CONSTRUCTING PAIRS OF DUAL BANDLIMITED FRAME WAVELETS IN $L^2(\mathbb{R}^n)$

JAKOB LEMVIG

ABSTRACT. Given a real, expansive dilation matrix we prove that any bandlimited function $\psi \in L^2(\mathbb{R}^n)$, for which the dilations of its Fourier transform form a partition of unity, generates a wavelet frame for certain translation lattices. Moreover, there exists a dual wavelet frame generated by a finite linear combination of dilations of ψ with explicitly given coefficients. The result allows a simple construction procedure for pairs of dual wavelet frames whose generators have compact support in the Fourier domain and desired time localization. The construction relies on a technical condition on ψ, and we exhibit a general class of function satisfying this condition.

1. INTRODUCTION

For $A \in GL_n(\mathbb{R})$ and $y \in \mathbb{R}^n$, we define the dilation operator on $L^2(\mathbb{R}^n)$ by $D_A f(x) = |\det A|^{1/2} f(Ax)$ and the translation operator by $T_y f(x) = f(x-y)$. Given a $n \times n$ real, expansive matrix A and a lattice of the form $\Gamma = P\mathbb{Z}^n$ for $P \in GL_n(\mathbb{R})$, we consider wavelet systems of the form

$$\{D_{A^j} T_\gamma \psi\}_{j\in\mathbb{Z},\gamma\in\Gamma},$$

where the Fourier transform of ψ has compact support. Our aim is, for any given real, expansive dilation matrix A, to construct wavelet frames with good regularity properties and with a dual frame generator of the form

$$\phi = \sum_{j=a}^{b} c_j D_{A^j} \psi \tag{1}$$

for some explicitly given coefficients $c_j \in \mathbb{C}$ and $a, b \in \mathbb{Z}$. This will generalize and extend the one-dimensional results on constructions of dual wavelet frames in [19, 22] to higher dimensions. The extension is non-trivial since it is unclear how to determine the translation lattice Γ and how to control the support of the generators in the Fourier domain. This will be done by considering suitable norms in $\mathbb{R}^n$ and non-overlapping packing of ellipsoids in lattice arrangements.

The construction of redundant wavelet representations in higher dimensions is usually based on extension principles [10, 11, 13–17, 20, 21]. By making use of extension principles one is restricted to considering expansive dilations A with integer coefficients. Our constructions work for any real, expansive dilation. Moreover, in the extension principle the number of generators often increases with the smoothness of the generators. We will construct pairs of dual wavelet frames generated by one smooth function with good time localization.

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It is a well-known fact that a wavelet frame need not have dual frames with wavelet structure. In [24] frame wavelets with compact support and explicit analytic form are constructed for real dilation matrices. However, no dual frames are presented for these wavelet frames. This can potentially be a problem because it might be difficult or even impossible to find a dual frame with wavelet structure. Since we exhibit *pairs* of dual wavelet frames, this issue is avoided.

The principal importance of having a dual generator of the form (1) is that it will inherit properties from ψ preserved by dilation and linearity, *e.g.* vanishing moments, good time localization and regularity properties. For a more complete account of such matters we refer to [19].

In the rest of this introduction we review basic definitions. A frame for a separable Hilbert space $\mathcal{H}$ is a countable collection of vectors $\{f_j\}_{j\in\mathbb{J}}$ for which there are constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1 \|f\|^2 \le \sum_{j\in\mathbb{J}} \left|\langle f, f_j\rangle\right|^2 \le C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If the upper bound holds in the above inequality, then $\{f_j\}$ is said to be a Bessel sequence with Bessel constant C_2. For a Bessel sequence $\{f_j\}$ we define the frame operator by

$$S\colon \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle f_j.$$

This operator is bounded, invertible, and positive. A frame $\{f_j\}$ is said to be *tight* if we can choose $C_1 = C_2$; this is equivalent to $S = C_1 I$ where I is the identity operator. Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be *dual* frames if

$$f = \sum_{j\in\mathbb{J}} \langle f, g_j\rangle f_j \quad \forall f \in \mathcal{H}.$$

It can be shown that two such Bessel sequences are indeed frames. Given a frame $\{f_j\}$, at least one dual always exists; it is called the canonical dual and is given by $\{S^{-1} f_j\}$. Only a frame, which is not a basis, has several duals.

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, \mathrm{e}^{-2\pi i\langle \xi, x\rangle}\, \mathrm{d}x$ with the usual extension to $L^2(\mathbb{R}^n)$.

Sets in $\mathbb{R}^n$ are, in general, considered equal if they are equal up to sets of measure zero. The boundary of a set E is denoted by ∂E, the interior by E°, and the closure by $\overline{E}$. Let $B \in GL_n(\mathbb{R})$. A *multiplicative tiling set* E for $\{B^j : j \in \mathbb{Z}\}$ is a subset of positive measure such that

$$\left|\mathbb{R}^n \setminus \bigcup_{j\in\mathbb{Z}} B^j(E)\right| = 0 \quad \text{and} \quad \left|B^j(E) \cap B^l(E)\right| = 0 \quad \text{for } l \ne j. \tag{2}$$

In this case we say that $\{B^j(E) : j \in \mathbb{Z}\}$ is an almost everywhere partition of $\mathbb{R}^n$, or that it tiles $\mathbb{R}^n$. A multiplicative tiling set E is *bounded* if E is a bounded set and $0 \notin \overline{E}$. By B-dilative periodicity of a function $f\colon \mathbb{R}^n \to \mathbb{C}$ we understand $f(x) = f(Bx)$ for a.e. $x \in \mathbb{R}^n$, and by a B-dilative partition of unity we understand $\sum_{j\in\mathbb{Z}} f(B^j x) = 1$; note that the functions in the "partition of unity" are not assumed to be non-negative, but can take any real or complex value.

A (full-rank) lattice Γ in $\mathbb{R}^n$ is a point set of the form $\Gamma = P\mathbb{Z}^n$ for some $P \in GL_n(\mathbb{R})$. The determinant of Γ is $d(\Gamma) = |\det P|$; note that the generating matrix P is not unique, and that $d(\Gamma)$ is independent of the particular choice of P.

2. The general form of the construction procedure

Fix the dimension $n \in \mathbb{N}$. We let $A \in GL_n(\mathbb{R})$ be expansive, *i.e.,* all eigenvalues of A have absolute value greater than one, and denote the transpose matrix by $B = A^t$. For any such dilation A, we want to construct a pair of functions that generate dual wavelet frames for some translation lattice. Our construction is based on the following result which is a consequence of the characterizing equations for dual wavelet frames by Chui, Czaja, Maggioni, and Weiss [9, Theorem 4].

Theorem 2.1. *Let $A \in GL_n(\mathbb{R})$ be expansive, let Γ be a lattice in $\mathbb{R}^n$, and let $\Psi = \{\psi_1, \dots, \psi_L\}$, $\tilde{\Psi} = \{\tilde{\psi}_1, \dots, \tilde{\psi}_L\} \subset L^2(\mathbb{R}^n)$. Suppose that the two wavelet systems $\{D_{A^j}T_\gamma\psi_l : j \in \mathbb{Z}, \gamma \in \Gamma, l = 1, \dots, L\}$ and $\{D_{A^j}T_\gamma\tilde{\psi}_l : j \in \mathbb{Z}, \gamma \in \Gamma, l = 1, \dots, L\}$ form Bessel families. Then $\{D_{A^j}T_\gamma\psi_l\}$ and $\{D_{A^j}T_\gamma\tilde{\psi}_l\}$ will be dual frames if the following conditions hold*

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} \hat{\tilde{\psi}}_l(B^j\xi)\overline{\hat{\psi}_l(B^j\xi)} = d(\Gamma) \qquad a.e.\ \xi \in \mathbb{R}^n, \tag{3}$$

$$\sum_{l=1}^{L} \hat{\tilde{\psi}}_l(\xi)\overline{\hat{\psi}_l(\xi+\gamma)} = 0 \qquad a.e.\ \xi \in \mathbb{R}^n \text{ for } \gamma \in \Gamma^* \setminus \{0\}. \tag{4}$$

Proof. By $\xi = B^j\omega$ for $j \in \mathbb{Z}$, condition (4) becomes

$$\sum_{l=1}^{L} \hat{\tilde{\psi}}_l(B^j\omega)\overline{\hat{\psi}_l(B^j\omega+\gamma)} = 0 \quad \text{a.e. } \omega \in \mathbb{R}^n \text{ for } \gamma \in \Gamma^* \setminus \{0\}. \tag{5}$$

We use the notation as in [9], thus $\Lambda(A,\Gamma) = \{\alpha \in \mathbb{R}^n : \exists (j,\gamma) \in \mathbb{Z} \times \Gamma^* : \alpha = B^{-j}\gamma\}$ and $I_{A,\Gamma}(\alpha) = \{(j,\gamma) \in \mathbb{Z} \times \Gamma^* : \alpha = B^{-j}\gamma\}$. Since $I_{A,\Gamma}(\alpha) \subset \mathbb{Z} \times (\Gamma^* \setminus \{0\})$ for any $\alpha \in \Lambda(A,\Gamma) \setminus \{0\}$, equation (5) yields

$$\frac{1}{d(\Gamma)} \sum_{(j,\gamma)\in I_{A,\Gamma}(\alpha)} \sum_{l=1}^{L} \hat{\tilde{\psi}}_l(B^j\omega)\overline{\hat{\psi}_l(B^j(\omega+B^{-j}\gamma))} = 0 \quad \text{a.e. } \omega \in \mathbb{R}^n$$

for $\alpha \neq 0$. By $I_{A,\Gamma}(0) = \mathbb{Z} \times \{0\}$, we can rewrite (3) as

$$\frac{1}{d(\Gamma)} \sum_{(j,\gamma)\in I_{A,\Gamma}(0)} \sum_{l=1}^{L} \hat{\tilde{\psi}}_l(B^j\omega)\overline{\hat{\psi}_l(B^j(\omega+B^{-j}\gamma))} = 1 \quad \text{a.e. } \omega \in \mathbb{R}^n,$$

using that $B^{-j}\gamma = 0$ for all $j \in \mathbb{Z}$. Gathering the two equations displayed above yields

$$\frac{1}{d(\Gamma)} \sum_{(j,\gamma)\in I_{A,\Gamma}(\alpha)} \sum_{l=1}^{L} \hat{\tilde{\psi}}_l(B^j\omega)\overline{\hat{\psi}_l(B^j(\omega+B^{-j}\gamma))} = \delta_{\alpha,0} \quad \text{a.e. } \omega \in \mathbb{R}^n,$$

for all $\alpha \in \Lambda(A,\Gamma)$. The conclusion follows now from [9, Theorem 4]. $\square$

The following result, Lemma 2.2, gives a sufficient condition for a wavelet system to form a Bessel sequence; it is an extension of [3, Theorem 11.2.3] from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^n)$.

Lemma 2.2. *Let $A \in GL_n(\mathbb{R})$ be expansive, Γ a lattice in $\mathbb{R}^n$, and $\phi \in L^2(\mathbb{R}^n)$. Suppose that, for some set $M \subset \mathbb{R}^n$ satisfying $\cup_{l\in\mathbb{Z}} B^l(M) = \mathbb{R}^n$,*

$$C_2 = \frac{1}{d(\Gamma)} \sup_{\xi \in M} \sum_{j\in\mathbb{Z}} \sum_{\gamma \in \Gamma^*} \left| \hat{\phi}(B^j\xi)\hat{\phi}(B^j\xi+\gamma) \right| < \infty. \tag{6}$$

Then the wavelet system $\{D_{A^j}T_\gamma \phi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ is a Bessel sequence with bound C_2. Further, if also

$$C_1 = \frac{1}{d(\Gamma)} \inf_{\xi\in M} \left(\sum_{j\in\mathbb{Z}} \left| \hat{\phi}(B^j\xi) \right|^2 - \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma^*\setminus\{0\}} \left| \hat{\phi}(B^j\xi)\hat{\phi}(B^j\xi+\gamma) \right| \right) > 0, \tag{7}$$

holds, then $\{D_{A^j}T_\gamma\phi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds C_1 and C_2.

Proof. The statement follows directly by applying Theorem 3.1 in [8] on generalized shift invariant systems to wavelet systems. In the general result for generalized shift invariant systems [8, Theorem 3.1], the supremum/infimum is taken over $\mathbb{R}^n$, but because of the B-dilative periodicity of the series in (6) and (7) for wavelet systems, it suffices to take the supremum/infimum over a set $M \subset \mathbb{R}^n$ that has the property that $\cup_{l\in\mathbb{Z}} B^l(M) = \mathbb{R}^n$ up to sets of measure zero. □

Theorem 2.1 and Lemma 2.2 are all we need to prove the following result on pairs of dual wavelet frames.

Theorem 2.3. *Let $A \in GL_n(\mathbb{R})$ be expansive and $\psi \in L^2(\mathbb{R}^n)$. Suppose that $\hat{\psi}$ is a bounded, real-valued function with* $\operatorname{supp}\hat{\psi} \subset \cup_{j=0}^{d} B^{-j}(E)$ *for some $d \in \mathbb{N}_0$ and some bounded multiplicative tiling set E for $\{B^j : j \in \mathbb{Z}\}$, and that*

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(B^j\xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{8}$$

Let $b_j \in \mathbb{C}$ for $j = -d, \dots, d$ and let $\overline{m} = \max\{j : b_j \neq 0\}$ and $\underline{m} = -\min\{j : b_j \neq 0\}$. Take a lattice Γ in $\mathbb{R}^n$ such that

$$\Big(\bigcup_{j=0}^{d} B^{-j}(E) + \gamma \Big) \cap \bigcup_{j=-\underline{m}}^{\overline{m}+d} B^{-j}(E) = \emptyset \quad \text{for all } \gamma \in \Gamma^* \setminus \{0\}, \tag{9}$$

and define the function ϕ by

$$\phi(x) = d(\Gamma) \sum_{j=-\underline{m}}^{\overline{m}} b_j \left|\det A\right|^{-j} \psi(A^{-j}x) \quad \text{for } x \in \mathbb{R}^n. \tag{10}$$

If $b_0 = 1$ and $b_j + b_{-j} = 2$ for $j = 1, 2, \dots, d$, then the functions ψ and ϕ generate dual frames $\{D_{A^j}T_\gamma\psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ and $\{D_{A^j}T_\gamma\phi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ for $L^2(\mathbb{R}^n)$.

Proof. On the Fourier side, the definition in (10) becomes

$$\hat{\phi}(\xi) = d(\Gamma) \sum_{j=-\underline{m}}^{\overline{m}} b_j \hat{\psi}(B^j\xi).$$

Since $\hat{\psi}$ by assumption is compactly supported in a "ringlike" structure bounded away from the origin, this will also be the case for $\hat{\phi}$. This property implies that

ψ and ϕ will generate wavelet Bessel sequences. The details are as follows. The support of $\hat{\psi}$ and $\hat{\phi}$ is

$$\operatorname{supp}\hat{\psi} \subset \bigcup_{j=0}^{d} B^{-j}(E), \qquad \operatorname{supp}\hat{\phi} \subset \bigcup_{j=-\underline{m}}^{\overline{m}+d} B^{-j}(E). \tag{11}$$

Note that $0 \le \underline{m}, \overline{m} \le d$. The sets $\{B^j(E) : j \in \mathbb{Z}\}$ tiles $\mathbb{R}^n$, whereby we see that

$$\left|\operatorname{supp}\hat{\psi}(B^j\cdot) \cap B^{-d}(E)\right| = 0 \quad \text{for } j < 0 \text{ and } j > d, \tag{12}$$

and,

$$\left|\operatorname{supp}\hat{\phi}(B^j\cdot) \cap B^{-d}(E)\right| = 0 \quad \text{for } j < -\underline{m} \text{ and } j > \overline{m} + d. \tag{13}$$

Since $\underline{m}, \overline{m} \ge 0$, condition (9) implies that $\hat{\psi}(B^j\xi)\hat{\psi}(B^j\xi + \gamma) = 0$ for $j \ge 0$ and $\gamma \in \Gamma^* \setminus \{0\}$. Therefore, using (12), we find that

$$\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma^*}\left|\hat{\psi}(B^j\xi)\hat{\psi}(B^j\xi+\gamma)\right| = \sum_{j=0}^{d}\left(\hat{\psi}(B^j\xi)\right)^2 < \infty \qquad \text{for } \xi \in B^{-d}(E).$$

An application of Lemma 2.2 with $M = B^{-d}(E)$ shows that ψ generates a Bessel sequence. Similar calculations using (13) will show that ϕ generates a Bessel sequence; in this case the sum over $\gamma \in \Gamma^*$ will be finite, but it will in general have more than one nonzero term.

To conclude that ψ and ϕ generate dual wavelet frames we will show that conditions (3) and (4) in Theorem 2.1 hold. By B-dilation periodicity of the sum in condition (3), it is sufficient to verify this condition on $B^{-d}(E)$. For $\xi \in B^{-d}(E)$ we have by (12),

$$\begin{aligned}
\frac{1}{d(\Gamma)}\sum_{j\in\mathbb{Z}}\overline{\hat{\psi}(B^j\xi)}\hat{\phi}(B^j\xi) &= \frac{1}{d(\Gamma)}\sum_{j=0}^{d}\hat{\psi}(B^j\xi)\hat{\phi}(B^j\xi)\\
&= \hat{\psi}(\xi)\left[b_0\hat{\psi}(\xi) + b_1\hat{\psi}(B\xi) + \cdots + b_d\hat{\psi}(B^d\xi)\right]\\
&\quad + \hat{\psi}(B\xi)\left[b_{-1}\hat{\psi}(\xi) + b_0\hat{\psi}(B\xi) + \cdots + b_{d-1}\hat{\psi}(B^d\xi)\right] + \cdots\\
&\quad + \hat{\psi}(B^d\xi)\left[b_{-d}\hat{\psi}(\xi) + \cdots + b_{-1}\hat{\psi}(B^{d-1}\xi) + b_0\hat{\psi}(B^d\xi)\right],
\end{aligned}$$

and further, by an expansion of these terms,

$$\begin{aligned}
&= \sum_{j,l=0}^{d} b_{l-j}\hat{\psi}(B^j\xi)\hat{\psi}(B^l\xi)\\
&= b_0\sum_{j=0}^{d}\hat{\psi}(B^j\xi)^2 + \sum_{\substack{j,l=0\\ j>l}}^{d}(b_{j-l}+b_{l-j})\hat{\psi}(B^j\xi)\hat{\psi}(B^l\xi).
\end{aligned}$$

Using that $b_0 = 1$ and $b_{j-l} + b_{l-j} = 2$ for $j \neq l$ and $j, l = 0, \dots, d$, we arrive at

$$\begin{aligned}\frac{1}{d(\Gamma)} \sum_{j\in\mathbb{Z}} \overline{\hat{\psi}(B^j\xi)}\hat{\phi}(B^j\xi) &= \sum_{j=0}^{d} \hat{\psi}(B^j\xi)^2 + \sum_{\substack{j,l=0\\ j>l}}^{d} 2\hat{\psi}(B^j\xi)\hat{\psi}(B^l\xi) \\ &= \Big(\sum_{j=0}^{d} \hat{\psi}(B^j\xi)\Big)^2 = \Big(\sum_{j\in\mathbb{Z}} \hat{\psi}(B^j\xi)\Big)^2 = 1,\end{aligned}$$

exhibiting that ψ and ϕ satisfy condition (3).

By (11) we see that condition (9) implies that the functions $\hat{\phi}$ and $\hat{\psi}(\cdot + \gamma)$ will have disjoint support for $\gamma \in \Gamma^* \setminus \{0\}$, hence (4) is satisfied. □

Remark 1. The use of the parameters b_j in the definition of the dual generator together with the condition $b_{-j} + b_j = 2$ was first seen in the work of Christensen and Kim [6] on pairs of dual Gabor frames. For other works on constructing pairs of dual Gabor frames, similar in spirit to this paper, we refer to [4, 5, 7, 18].

We can restate Theorem 2.3 for wavelet systems with standard translation lattice $\mathbb{Z}^n$ and dilation $\widetilde{A} = P^{-1}AP$, where $P \in GL_n(\mathbb{R})$ is so that $\Gamma = P\mathbb{Z}^n$. The result follows directly by an application of the relations $D_{\widetilde{A}^j}D_P = D_P D_{A^j}$ for $j \in \mathbb{Z}$ and $D_P T_{Pk} = T_k D_P$ for $k \in \mathbb{Z}^n$, and the fact that D_P is unitary as an operator on $L^2(\mathbb{R}^n)$.

Corollary 2.4. *Suppose ψ, $\{b_j\}$, A and Γ are as in Theorem 2.3. Let $P \in GL_n(\mathbb{R})$ be such that $\Gamma = P\mathbb{Z}^n$, and let $\widetilde{A} = P^{-1}AP$. Then the functions $\tilde{\psi} = D_P\psi$ and $\tilde{\phi} = D_P\phi$, where ϕ is defined in (10), generate dual frames $\{D_{\widetilde{A}^j}T_k\tilde{\psi}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ and $\{D_{\widetilde{A}^j}T_k\tilde{\phi}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ for $L^2(\mathbb{R}^n)$.*

The following Example 1 is an application of Theorem 2.3 in $L^2(\mathbb{R}^2)$ for the quincunx matrix. In particular, we construct a partition of unity of the form (8) for the quincunx matrix.

Example 1. The quincunx matrix is defined as

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and its action on $\mathbb{R}^2$ corresponds to a counter clockwise rotation of 45 degrees and a dilation by $\sqrt{2}I_{2\times 2}$. Define the tent shaped, piecewise linear function g by

$$g(x_1,x_2) = \begin{cases} -1 + 2x_1 + 2x_2, & \text{for } (x_1,x_2) \in J_1, \\ 2x_2, & \text{for } (x_1,x_2) \in J_2, \\ 2x_1, & \text{for } (x_1,x_2) \in J_3, \\ 2 - 2x_1, & \text{for } (x_1,x_2) \in J_4, \\ 2 - 2x_2, & \text{for } (x_1,x_2) \in J_5, \\ 0 & \text{otherwise,} \end{cases}$$

where the sets J_i are the triangular domains sketched in Figure 1. Note that the value at "the top of the tent" is $g(1/2, 1/2) = 1$. Define $\hat{\psi}$ as a mirroring of g in the

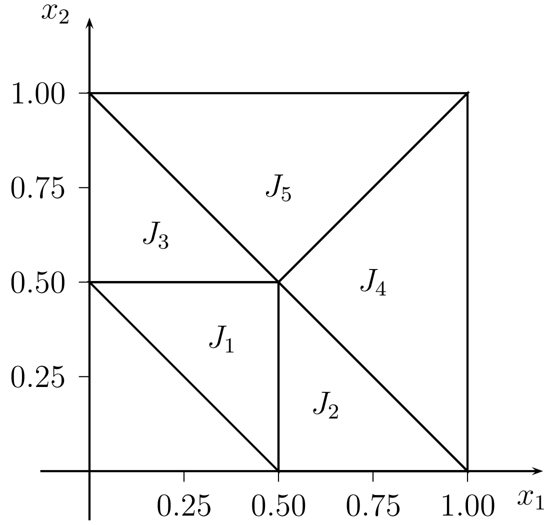

FIGURE 1. Sketch of the triangular domains J_i, $i = 1, 2, 3, 4, 5$.

x_1 axis and the x_2 axis:

$$\hat{\psi}(\xi_1, \xi_2) = \begin{cases} g(\xi_1, \xi_2) & \text{for } (\xi_1, \xi_2) \in [0, \infty) \times [0, \infty)\,, \\ g(\xi_1, -\xi_2) & \text{for } (\xi_1, \xi_2) \in [0, \infty) \times (-\infty, 0)\,, \\ g(-\xi_1, \xi_2) & \text{for } (\xi_1, \xi_2) \in (-\infty, 0) \times [0, \infty)\,, \\ g(-\xi_1, -\xi_2) & \text{for } (\xi_1, \xi_2) \in (-\infty, 0) \times (-\infty, 0)\,. \end{cases}$$

Since the transpose B of the quincunx matrix also corresponds to a rotation of 45 degrees (but clockwise) and a dilation by $\sqrt{2} I_{2\times 2}$, we see that $\sum_{j\in\mathbb{Z}} \hat{\psi}(B^j \xi) = 1$.

We are now ready to apply Theorem 2.3 with $E = [-1,1]^2 \setminus B^{-1}([-1,1]^2) = [-1,1]^2 \setminus I_1$ and $d = 2$; the set E is the union of the domians J_4 and J_5 and their mirrored versions. We choose $b_{-2} = b_{-1} = 0$ and $b_1 = b_2 = 2d(\Gamma)$, hence $\underline{m} = 0$ and $\overline{m} = 2$. Therefore,

$$\bigcup_{j=0}^{d} B^{-j}(E),\ \bigcup_{j=-\underline{m}}^{\overline{m}+d} B^{-j}(E) \subset [-1,1]^2\,,$$

that shows that we can take $\Gamma^* = 2\mathbb{Z}^2$ or $\Gamma = 1/2\mathbb{Z}^2$, since $([-1,1]^2 + \gamma) \cap [-1,1]^2 = \emptyset$ whenever $0 \neq \gamma \in 2\mathbb{Z}^2$. Defining the dual generator according to (16) yields

$$\phi(x) = (1/4)\psi(x) + (1/4)\psi(A^{-1}x) + (1/8)\psi(A^{-2}x); \tag{14}$$

using that $d(\Gamma) = 1/4$, and we remark that $\hat{\phi}$ is a piecewise linear function since this is the case for $\hat{\psi}$. The conclusion from Theorem 2.3 is that ψ and ϕ generate dual frames $\{D_{A^j} T_{k/2} \psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{A^j} T_{k/2} \phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R}^2)$.

The frame bounds can be found using Lemma 2.2 since the series (6) and (7) are finite sums on E; for $\{D_{A^j} T_{k/2} \psi\}$ one finds $C_1 = 4/3$ and $C_2 = 4$.

When the result on constructing pairs of dual wavelet frames is written in the generality of Theorem 2.3, it is not always clear how to choose the set E and the lattice Γ. In Example 1 we showed how this can be done for the quincunx dilation matrix and constructed a pair of dual frame wavelets. In Section 3 and Theorem 3.3 we specify how to choose E and Γ for general dilations. The issue of exhibiting functions ψ satisfying the condition (8) is addressed in Section 4.

In one dimension, however, it is straightforward to make good choices of E and Γ as is seen by the following corollary of Theorem 2.3. The corollary unifies the construction procedures in Theorem 2 and Proposition 1 from [19] in a general procedure.

Corollary 2.5. *Let $d \in \mathbb{N}_0$, $a > 1$, and $\psi \in L^2(\mathbb{R})$. Suppose that $\hat{\psi}$ is a bounded, real-valued function with* $\operatorname{supp} \hat{\psi} \subset [-a^c, -a^{c-d-1}] \cup [a^{c-d-1}, a^c]$ *for some $c \in \mathbb{Z}$, and that*

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(a^j \xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{15}$$

Let $b_j \in \mathbb{C}$ for $j = -d, \dots, d$, let $m = -\min\{j : \{b_j \neq 0\}\}$, and define the function ϕ by

$$\phi(x) = \sum_{j=-m}^{d} b_j a^{-j} \psi(a^{-j}x) \quad \text{for } x \in \mathbb{R}. \tag{16}$$

Let $b \in (0, a^{-c}(1+a^m)^{-1}]$. If $b_0 = b$ and $b_j + b_{-j} = 2b$ for $j = 1, 2, \dots, d$, then ψ and ϕ generate dual frames $\{D_{a^j} T_{bk} \psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j} T_{bk} \phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof. In Theorem 2.3 for $n = 1$ and $A = a$ we take $E = [-a^c, -a^{c-1}] \cup [a^{c-1}, a^c]$ as the multiplicative tiling set for $\{a^j : j \in \mathbb{Z}\}$. The assumption on the support of $\hat{\psi}$ becomes

$$\operatorname{supp} \hat{\psi} \subset \bigcup_{j=0}^{d} a^{-j}(E) = [-a^c, -a^{c-d-1}] \cup [a^{c-d-1}, a^c].$$

Moreover, since

$$\bigcup_{j=0}^{d} a^{-j}(E) \subset [-a^c, a^c], \qquad \bigcup_{j=-m}^{2d} a^{-j}(E) \subset \left[-a^{c+m}, a^{c+m}\right],$$

and

$$([-a^c, a^c] + \gamma) \cap \left[-a^{c+m}, a^{c+m}\right] = \emptyset \quad \text{for } |\gamma| \geq a^c + a^{c+m} = a^c(1+a^m),$$

the choice $\Gamma^* = b^{-1}\mathbb{Z}$ for $b^{-1} \geq a^c(1+a^m)$ satisfies equation (9). This corresponds to $\Gamma = b\mathbb{Z}$ for $0 < b \leq a^{-c}(1+a^m)^{-1}$. □

The assumptions in Corollary 2.5 imply that $m \in \{0, 1, \dots, d\}$; we note that in case $m = 0$, the corollary reduces to [19, Theorem 2].

3. A special case of the construction procedure

We aim for a more automated construction procedure than what we have from Theorem 2.3, in particular, we therefore need to deal with good ways of choosing E and Γ. The basic idea in this automation process will be to choose E as a dilation of the difference between I_* and $B^{-1}(I_*)$, where I_* is the unit ball in a norm in which the matrix $B = A^t$ is expanding "in all directions"; we will make this statement precise in Section 3.1. This idea is instrumental in the proof of Theorem 3.3.

3.1. **Some results on expansive matrices.** We need the following well-known equivalent conditions for a (non-singular) matrix being expansive.

Proposition 3.1. *For $B \in GL_n(\mathbb{R})$ the following assertions are equivalent:*

(i) B is expansive, i.e., all eigenvalues λ_i of B satisfy $|\lambda_i| > 1$.

(ii) For any norm $|\cdot|$ on $\mathbb{R}^n$ there are constants $\lambda > 1$ and $c \geq 1$ such that

$$|B^j x| \geq 1/c\lambda^j\,|x| \qquad \text{for all } j \in \mathbb{N}_0,$$

for any $x \in \mathbb{R}^n$.

(iii) There is a Hermitian norm $|\cdot|_*$ on $\mathbb{R}^n$ and a constant $\lambda > 1$ such that

$$|B^j x|_* \geq \lambda^j\,|x|_* \qquad \text{for all } j \in \mathbb{N}_0,$$

for any $x \in \mathbb{R}^n$.

(iv) $\mathcal{E} \subset \lambda\mathcal{E} \subset B\mathcal{E}$ for some ellipsoid $\mathcal{E} = \{x \in \mathbb{R}^n : |Px| \leq 1\}$, $P \in GL_n(\mathbb{R})$, and $\lambda > 1$.

By Proposition 3.1 we have that for a given expansive matrix B, there exists a scalar product with the induced norm $|\cdot|_*$ so that

$$|Bx|_* \geq \lambda\,|x|_* \quad \text{for } x \in \mathbb{R}^n,$$

holds for some $\lambda > 1$. We say that $|\cdot|_*$ is a norm associated with the expansive matrix B. Note that such a norm is not unique; we will follow the construction as in the proof of [2, Lemma 2.2], so let c and λ be as in (ii) in Proposition 3.1 for the standard Euclidean norm with $1 < \lambda < |\lambda_i|$ for $i = 1, \dots, n$, where λ_i are the eigenvalues of B. For $k \in \mathbb{N}$ satisfying $k > 2\ln c/\ln\lambda$ we introduce the symmetric, positive definite matrix $K \in GL_n(\mathbb{R})$:

$$K = I + (B^{-1})^t B^{-1} + \cdots + (B^{-k})^t B^{-k}. \tag{17}$$

The scalar product associated with B is then defined by $\langle x, y\rangle_* = x^t K y$. It might not be effortless to estimate c and λ for some given B, but it is obvious that we just need to pick $k \in \mathbb{N}$ such that $B^t K B - \lambda^2 K$ becomes positive semi-definite for some $\lambda > 1$ since this corresponds to $\langle KBx, Bx\rangle \geq \lambda^2 \langle Kx, x\rangle$, that is, $|Bx|_*^2 \geq \lambda^2\,|x|_*^2$ for all $x \in \mathbb{R}^n$.

We let I_* denote the unit ball in the Hermitian norm $|\cdot|_* = |K^{1/2}\cdot|$ associated with B, *i.e.,*

$$I_* = \{x \in \mathbb{R}^n : |x|_* \leq 1\} = \left\{x \in \mathbb{R}^n : |K^{1/2}x| \leq 1\right\} = \left\{x \in \mathbb{R}^n : x^t K x \leq 1\right\}, \tag{18}$$

and we let O_* denote the annulus

$$O_* = I_* \setminus B^{-1}(I_*).$$

The ringlike structure of O_* is guaranteed by the fact that B is expanding in all directions in the $|\cdot|_*$ norm, *i.e.,*

$$I_* \subset \lambda I_* \subset B(I_*), \qquad \lambda > 1, \tag{19}$$

which is (iv) in Proposition 3.1. We note that by an orthogonal substitution I_* takes the form $\{x \in \mathbb{R}^n : \mu_1\tilde{x}_1^2 + \cdots + \mu_n\tilde{x}_n^2 \leq 1\}$, where μ_i are the positive eigenvalues of K and $x = Q\tilde{x}$ with the ith column of $Q \in O(n)$ comprising of the ith eigenvector of K. The annulus O_* is a bounded multiplicative tiling set for $\{B^j : j \in \mathbb{Z}\}$. This is a consequence of the following result.

Lemma 3.2. *Let $B \in GL_n(\mathbb{R})$ be an expansive matrix. For $x \neq 0$ there is a unique $j \in \mathbb{Z}$ so that $B^j x \in O_*$; that is,*

$$\mathbb{R}^n \setminus \{0\} = \bigcup_{j \in \mathbb{Z}} B^j(O_*) \quad \text{with disjoint union.} \tag{20}$$

Proof. From equation (19) we know that $\{B^l(I_*)\}_{l\in\mathbb{Z}}$ is a nested sequence of subsets of $\mathbb{R}^n$, thus

$$B^l(I_*) \setminus B^{l-1}(I_*) = B^l(O_*), \quad l \in \mathbb{Z},$$

are disjoint sets. Since $|B^{-j}x|_* \leq \lambda^{-j}|x|_*$ and $|B^j x|_* \geq \lambda^j |x|_*$ for $j \geq 0$ and $\lambda > 1$, we also have

$$\begin{aligned}\bigcup_{m=-l+1}^{l} B^m(O_*) &= B^l(I_*) \setminus B^{-l}(I_*) = \left\{x \in \mathbb{R}^n : |B^{-l}x|_* \leq 1 \text{ and } |B^l x|_* > 1\right\} \\ &\supset \left\{x \in \mathbb{R}^n : \lambda^{-l}|x|_* \leq 1 \text{ and } \lambda^l |x|_* > 1\right\} = \left\{x \in \mathbb{R}^n : \lambda^{-l} < |x|_* \leq \lambda^l\right\}.\end{aligned}$$

Taking the limit $l \to \infty$ we get (20). □

Example 2. Let the following dilation matrix be given

$$A = \begin{pmatrix} 3 & -3 \\ 1 & 0 \end{pmatrix}. \tag{21}$$

Here we are interested in the transpose matrix $B = A^t$ with eigenvalues $\mu_{1,2} = 3/2 \pm i\sqrt{3}/2$, hence B is an expansive matrix with $|\mu_{1,2}| = \sqrt{3} > 1$. The dilation matrix B is not expanding in the standard norm $|\cdot|_2$ in $\mathbb{R}^n$, *i.e.*, $I_2 \not\subset B(I_2)$, as shown by Figure 2. In order to have B expanding the unit ball we need to use the

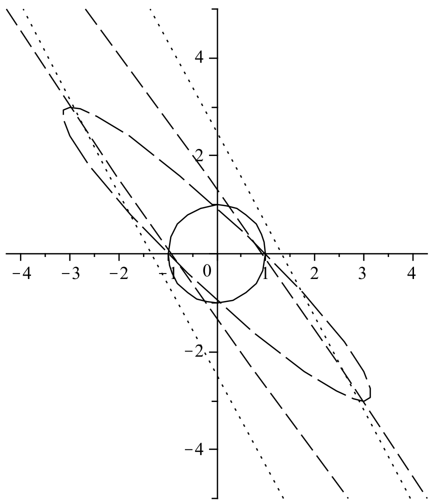

FIGURE 2. Boundaries of the sets I_2, $B(I_2)$, $B^2(I_2)$, and $B^3(I_2)$ marked by solid, long dashed, dashed, and dotted lines, respectively. Note that $I_2 \setminus B(I_2)$ is non-empty, and even $I_2 \setminus B^2(I_2)$ is non-empty.

Hermitian norm from (iii) in Proposition 3.1 associated with B. In (17) we take $k = 2$ so that the real, symmetric, positive definite matrix K is

$$K = I + (B^{-1})^t B^{-1} + (B^{-2})^t B^{-2} = \begin{pmatrix} 28/9 & 16/9 \\ 16/9 & 8/3 \end{pmatrix},$$

and let $\langle x,y\rangle_* := x^t K y$. The choice $k=2$ suffices since it makes $B^tKB-\lambda^2K$ semi-positive definite for $\lambda = 1.03$ and thus

$$|Bx|_* \geq \lambda\,|x|_*\,, \qquad x \in \mathbb{R}^2,$$

holds for $\lambda = 1.03$.

Figure 3 and 4 illustrate that B indeed expands the Hermitian norm unit ball I_* in all directions. We also remark that the Hermitian norm with $k=1$ will not make

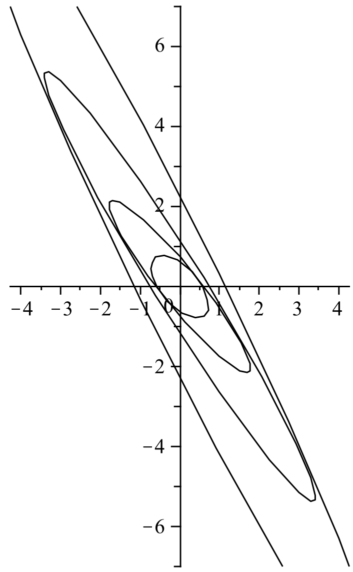

FIGURE 3. The unit ball I_* in the Hermitian norm $|\cdot|_*$ associated with B and its dilations $B(I_*), B^2(I_*), B^3(I_*)$. Only the boundaries are marked.

the dilation matrix B expanding in $\mathbb{R}^n$; in this case we have a situation similar to Figure 2.

3.2. **A crude lattice choice.** Let us consider the setup in Theorem 2.3 with the set $E = B^c(O_*)$ for some $c\in\mathbb{Z}$, where the norm $|\cdot|_* = |K^{1/2}\cdot|$ is associated with B. Let μ be the smallest eigenvalue of K such that $\ell = \sqrt{1/\mu}$ is the largest semi-principal axis of the ellipsoid I_*, *i.e.*, $\ell = \max_{x\in I_*} |x|_2$. Then we can take any lattice $\Gamma = P\mathbb{Z}^n$, where P is a non-singular matrix satisfying

$$\|P\|_2 \leq \frac{1}{\ell\,\|A^c\|_2\,(1+\|A^{\underline{m}}\|_2)}, \tag{22}$$

as our translation lattice in Theorem 2.3. To see this, recall that we are looking for a lattice Γ^* such that, for $\gamma\in\Gamma^*\setminus\{0\}$,

$$\operatorname{supp}\hat{\phi} \cap \operatorname{supp}\hat{\psi}(\cdot\pm\gamma) = \emptyset. \tag{23}$$

For our choice of E we find that $\operatorname{supp}\hat{\phi}\subset B^{c+\underline{m}}(I_*)$ and $\operatorname{supp}\hat{\psi}\subset B^c(I_*)$. Since

$$\left|B^{c+\underline{m}}x\right|_2 \leq \left\|B^{c+\underline{m}}\right\|_2 |x|_2 \leq \left\|B^{c+\underline{m}}\right\|_2 \ell \qquad \text{for any } x\in I_*,$$

and similar for $B^c x$, we have the situation in (23) whenever $|\gamma|_2 \geq \ell(\|A^c\|_2 + \|A^{c+\underline{m}}\|_2)$. Here we have used that for the 2-norm $\|A\|_2 = \|B\|_2$. For $z\in\mathbb{Z}^n$ we have

$$|z|_2 \leq \|P^t\|_2\,|(P^t)^{-1}z|_2 = \|P\|_2\,|(P^t)^{-1}z|_2,$$

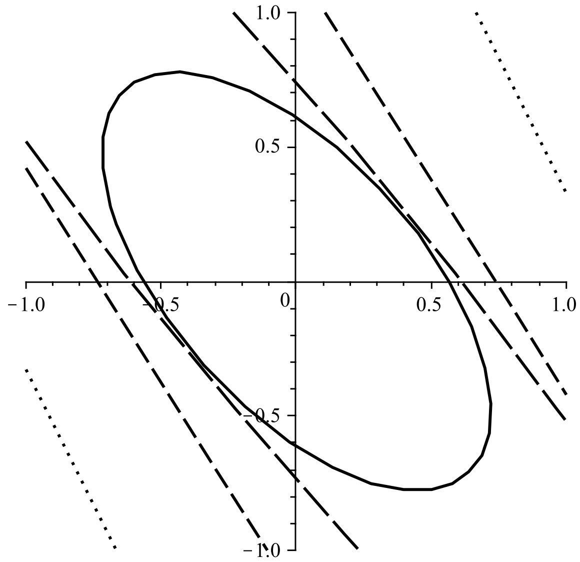

FIGURE 4. A zoom of Figure 3. Boundaries of the sets I_*, $B(I_*)$, $B^2(I_*)$, and $B^3(I_*)$ marked by solid, long dashed, dashed, and dotted lines, respectively.

therefore, by $|z|_2 \geq 1$ for $z \neq 0$, we have

$$\left|(P^t)^{-1}z\right|_2 \geq \frac{1}{\|P\|_2} \qquad \text{for } z \in \mathbb{Z} \setminus \{0\}.$$

Now, by assuming that P satisfies (22), we have

$$|\gamma|_2 = |(P^t)^{-1}z|_2 \geq 1/\|P\|_2 \geq \ell \, \|A^c\|_2 \, (1 + \|A^{\underline{m}}\|_2) \geq l(\|A^c\|_2 + \left\|A^{c+\underline{m}}\right\|_2)$$

for $0 \neq \gamma = (P^t)^{-1}z \in (P^t)^{-1}\mathbb{Z}^n = \Gamma^*$, hence the claim follows.

A lattice choice based on (22) can be rather crude, and produces consequently a wavelet system with unnecessarily many translates. From equation (22) it is obvious that any lattice $\Gamma = P\mathbb{Z}^n$ with $\|P\|$ sufficiently small will work as translation lattice for our pair of generators ψ and ϕ. Hence, the challenging part is to find a sparse translation lattice whereby we understand a lattice Γ with large determinant $d(\Gamma) := |\det P|$. In the dual lattice system this corresponds to a dense lattice Γ^* with small volume $d(\Gamma^*)$ of the fundamental parallelotope I_{Γ^*} since $d(\Gamma)d(\Gamma^*) = 1$. In Theorem 3.3 in the next section we make a better choice of the translation lattice compared to what we have from (22).

Using a crude lattice approach as above, we can easily transform the translation lattice to the integer lattice if we allow multiple generators. We pick a matrix P that satisfies condition (22) and whose inverse is integer valued, *i.e.*, $Q := P^{-1} \in GL_n(\mathbb{Z})$. The conclusion from Theorem 2.3 is that $\{D_{A^j}T_{Q^{-1}k}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ and $\{D_{A^j}T_{Q^{-1}k}\phi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ are dual frames. The order of the quotient group $Q^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ is $|\det Q|$, so let $\{d_i : i = 1, \dots, |\det Q|\}$ denote a complete set of representatives of the quotient group, and define

$$\Psi = \{T_{d_i}\psi : i = 1, \dots, |\det Q|\}\,, \quad \Phi = \{T_{d_i}\phi : i = 1, \dots, |\det Q|\}\,.$$

Since $\{D_{A^j}T_{Q^{-1}k}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n} = \{D_{A^j}T_k\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,\psi\in\Psi}$ and likewise for the dual frame, the statement follows.

3.3. **A concrete version of Theorem 2.3.** We list some standing assumptions and conventions for this section.

General setup. We assume $A \in GL_n(\mathbb{R})$ is expansive. Let $|\cdot|_* = \langle \cdot, \cdot \rangle_*^{1/2}$ be a Hermitian norm as in (iii) in Proposition 3.1 associated with $B = A^t$, let I_* denote the unit ball in the $|\cdot|_*$-norm, and let $K \in GL_n(\mathbb{R})$ be the symmetric, positive definite matrix such that $\langle x, y \rangle_* = y^t K x$. Let $\Lambda := \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, where $\{\lambda_i\}$ are the eigenvalues of K, and let $Q \in O(n)$ be such that the spectral decomposition of K is $Q^t K Q = \Lambda$.

The following result is a special case of Theorem 2.3, where we, in particular, specify how to choose the translation lattice Γ. Since we in Theorem 3.3 define Γ, it allows for a more automated construction procedure.

Theorem 3.3. *Let A, I_*, K, Q, Λ be as in the general setup. Let $d \in \mathbb{N}_0$ and $\psi \in L^2(\mathbb{R}^n)$. Suppose that $\hat{\psi}$ is a bounded, real-valued function with* $\operatorname{supp} \hat{\psi} \subset B^c(I_*) \setminus B^{c-d-1}(I_*)$ *for some $c \in \mathbb{Z}$, and that (8) holds. Take $\Gamma = (1/2)A^c Q \sqrt{\Lambda} \mathbb{Z}^n$. Then the function ψ and the function ϕ defined by*

$$\phi(x) = d(\Gamma) \left[\psi(x) + 2 \sum_{j=0}^{d} |\det A|^{-j} \psi(A^{-j} x) \right] \quad \text{for } x \in \mathbb{R}^n, \tag{24}$$

generate dual frames $\{D_{A^j} T_\gamma \psi\}_{j \in \mathbb{Z}, \gamma \in \Gamma}$ and $\{D_{A^j} T_\gamma \phi\}_{j \in \mathbb{Z}, \gamma \in \Gamma}$ for $L^2(\mathbb{R}^n)$

Remark 2. Note that $d(\Gamma) = 2^{-n} |\det A|^c (\lambda_1 \cdots \lambda_n)^{1/2}$ and $\sqrt{\Lambda} = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$.

Proof. The annulus O_* is a bounded multiplicative tiling set for $\{B^j : j \in \mathbb{Z}\}$ by Lemma 3.2, hence this is also the case for $B^c(O_*)$ for $c \in \mathbb{Z}$. The support of $\hat{\psi}$ is $\operatorname{supp} \hat{\psi} \subset B^c(I_*) \setminus B^{c-d-1}(I_*) = \cup_{j=0}^d B^{c-j}(O_*)$. Therefore we can apply Theorem 2.3 with $E = B^c(O_*)$, $b_j = 2$ and $b_{-j} = 0$ for $j = 1, \dots, d$ so that $\underline{m} = 0$ and $\overline{m} = d$. The only thing left to justify is the choice of the translation lattice Γ. We need to show that condition (9) with $\underline{m} = 0$ and $\overline{m} = d$ in Theorem 2.3 is satisfied by $\Gamma^* = 2B^c Q \Lambda^{-1/2} \mathbb{Z}^n$. By the orthogonal substitution $x = Q\tilde{x}$ the quadratic form $x^t K x$ of equation (18) reduces to

$$\lambda_1 \tilde{x}_1^2 + \dots + \lambda_n \tilde{x}_n^2,$$

where $\lambda_i > 0$, hence in the $\tilde{x} = Q^t x$ coordinates I_* is given by

$$\tilde{I}_* = \left\{ \tilde{x} \in \mathbb{R}^n : \left(\frac{\tilde{x}_1}{1/\sqrt{\lambda_1}} \right)^2 + \dots + \left(\frac{\tilde{x}_n}{1/\sqrt{\lambda_n}} \right)^2 < 1 \right\}$$

which is an ellipsoid with semi axes $\frac{1}{\sqrt{\lambda_1}}, \dots, \frac{1}{\sqrt{\lambda_n}}$. Therefore, in the $\tilde{x}$ coordinates,

$$(\tilde{I}_* + \gamma) \cap \tilde{I}_* = \emptyset \qquad \text{for } 0 \neq \gamma \in 2\Lambda^{-1/2} \mathbb{Z}^n,$$

or, in the x coordinates,

$$(I_* + \gamma) \cap I_* = \emptyset \qquad \text{for } 0 \neq \gamma \in 2Q\Lambda^{-1/2} \mathbb{Z}^n.$$

By applying B^c to this relation it becomes

$$\big(B^c(I_*) + \gamma \big) \cap B^c(I_*) = \emptyset \qquad \text{for } 0 \neq \gamma \in \Gamma^* = 2B^c Q \Lambda^{-1/2} \mathbb{Z}^n, \tag{25}$$

whereby we see that condition (9) is satisfied with $\underline{m} = 0$ and $\Gamma^* = 2B^c Q \Lambda^{-1/2} \mathbb{Z}^n$. The dual lattice of Γ^* is $\Gamma = 1/2 A^{-c} Q \Lambda^{1/2} \mathbb{Z}^n$. It follows from Theorem 2.3 that ψ and ϕ generate dual frames for this choice of the translation lattice. $\square$

The frame bounds for the pair of dual frames $\{D_{A^j}T_\gamma\psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ and $\{D_{A^j}T_\gamma\phi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ in Theorem 3.3 can be given explicitly as

$$C_1 = \frac{1}{d(\Gamma)} \inf_{\xi\in B^{c-d}(O_*)} \sum_{j=0}^{d} \Big(\hat{\psi}(B^j\xi)\Big)^2, \qquad C_2 = \frac{1}{d(\Gamma)} \sup_{\xi\in B^{c-d}(O_*)} \sum_{j=0}^{d} \Big(\hat{\psi}(B^j\xi)\Big)^2,$$

and

$$C_1 = \frac{1}{d(\Gamma)} \inf_{\xi\in B^{c-d}(O_*)} \sum_{j=-d}^{d} \Big(\hat{\phi}(B^j\xi)\Big)^2, \qquad C_2 = \frac{1}{d(\Gamma)} \sup_{\xi\in B^{c-d}(O_*)} \sum_{j=-d}^{d} \Big(\hat{\phi}(B^j\xi)\Big)^2,$$

respectively. The frame bounds do not depend on the specific structure of Γ, but only on the determinant of Γ; in particular, the condition number C_2/C_1 is independent of Γ.

To verify these frame bounds, we note that equation (25) together with the fact $\operatorname{supp}\hat{\psi}$, $\operatorname{supp}\hat{\phi} \subset B^c(I_*)$ imply that

$$\hat{\psi}(\xi)\hat{\psi}(\xi+\gamma) = \hat{\phi}(\xi)\hat{\phi}(\xi+\gamma) = 0 \qquad \text{for a.e. } \xi\in\mathbb{R}^n \text{ and } \gamma\in\Gamma^*\setminus\{0\}.$$

Therefore, by equations (12) and (13) with $E = B^c(O_*)$, $\underline{m} = 0$ and $\overline{m} = d$, we have

$$\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma^*} \left|\hat{\psi}(B^j\xi)\hat{\psi}(B^j\xi+\gamma)\right| = \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(B^j\xi)\right|^2 = \sum_{j=0}^{d}\Big(\hat{\psi}(B^j\xi)\Big)^2,$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma^*} \left|\hat{\phi}(B^j\xi)\hat{\phi}(B^j\xi+\gamma)\right| = \sum_{j\in\mathbb{Z}} \left|\hat{\phi}(B^j\xi)\right|^2 = \sum_{j=-d}^{d}\Big(\hat{\phi}(B^j\xi)\Big)^2,$$

for $\xi\in B^{c-d}(O_*)$. The stated frame bounds follow from Lemma 2.2.

Example 3. Let A and K be as in Example 2. The eigenvalues of K are $\lambda_1 = (26+2\sqrt{65})/9 \approx 4.7$ and $\lambda_2 = (26-2\sqrt{65})/9 \approx 1.1$. Let the normalized (in the standard norm) eigenvectors of K be columns of $Q\in O(2)$ and $\Lambda = \operatorname{diag}(\lambda_1,\lambda_2)$, hence $Q^tKQ = \Lambda$. By the orthogonal transformation $x = Q\tilde{x}$ the Hermitian norm unit ball I_* becomes

$$\tilde{I}_* = \left\{\tilde{x}\in\mathbb{R}^2 : \left(\frac{\tilde{x}_1}{1/\sqrt{\lambda_1}}\right)^2 + \left(\frac{\tilde{x}_2}{1/\sqrt{\lambda_2}}\right)^2 < 1\right\} \subset I_2$$

which is an ellipse with semimajor axis $1/\sqrt{\lambda_2}\approx 0.95$ and semiminor axis $1/\sqrt{\lambda_1}\approx 0.46$. Since $\Lambda^{-1/2} = \operatorname{diag}(1/\sqrt{\lambda_1}, 1/\sqrt{\lambda_2})$, we have

$$\left|(\tilde{I}_*+\gamma)\cap\tilde{I}_*\right| = 0 \qquad \text{for } 0\neq\gamma\in 2\Lambda^{-1/2}\mathbb{Z}^2.$$

By the orthogonal substitution back to x coordinates, we get

$$|(I_*+\gamma)\cap I_*| = 0 \qquad \text{for } 0\neq\gamma\in 2Q\Lambda^{-1/2}\mathbb{Z}^2.$$

Suppose that $\hat{\psi}$ is a bounded, real-valued function with $\operatorname{supp}\hat{\psi}\subset B^c(I_*)\backslash B^{c-d-1}(I_*)$ for $c=1$ that satisfies the B-dilative partition (8). Since $c=1$ we need to take $\Gamma^* = 2B^1Q\Lambda^{-1/2}\mathbb{Z}^2$ and $\Gamma = 1/2A^{-1}Q\Lambda^{1/2}\mathbb{Z}^2$, see Figure 5 and 6.

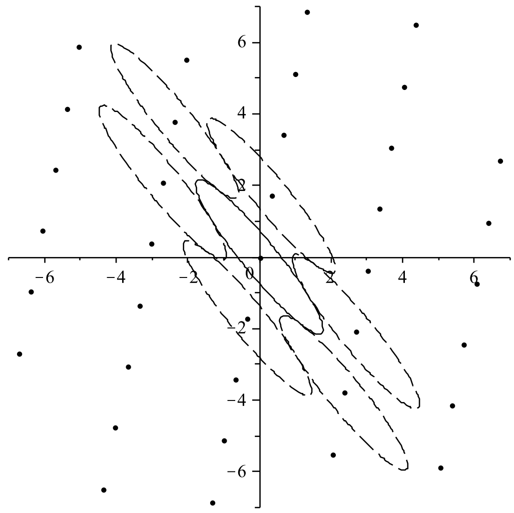

FIGURE 5. The dual lattice $\Gamma^* = 2B^cQ\Lambda^{-1/2}\mathbb{Z}^2$ for $c = 1$ is shown by dots, and the boundary of the set $B^c(I_*)$ by a solid line. Boundaries of the set $B^c(I_*)$ translated to several different $\gamma \in \Gamma^* \setminus \{0\}$ are shown with dashed lines. Recall that $\operatorname{supp}\hat{\psi}, \operatorname{supp}\hat{\phi} \subset B^c(I_*)$, hence $\operatorname{supp}\hat{\phi} \cap \operatorname{supp}\hat{\psi}(\cdot + \gamma) = \emptyset$ for $\gamma \in \Gamma^* \setminus \{0\}$.

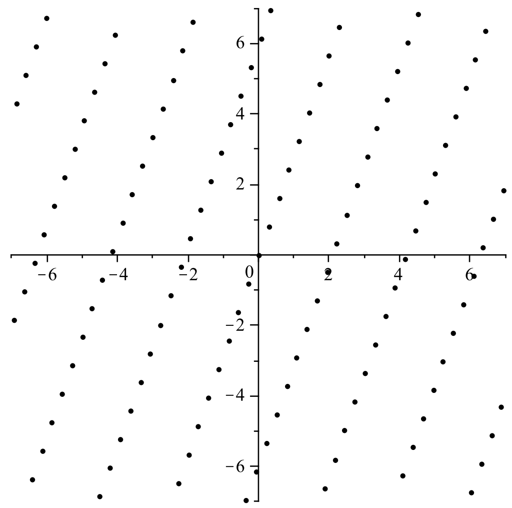

FIGURE 6. The translation lattice $\Gamma = (1/2)A^cQ\Lambda^{1/2}\mathbb{Z}^2$ for $c = 1$.

3.4. **An alternative lattice choice.** Let the setup up and assumptions be as in Theorem 3.3, except for the lattice Γ which we want to choose differently. As in Section 3.2 the dual lattice Γ^* needs to satisfy (23) for $\gamma \in \Gamma^* \setminus \{0\}$. We want to choose Γ^* as dense as possible since this will make the translation lattice Γ as sparse as possible and the wavelet system with as few translates as possible. Since $\operatorname{supp}\hat{\psi}, \operatorname{supp}\hat{\phi} \subset B^c(I_*)$, we are looking for lattices Γ^* that packs the ellipsoids $B^c(I_*) + \gamma$, $\gamma \in \Gamma^*$, in a non-overlapping, optimal way. By the coordinate transformation $\hat{x} = \Lambda^{-1/2}Q^tB^{-c}x$, the ellipsoid $B^c(I_*)$ turns into the standard unit ball I_2 in $\mathbb{R}^n$. This calculations are as follows.

$$B^c(I_*) = \left\{B^cx : |x|_*^2 \le 1\right\} = \left\{x : |K^{1/2}B^{-c}x|_2^2 \le 1\right\}$$

$$= \left\{ x : \left| K^{1/2} B^{-c} B^{c} Q \Lambda^{-1/2} \hat{x} \right|_2^2 \le 1 \right\}$$
$$= \left\{ x : \left\langle \hat{x}, \Lambda^{-1/2} Q^t K Q \Lambda^{-1/2} \hat{x} \right\rangle_2 \le 1 \right\} = \left\{ x : \left| \hat{x} \right|_2^2 \le 1 \right\},$$

and we arrive at a standard sphere packing problem with lattice arrangement of non-overlapping unit n-balls. The proportion of the Euclidean space $\mathbb{R}^n$ filled by the balls is called the density of the arrangement, and it is this density we want as high as possible.

Taking Γ as in Theorem 3.3 corresponds to a square packing of the unit n-balls $I_2 + k$ by the lattice $2\mathbb{Z}^n$, *i.e.*, $k \in 2\mathbb{Z}^n$. The density of this packing is $V_n 2^{-n}$, where V_n is the volume of the n-ball: $V_{2n} = \pi^n/(n!)$ and $V_{2n+1} = (2^{2n+1} n! \pi^n)/(2n+1)!$. This is not the densest packing of balls in $\mathbb{R}^n$ since there exists a lattice with density bigger than $1.68 n 2^{-n}$ for each $n \neq 1$ [12]; a slight improvement of this lower bound was obtained in [1] for $n > 5$. Moreover, the densest lattice packing of hyperspheres is known up to dimension 8, see [23]; it is precisely this dense lattice we want to use in place of $2\mathbb{Z}^n$ (at least whenever $n \le 8$).

In $\mathbb{R}^2$ Lagrange proved that the hexagonal packing, where each ball touches 6 other balls in a hexagonal lattice, has the highest density $\pi/\sqrt{12}$. Hence using $P\mathbb{Z}^2$ with

$$P = \begin{pmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{pmatrix}$$

instead of $2\mathbb{Z}^2$ improves the packing by a factor of

$$\frac{\pi/\sqrt{12}}{\pi/2^2} = 4/\sqrt{12} = 2/\sqrt{3}.$$

It is easily seen that this factor equals the relation between the area of the fundamental parallelogram of the two lattices $|\det 2I_{2\times 2}| \,/\, |\det P|$. In Figure 5 we see that each ellipse only touches 4 other ellipses corresponding to the square packing $2\mathbb{Z}^n$; in the optimal packing each ellipse touch 6 others. In $\mathbb{R}^3$ Gauss proved that the highest density is $\pi/\sqrt{18}$ obtained by the hexagonal close and face-centered cubic packing; here each ball touches 12 other balls.

4. Dilative partition of unity

With Theorem 3.3 at hand the only issue left is to specify how to construct functions satisfying the partition of unity (8) for any given expansive matrix. In the two examples of this section we outline possible ways of achieving this.

4.1. **Constructing a partition of unity.** As usual we fix the dimension $n \in \mathbb{N}$ and the expansive matrix $B \in GL_n(\mathbb{R})$. In the examples in this section we construct functions satisfying the assumptions in Theorem 3.3, that is, a real-valued function $g \in L^2(\mathbb{R}^n)$ with $\operatorname{supp} g \subset B^c(I_*) \setminus B^{c-d-1}(I_*)$ for some $c \in \mathbb{Z}$ and $d \in \mathbb{N}_0$ so that the B-dilative partition

$$\sum_{j\in\mathbb{Z}} g(B^j \xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{26}$$

holds.

In the construction we will use that the radial coordinate of the surface of the ellipsoid $\partial B^j(I_*)$, $j \in \mathbb{Z}$, can be parametrized by the $n-1$ angular coordinates $\theta_1, \dots, \theta_{n-1}$. The radial coordinate expression will be of the form $h(\theta_1, \dots, \theta_{n-1})^{-1/2}$

for some positive, trigonometric function h, where h is bounded away from zero and infinity with the specific form of h depending on the dimension n and the length and orientation of the ellipsoid axes.

We illustrate this with the following example in $\mathbb{R}^4$. We want to find the radial coordinate r of the ellipsoid

$$\left\{x \in \mathbb{R}^4 : (x_1/\ell_1)^2 + (x_2/\ell_2)^2 + (x_3/\ell_3)^2 + (x_4/\ell_4)^2 = 1\right\}, \qquad \ell_i > 0,\ i = 1, 2, 3, 4,$$

as a function the angular coordinates θ_1, θ_2 and θ_3. We express $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ in the hyperspherical coordinates $(r, \theta_1, \theta_2, \theta_3) \in \{0\} \cup \mathbb{R}_+ \times [0, \pi] \times [0, \pi] \times [0, 2\pi)$ as follows:

$$x_1 = r\cos\theta_1, \qquad x_2 = r\sin\theta_1\cos\theta_2,$$
$$x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, \qquad x_4 = r\sin\theta_1\sin\theta_2\sin\theta_3.$$

Then we substitute x_i, $i = 1, \ldots, 4$, in the expression above and factor out r^2 to obtain $r^2 f(\theta_1, \theta_2, \theta_3) = 1$, where

$$\begin{aligned} f(\theta_1, \theta_2, \theta_3) = {} & \ell_1^{-2}\cos^2\theta_1 + \ell_2^{-2}\sin^2\theta_1\cos^2\theta_2 \\ & + \ell_3^{-2}\sin^2\theta_1\sin^2\theta_2\cos^2\theta_3 + \ell_4^{-2}\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3. \end{aligned} \tag{27}$$

The conclusion is that $r = r(\theta_1, \theta_2, \theta_3) = f(\theta_1, \theta_2, \theta_3)^{-1/2}$.

Example 4. For $d = 1$ in Theorem 3.3 we want $g \in C_0^s(\mathbb{R}^n)$ for any given $s \in \mathbb{N} \cup \{0\}$. The choice $d = 1$ will fix the "size" of the support of g so that $\operatorname{supp} g \subset B^c(I_*) \setminus B^{c-2}(I_*)$ for some $c \in \mathbb{Z}$. Now let $r_1 = r_1(\theta_1, \ldots, \theta_{n-1})$ and $r_2 = r_2(\theta_1, \ldots, \theta_{n-1})$ denote the radial coordinates of the surface of the ellipsoids $\partial B^{c-1}(I_*)$ and $\partial B^c(I_*)$ parametrized by $n-1$ angular coordinates $\theta_1, \ldots, \theta_{n-1}$, respectively.

Let f be a continuous function on the annulus $S = \overline{B^c(O_*)}$ satisfying $f|_{\partial B^{c-1}(I_*)} = 1$ and $f|_{\partial B^c(I_*)} = 0$. Using the parametrizations r_1, r_2 of the surfaces of the two ellipsoids and fixing the $n-1$ angular coordinates we realize that we only have to find a continuous function $f : [r_1, r_2] \to \mathbb{R}$ of one variable (the radial coordinate) satisfying $f(r_1) = 1$ and $f(r_2) = 0$. For example the general function $f \in C^0(S)$ of d variables can be any of the functions below:

$$f(x) = f(r, \theta_1, \ldots, \theta_{n-1}) = \frac{r_2 - r}{r_2 - r_1}, \tag{28a}$$

$$f(x) = f(r, \theta_1, \ldots, \theta_{n-1}) = \frac{(r_2 - r)^2}{(r_2 - r_1)^3}(2(r - r_1) + r_2 - r_1), \tag{28b}$$

$$f(x) = f(r, \theta_1, \ldots, \theta_{n-1}) = \tfrac{1}{2} + \tfrac{1}{2}\cos\pi(\tfrac{r - r_1}{r_2 - r_1}), \tag{28c}$$

where $r = |x| \in [r_1, r_2]$, $\theta_1, \ldots, \theta_{n-2} \in [0, \pi]$, and $\theta_{n-1} \in [0, 2\pi)$; recall that $r_1 = r_1(\theta_1, \ldots, \theta_{n-1})$ and $r_2 = r_2(\theta_1, \ldots, \theta_{n-1})$. In definitions (28b) and (28c) the function f even belongs to $C^1(S)$.

Define $g \in L^2(\mathbb{R})$ by:

$$g(x) = \begin{cases} 1 - f(Bx) & \text{for } x \in B^{c-1}(I_*) \setminus B^{c-2}(I_*), \\ f(x) & \text{for } x \in B^c(I_*) \setminus B^{c-1}(I_*), \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

This way g becomes a B-dilative partition of unity with $\operatorname{supp} g \subset B^c(I_*) \setminus B^{c-2}(I_*)$, so we can apply Theorem 3.3 with $\hat{\psi} = g$ and $d = 2$.

We can simplify the expressions for the radial coordinates r_1, r_2 of the surface of the ellipsoids $\partial B^{c-1}(I_*)$ and $\partial B^c(I_*)$ from the previous example by a suitable coordinate change. The idea is to transform the ellipsoid $B^{c-1}(I_*)$ to the standard unit ball I_2 by a first coordinate change $\tilde{x} = \Lambda^{1/2} Q^t B^{-c+1} x$. This will transform the outer ellipsoid $B^c(I_*)$ to another ellipsoid. A second and orthogonal coordinate transform $\hat{x} = Q_\prime^t \tilde{x}$ will make the semiaxes of this new ellipsoid parallel to the coordinate axes, leaving the standard unit ball I_2 unchanged. Here $Q_\prime$ comes from the spectral decomposition of $A^{-1}B^{-1}$, *i.e.*, $A^{-1}B^{-1} = Q_\prime^t \Lambda_\prime Q_\prime$. In the $\hat{x}$ coordinates $r_1 = 1$ is a constant and $r_2 = f^{-1/2}$ with f of the form (27) for $n = 4$ and likewise for $n \neq 4$.

In the construction in Example 4 we assumed that $d = 1$. The next example works for all $d \in \mathbb{N}$; moreover, the constructed function will belong to $C_0^\infty(\mathbb{R}^n)$.

Example 5. For sufficiently small $\delta > 0$ define $\Delta_1, \Delta_2 \subset \mathbb{R}^n$ by

$$\begin{aligned} \Delta_1 &= B^{c-d-1}(I_*) + B(0,\delta), \\ \Delta_2 + B(0,\delta) &= B^c(I_*). \end{aligned}$$

This makes $\Delta_2 \setminus \Delta_1$ a subset of the annulus $B^c(I_*) \setminus B^{c-d-1}(I_*)$; it is exactly the subset, where points less than δ in distance from the boundary have been removed, or in other words

$$\Delta_2 \setminus \Delta_1 + B(0,\delta) = B^c(I_*) \setminus B^{c-d-1}(I_*).$$

For this to hold, we of course need to take $\delta > 0$ sufficiently small, *e.g.* such that $\Delta_1 \subset r\Delta_1 \subset \Delta_2$ holds for some $r > 1$.

Let $h \in C_0^\infty(\mathbb{R}^n)$ satisfy $\operatorname{supp} h = B(0,1)$, $h \geq 0$, and $\int h \,\mathrm{d}\mu = 1$, and define $h_\delta = \delta^{-d} h(\delta^{-1}\cdot)$. By convoluting the characteristic function on $\Delta_2 \setminus \Delta_1$ with h_δ we obtain a smooth function living on the annulus $B^c(I_*) \setminus B^{c-d-1}(I_*)$. So let $p \in C_0^\infty(\mathbb{R}^n)$ be defined by

$$p = h_\delta * \chi_{\Delta_2 \setminus \Delta_1},$$

and note that $\operatorname{supp} p = B^c(I_*) \setminus B^{c-d-1}(I_*)$ since $\operatorname{supp} h_\delta = B(0,\delta)$. Normalizing the function p in a proper way will give us the function g we are looking for. We will normalize p by the function w:

$$w(x) = \sum_{j \in \mathbb{Z}} p(B^j x).$$

For a fixed $x \in \mathbb{R}^n \setminus \{0\}$ this sum has either d or $d+1$ nonzero terms, and w is therefore bounded away from 0 and ∞:

$$\exists c, C > 0 : c < w(x) < C \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\},$$

hence we can define a function $g \in C_0^\infty(\mathbb{R}^n)$ by

$$g(x) = \frac{p(x)}{w(x)} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}, \quad \text{and,} \quad g(0) = 0. \tag{30}$$

The function g will be an almost everywhere B-dilative partition of unity as is seen by using the B-dilative periodicity of w:

$$\sum_{j \in \mathbb{Z}} g(B^j x) = \sum_{j \in \mathbb{Z}} \frac{p(B^j x)}{w(B^j x)} = \sum_{j \in \mathbb{Z}} \frac{p(B^j x)}{w(x)} = \frac{1}{w(x)} \sum_{j \in \mathbb{Z}} p(B^j x) = 1.$$

Since p is supported on the annulus $B^c(I_*)\backslash B^{c-d-1}(I_*)$, we can simplify the definition in (30) to get rid of the infinite sum in the denominator; this gives us the following expression

$$g(x) = p(x)/\sum_{j=-d}^{d} p(B^j x) \qquad \text{for } x \in \mathbb{R}^n \setminus \{0\}.$$

We can obtain a more explicit expression for p by the following approach. Let $r_1 = r_1(\theta_1, \dots, \theta_{n-1})$ and $r_2 = r_2(\theta_1, \dots, \theta_{n-1})$ denote the radial coordinates of the surface of the ellipsoids $\partial B^{c-d-1}(I_*)$ and $\partial B^c(I_*)$ parametrized by $n-1$ angular coordinates $\theta_1, \dots, \theta_{n-1}$, respectively. Finally, let $p \in C_0^\infty(\mathbb{R}^n)$ be defined by

$$p(x) = \eta(|x| - r_1)\,\eta(r_2 - |x|), \quad \text{with } r_1 = r_1(\theta_1, \dots, \theta_{n-1}) \text{ and } r_2 = r_2(\theta_1, \dots, \theta_{n-1})$$

where $\theta_1, \dots, \theta_{n-1}$ can be found from x, and

$$\eta(x) = \begin{cases} \mathrm{e}^{-1/x} & x > 0, \\ 0 & x \le 0. \end{cases}$$

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Current address: Institute of Mathematics, University of Osnabrück, 49069 Osnabrück, Germany. Telephone: +49 (0)541 969-3580, Fax: +49 (0)541 969-2770

E-mail address: `jlemvig@uni-osnabrueck.de`

Department of Mathematics, Matematiktorvet Building 303, Technical University of Denmark, DK-2800 Kgs. Lyngby, Denmark