Generalized Normal Solution of Degenerated System of Equations / Inequalities

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Abstract

The regularization problem of degenerated system of equations / inequalities in Banach spaces is considered. Our approach is based on the explicit parametrization of input data and on the utilization of the multy-valued mapping techniques. We suggest an extended minimization method that resolves both regularization and data correction problems simultaneously. According to the method an ill-posed problem should be replaced by a search of the minimal norm element in the join of solution sets of the family of problems that are equivalent to the initial one with respect to input data accuracy. We named this element as generalized normal solution (GNS) of a given ill-posed problem. Theoretical results on regularization property of this method as well as problems of its approximation and numeric implementations via linearization and normal spline collocation methods are presented here.

Keywords: **degenerated equations, inequalities, ill-posed problems, regularization, multy-valued mappings, linearization, normal splines.**

1. Introduction

If we increase complexity of mathematical models then we can get an ill-posed numerical problem connected with them. There are rather developed regularization theory and numeric implementation technique for linear problems, see A.N.Tikhonov and V.Ya.Arsenin [17], V.K.Ivanov et al. [11], V.A.Morozov [14]. But in non-linear cases it is not the case. The most known regularization methods for ill-posed non-linear operator equations are the Ivanov's quasi-decision method and Tikhonov's regularization. We note, following to Morozov [14], the completed theory of non-linear problems regularization is absent up to day.

In 1980 Tikhonov formulated in [16] a new regularization concept for the linear equations system. According to the concept an ill-posed computational problem should be replaced by a search of the minimal norm element in the join of solution sets of the family of problems which are equivalent to the initial one with respect to input data accuracy. Tikhonov restricted oneself to linear equations with disturbed matrices. In this case numeric implementations coincided with the generalized discrepancy method suggested by V.Goncharsky et al. [4]. However this concept yields nontrivial advance in nonlinear ill-posed problems. It was obtained in our works [5,7,9] for nonlinear equations / inequalities systems and extremum problems due to the explicit parametrization of input data and to the utilization of the multy-valued mapping technique. It allows to state numeric problems that take into account the detailed data structure and gives a solution of the data correction problem. The corresponding regularization problem was named a generalized normal solution (GNS) of the initial ill-posed one. The explicit parametrization of input data has allowed to specify concepts of degeneration, singularity and ill-posedness of equations, as well as inequalities, in terms of the theory of multiple-valued mappings.

Here we describe the extended minimization method for the GNS construction. The seeking variables in the method are initial desired ones and input date. Therefore both regularization and data correction problems are being resolved simultaneously. The consequent numeric problem can be rather complicated and we give a GNS approximation method. It included in separation of given degenerated system some regular part and minimization of a penalty function of the complement subsystem under separated regular constraints. If we've got a close GNS approximation then we can to precise it thereby the linearization method. This approach is illustrated on example of singular ordinary differential equations.

2. Problem statement and definitions

We consider the problem to solve the system

$F(x, y) = 0, \quad g(x, y) \le 0$, (1)

with respect to x when y is input data. Transformations F and g act from the product of a Banach space X and a metric compact Y. Ranges of values are a Banach space Z and the arithmetic space R^m respectively (inequality subsystem is a finite one). In the case of differential equations it is useful to assume that x belongs to some linear manifold $D \subset X$ (possible D=X). Both transformations F and g are assumed being continuous ones.

Most advanced models in our investigations are:

a. b.v. / i.v. problems in degenerated (singular) ODE systems

$F\left(x'(t), x(t), t, y\right) = 0, \quad a \le t \le b$; (1a)

b. finite dimensional system (possibly inconsistent)

$g_i\left(x, y\right) \le 0, \quad 1 \le i \le m$, (1b)

especially Afriat systems of Inverse Rational Consumption Problem, see H.Varian [18] and our paper [8].

Denote a solution set of (1) by

$P(y) = \{x \in D : (1)\}$, (2)

such that P will be a multi-valued mapping of Y onto set of closed subset of D. Introduction of this mapping allows to specify concept well-posedness (or regularity) of a solution problem of the system (1) and also ill-posedness, generalizing concept of degeneracy (or singularity) of equations systems and degeneracy of inequalities systems. Namely, the solution problem of the system (1) is well-posed one in sense of Hadamard (classical correct or regular) if the mapping P is one-valued continuous function in some neighborhood of exact data y^0 . Consequently, this problem will be ill-posed (irregular, singular or degenerated) one if P hasn't this property.

In the case when (1) is an equations system (inequalities absent) with differentiable transformation F sufficient regularity conditions are provided by conditions of the implicit function theorem. However when inequalities present such simple analytical regularity conditions absent. We say, an inequalities system is degenerated one if it can be inconsistent under disturbed data.

In order to specify the regularity problem we precise the solution problem of the degenerated system (1) as the seeking of its solution that is closest one to some given element $z \in D$ (trial solution). The natural path of the closeness measure changing based on the Tikhonov stabilizing functional (stabilizer) construction. Its main properties are lower semicontinuity and compactness of level sets. We note this functional as $\Omega(x)$ and assume it is defined for $x \in D$. Usually D is a Hilbert space H that compact embedded in X, then $\Omega(x) = \|x\|_H^2$. We precise the main computing problem for (1) as the y-parametric Normal Solution (NS) problem for the system (1)

$$M(y) = Arg\min\{\Omega(x-z) : x \in P(y)\}. \quad (3)$$

The main assumption of the regularization theory is: an initial numeric problem with exact data is a resolvable one. It means the system (1) with data y^0 is consistent one or the set $P(y^0)$ is nonempty. Obviously, this assumption implies that the exact problem (3) is solved. According to the remark made above classical correctness of the problem (3) means uniqueness and continuity of mapping M. However, uniqueness can be difficulty achievable and it is an excessiveness property for practical purposes.

Denote the semi-deviation of a set A from a set B in some metric space by

$$\beta(A,B) = \sup\{\rho(a,B) : a \in A\}.$$

Therefore the Hadamard distance between A and B will be $h(A,B)$= max{ $\beta(A,B)$, $\beta(B,A)$ }

We introduced in [5,7,9] the next notions:

Definition 1. *The mapping P is said to be a compact* β *-continuous one at point y if for any compact K the mapping* $P \cap K$ *is* β *-continuous one at y, i.e.*

$$\lim_{y' \to y} \beta\left(P(y'), P(y)\right) = 0$$
.

This property is called also *upper semi-continuous* of multy-valued mapping *P*. Continuity of transformations *F* and *g* provides in common case validity of this property for mapping (2), see V.Fedorov [3; lemma 1.3].

Definition 2. *The mapping P is said to be regular one on Y if for any compact K the mapping* $P \mathrm{I} K$ *is h-continuous one on Y, i.e.*

$$\lim_{y' \to y} h\left(P(y'), P(y)\right) = 0, \quad \forall y \in Y$$
.

This property is called also *Hadamard-continuous* of mapping *P*. In finite-dimensional case it holds when Jacobeans of active subsystems of (1) is non-degenerate. Consequently, this property of mapping (2) is a generalization of the implicit function theorem conditions for inequality systems. Also it holds (in Banach space) when *P* is β -continuous one-valued mapping.

Definition 3. *The problem (3) is said to be semi-correct one on Y if its solution set* $M(y)$ *is non-empty and compact set for* $\forall y \in Y$ *, and the mapping M is* β *- continuous on Y.*

This notion is a generalization of classical (Hadamard's) correctness and coincides with it if *M* in addition is one-valued mapping. New notion relaxes the regularization problem if the solution set $M(y)$ has a small diameter. The common condition of semi-correctness of (3) yields the next

Theorem 1. *If* the *system (1) with exact data is consistent one and P generated by the system is regular mapping on Y, then the NS problem (3) is semi-correct one on Y.*

This theorem is a simple corollary of known property of marginal mappings (see Aubin J.P. and Ekland I. [1; ch. 3, st. 23]) and compact property of stabilizer $\Omega(x)$.

Corollary: *If M is one-valued mapping (ex. if P is convex), then the NS problem (3) is Hadamard-correct one.*

However, if the system (1) degenerates, then its mapping *P* won?t be regular one and we have to state another correct or semi-correct problem such that solution of new problem will approximate the solution of the main problem (3). Theorem 1 gives an effective direction for regularizing problems construction. The very productive regularization method for common non-linear systems is the next one.

1. **Generalized NS problem (extended minimization)**

Let $\tilde{y}$ be a data realization, $Q_\delta\left(y^0\right)$ be a set of possible data realizations, where parameter δ is the estimation of error level. Usually this set is a sphere which diameter is δ . Then the similar set

$Q_\delta(\tilde{y})$ is a set of equivalent data. We stated in [9] the next generalized normal solution (GNS) problem for (1): to minimize functional $\Omega(x-z)$ provided that $\{x \in P(y), y \in Q_\delta(\tilde{y})\}$ c

$$N_\delta(\tilde{y}) = Arg\min\{\Omega(x): x \in P(y), y \in Q_\delta(\tilde{y})\}. \quad (4)$$

Let's note, this problem is posed as extended minimization in the product of spaces X and Y. We have to determine both a solution *x* and an equivalent data *y*. It is useful to represent the left side of (4) as the pair of components

$$N_\delta(\tilde{y}) = \{R_\delta(\tilde{y}) \subset X; S_\delta \subset Y\}.$$

Here $R_\delta(\tilde{y})$ is the solution-set and S_δ is the corrected data set. Obviously, if the GNS problem (4) is resolvable one and $y^\delta \in S_\delta$ is corrected data, then the NS problem (3) is also resolvable one. We have got in [9] the next result on the GNS problem.

Theorem 2. *If the system (1) with exact data y^0 is consistent, then the GNS problem (4) is solvable under each possible data realization $\tilde{y}$, its solution set $R_\delta(\tilde{y})$ is compact one and the next limit holds:*

$$\lim_{\delta \downarrow 0} \sup\{\beta(R_\delta(\tilde{y}), M(y^0)) : \tilde{y} \in Q_\delta(y^0)\} = 0. \quad (5)$$

This limit means that x-components R_δ of the GNS problem solution approximate normal solutions of the exact system (1). If such NS is unique, then transformation R_δ is a Tikhonov's regularization operator. Thus, the GNS problem (4) resolves both regularization and data correction problems. We call the transition from an ill-posed problem (3) to the GNS problem as *the extended minimization method.*

In common case an effective numerical method for solving of the problem (4) is the *linearization method* (LM), see Pshenichnyj B.N. and Danilin Yu.M. [15]. Previously the initial problem should be discretizated some way. Peculiarity of the problem (minimizing functional is quadratic one and admissible data region is small) gives hope for correct fulfillness and efficiency of the method. The core of the LM is the quadratic programming (QP) problem. We have constructed (1984, [5]) the QP algorithm for linear NS problems in Hilbert space (connected with integral equations) which allows to restrict oneself to partial discretization of initial or approximating QP problems. However, the extended minimization method can be rather complicated. Its successful implementation can be fulfilled when we have a close solution approximation. Since the GNS problem approximates the normal solution of the exact initial system (1) we may to restrict ourselves to approximation of the NS problem (3).

2. Approximation of the NS problem

Let the system (1) be irregular on the whole and let it is represented as two sub-systems

$F_0(x,y)=0, \quad g_0(x,y)\le 0$, (6_0)

$F_1(x,y)=0, \quad g_1(x,y)\le 0$, (6_1)

such that (6_0) is a regular one. It means the set of solutions

$P_0(y)=\{x\in D:(6_0)\}$

generates the regular mapping P_0. Possible, $F_0\equiv 0$, $g_0\equiv 0$, $F_1=F$, $g_1\equiv g$.

Let's introduce certain penalty functions of (6_1), for example

$$f(x,y)=\|F_1(x,y)\|_Z^2+\|g_1^+(x,y)\|_{(m)}^2 .$$

Here positive cutting $g_1^+=\max\{g_1,0\}$ is used.

We assume further *the mapping* P_0 *and the function* $f(x,y)$ *are a locally Lipschitzian one.* Concerning the mapping P_0 it means that for any r>0 (such that the set $P_r(y)=\{x\in P(y):\|x-z\|_H\le r\}$ is nonempty one) there is a number $l_P(r)>0$ such that for any $y,y'\in Q_\delta(y^0)$ the inequality

$$h(P_r(y),P_r(y'))\le l_P(r)\rho(y,y') \quad (7)$$

is fulfilled. Consequently, for the function *f(x,y)* and for any bounded part $D^0\subset D$ there are such positive numbers *a*, *b*, l_f that under $(x,x')\in D^0,\ (y,y')\in Y$ the inequaliti

$$|f(x,y)-f(x',y)|\le l_f\|x-x'\|_H, \quad |f(x,y)-f(x,y')|\le(a+b\Omega(x-z))\rho(y,y') \quad (8)$$

are fulfilled. Also we note, the continuous and convex *functional* $\Omega(x)$ *is a locally Lipschitzian one*. It means there is such number l_s>0 that under $(x,x')\in D^0$ the inequality

$$\left|\Omega(x-z)-\Omega(x'-z)\right| \leq \left|l_s\right| \left\|x-x'\right\|_H \quad (9)$$

holds.

We state the next extremum problem:

$$M_0(y)=Arg\min\{f(x,y):x\in P_0(y)\}. \quad (10)$$

Obviously, if the initial system (1) is consistent one and the problem (10) is resolvable one then $M_0(y)=P(y)$. But when the NS problem (3) is an ill-posed one the problem (10) is the same

Let's introduce the Tikhonov smoothing functional

$$T_\alpha(x,y)=f(x,y)+\alpha\Omega(x), \quad \alpha>0, \quad (11)$$

and state the problem

$$M_{0\alpha}(y)=Arg\min\{T_\alpha(x,y):x\in P_0(y)\}. \quad (12)$$

This problem unlike of the GNS one is posed in *X*-space while *y* is a fixed parameter.

Theorem 3, formulated below, shows that the problem (12) as well as the GNS approximates the NS problem. Consequently transition from the problem (3) to (12) can be considered as the *Regularized Partial Penalty Function* (RPPF) method for non-regular systems of equations / inequalities**.** Numerical implementation for the RPPF method in common case can be fulfilled effectively by the LM as well as for the GNSP (4). But the RPPF is rather simpler. Having a solution of the (12) one can to solve the more advanced problem (4).

Theorem 3. *If the mapping* P_0 *is regular and locally Lipschitzian and the penalty function* $f(x,y)$ *is locally Lipschitzian, then the problem (12) is a compact solvable under all* $\alpha>0$. *Moreover, if* $\alpha=\alpha(\delta)$ *such that*

$$\lim_{\delta\downarrow 0}\alpha(\delta)=\lim_{\delta\downarrow 0}\frac{\delta}{\alpha(\delta)}=0, \quad (13)$$

then the next limit holds:

$$\lim_{\delta\downarrow 0}\sup\left\{\beta\left(M_{0\alpha(\delta)}(y),\ M\left(y^0\right)\right):y\in Q_\delta\left(y^0\right)\right\}=0. \quad (14)$$

This theorem is closed on a content and proof techniques to the theorem 2 from Vasil'ev F. [19, chapter

2, §5]. There the problem of minimization of approximate function on exact set (P is a constant mapping) is considered in traditional Tikhonov's scheme. Perturbation of admissible set P leads to complication of the theorem 3 proof. Now in addition to proving scheme of [19] we have to consider projection of a point $x_\alpha \in M_\alpha(y) \subset P(y),$ on exact set $P(y^0)$ and projection of $x^0 \in M(y^0) \subset P(y^0)$ on *P(y)*. But parametrization of input data allows to overcome new obstacles.

Thus, the limit equality (14) means the problem (12) as well as the GNS approximates the NS problem. The supremum on admissible data realizations means stability of (12) and, in accordance of definition 3, its semi-correctness.

It is necessary to note, the problem of choosing of regularization parameter α in our method as well as in any Tikhonov-type regularization for nonlinear problems, see [14, 19], has not exact decision. The given in the theorem 8 rule (13) for concordance of artificial regularization parameter α with a level of input data errors δ is not effective from a point of view of calculus mathematics. Really, for a given level of errors δ any parameter $\alpha > 0$ can be considered as a realization of a relation $\alpha(\delta)$, satisfying to conditions (13). Such problem is typical in regularization theory. A practical value of such relations is heuristic. First from relations (13) requires a diminution of a parameter α with a diminution of δ, but second requires moderation in it. The final conclusion about magnitude α in an approximating problem (12) requires the additional analysis outside of a used model. For our approach it is essential that we can get an appropriate approximation of the normal solution of (15) and then to apply the LM for precizing this approximation.

5. Example: Singular Problem in ODE

Here we'll illustrate the RPPF method on example of initial value problem for a singular ODE system (1a) in R^n :

$$\begin{cases} F(x'(t), x(t), t, y) = 0, \quad a \le t \le b; \\ x(a) = x^0. \end{cases} \qquad (15)$$

Consideration of boundary value problem for equation (1a) is similar. The method is being realized under fixed input data, therefore we omit the *y* below.

Denote $x' = u$ and pass from the system (15) to equivalent differential-algebraic system with respect to variables $(x,\ u) \in R^{2n}$:

$$x'(t) = u(t), \quad x(a) = x^0 \ ; \qquad (16)$$

$$F(u(t), x(t), t) = 0 \ . \qquad (17)$$

Its solution *{x (t), u (t)}* exists under exact data (the main assumption) and belongs to the space $X = C^1[a,b] \times C[a,b]$.

Correctness of such problems, when the partial Jacobean $F'_u(u,x,t)$ is degenerated, had investigated by R.März [13]. Rather complicated technique for projection and matrix pseudoinversion had used there. The most extended approach for numeric implementation of singular problem (12) is based on combination of implicit difference schemes and Newton method, see [12, 13].

In order to apply the RPPF method it is convenient to assign as *D* the Hilbert-Sobolev space H^2 [*a,b*] with norm

$$\|x\| = \left[\|x(a)\|_n^2 + \int_a^b \|x''(s)\|_n^2 ds \right]^{1/2}.$$

This space is compact imbedded in *C*[*a,b*] such that $\Omega(x) = \|x\|^2$. Obviously, subsystem (16) is regular one, respectively, we introduce the penalty function of the complementary part (17) as

$$f(x,u) = \int_a^b \|F(u(t), x(t), t)\|_n^2 dt \quad . (18)$$

By this all components of the RPPF method (11), (12) are defined. The correct optimal problem connected with the initial ill-posed one is to minimize the smoothing functional

$$T_\alpha(x,u) = f(x,u) + \alpha \|x\|^2, \quad \alpha > 0 \quad , (19)$$

under conditions (16).

However, let's consider detailed scheme of RPPF for singular problem (15). It concluded in minimizing the functional

$$T_\alpha(x,u) = \int_a^b \left[\|F(u(t), x(t), t)\|_n^2 + \alpha \|x''(t)\|_n^2 \right] dt, \quad \alpha > 0 \quad , (20)$$

under conditions (16). Note, the construction of functional (20) non-completely coincides with abstract formula (19). We omitted in (20) inessential constant $\|x(a)\|_n^2$.

This problem is an ordinary optimal control one with simplest constraints (16). According to the theorem 3 its solution approximates the normal solution of initial singular problem (15) under the circumstances (13). Having appropriate approximation of the NS problem's solution we can apply to the last one the linearization method. The main component of the LM for the system (15) is the NS problem for singular linear system

$$A(t)x'(t) + B(t)x(t) = f(t), \quad a \leq t \leq b,$$

with initial (for problem (15)) condition $x(a) = x^0$. The effective method for such problems is the normal spline-collocation one developed in our works [5, 6, 10].

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