Nonlinear Forced Response of Nonuniform Beam with Rectangular Cross-Section and Parabolic Thickness Variation

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Abstract
This paper presents the primary resonance of single mode of forced and undamped bending vibrations of nonuniform beam with rectangular cross-section, constant width and convex parabolic thickness variation. The case of nonlinear curvature is considered. We apply the method of multiple scales directly to the nonlinear partial-differential equation of motion and boundary conditions. The frequency-response is analytically determined.

1 Introduction

Nayfeh and Nayfeh [5] constructed the nonlinear mode shapes and natural frequencies for a class of one-dimensional continuous systems with weak cubic geometric and inertia nonlinearities. The methode of multiple scales is applied directly to the partial-differential equation and boundary conditions of nonlinear uniform beam.

Nayfeh and Nayfeh [6] used two approaches, the method of multiple scales to treat directly the governing partial-differential equation, respectively a Galerkin procedure and the method of multiple scales, to determine the nonlinear modes and natural frequencies of simply supported uniform Euler-Bernoulli beam resting on an elastic foundation with distributed quadratic and cubic nonlinearities.

Caruntu [3] considered the case of nonlinear free vibrations of nonuniform beam with rectangular cross-section, constant width, parabolic thickness variation and a sharp end. This beam is a good approximation of a cylindrical gear tooth. This cantilever beam was studied in the case of geometrical nonlinerities. In the absence of internal resonance the nonlinear modes are taken to be perturbed versions of the linear modes. Therefore the nonlinear planar mode shapes and natural frequencies of a gear tooth with a sharp end are analytically determined.

Caruntu [4] has presented numerical determinations in the linear case of free bending vibrations of this beam using the factorization method.

2 The nonlinear partial-differential equation of forced motion of nonuniform beam

We consider the case of nonlinear forced vibrations of nonuniform beam, the case of nonlinear curvature k (geometrical nonlinearities)

$$k \cong y''\left(1-\frac{3}{2}y'^{2}\right), \tag{1}$$

where y is considered the transverse displacement. Caruntu [3] have presented the partial-differential equation of nonlinear free bending vibrations in this case

$$\frac{1}{A(x^*)}\frac{\partial^2}{\partial x^{*2}}\left[EI(x^*)\frac{\partial^2 y}{\partial x^{*2}}\right]+\frac{3E}{A(x^*)}\left\{\frac{\partial^2}{\partial x^{*2}}\left[I(x^*)\frac{\partial^2 y}{\partial x^{*2}}\left(\frac{\partial y}{\partial x^*}\right)^2\right]+\frac{\partial}{\partial x^*}\left[I(x^*)\left(\frac{\partial^2 y}{\partial x^{*2}}\right)^2\frac{\partial y}{\partial x^*}\right]\right\}+\rho_0\frac{\partial^2 y}{\partial t^{*2}}=0 \tag{2}$$

where the cross-sectional area A, the mass density ρ_0, the cross-sectional area of beam, the Young's modulus E and the moment of inertia I are considered.

This paper deals with the undamped forced vibrations. For simplicity is considered only the case of a single frequency excitation Ω^*

$$\frac{1}{A(x^*)}\frac{\partial^2}{\partial x^{*2}}\left[EI(x^*)\frac{\partial^2 y}{\partial x^{*2}}\right]+\frac{3E}{A(x^*)}\left\{\frac{\partial^2}{\partial x^{*2}}\left[I(x^*)\frac{\partial^2 y}{\partial x^{*2}}\left(\frac{\partial y}{\partial x^*}\right)^2\right]+\frac{\partial}{\partial x^*}\left[I(x^*)\left(\frac{\partial^2 y}{\partial x^{*2}}\right)^2\frac{\partial y}{\partial x^*}\right]\right\}+\rho_0\frac{\partial^2 y}{\partial t^{*2}}=F(x^*)\cos(\Omega^* t^*) \tag{3}$$

We introduce the nondimensional quantities defined by:

$$x=\frac{x^*}{l}\quad,\quad t=t^*\frac{1}{l^2}\sqrt{\frac{EI_0}{\rho_0 A_0}}\quad,\quad w=\frac{y}{W}\quad,\quad \Omega=\Omega^* l^2\sqrt{\frac{\rho_0 A_0}{EI_0}}\,, \tag{4}$$

where W is the characteristic transverse displacement, usually taken as l (the reference length). The partial-differential equation (3) becomes

$$\frac{1}{A(x)}\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 w}{\partial x^2}\right]+\frac{\alpha^*}{A(x)}\left\{\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2\right]+\frac{\partial}{\partial x}\left[I(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial w}{\partial x}\right]\right\}+\frac{\partial^2 w}{\partial t^2}=f(x)\cos(\Omega t)\,, \tag{5}$$

where

$$\alpha^*=\frac{3W^2}{l^2} \tag{6}$$

Considering the linear and the nonlinear operators

$$L_2[w(x,t)]=\frac{1}{A(x)}\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 w}{\partial x^2}\right], \tag{7}$$

$$N[w(x,t)]=\frac{1}{A(x)}\left\{\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2\right]+\frac{\partial}{\partial x}\left[I(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial w}{\partial x}\right]\right\} \tag{8}$$

the nonlinear partial-differential equation of single-mode forced motion of the presented nonlinear beam, becomes

$$L_2[w(x,t)]+\alpha^* N[w(x,t)]+\frac{\partial^2 w}{\partial t^2}=f(x)\cos(\Omega t) \tag{9}$$

3 The problem formulation

The nonuniform beam with constant width, parabolic thickness variation and a sharp end case is considered (see Fig.1), for the reason that we have already determined [4] the natural frequencies and the mode shapes:

$$a(x) = a_0 \quad , \quad b(x) = b_0 \left(1 - x^2\right) \quad , \quad x \in [x_0, 1) \quad , \quad x_0 \in [-1, 1) \tag{10}$$

The cross-sectional area and the moment of inertia at *x=0* , and the dimensionless forms of them, are

$$A_0 = a_0 b_0 \quad , \quad I_0 = \frac{a_0 b_0}{12} \quad , \quad A(x) = 1 - x^2 \quad , \quad I(x) = \left(1 - x^2\right)^3 \tag{11}$$

Fig.1 Nonuniform beam of rectangular cross-section with a sharp end

We examine the primary resonance (the case of nonlinear curvature of a cantilever beam, Fig.1) of the problem

$$L_2[w(x,t)] + \alpha^* N[w(x,t)] + \frac{\partial^2 w}{\partial t^2} = f(x)\cos(\Omega t) \tag{12}$$

$$w = \frac{\partial w}{\partial x} = 0 \text{ at } x = x_0 \quad \text{and} \quad w \text{ finite at } x = 1 \ , \tag{13}$$

4 Primary Resonance

We apply the method of multiple scales directly to the governing partial-differential system (12), (13). Introducing a small dimensionless parameter *e* as a bookkeeping device we obtain the problem:

$$L_2[w(x,t)] + \varepsilon \cdot \alpha^* N[w(x,t)] + \frac{\partial^2 w}{\partial t^2} = \varepsilon f(x)\cos(\Omega t) \tag{14}$$

$$w = \frac{\partial w}{\partial x} = 0 \text{ at } x = x_0 \quad \text{and} \quad w \text{ finite at } x = 1 \ . \tag{15}$$

The scheme for ordering the terms is consistent with our notions of primary resonance (we anticipate that a small-amplitude excitation produces large-amplitude response). A first-order uniform expansion is considered as

$$w(x,t,\varepsilon) = w_0(x, T_0, T_1) + \varepsilon \cdot w_1(x, T_0, T_1) \ , \tag{16}$$

where $T_0 = t$ is a fast scale and $T_1 = \varepsilon t$ is a slow scale. The time derivative becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon \cdot D_1 \quad \text{where} \quad D_n = \frac{\partial}{\partial T_n} \ . \tag{17}$$

Using the equation (16), and equating coefficients of like powers of *e* , from (14), we obtain:

Order e0

$$D_0^2 w_0 + L_2[w_0] = 0 \ , \tag{18}$$

$$w_0 = \frac{\partial w_0}{\partial x} = 0 \ \text{ at } x = x_0 \quad \text{and} \quad w_0 \ \text{ finite at } x = 1 . \tag{19}$$

Order e

$$D_0^2 w_1 + L_2[w_1] = -2 D_0 D_1 w_0 - \alpha^* N(w_0) + f(x)\cos(\Omega t), \tag{20}$$

$$w_1 = \frac{\partial w_1}{\partial x} = 0 \ \text{ at } x = x_0 \quad \text{and} \quad w_1 \ \text{ finite at } x = 1 . \tag{21}$$

We write the solution of equation (18)

$$w_{0k}(x, T_0, T_1) = \varphi_k(x)\left[A_k(T_1)e^{i\omega T_0} + \overline{A_k}(T_1)e^{-i\omega T_0}\right], \tag{22}$$

where A_k is undetermined at this moment of approximation. The linear free vibrations case is presented by Caruntu [3] with numerical determinations in [4].

Instead of using the frequency of the excitation W as a parameter, we introduce a detuning parameter *s*, which quantitatively describes the nearness of W to ω_k. Accordingly we write

$$\Omega = \omega_k + \varepsilon\sigma \tag{23}$$

Substituting (22) into (20) we obtain

$$D_0^2 w_1 + L_2[w_1] = -2i\omega_k \varphi_k(x)\left[A_k' e^{i\omega_k T_0} - \overline{A_k'} e^{-i\omega_k T_0}\right] -$$

$$-\alpha^* \cdot \left[\omega_k^2 N_1(\varphi_k) + N_2(\varphi_k)\right]\left(A_k e^{i\omega_k T_0} + \overline{A_k} e^{-i\omega_k T_0}\right)^3 + f(\varphi_k)\cos(\omega_k T_0 + \sigma T_1). \tag{24}$$

Using (10), (7) and (8), we have denoted the nonlinear operators

$$N_1(\varphi_k) = \varphi_k(\varphi_k')^2, \tag{25}$$

$$N_2(\varphi_k) = (1 - x^2)^2\left[3\varphi_k'\varphi_k''\varphi_k''' + 3(\varphi_k'')^3\right] - 30x(1 - x^2)\varphi_k'(\varphi_k'')^2 . \tag{26}$$

This inhomogeneous equation (24) and the condition (21), have a solution only if a solvability condition is satisfied.

It means that, the right hand sides of (24) be orthogonal to every solution of the homogeneous problem

$$-2i\omega_k g_{1kk}\left(A_k' e^{i\omega_k T_0} - \overline{A_k'} e^{-i\omega_k T_0}\right) - \alpha^*\left(\omega_k^2 g_{2kk} + g_{3kk}\right)\left(A_k e^{i\omega_k T_0} + \overline{A_k} e^{-i\omega_k T_0}\right)^3 + \frac{1}{2} f_k\left(e^{i(\omega_k T_0 + \sigma T_1)} + e^{-i(\omega_k T_0 + \sigma T_1)}\right) = 0 , \tag{27}$$

where

$$g_{1kk} = \langle \varphi_k, \varphi_k \rangle \quad , \quad g_{2kk} = \langle \varphi_k, N_1(\varphi_k) \rangle \quad , \quad g_{3kk} = \langle \varphi_k, N_2(\varphi_k) \rangle \quad , \quad f_k = (\varphi_k, f(\varphi_k)) \tag{28}$$

The inner product between $\varphi_m(x)$ and $\varphi_n(x)$ is defined by:

$$\langle \varphi_m(x), \varphi_n(x) \rangle = \int_{x_0}^{1} (1 - x^2)\varphi_m(x)\varphi_n(x)\,\mathrm{d}x . \tag{29}$$

Because the operator L_2 is self-adjoint with given boundary conditions defined, the eigenfunctions $\varphi_m(x)$ corresponding to different eigenvalues ω_m, are orthogonal. Secular terms will be eliminated if we choose A_k to be a solution of

$$-2i\omega_k g_{1kk} A_k' - 3\alpha^*\left(\omega_k^2 g_{2kk} + g_{3kk}\right)A_k^2 \overline{A_k} + \frac{1}{2} f_k e^{i\sigma T_1} = 0 \quad . \tag{30}$$

This equation governs the amplitude and phase evolution. Expressing A_k in the polar form

$$A_k = \frac{1}{2} a_k e^{i\beta_k} \quad , \tag{31}$$

where a_k and β_k (first approximations to the amplitude and phase of the motion) are real, and separating equation (30) into real and imaginary parts, we obtain for the amplitude and for the phase:

$$a_k' = \frac{1}{2\omega_k g_{1kk}} f_k \sin\left(\sigma T_1 - \beta_k\right) \tag{32}$$

$$\omega_k a_k \beta_k' = \frac{3\alpha^*}{8}\left(\omega_k^2 \frac{g_{2kk}}{g_{1kk}} + \frac{g_{3kk}}{g_{1kk}}\right)a_k^3 + \frac{1}{2} f_k \cos\left(\sigma T_1 - \beta_k\right) \quad . \tag{33}$$

Equations (33) can be be transformed into an autonomous system (T_1 does not appear explicitly) by letting

$$\gamma_k = \sigma T_1 - \beta_k \tag{34}$$

The result is

$$a_k' = \frac{1}{2\omega_k g_{1kk}} f_k \sin\gamma_k \tag{35}$$

$$\gamma_k' = \sigma - \frac{3\alpha^*}{8\omega_k g_{1kk}}\left(\omega_k^2 \frac{g_{2kk}}{g_{1kk}} + \frac{g_{3kk}}{g_{1kk}}\right)a_k^2 + \frac{f_k}{2\omega_k g_{1kk} a_k}\cos\gamma_k \quad . \tag{36}$$

The steady-state motion is obtained considering

$$\frac{1}{2\omega_k g_{1kk}} f_k \sin\gamma_k = 0 \tag{37}$$

$$\sigma = \frac{3\alpha^*}{8\omega_k g_{1kk}}\left(\omega_k^2 \frac{g_{2kk}}{g_{1kk}} + \frac{g_{3kk}}{g_{1kk}}\right)a_k^2 - \frac{f_k}{2\omega_k g_{1kk} a_k}\cos\gamma_k \quad . \tag{38}$$

We have the solutions of (37)

$$\gamma_k = n\pi \quad , \quad n \text{ integer} \tag{39}$$

then

$$\sigma = \frac{3\alpha^*}{8\omega_k g_{1kk}}\left(\omega_k^2 \frac{g_{2kk}}{g_{1kk}} + \frac{g_{3kk}}{g_{1kk}}\right)a_k^2 \mp \frac{f_k}{2\omega_k g_{1kk} a_k} \tag{40}$$

If the coefficients g_{1kk}, g_{2kk}, g_{3kk} are determined for a considered ω_k (using [4]), then the frequency-response curve can be ploted (a_k as a function of σ). Each point of this curve corresponds to a singular point in a different state plane; there is one state plane for each parameter f_k. The backbone curve is the parabola

$$\sigma = \frac{3\alpha^*}{8\omega_k g_{1kk}} \left(\omega_k^2 \frac{g_{2kk}}{g_{1kk}} + \frac{g_{3kk}}{g_{1kk}} \right) a_k^2 \tag{41}$$

In the absence of damping, the peak amplitude is infinite, and the frequency-response consists of two branches having as their asymptote the curve (41). Relation (39) shows that the response is either in phase or *1800* out of phase with the excitation.

5 Conclusion

Using the method of multiple scales, the primary resonances in the case of single-mode forced motion of a cantilever beam with a constant width, parabolic thickness variation and a sharp end, in the case of nonlinear curvature, are studied. This method is applied directly to the nonlinear partial differential equation and boundary conditions. The first four linear mode shapes and natural frequencies are already determined by Caruntu [4] using the factorization method. This beam is a good approximation of a gear tooth with a sharp end (the case of a gear with a small number of teeth) . The frequency-response curves are analytically determined. The significance of this paper is represented by the analytical results; in this way we can have numerical determinations.

Acknowledgements:

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