properties of the hereditary element

$$\Pi^{\tilde{c}} = \sum_{\alpha=0}^{\alpha=n}\int_{0}^{q^{\alpha}} \mathrm{P}_{\alpha}^{\tilde{c}}\, dq^{\alpha} = -\sum_{\nu=1}^{\nu=N}\sum_{k=1}^{k=K_\nu} c_{(\nu,\nu+1)k}\int_{0}^{q^{\alpha}}\left\{\int_{0}^{t}\mathsf{R}_{(\nu,\nu+1)k}(t-\tau)\left[\rho_{(\nu,\nu+1)k}(\tau)-\rho_{(\nu,\nu+1)k0}-a_{(\nu,\nu+1)k}\left(q^{0}(\tau)\right)\right]d\tau\right\}\frac{\left(\vec{\rho}_{(\nu,\nu+1)k}, d\vec{r}_{(\nu)k}\right)}{\left|\vec{\rho}_{(\nu,\nu+1)k}\right|} \quad (60)$$

By using derivatives of these funtions as a rheological potential we can express a generalized rheological reactions:

$$\mathrm{P}_{\alpha}^{c} = -\frac{\partial \Pi^{c}}{\partial q^{\alpha}} \qquad \mathrm{P}_{\alpha}^{\tilde{c}} = -\frac{\partial \Pi^{\tilde{c}}}{\partial q^{\alpha}} \quad (61)$$

EXAMPLE 4. Rheological rheonomic oscillator [27] is presented on the figure No. 4 and 5. Genralized coordinte is $x(t)$, and rheonomic coordinte is change $x_0(t)$ of lenght:

Initial equation is:

$$m\ddot{x} + P(t) = F(t)$$

For the standard hereditary element stress-strain relation is:

$$n\dot{P}(t) + P(t) = nc\left[\dot{x}(t) - \dot{x}_0(t)\right] + \tilde{c}\left[x(t) - x_0(t)\right]$$

Dynamic equation of the rheological-rheonomic oscillator is:

$$nm\dddot{x}(t) + m\ddot{x}(t) + nc\dot{x}(t) + \tilde{c}x(t) = n\left[\dot{F}(t) + c\dot{x}_0(t)\right] + \left[F(t) + \tilde{c}x_0(t)\right]$$

and rheonomic reaction is:

$$P(t) = c\left\{\left[x(t) - x_0(t)\right] - \int_{0}^{t}\mathsf{R}(t-\tau)\left[x(\tau) - x_0(\tau)\right]d\tau\right\}$$

Potential function of the hereditary element and rheological reactions which correspond to elastic properties are:

$$\Pi^{c} = \frac{1}{2}c\left[x(t) - x_0(t)\right]^2$$

$$\mathrm{P}_{0}^{c} = -\frac{\partial \Pi^{c}}{\partial x_0} = c\left[x(t) - x_0(t)\right] \qquad \mathrm{P}_{1}^{c} = -\frac{\partial \Pi^{c}}{\partial x_1} = -c\left[x(t) - x_0(t)\right]$$

Rheological potential function of the hereditary element and rheological reactions which correspond to the rheological properties are:

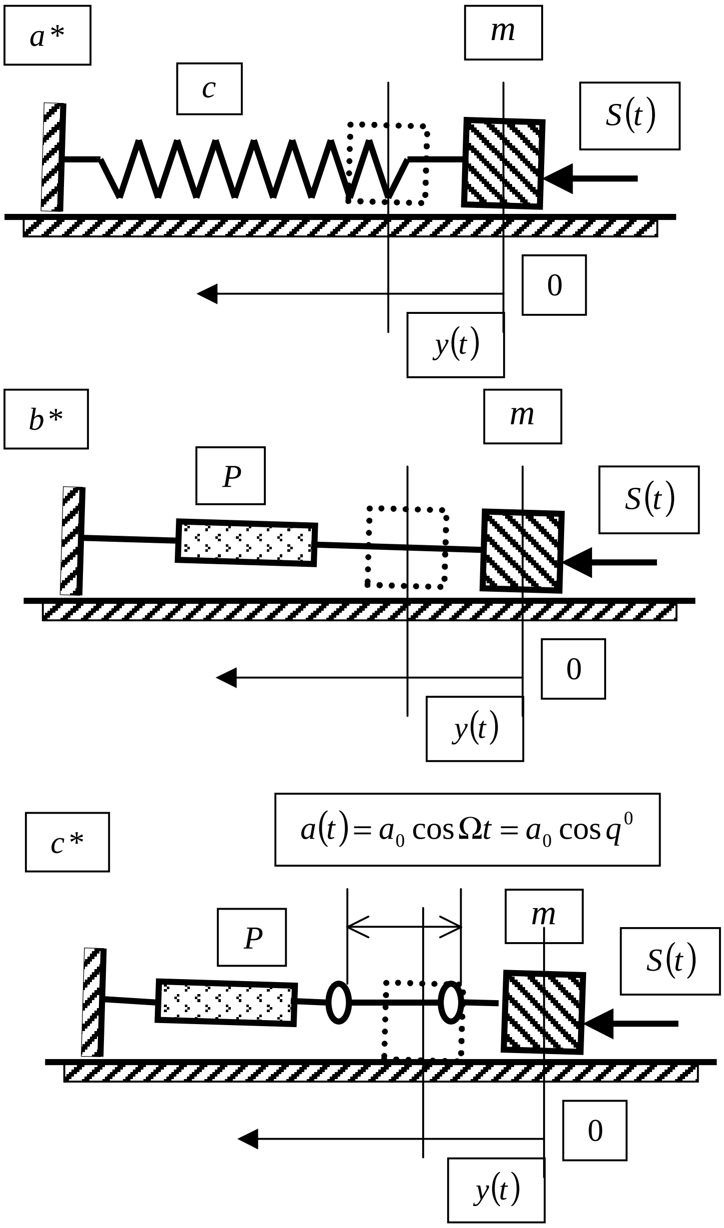


Figure No. 5. Classical ($a*$) , hereditary ($b*$) and hereditary-rheonomic $(c*)$ oscillator, excited by $S(t)$

$$\Pi^{\tilde{c}} = c\int_0^{x-x_0}\left\{\int_0^t \mathsf{R}(t-\tau)[x(\tau)-x_0(\tau)]d\tau\right\}dd[x(t)-x_t(\tau)]$$

$$Q_0^{\tilde{c}}(t) = -\frac{\partial \Pi^{\tilde{c}}}{\partial x_0} = -c\left\{\int_0^t \mathsf{R}(t-\tau)[x(\tau)-x_0(\tau)]d\tau\right\}$$

$$Q_1^{\tilde{c}}(t) = -\frac{\partial \Pi^{\tilde{c}}}{\partial x} = c\left\{\int_0^t \mathsf{R}(t-\tau)[x(\tau)-x_0(\tau)]d\tau\right\}$$

Kinetic Energy is:

$$E_k = \frac{1}{2} m\dot{x}^2$$

EXAMPLE 5. Rheological pendulum (see Figure No.6) [28] with a thread whcih increases its length on one of its segments (for example the unfolding of unstreachable segment and tied to the segment by a hereditary thread). The system has three degrees of motions which are defined by coordiantes: rheonomic coordinate $\rho_0(t)$ and generalized coordinates, ρ and φ. Rheoplogical conncection does not decrease the number of degrees of freedom. The rheonomic constraint decreases the number of degrees of motion freedom and that is because it assigns one degree of motion as enforcment.

The potential function of the herediatary element and rheological reaction component which correspond to elastic properties are:

$$\Pi^c = \frac{1}{2} c[\rho(t) - \rho_0(t)]^2$$

$$\mathrm{P}_0^c = -\frac{\partial \Pi^c}{\partial \rho_0} = c[\rho(t) - \rho_0(t)] \qquad \mathrm{P}_1^c = -\frac{\partial \Pi^c}{\partial \rho} = -c[\rho(t) - \rho_0(t)] \qquad \mathrm{P}_\varphi^c = -\frac{\partial \Pi^c}{\partial \varphi} = 0$$

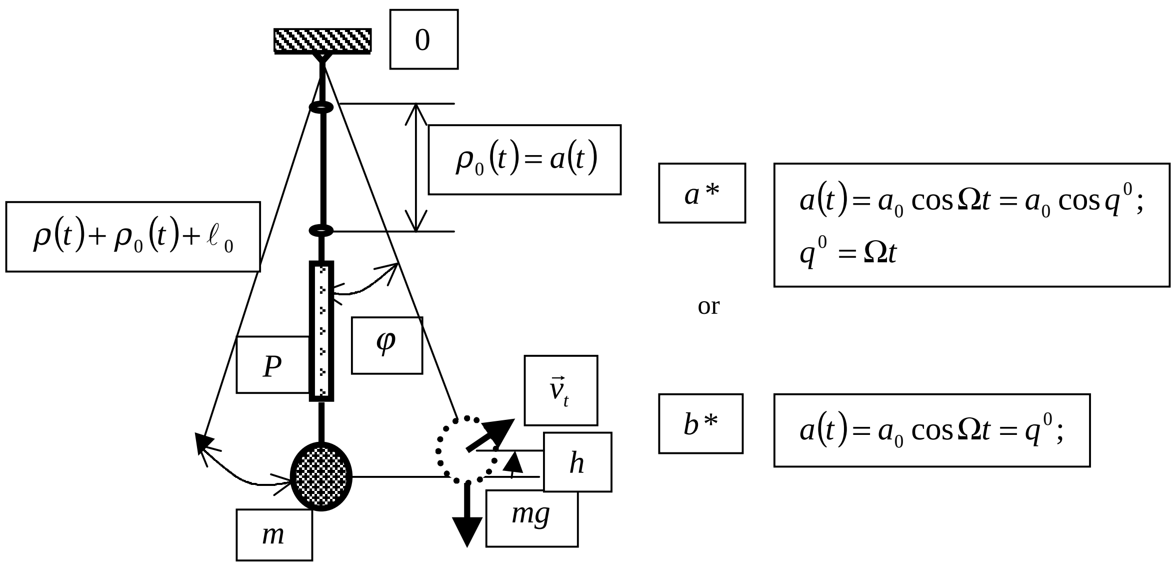


Figure No. 6. Rheological-rheonomic hereditary pendulum (Pendulum with a thread whcih increases its length on one of its segments (for example the unfolding of unstreachable segment and tied to the segment by a hereditary thread))

Rheological potential as a function of the hereditary element and rheological reactions which correspond to the rheological properties of the hereditary element are:

$$\Pi^{\tilde{c}} = c \int_0^{\rho - \rho_0} \left\{ \int_0^t \mathsf{R}(t-\tau)[\rho(\tau) - \rho_0(\tau)] d\tau \right\} d[\rho(t) - \rho_0(\tau)]$$

$$Q_{\rho_0}^{\tilde{c}}(t) = -\frac{\partial \Pi^{\tilde{c}}}{\partial \rho_0} = -c \left\{ \int_0^t \mathsf{R}(t-\tau)[\rho(\tau) - \rho_0(\tau)] d\tau \right\}$$

$$Q_{\rho}^{\tilde{c}}(t) = -\frac{\partial \Pi^{\tilde{c}}}{\partial \rho} = c\left\{\int_0^t \mathbf{R}(t-\tau)[\rho(\tau) - \rho_0(\tau)]d\tau\right\}$$

$$Q_{\varphi}^{\tilde{c}}(t) = -\frac{\partial \Pi^{\tilde{c}}}{\partial \varphi} = 0$$

Kinetic energy is:

$$E_k = \frac{1}{2}m\left\{[\ell_0 + \rho_0(t) + \rho(t)]^2\dot{\varphi}^2 + [\dot{\rho}_0(t) + \dot{\rho}(t)]^2\right\}$$

The expanded system of equations is:

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^{\alpha}} - \frac{\partial E_k}{\partial q^{\alpha}} - \mathrm{Q}_{\alpha} - \mathrm{Q}_{\varepsilon}^{f} - \mathrm{P}_{\alpha} - \mathrm{Q}_{\alpha}^{*} = 0 \ , \quad \alpha = \rho_0, \rho, \varphi;$$

By using previouis equations for each of the coordiantes we determine an equation:

EXAMPLE 6: Motion in the plane of two material particles tied to one another by a hereditary element and a rheonomical constraint as a connection in a que (series). We are using the earlier discussed example.

For a system of two material partcicles on interdistance of ρ in the plane of the system motion of relations (16**), by eliminatng the reaction of the hereditary element we obtain the following relation between internal coordinates ρ and φ of the system.

$$\cos\varphi \frac{d^2}{dt^2}\{\rho\sin\varphi\} = \sin\varphi\frac{d^2}{dt^2}\{\rho\cos\varphi\} - \frac{m_2}{m_1}F_{01}\sin\beta\cos\Omega_1 t \qquad \text{(a)}$$

$$[2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}] = -\frac{m_2}{m_1}F_{01}\sin\beta\cos\Omega_1 t \qquad \text{(b)}$$

Solution of the last differential equations (b) which can be observedas a differential equation of the first order by $\dot{\varphi}(t)$, so that we obtain:

$$\dot{\varphi}(t) = \dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^2} - \frac{m_2}{m_1}F_{01}\sin\beta\left[\frac{1}{\rho(t)}\int_0^t \rho(\tau)\cos\Omega_1\tau d\tau\right] \qquad \text{(c)}$$

The last differential equation (c) gives the connection between angular velocity $\dot{\varphi}(t)$ of the relative rotating of the second material particles around the first one in the plane of dynamics of that system of material particles and distnace ρ between those particles. We can see that angular velocity $\dot{\varphi}(t)$ of the relative rotating of the second material paritcle around the first consists of two parts, and that one of the components is opposite proportional to the square of there interdistance ρ, and that the second component is in an integral form and depends on the external enforcement excitation force.The first component corresponds to the case of the own-free motion of material particles conncected with a hereditary element, when there is no external enforcement force, while the other is the result of enforced relative rotation as a consequence of external enforcement force.

The square relative velocity of the relative rotation of the second mass particle around first one is:

$$v_r^2 = \dot{\rho}^2 + \rho^2\omega^2 = [\dot{\rho}(t)]^2 + [\rho(t)]^2\left\{\dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^2} - \frac{m_2}{m_1}F_{01}\sin\beta\left[\frac{1}{\rho(t)}\int_0^t \rho(\tau)\cos\Omega_1\tau d\tau\right]\right\}^2 \qquad \text{(d)}$$

By introducing the expression (d) for the square of the relative velocity of the second material particle in relation to the first one in the expression (18) for force $\mathrm{P}_{1,2}(t) = P(t)$ of interaction of material particles, and then the obtained expression we introduce into the esxpression for force of interaction of material particles into the integral realtion (2) of rheological hereditary conncection in series with the rheonomic costraint for the case of enforced motion of the system in the plane of rhological -rheonomic connection which comes down to the following integro-differential relation which has the form:

$$[\rho(t)-\ell_0-\rho_0(t)]-\int_0^t \mathrm{R}(t-\tau)[\rho(\tau)-\ell_0-\rho_0(\tau)]d\tau =$$

$$=-\frac{m_1 m_2}{c(m_1+m_2)}\left\{\ddot{\rho}(t)-[\rho(t)]^2\left\{\dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^2}-\frac{m_2}{m_1}F_{01}\sin\beta\left[\frac{1}{\rho(t)}\int_0^t\rho(\tau)\cos\Omega_1\tau d\tau\right]\right\}^2\right\} \qquad \text{(e)}$$

This rheological-rheonomic connection is an integro-differential nonlinear equation from which we deteremine the relative distance $\rho(t)$ of material particles as the function of time, in which way we in fact solved the basic problem.

Everything else is simple.

Instead of the previous connection we can write:

* for the standard hereditary element in series (que) with a reheonomical segment of the thread:

$$n\dot{\mathrm{P}}(t)+\mathrm{P}(t)=nc[\dot{\rho}(t)-\dot{\rho}_0(t)]+\tilde{c}[\rho(t)-\ell_0-\rho_0(t)]$$

While from dynamic equations of force of material particles interaction it is:

$$\mathrm{P}(t)=-\frac{m_1 m_2}{(m_1+m_2)}\left\{\ddot{\rho}(t)-\rho(t)\left\{\dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^2}-\frac{m_2}{m_1}F_{01}\sin\beta\left[\frac{1}{\rho(t)}\int_0^t\rho(\tau)\cos\Omega_1\tau d\tau\right]\right\}^2\right\}$$

By eliminating the force $\mathrm{P}(t)$ from the last two realtions we obtain:

$$-\frac{m_1 m_2}{(m_1+m_2)}\left\{n\dddot{\rho}(t)+\ddot{\rho}(t)-n\dot{\rho}(t)\left\{\dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^2}-\frac{m_2}{m_1}F_{01}\sin\beta\left[\frac{1}{\rho(t)}\int_0^t\rho(\tau)\cos\Omega_1\tau d\tau\right]\right\}^2\right\}+$$

$$+\frac{m_1 m_2}{(m_1+m_2)}\left\{-[\rho(t)]\left\{\dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^2}-\frac{m_2}{m_1}F_{01}\sin\beta\left[\frac{1}{\rho(t)}\int_0^t\rho(\tau)\cos\Omega_1\tau d\tau\right]\right\}^2\right\}+$$

$$+\frac{m_1 m_2}{(m_1+m_2)}2n[\rho(t)]\left\{\dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^2}-\frac{m_2}{m_1}\frac{1}{\rho(t)}F_{01}\sin\beta\int_0^t\rho(\tau)\cos\Omega_1\tau d\tau\right\}\left\{2\dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^3}\dot{\rho}(t)+\frac{m_2}{m_1}F_{01}\sin\beta\frac{d}{dt}\left[\frac{1}{\rho(t)}\int_0^t\rho(\tau)\cos\Omega_1\tau d\tau\right]\right\}=$$

$$=nc[\dot{\rho}(t)-\dot{\rho}_0(t)]+\tilde{c}[\rho(t)-\ell_0-\rho_0(t)]$$

For the case of free motion without the effects of an external force, only under the influence of the initial disturbance of equilibrium system position, and the excitation by a rheonomic constraint-conncection the previous equation for determining the distance between the material particles is:

$$-\frac{m_1 m_2}{(m_1+m_2)}\left\{n\dddot{\rho}(t)+\ddot{\rho}(t)+[\rho(t)-3n\dot{\rho}(t)]\left\{\dot{\varphi}_0\frac{\rho_0^2}{[\rho(t)]^2}\right\}^2\right\}=nc[\dot{\rho}(t)-\dot{\rho}_0(t)]+\tilde{c}[\rho(t)-\ell_0-\rho_0(t)]\text{or}$$

$$n\dddot{\rho}(t)+\ddot{\rho}(t)+n\frac{m_1+m_2}{m_1 m_2}c[\dot{\rho}(t)-\dot{\rho}_0(t)]+\frac{m_1+m_2}{m_1 m_2}[\rho(t)-\ell_0-\rho_0(t)]+\dot{\varphi}_0^2\frac{\rho_0^4}{[\rho(t)]^4}[3n\dot{\rho}(t)-\rho(t)]=0$$

V. CONCLUDING REMARKS

In this paper we derive the covariant differential equations of the disrete hereditary systems with rheonomic constraints between mass particles and hereditary element. In this paper we show applications of the rheonomic coordinate method to the dynamiccs of the discrete hereditary system. By introducing rheonomic coordinate in the sense of V. Vuji~i} we derive an extended system of differenttial equations in covariant form and coresponding generalizer reaktive forces for generalized coordinates in extendend form. We consider some special examples of the discrete hereditary systems. We introduce expressions of the rheological potential which corresponds to elastic and hereditary properties.

REFERENCES

[1] An|eli} T., Stojanovi}, R., Racionalna mehanika, Zavod za izdavawe uxbenika Srbije, Beograd, 1965., str. 588.

[2] Butenin, M. V., Fufaev, N. N., (1991), *Vvedenie v analiti~eskuÓ mehaniku*, Moskva, Nauka, 1991, 256 str.

[3] Vasilenko Á V., *Teori® kolebaniy*, Kiev, V [, 1992, 430 str.

[4] VerlanÃ A.F., Sizikov V.S., (1986), *IntegralÃnÀe uravneni®: MetodÀ, algoritmÀ, programmÀ*, Kiev, Naukova Dumka, 1986, 534 str.

[5] Vuji~i}, A. V., Kovarijantna dinamika, Matemati~ki institut SANU, specijalna izdawa, 14, Beograd, 1981., str.136,

[6] Vuji~i}, A. V., *Dynamics of Rheonomic Systems*, Math. inst. SANU, Sp. Ed. Beograd, 1990, pp. 96.

[7] Vuji~i}, A.V. and Hedrih (Stevanovi}), K., (1993), The rheonomic constraints force, Facta Universitatis, Series Mechanics, Automatic Control and Robotics, Vol. 1, No. 3, 1993, pp. 313-322.

[8] Vuji~i}, A.V., One aproach to the problem of two bodies, Facta Universitatis, Series Mechanics, Automatic Control and Robotics, Vol. 2, No 8, 1998, pp. 629-634.

[9] Vuji~i}, A. V., Preprincipi mehanike, Zavod za uxbenike i nastavna sredsttva, 1998, str. 213.

[10] Gantmaher, F. R., Analiti~ka mehanika, prevod s Ruskog Vuji~i}, A. V.,, Zavod za izdavawe uxbenika Srbije, Beograd, 1963, str. 215.

[11] Gantmaher, F. R., *Lekcii po analiti~eskoy mehanike*, IzdatelÃstvo Nauka, 1966, str. 300.

[12] Goldstein, H., Classical Mechanics, Addison-Wesley Publ. Comp. 1980, pp. 672.

[13] Goroshko, O.A., Puchko, N.P., Lagrangian equations for the multibodies hereditary systems, Facta Universitatis, Series Mechanics, Automatic Control and Robotics, Vol. 2, No 7, 1997, pp. 209-222.

[14] Goro{ko O. A., Pu~ko N. P. , (1997), *FormiÀ uravneniy Lagran`a dl® nasledstvennÀh mnogomassovÀh sistem,* Proceedings of the YUCTAM, 1997, Beograd, pp. 123-137.

[15] Jankovi}, S., Poti}, P., Hedrih (Stevanovi}), K., Parcijalne diferencijalne jedna~ine i intergralne jedna~ine sa primenama u in`ewerstvu, Univerzitet u Ni{u, 1999, str. 347.

[16] Pars L.A., A treatise on Analytical Dynamics, Heinemann, London.

[17] Pars, L., Analiti~eskaÂ dinamika, Nauka, Moskva, 1971, str. 636. (prevod s engleskog Lur*Ãe, K.A.)*

[18] Rabotnov Á M., (1977), *ÝlementÀ nasledstvenoy mehaniki tverdih tel*, Moskva, Izd. Nauka, 1977, 384 str.

[19] Ra{kovi}, D., Mehanika III - Dinamika, Nau~na kwiga, Beograd, 1963, 1972. str. 424.

[20] Ra{kovi}, D., Analiti~ka mehanika, Ma{inski fakultet, Kragujevac, 1974, str. 147.
[21] Ra{kovi}, D., Teorija oscilacija, Nau~na kwiga, Beograd, 1965, str. 503. [22] Ra{kovi}, D., Osnovi tenzorskog ra~una, Ma{inski fakultet, Kragujevac, 1974.
[23] R`anicin, A.R., *NekotorÀe voprosÀ mehaniki sistem deformiruÓçihsÔ vo vremeni*, Moskva, GITTL, 1949, 248 str.2. Rzhanitsin,
[24] SavÀn, G. N., Ruçickiy, Â. Â., (1976), *∃lementÀ mehaniki nasledstvennÀh sred*, Kiev, V. [., 1976, 252 str.
[25] Stojanovi}, R., Uvod u nelinearnu mehaniku kontinuuma, Zavod za izdavanje uxbenika SR Srbije, Beograd, 1965, p. 93.
[26] Uitteker E. T., *Analiti~eska® dinamika*, Moskva, GITTL, 1937, 500 str. (prev. s engl. Whittaker, E.T.)
[27] Hedrih (Stevanovi}), K., Modeliranje hidromehani~kih naslednih sistema, Predavanje po pozivu, HIPNEF "98, SMEITS, 1998., str. 15-18.
[28] Hedrih (Stevanovi}), K., Thermorheological Hereditary Pendulum, (Ref. No. TVP-11) Thermal Stresses 99, Cracow 1999, pp.199-302.
[29] Hedrih (Stevanovi}), K., Izabrana poglavqa teorije elasti~nosti, Ma{inski fakultet, Ni{, 1977, 1988, str. 424.
[30] Hedrih (Stevanovi}), K., Zbirka re{enih zadataka iz teorije elasti~nosti, Prilog, Nau~na kwiga, Beograd, 1991, str. 392.
[31] Hedrih (Stevanovi}), K., Izabrana poglavqa teorije nelinearnih oscilacija, Univerzitet u Ni{u, 1975, str. 180.
[32] Hedrih (Stevanovi}), K., Filipovski, A., Longitudinal Vibrations of Rheological Rod with Variable Cross Section, Comunications in Nonlinear Sciences and Numerical Simulations, Shanghai Insttitute of Applied Mathematics and Mechanics, China, 1999, (to appear)
[33] Hedrih (Stevanovi}), K., Mechanica of discrete piezorheological hereditary systems, Abstract reference number 003, LDS'99, http://www.sparks.co.uk/lds99, The third International Conference on Low Dimensional Structures and Devices, Renaissance Antalya Resort, Antalya, Turkey, 1999.
[34] Hedrih (Stevanovi}), K., N. Multybody Heredetary Systems, Section S4.3, GAMM 99, Web: htt:/www.lpmm.univ-metz.fr/gamm99, University of Metz, France, 1999.