Solutions of weakly nonlinear systems bounded on R in resonance

A.A.Boichuk

Abstract
Conditions for the existence of solutions bounded on R are obtained for a weakly nonlinear ordinary differential system in assumption that the operator L defined by linear ($\varepsilon = 0$) homogeneous system is a Fredholm operator and the corresponding inhomogeneous linear system has an $r-$ parametric set of solutions bounded on R. If L is a Fredholm operator with index zero and in case $r = 1$ we obtain the well-known result of K.Palmer.

1 Introduction

Linear systems. Let us denote by $BC(J)$ the Banach space of continuous vector functions $x : J \to R^n$ bounded on an interval J with norm $\|x\| = \sup_{t\in J} |x(t)|$, and by $BC^1(J)$ the Banach space of vector functions $x : J \to R^n$, continuously differentiable on J and bounded together with their derivative and with norm $\|x\| = \sup_{t\in J} |x(t)| +\sup_{t\in J} |\dot{x}(t)|$. Let us consider the system

$$\dot{x} = A(t)x \tag{1}$$

with an $n \times n$ - matrix $A(t)$, whose components are real functions, continuous and bounded on the whole line $R = (-\infty, +\infty)$: $A(\cdot) \in BC(R)$. It is known [1, 2] that the system (1) is an exponential-dichotomy (e-dichotomy) on an interval J if there exists a projector $P(P^2 = P)$ and constants $K \geq 1, \alpha > 0$ such that

$$\|X(t)PX^{-1}(s)\| \leq Ke^{-\alpha(t-s)}, t \geq s$$

$$\|X(t)(I - P)X^{-1}(s)\| \leq Ke^{-\alpha(s-t)}, s \geq t$$

for all $t, s \in J$; $X(t)$ is the normal ($X(0) = I$) fundamental matrix of system (1).

Let's consider the problem about solutions $x : R \to R^n, x(\cdot) \in BC^1(R)$ bounded on R of the inhomogeneous system

$$\dot{x} = A(t)x + f(t), \quad f(\cdot) \in BC(R) \tag{2}$$

In the case where the homogeneous system (1) is an e-dichotomy on R, and so system (1) has only trivial solution bounded on R the inhomogeneous system (2) has a unique solution bounded on R for each $f(\cdot) \in BC(R)$. The resonance case where system (1) has nontrivial solutions bounded on R is considerably less investigated. The well-known result of K.Palmer [2, p.245], giving sufficient conditions for the Fredholm property of the considered problem is formulated as follows:

Lemma 1. Let system (1) be an e-dichotomy on both half-lines $R_+ = [0, +\infty)$ and $R_- = (-\infty, 0]$. Then an operator $L : \quad BC^1(R) \to BC(R)$ defined by

$$(Lx)(t) = \dot{x}(t) - A(t)x(t) \tag{3}$$

is a Fredholm operator and $f \in Im(L)$ if and only if

$$\int_{-\infty}^{\infty} \psi^*(t) f(t) dt = 0, \tag{4}$$

for all solutions $\psi(t)$ bounded on R of the system

$$\dot{x} = -A^*(t)x, \tag{5}$$

adjoint to (1). The index of L is $indL = dimV + dimW - n$, where V and W are the stable and unstable subspaces for system (1) [3, p.389].

2 Actual contents

Let us define more exactly some results of this Palmer's lemma, which will be used below for the investigation of weakly nonlinear systems. The general solution of (2), bounded on both half-lines R_+ and R_- is given by

$$x(t,\xi) = \begin{cases} X(t)P\xi + \int_0^t X(t)PX^{-1}(s)f(s)ds - \\ \int_t^\infty X(t)(I-P)X^{-1}(s)f(s)ds, t \geq 0; \\ \\ X(t)(I-Q)\xi + \int_{-\infty}^t X(t)QX^{-1}(s)f(s)ds - \\ \int_t^0 X(t)(I-Q)X^{-1}(s)f(s)ds, t \leq 0. \end{cases} \quad (6)$$

Solution (6) will be bounded on R only if the vector constant $\xi \in R^n$ satisfies the condition

$$P\xi - \int_0^\infty (I-P)X^{-1}(s)f(s)ds = (I-Q)\xi + \int_{-\infty}^0 QX^{-1}(s)f(s)ds$$

so that the constant $\xi \in R^n$ is determined from the algebraic system

$$[P-(I-Q)]\xi = \int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^\infty (I-P)X^{-1}(s)f(s)ds \quad (7)$$

Let $D = P-(I-Q)$ be an $(n\times n)-$ dimensional matrix, and let D^+ be an $(n\times n)-$ dimensional matrix which is a pseudoinverse on Moore-Penrose to D [4, 5]. By $P_{N(D)}$ $(P_{N(D^*)})$ we shall denote an $(n\times n)-$ dimensional matrix - orthoprojector: $P^2_{N(D)} = P_{N(D)} = P^*_{N(D)}$, $(P^2_{N(D^*)} = P_{N(D^*)} = P^*_{N(D^*)})$, projecting R^n onto the null - space $N(D) = kerD$ of the matrix D, (onto the null-space $N(D^*) = kerD^*$ of the matrix D^* transpose to D).
The system (2) has solutions bounded on R only if the algebraic system (7) is solvable over $\xi \in R^n$. For this it is necessary and sufficient, that the right-hand side of the system (7) belong to the orthogonal complement $N^\perp(D^*) = R(D)$ to subspace $N(D^*)$. It follows that

$$P_{N(D^*)}\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^\infty (I-P)X^{-1}(s)f(s)ds\} = 0. \quad (8)$$

Thus the general solution of the system (2) bounded on R has a form (6) with constant $\xi \in R^n$, which is determined from (7) as follows

$$\xi = D^+\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^\infty (I-P)X^{-1}(s)f(s)ds\} + P_{N(D)}c, \ \ \forall c \in R^n. \quad (9)$$

In other words: $f(\cdot) \in Im(L)$ only if the condition (8) is satisfied and thus the general solution of the system (2) bounded on the whole line R has a form $x(t,c) =$

$$
= \begin{cases} X(t)PP_{N(D)}c + \int_0^t X(t)PX^{-1}(s)f(s)ds - \\ - \int_t^{\infty} X(t)(I-P)X^{-1}(s)f(s)ds + \\ + X(t)PD^{+}\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds\}, \\ t \geq 0; \\ X(t)(I-Q)P_{N(D)}c + \int_{-\infty}^t X(t)QX^{-1}(s)f(s)ds - \\ - \int_t^0 X(t)(I-Q)X^{-1}(s)f(s)ds + \\ + X(t)(I-Q)D^{+}\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds\}, \\ t \leq 0. \end{cases}
$$

Since $DP_{N(D)} = 0$ [5, p.90], then $PP_{N(D)} = (I-Q)P_{N(D)}$. Let $dimN(L) = r$, then $r = rang[PP_{N(D)}] = rang[(I-Q)P_{N(D)}]$ and vice versa. Let $[PP_{N(D)}]_r = [(I-Q)P_{N(D)}]_r$ be an $(n \times r)-$ dimensional matrix, whose columns are complete set of r linearly - independent columns of the matrix $PP_{N(D)} = (I-Q)P_{N(D)}$. Then

$$
X_r(t) = X(t)[PP_{N(D)}]_r = X(t)[(I-Q)P_{N(D)}]_r
$$

is an $(n \times r)$ - dimensional matrix, whose columns are complete set of r linearly - independent solutions of the system (2) bounded on R . Therefore the general solution of the system (2) bounded on R can be written as

$$
x(t, c_r) = X_r(t)c_r + (G[f])(t), \quad \forall c_r \in R^r, \tag{10}
$$

where : $(G[f])(t) =$

$$
= \begin{cases} \int_0^t X(t)PX^{-1}(s)f(s)ds - \int_t^{\infty} X(t)(I-P)X^{-1}(s)f(s)ds + \\ + X(t)PD^{+}\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds\}, \\ t \geq 0; \\ \int_{-\infty}^t X(t)QX^{-1}(s)f(s)ds - \int_t^0 X(t)(I-Q)X^{-1}(s)f(s)ds + \\ + X(t)(I-Q)D^{+}\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds\}, \\ t \leq 0; \end{cases}
$$

is the generalized Green operator for the problem of solutions of the system (2) bounded on the whole line R .

Since $P_{N(D^*)}D = 0$ [5, p.90], we have $P_{N(D^*)}Q = P_{N(D^*)}(I-P)$. Therefore the condition (8) is equivalent to one of conditions

$$
P_{N(D^*)}\int_{-\infty}^{\infty} QX^{-1}(s)f(s)ds = 0, \;\; P_{N(D^*)}\int_{-\infty}^{\infty}(I-P)X^{-1}(s)f(s)ds = 0. \tag{11}
$$

Let $d = rang[P_{N(D^*)}(I-P)] = rang[P_{N(D^*)}Q] = dim N(L^*)$, then each of conditions (11) consists only from d linearly - independent conditions. Really, let $[Q^*P_{N(D^*)}]_d$ $({}_d[P_{N(D^*)}Q])$ be an $n \times d$ ($d \times n$) - dimensional matrix whose columns (rows) are d - linearly-independent columns (rows) of the matrix $[Q^*P_{N(D^*)}]$ ($[P_{N(D^*)}Q]$). Note that $X^{*-1}(t)$ is the fundamental matrix of the system (5), which is an e-dichotomy on R_+ with a projector $I - P^*$ and on R_- with a projector $I - Q^*$ [2, p.246]. Then, as above

$$H_d(t) = X^{*-1}(t)[Q^*P_{N(D^*)}]_d = X^{*-1}(t)[(I-P^*)P_{N(D^*)}]_d$$

is an $n \times d-$ dimensional matrix whose columns are complete set of d linearly - independent solutions bounded on R of the system (5), adjoint to (1); hence

$$H_d^*(t) ={}_d [P_{N(D^*)}Q]X^{-1}(t) ={}_d [P_{N(D^*)}(I-P)]X^{-1}(t)$$

is an $d \times n-$ dimensional matrix, whose rows are complete set of d linearly - independent solutions of the system (5) bounded on R . Thus Lemma 1 can be formulated as follows.

Lemma 2. Let system (1) be an e-dichotomy on R_+ and R_- with projectors P and Q, respectively. Then:

a) an operator L is a Fredholm;

b) the homogeneous system (1) has $r-$ parametric set ($r = rang[PP_{N(D)}] = rang[(I-Q)P_{N(D)}]$) of solutions bounded on R: $X_r(t)c_r, \ \ \forall c_r \in R^r$;

c) the system (5) adjoint to (1) has $d-$ parametric set ($d = rang[P_{N(D^*)}(I-P)] = rang[P_{N(D^*)}Q]$) of solutions bounded on R : $H_d(t)c_d, \ \ \forall c_d \in R^d$;

d) $f \in Im(L)$ in only case when:

$$\int_{-\infty}^{\infty} H_d^*(s)f(s)ds = 0; \tag{12}$$

the inhomogeneous system (2) has an $r-$ parametric set of solutions (10) bounded on R;

e) $indL = rang[P_{N(D^*)}(I-P)] - rang[PP_{N(D)}] =$

$$rang[P_{N(D^*)}Q] - rang[(I-Q)P_{N(D)}] = d - r.$$

Nonlinear systems. For weakly nonlinear system

$$\dot{x} = A(t)x + f(t) + \varepsilon Z(x,t,\varepsilon), \tag{13}$$

let us find conditions for the existence of solutions bounded on R

$$x = x(t,\varepsilon) : x(\cdot,\varepsilon) : R \to R^n, x(\cdot,\varepsilon) \in BC^1(R), x(t,\cdot) \in C[0,\varepsilon_0],$$

which turns, for $\varepsilon = 0$, into one of generating solutions $x_0(t,c_r)$ (10) of the system (2). The nonlinear vector function $Z(x,t,\varepsilon)$ is such that:

$$Z(\cdot,t,\varepsilon) \in C^1[\ \|x - x_0\| \leq q\ ]; \quad Z(x,\cdot,\varepsilon) \in BC(R); \quad Z(x,t,\cdot) \in C[0,\varepsilon_0].$$

Theorem 1 (necessary condition).
Assume, that the system (1) is an e-dichotomy on R_+ and R_- with projectors P and Q, respectively. Let the system (13) have solution bounded on R $x(t,\varepsilon) : x(\cdot,\varepsilon) : R \to R^n, x(\cdot,\varepsilon) \in BC^1(R), x(t,\cdot) \in C[0,\varepsilon_0]$, and $x(t,\varepsilon)$ turns, for $\varepsilon = 0$, into one of generating solutions $x_0(t,c_r)$(10) of the system (2) with the vector constant $c_r = c_r^* \in R^r$. Then the vector c_r^* satisfies the equation

$$F(c_r^*) = \int_{-\infty}^{\infty} H_d^*(s) Z(x_0(s,c_r^*),s,0)ds = 0. \tag{14}$$

Proof. The condition (12) of the existence of generating solutions bounded on R $x_0(t,c_r^*)$ is assumed to be fulfilled. Considering the nonlinearity in (13) as nonhomogeneity and applying lemma 2 to (13),we obtain the following condition

$$\int_{-\infty}^{\infty} H_d^*(s) Z(x(s,\varepsilon),s,\varepsilon)ds = 0.$$

Passing to the limit as $\varepsilon \to 0$ in an integral we come to required condition (14).

By analogy to a case of the periodic problem [5, p.184] if is natural to call the equation (14) the equation for generating amplitudes of the problem about solutions of the system (13) bounded on the whole line R . If the equation (14) has a solution, the vector constant $c_r^* \in R^r$ determines that generating solution $x_0(t,c_r^*)$ to which the solution bounded on R $x = x(t,\varepsilon)$:

$x(\cdot,\varepsilon): R \to R^n, x(\cdot,\varepsilon) \in BC^1(R), x(t,\cdot) \in C[0,\varepsilon_0], x(t,0) = x_0(t,c_r^*)$ of the original problems (13) may correspond. However, if the equation (14) has no solution, the problem (13) has no solution bounded on R in considered space. Since here and below all expressions are obtained in the real form, we speak about real solutions of the equation (14), which may be algebraic or transcendental.
By changing the variables in (13) according to the relation

$$x(t,\varepsilon) = x_0(t,c_r^*) + y(t,\varepsilon),$$

we arrive at the problem of finding sufficient conditions for the existence of solution bounded on R $y = y(t,\varepsilon): y(\cdot,\varepsilon): R \to R^n, y(\cdot,\varepsilon) \in BC^1(R), y(t,\cdot) \in C[0,\varepsilon_0], y(t,0) = 0$ for the problem:

$$\dot{y} = A(t)y + \varepsilon Z(x_0(t,c_r^*) + y, t, \varepsilon). \tag{15}$$

Taking into account the continuous differentiability of a vector function $Z(x,t,\varepsilon)$ in x and its continuity in ε in a neighborhood of a point $x_0(t,c_r^*), \varepsilon = 0$, we can select a term linear in y and terms of zero order in ε :

$$Z(x_0(t,c_r^*) + y, t, \varepsilon) = f_0(t,c_r^*) + A_1(t)y + R(y(t,\varepsilon),t,\varepsilon), \tag{16}$$

where

$$f_0(t,c_r^*) = Z(x_0(t,c_r^*),t,0), \quad f_0(\cdot,c_r^*) \in BC(R);$$

$$A_1(t) = A_1(t,c_r^*) = \frac{\partial Z(x,t,0)}{\partial x}\,|_{x=x_0(t,c_r^*)}, \quad A_1(\cdot) \in BC(R);$$

$$R(0,t,0) = 0, \frac{\partial R(0,t,0)}{\partial y} = 0, \quad R(y,\cdot,\varepsilon) \in BC(R).$$

.

Regarding formally the nonlinearity $Z(x_0 + y, t, \varepsilon)$ in the system (15) as nonhomogeneity and applying lemma 2 to (15), we obtain the following representation of a solution of the system (15) bounded on R

$$y(t,\varepsilon) = X_r(t)c + y^{(1)}(t,\varepsilon).$$

In this expression the unknown vector of constants $c = c(\varepsilon) \in R^r$ is determined from the condition type (12) of the existence of such solution for the system (15) :

$$B_0 c = -\int_{-\infty}^{\infty} H_d^*(\tau)[A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)]d\tau, \qquad (17)$$

where:

$$B_0 = \int_{-\infty}^{\infty} H_d^*(\tau)A_1(\tau)X_r(\tau)d\tau =$$

$$=_d [P_{N(D^*)}Q] \int_{-\infty}^{\infty} X^{-1}(\tau)A_1(\tau)X(\tau)d\tau[(I-Q)P_{N(D)}]_r$$

is an $(d \times r)$ - dimensional matrix $(r = rang[PP_{N(D)}] = rang[(I-Q)P_{N(D)}],\ \ d = rang[P_{N(D^*)}(I-P)] = rang[P_{N(D^*)}Q]\ \ )$.

The unknown vector function $y^{(1)}(t,\varepsilon)$ is determined by the help of the generalized Green operator (10) from the relation:

$$y^{(1)}(t,\varepsilon) = \varepsilon\,(G\,[Z(x_0(t,c_r^*) + y, t, \varepsilon)])\,(t),$$

Let $P_{N(B_0)}$ be an $(r \times r)-$ dimensional matrix - orthoprojector: $R^r \to N(B_0)$, and let $P_{N(B_0^*)}$ be a $(d \times d)-$ dimensional matrix - orthoprojector: $R^d \to N(B_0^*)$. The equation (17) is solvable with respect to $c \in R^r$ if and only if

$$P_{N(B_0^*)} \int_{-\infty}^{\infty} H_d^*(\tau)[A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)]d\tau = 0. \qquad (18)$$

If $P_{N(B_0^*)}\ {}_d[P_{N(D^*)}Q] = 0$, then the condition (18) is always hold. If, in addition, $P_{N(B_0)} = 0$, then the equation (17) is uniquely solvable with respect to $c \in R^r$. For finding solutions of the problem (15) bounded on $R \quad y = y(t,\varepsilon) : y(\cdot,\varepsilon) : R \to R^n, y(\cdot,\varepsilon) \in BC^1(R), y(t,\cdot) \in C[0,\varepsilon_0], y(t,0) = 0$ we arrive at the following operator system, which is equivalent to (15) on considered space of functions

$$y(t,\varepsilon) = X_r(t)c + y^{(1)}(t,\varepsilon), \qquad (19)$$

$$c = -B_0^+ \int_{-\infty}^{\infty} H_d^*(\tau)[A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)]d\tau,$$

$$y^{(1)}(t,\varepsilon) = \varepsilon\left(G\left[Z(x_0(t,c_r^*) + y, t, \varepsilon)\right]\right)(t)$$

The operator system (19) belongs to the class of systems [5, p.188], for which solvability is applicable a simple iteration method convergent for $\varepsilon \in [0,\varepsilon_*] \subseteq [0,\varepsilon_0]$. Really, the system (19) can be rewritten as:

$$z = L^{(1)}z + Fz, \tag{20}$$

where:$z = col(y(t,\varepsilon), c(\varepsilon), y^{(1)}(t,\varepsilon)) - (2n+r)-$ dimensional column vector; $L^{(1)}$ and F are linear and nonlinear operators bounded on R :

$$L^{(1)} = \begin{pmatrix} 0 & X_r & I_n \\ 0 & 0 & L_1 \\ 0 & 0 & 0 \end{pmatrix}; L_1 * = -B_0^+ \int_{-\infty}^{\infty} H_d^*(\tau) A_1(\tau) * d\tau;$$

$$Fz = col\left[0, \int_{-\infty}^{\infty} H_d^*(\tau) R(y(\tau,\varepsilon),\tau,\varepsilon) d\tau, \varepsilon G\left[Z(x_0(\tau,c_r^*) + y, \tau, \varepsilon)\right]\right]$$

By virtue of a structure of an operator $L^{(1)}$ with zero blocks at the principal diagonal and below, the system (20) may be transformed to the form

$$z = \tilde{L}Fz, \quad \tilde{L} = (I_s - L^{(1)})^{-1}, \quad s = 2n + r, \tag{21}$$

for the solution of which one of variants of a fixed point principle [6] is applicable for sufficiently small $\varepsilon \in [0,\varepsilon_*]$. Using a simple iteration method for finding a solution of the operator systems (19), and hence for finding solution of the original system (13) bounded on R , we arrive at the following result [7].

Theorem 2 (sufficient condition). Assume that the weakly nonlinear system (13) satisfies the conditions stated above, and thus the corresponding generating linear system (2) has an r - parameter set of generating solutions $x_0(t,c_r)$ (10) bounded on R. Then, for every value of the vector $c_r = c_r^* \in R^r$ that satisfies the equation for generating amplitudes (14), provided that the condition

$$P_{N(B_0)} = 0, \; P_{N(B_0^*)\ d}[P_{N(D^*)}Q] = 0, \tag{22}$$

is satisfied, there exists a unique solution bounded on R of the system (13). This solution $x(t,\varepsilon):\; x(t,\cdot) \in C[0,\varepsilon_0]$ turns, for $\varepsilon = 0$, into the generation solution $x(t,0) = x_0(t, c_r^*)$ (10) and can be determined by a simple iteration method convergent for $\varepsilon \in [0, \varepsilon_*] \subseteq [0, \varepsilon_0]$:

$$y_{k+1}^{(1)}(t, \varepsilon) = \varepsilon \left(G \left[Z(x_0(\tau, c_r^*) + y_k, \tau, \varepsilon)\right]\right)(t)$$

$$c_{k+1} = -B_0^+ \int_{-\infty}^{\infty} H_d^*(\tau)[A_1(\tau) y_{k+1}^{(1)}(\tau, \varepsilon) + R(y_k(\tau, \varepsilon), \tau, \varepsilon)] d\tau,$$

$$y_{k+1}(t, \varepsilon) = X_r(t) c_{k+1} + y_{k+1}^{(1)}(t, \varepsilon),$$

$$x_k(t, \varepsilon) = x_0(t, c_r^*) + y_k(t, \varepsilon), \quad k = 0, 1, 2, ...; \quad y_0(t, \varepsilon) = 0.$$

3 Conclusion

Necessary estimates for ε_* and for the error of approximation of iteration process can be obtained in the standard way [6].

The condition (22) means [5] that the constant $c_r^* \in R^r$ is a simple root of the equation (14) for generating amplitudes of the problem about solutions of the system (13) bounded on the whole line R.

If L is a Fredholm operator with index zero and in case $r = 1$ from this theorem we obtain the well-known result of K. Palmer [2, p.248].

References

[1] Coppel W.A. *Dichotomies in stability theory,* Lecture Notes in Math.,**629**, Springer-Verlag, Berlin, (1978).

[2] Palmer K.J."Exponential dichotomies and transversal homoclinic points", J.Differential Equations,**55**, 225-256 (1984).

[3] Sacker R.J."The splitting index for linear differential systems", J.Differential Equations, **33**,368-405 (1979).

[4] Rao C.R., Mitra S.K. *Generalized inverse of matrices and its applications,*. – New York: Wiley, (1971).

[5] Boichuk A.A., Zhuravlev V.F., Samoilenko A.M.,*Generalized inverse operators and Noether boundary value problems,* Inst.of Mathematics NAS of Ukraine, Kyiv, Ukraine (1995). (in Russian).

[6] Krasnosel'skii M.A., Vainikko G.M.,Zabreiko P.P.,Rutitskii Ya.B. and Stetsenko V.Ya., *Approximate solution of operator equation,* Moscow, Nauka,(1968). (in Russian).

[7] Boichuk A.A."Solutions of weakly nonlinear differential equations bounded on the whole line", Nonlinear Oscillations, **2**,No 1, 3-10 (1999).

Author

Boichuk, Alexander
National Academy of Sciences of Ukraine
Institute of Mathematics
Tereshchenkivs'ka str.3,
252601, Kyiv-4,
Ukraine
E-mail: boichuk@imath.kiev.ua