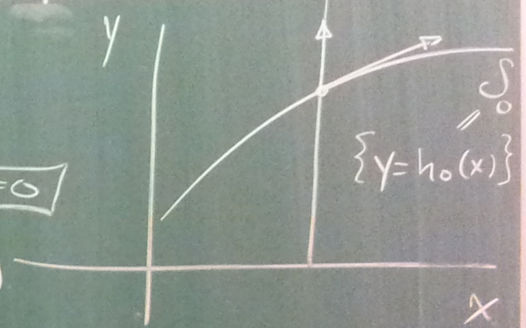
$\boxed{\varepsilon = 0}$

$$\left.\begin{aligned} x' &= 0 \\ y' &= \underbrace{g(x,y,0)}_{g_0(x,y)} \end{aligned}\right\} \Rightarrow \begin{aligned} x' &= 0 \\ y' &= g_0 \end{aligned} \left\{ \underbrace{z}_{\binom{x}{y}}{}' = \underbrace{G_0}_{\binom{0}{g_0}} \right.$$

$x \in \mathbb{R}^m,\ y \in \mathbb{R}^n$

$$\begin{aligned} \mathring{x} &= f(x,y,\varepsilon) \\ \varepsilon \mathring{y} &= g(x,y,\varepsilon) \end{aligned} \qquad \begin{aligned} x' &= \varepsilon f(x,y,\varepsilon) \\ y' &= g(x,y,\varepsilon) \end{aligned}$$



$\boxed{\varepsilon=0}$ tgt directions @ $O(1)$

$\boxed{\text{Eigenspaces of } Dg}$

$$\underbrace{D_z G_0}_{\text{Jacobian}} = \begin{pmatrix} \boxed{0} & 0 \\ D_x g_0 & \boxed{D_y g_0} \end{pmatrix}$$

ⓐ Spectrum of Jacobian : $\sigma(D_z G_0) = \underbrace{\{0\}}_{\text{tgt directions @ } O(1)} \cup \sigma\underbrace{(D_y g_0)}_{\text{"tgt to fibers" directions @ } O(1)}$

"tgt to fibers" = vertical, directions @ $O(1)$ $\overset{\varepsilon=0}{\leftrightarrow}$

ⓑ $\mathbb{E}_f(z)$ = eigenspace assoc'd with $D_y g_0 = \mathrm{Span}\begin{pmatrix} 0 \\ I \end{pmatrix}$

⊛ $D_y G_0 \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ D_x g_0 & D_y g_0 \end{pmatrix}\begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ D_y g_0 \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} D_y g_0$

(same) basis

$\Rightarrow \sigma\left(D_y g_0^{(z)} \big|_{\mathbb{E}_f(z)} \right) = \sigma(D_y g_0(z))$.

⊛ $P^{-1} A P = J$

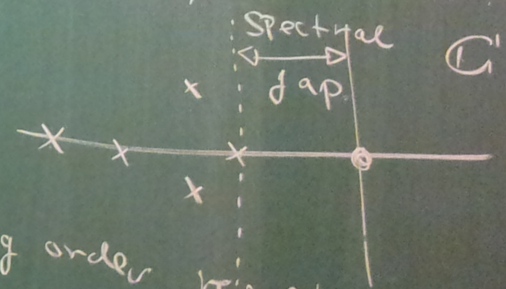
$\Rightarrow AP = PJ$

ⓒ $\mathbb{E}_S(z)$ = e'space assoc'd with $\{0\} = \mathrm{Ker}(D_y G_0(z)) = \mathrm{Span}\begin{pmatrix} I \\ Dh_0 \end{pmatrix}$

⊛ $D_z G_0 \begin{pmatrix} I \\ Dh_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ D_x g_0 & D_y g_0 \end{pmatrix} \begin{pmatrix} I \\ Dh_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \underbrace{D_x g_0 + D_y g_0 \, Dh_0}_{?} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

Know $S_0 = \{ g_0(z) = 0 \} \Rightarrow g_0(x, h_0(x)) = 0 \xRightarrow{D_x} D_x g_0 + D_y g_0 \, Dh_0 = 0$

⊙ Normal Hyperbolicity : $0 \notin \sigma(D_y g_0)$



⊛ Does NOT hold for $\varepsilon \neq 0$! IOW, it's a leading order result

$\boxed{\text{New basis}}$

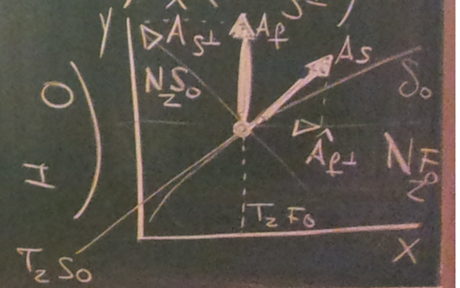
$z(x) = (x, h_0(x)) \in S_0$

$\underbrace{A(z)}_{(n+m)\times(n+m)} = \begin{pmatrix} I & 0 \\ Dh(x) & I \end{pmatrix} = (\underset{(n+m)\times m}{A_s}, \underset{(n+m)\times n}{A_f}) \Leftrightarrow \overbrace{A^{-1}}^{\hat{A}} = \begin{pmatrix} I & 0 \\ -Dh & I \end{pmatrix} = \begin{pmatrix} \hat{A}_{f\perp} \\ \hat{A}_{s\perp} \end{pmatrix} \begin{array}{l} m \times (n+m) \\ n \times (n+m) \end{array}$

every column is a vector

⊛ $\hat{A} A = I \Rightarrow \begin{pmatrix} \hat{A}_{f\perp} \\ \hat{A}_{s\perp} \end{pmatrix} (A_s, A_f) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{A}_{f\perp} A_s & \hat{A}_{f\perp} A_f \\ \hat{A}_{s\perp} A_s & \hat{A}_{s\perp} A_f \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$

$\Rightarrow \left| \begin{array}{l} \hat{A}_{f\perp} A_f = 0 \Rightarrow \text{row span}(\hat{A}_{f\perp}) \perp \text{col span}(A_f) \\ \hat{A}_{s\perp} A_s = 0 \Rightarrow \text{row span}(\hat{A}_{s\perp}) \perp \text{col span}(A_s) \end{array} \right.$

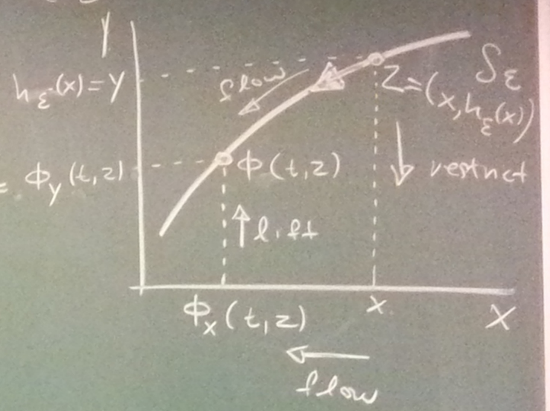


Invariance Equation

Let the flow corresponding to $G = \begin{pmatrix} \varepsilon f \\ g \end{pmatrix}$ be denoted by $\phi(t,z)$

$$\Rightarrow \phi(\underset{\substack{\uparrow \\ \text{time}}}{t}, \underset{\hookrightarrow \text{IC}}{z}) = \begin{pmatrix} \phi_x(t,z) \\ \phi_y(t,z) \end{pmatrix}.$$ Then, on S_ε, we have

$$h_\varepsilon(\phi_x(t,z)) = \phi_y(t,z)$$



$$\phi_y(t,z) - h_\varepsilon(\phi_x(t,z)) = 0$$

$$\overset{(\cdot)'}{\Longrightarrow} \underbrace{(\phi_y(t,z))'}_{y'} - D_x h_\varepsilon(\phi_x(t,z)) \underbrace{(\phi_x(t,z))'}_{x'} = 0$$

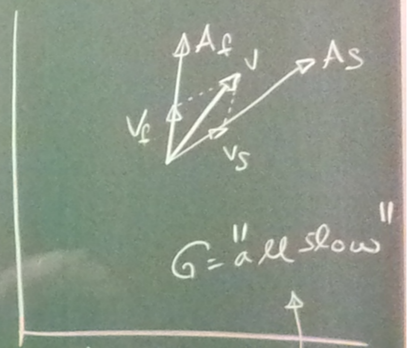
$$\Rightarrow g(\overbrace{\phi(t,z)}^{\in S_\varepsilon}, \varepsilon) - \varepsilon D_x h_\varepsilon(\overbrace{\phi_x(t,z)}^{\text{vertical proj from } S_\varepsilon}) f(\overbrace{\phi(t,z)}^{\in S_\varepsilon}, \varepsilon) = 0$$

$$\Rightarrow g(z(x), \varepsilon) - \varepsilon D_x h_\varepsilon(x) f(z(x), \varepsilon) = 0$$

$\hookrightarrow$ any point on x-coord. plane

$$\Rightarrow g - \varepsilon D_x h_\varepsilon f = 0 \Rightarrow \underbrace{(-Dh_\varepsilon, I)}\begin{pmatrix} \varepsilon f \\ g \end{pmatrix} = 0 \Rightarrow \hat{A}_{s\perp} G = 0$$

$\hookrightarrow G(z(x)) \in T_{z(x)} S_\varepsilon$



(*) $$V = V_s + V_f = A_s \tilde{V}_s + A_f \tilde{V}_f = (A_s, A_f)\begin{pmatrix} \tilde{V}_s \\ \tilde{V}_f \end{pmatrix} = A \tilde{V} \Rightarrow \tilde{V} = \overbrace{A^{-1}}^{\hat{A}} V = \begin{pmatrix} \hat{A}_{f\perp} \\ \hat{A}_{s\perp} \end{pmatrix} V = \begin{pmatrix} \hat{A}_{f\perp} V \\ \hat{A}_{s\perp} V \end{pmatrix}$$

$$\Rightarrow V = A_s \hat{A}_{f\perp} V + A_f \boxed{\hat{A}_{s\perp} V}$$

Asymptotics of h_ε

Write $h_\varepsilon(x) = h_0(x) + \varepsilon h_1(x) + \varepsilon^2 h_2(x) + \dots$

→ Determine coeff's $h_0, h_1, h_2, \dots$

Inv Eq $\Rightarrow \left[(-Dh_0, I) + \varepsilon(-Dh_1, 0) + \dots\right]\left[\begin{pmatrix}0\\ g_0\end{pmatrix} + \varepsilon\begin{pmatrix}f_0\\ g_1\end{pmatrix} + \dots\right] = 0$

$\Rightarrow$ $\underline{O(1)}$ $\quad (-Dh_0, I)\begin{pmatrix}0\\ g_0\end{pmatrix} = 0$

$\underline{O(\varepsilon)}$ $\quad (-Dh_0, I)\begin{pmatrix}f_0\\ g_1\end{pmatrix} + (-Dh_1, 0)\begin{pmatrix}0\\ g_0\end{pmatrix} = 0$

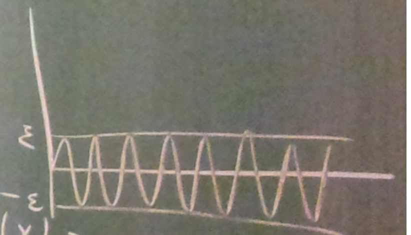
...

→ $g_0 = 0 \Rightarrow S_0 \rightsquigarrow y = h_0(x)$

→ $-Dh_0 f_0 + g_1 = 0 \xrightarrow{\text{below}} -Dh_0 f_0 + D_\varepsilon g_0 - (D_y g_0)\boxed{h_1} = 0 \Rightarrow h_1 = (D_y g_0)^{-1}\underbrace{\left[D_\varepsilon g_0 - Dh_0 f_0\right]}$

(*) $g(x, h_\varepsilon(x), \varepsilon) = g(x, h_0(x) + \varepsilon h_1(x) + \dots, \varepsilon)$

$= \underbrace{g_0(x, h_0(x))}_{0} + \varepsilon\left[D_\varepsilon g(x, h_0(x), 0) + D_y g(x, h_0(x), 0)\, h_1(x)\right] + \dots$



$f_\varepsilon(x) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right)$

$f'_\varepsilon(x) = \cancel{\varepsilon}\frac{1}{\cancel{\varepsilon}}\cos\left(\frac{x}{\varepsilon}\right)$

$\boxed{\text{Asymptotics of } h_\varepsilon}$ Write $h_\varepsilon(x) = h_0(x) + \varepsilon h_1(x) + \varepsilon^2 h_2(x) + \dots$

→ Determine coeff's $h_0, h_1, h_2, \dots$

Inv Eq $\Rightarrow \left[(-Dh_0, I) + \varepsilon(-Dh_1, 0) + \dots \right]\left[\begin{pmatrix} 0 \\ g_0 \end{pmatrix} + \varepsilon \begin{pmatrix} f_0 \\ g_1 \end{pmatrix} + \dots \right] = 0$

$\Rightarrow$ $\underline{O(1)}$ $\underbrace{(-\cancel{Dh_0}, I)}^{\text{whatever} \rightsquigarrow (0,I)} \begin{pmatrix} 0 \\ g_0 \end{pmatrix} = 0 \rightsquigarrow g_0 = 0 \rightsquigarrow h_0 \rightsquigarrow$ <u>leading order</u> δ_ε

$\underline{O(\varepsilon)}$ $(-D\overset{\checkmark}{h_0}, I)\begin{pmatrix} f_0 \\ g_1 \end{pmatrix} + (-\overset{*}{\cancel{Dh_1}}, 0)\begin{pmatrix} 0 \\ g_0 \end{pmatrix} = 0 \rightsquigarrow h_1 \rightsquigarrow S_\varepsilon$ up to & incl $O(\varepsilon)$ term

although $(-Dh_0 + \varepsilon *, I)$ is leading order

$\boxed{\text{Conclusion}}$ $A^{(\text{approx})}_{S^\perp}$ = correct to $O(\varepsilon^k) \rightsquigarrow h_0, h_1, \dots, h_{k+1}$

$\boxed{\text{QSSA}}$

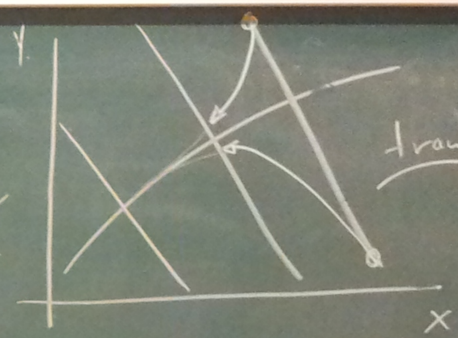
$\dot{x} = f(x,y,\varepsilon) \rightsquigarrow$ (reduced model) $\boxed{\dot{x} = f(x, h_\varepsilon(x), \varepsilon)} = \boxed{f_0(x, h_0(x))} + O(\varepsilon)$

$\dot{y} = \frac{1}{\varepsilon} g(x,y,\varepsilon) \rightsquigarrow$ (slaved) $y = h_\varepsilon(x)$

$\rightarrow g(x,y,\varepsilon) = 0 \Rightarrow g(x, h_{QSSA}(x), \varepsilon) = 0 \Rightarrow \overbrace{(0, I)}^{\text{LO incorrect}} \overbrace{\begin{pmatrix} \varepsilon f \\ g \end{pmatrix}}^{G} = 0 \rightsquigarrow h_{QSSA} = \overset{\checkmark}{h_0} + \overset{X}{\varepsilon h_{1,QSSA}}$

$$\dot x = f(x,y,\varepsilon) - \frac{1}{\varepsilon} g(x,y,\varepsilon) \rightsquigarrow \text{(below)}$$

$$\dot y = \frac{1}{\varepsilon} g(x,y,\varepsilon) \rightsquigarrow \text{Slave } y = h_{QSSA}(x) = h_0 + \dots$$



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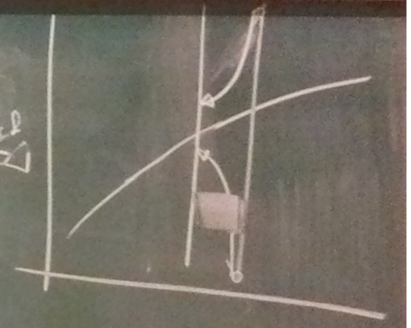


$$\overset{\circ}{(\underbrace{x+y})} = f(\overset{\bar x - y}{\tilde{x}}, y, \varepsilon)$$

$$\Rightarrow \bar x' = \bar f(\bar x, y, \varepsilon)$$

$$\dot y = \frac{1}{\varepsilon} \bar g(\bar x, y, \varepsilon)$$

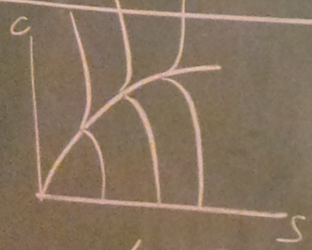
$\boxed{\text{t QSSA = total QSSA}}$

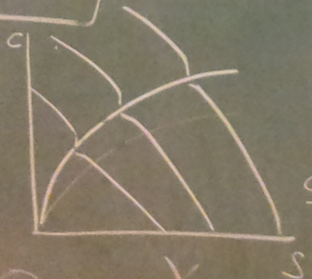
$$\dot x = f(x,y,\varepsilon) - \frac{1}{\varepsilon} g(x,y,\varepsilon)$$

$$= f(x, h_{QSSA}(x), \varepsilon) - \frac{1}{\varepsilon} g(x, h_{QSSA}(x), \varepsilon)$$

$$= f_0(x, h_0(x)) - \frac{1}{\varepsilon} \cancel{g_0(x, h_0(x))} - \frac{1}{\varepsilon}\left[\varepsilon D_y g_0 h_1 + \varepsilon D_\varepsilon g_0\right] + O(\varepsilon^2)$$

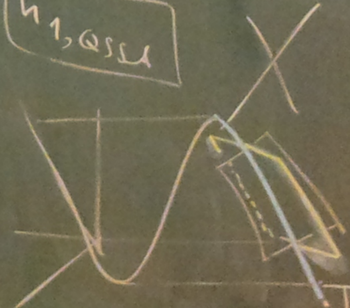
$$\Rightarrow \dot x = f_0 - D_\varepsilon g_0 - (D_y g_0) \boxed{h_{1,QSSA}}$$

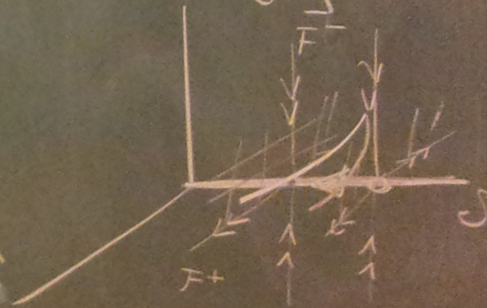




	$\boxed{\text{QSSA}}$	$\boxed{\text{tQSSA}}$
	(S, c)	$(S+c, c)$
Constr	$\dot c = 0$	$\dot c = 0$
RM	$\dot S = \dots$	$\dot{(S+c)} = \dots$

$$S + E \rightleftarrows C \to P + E$$





T.J. Kaper, AMS 1999

Lie bracket

Let u, v be vector fields in $\mathbb{R}^{n+m}$. Then,

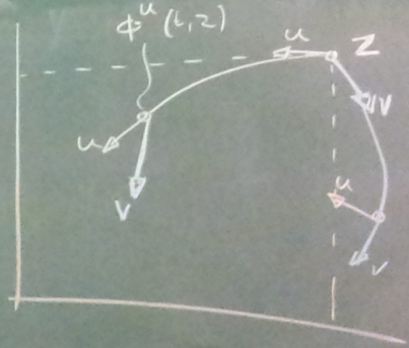
$$[u,v] = \underbrace{\overset{(n+m)\times(n+m)}{(D_z v)}}_{\frac{\partial v}{\partial u}} \overset{(n+m)\times 1}{u} - (D_z u)\, v = \text{"} u(v) - v(u) \text{"}$$

$\frac{\partial v}{\partial u}$ = directional derivative.

$$\lim_{t \to 0} \frac{v(\phi^u(t,z)) - v(z)}{t}$$

$x \in \mathbb{R}^m,\ y \in \mathbb{R}^n$

$\dot{x} = f(x,y,\varepsilon)$	$x' = \varepsilon f(x,y,\varepsilon)$
$\varepsilon \dot{y} = g(x,y,\varepsilon)$	$y' = g(x,y,\varepsilon)$



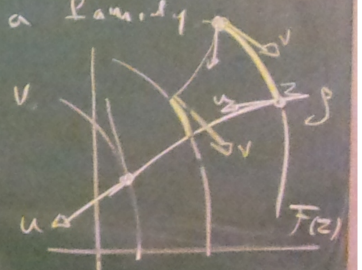
Facts

#1 Let S be a manifold invariant under the flow$\underline{\underline{s}}$ generated by u & v ($\Leftrightarrow$ $u(z) \in T_z S$ & $v(z) \in T_z S$), $z \in S$.

Then, S is invariant under $[u,v]$ as well ($\Leftrightarrow$ $[u,v] \in T_z S$).

#2 Let S be as above (invariant under u) & $\{F(z)\}_{z \in S}$ be a family which is invariant (as a family) under u & invidually under v.

Then, $\{F(z)\}_{z \in S}$ is invariant under $[u,v]$.



Conclusion

Let $A = (\underbrace{A_s}_{\in T_z S}, \underbrace{A_f}_{\in T_z F_{(z)}})$ be a (local) basis for $T_z\mathbb{R}^N$.

Then,

$$\underbrace{\hat A}_{A^{-1}}[A,g] = \begin{pmatrix}\hat A_{f\perp}\\ \hat A_{s\perp}\end{pmatrix}\left([A_s,g],\,[A_f,g]\right) = \begin{pmatrix} \overset{\perp\text{fast}}{\hat A_{f\perp}}\,\overset{\text{slow}}{[A_s,g]} & \cancelto{0}{\hat A_{f\perp}[A_f,g]} \\ \cancelto{0}{\underset{\perp\text{slow}}{\hat A_{s\perp}}\,\underset{\text{slow}}{[A_s,g]}} & \hat A_{s\perp}[A_f,g]\end{pmatrix}$$

$$\Rightarrow \Lambda = \hat A\,[A,g] = \begin{pmatrix}\Lambda^s_s & \cancelto{0}{\Lambda^s_f}\\ \cancelto{0}{\Lambda^f_s} & \Lambda^f_f\end{pmatrix} \quad \text{block-diagonal matrix.}$$

CSP

$\underbrace{G}_{\text{full vector field}} = A w \Rightarrow$ evolution eqs for w $\Rightarrow$ start $w = \hat A G$

$$\Rightarrow w' = (\hat A G)' = \hat A G' + \hat A' G = \hat A (D_z G)\,\underbrace{z'}_{\underbrace{G}_{Aw}} - \hat A A' \underbrace{\hat A G}_{w}$$

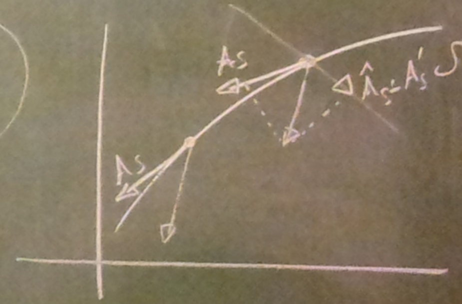
$$\hat A A = I \overset{(\cdot)'}{\Rightarrow} \hat A' A + \hat A A' = 0 \Rightarrow \hat A' = -\hat A A' \hat A$$

$$= \hat A (D_z G) A w - \hat A (D_z A)\,\cancelto{G}{z'}\,w = \hat A\left[(D_z G)A - (D_z A)G\right]w$$

$$\Rightarrow \boxed{w' = \hat A[A,G]\,w = \Lambda w} \rightsquigarrow \begin{pmatrix}w^s\\ w^f\end{pmatrix}' = \begin{pmatrix}\Lambda^s_s & \cancelto{0}{\Lambda^s_f}\\ \cancelto{0}{\Lambda^f_s} & \Lambda^f_f\end{pmatrix}\begin{pmatrix}w^s\\ w^f\end{pmatrix} \Rightarrow \begin{aligned}(w^s)' &= \Lambda^s_s w^s\\ (w^f)' &= \Lambda^f_f w^f \rightsquigarrow \underbrace{\{w^f = 0\}}_{S} \text{ invariant}\end{aligned}$$

$\boxed{\text{ILDM}}$ $\quad G' = (D_z G) z' = \underbrace{(D_z G)}_{\text{Jacobian}} G$

Recall $\quad \Lambda = \hat{A}\,[A, G] = \hat{A}\,(D_z G)A - \hat{A}\,(D_z A)G \Rightarrow \underbrace{\hat{A}\,(D_z G)A}_{\text{ILDM: block-diagonal form}} = \Lambda + \hat{A}\,\underbrace{(D_z A)G}_{A'}$



Thus, $\quad \hat{A}\,(D_z G)A = \begin{pmatrix} \Lambda_s^s & 0 \\ 0 & \Lambda_f^f \end{pmatrix} + \begin{pmatrix} \hat{A}_{f\perp} \\ \hat{A}_{s\perp} \end{pmatrix} \begin{pmatrix} A_s', & A_f' \end{pmatrix}$

$$= \begin{pmatrix} * & * \\ \boxed{\hat{A}_{s\perp} A_s'} & * \end{pmatrix}$$

$\boxed{\text{CSP}}$

Start with $A \Rightarrow$ update $\underbrace{\tilde{A}}_{\text{new}} = \underbrace{A}_{\text{old}}(I - L)$, with $L = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}$

Calculate $\underbrace{\tilde{\Lambda}^f_s}_{\text{new}} = L\underbrace{\Lambda^s_s}_{O(\varepsilon)} - L\underbrace{\Lambda^s_f}_{O(\varepsilon)}L + \boxed{\underbrace{\Lambda^f_s - \Lambda^f_f L}_{= 0}} - (D_z L)G + O(\varepsilon)$

0 Ideally.