CONSTRUCTING PAIRS OF DUAL BANDLIMITED FRAMELETS WITH DESIRED TIME LOCALIZATION

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Abstract. For sufficiently small translation parameters, we prove that any bandlimited function ψ, for which the dilations of its Fourier transform form a partition of unity, generates a wavelet frame with a dual frame also having the wavelet structure. This dual frame is generated by a finite linear combination of dilations of ψ with explicitly given coefficients. The result allows a simple construction procedure for pairs of dual wavelet frames whose generators have compact support in the Fourier domain and desired time localization. The construction is based on characterizing equations for dual wavelet frames and relies on a technical condition. We exhibit a general class of function satisfying this condition; in particular, we construct piecewise polynomial functions satisfying the condition.

1. Introduction

Let $\psi \in L^2(\mathbb{R})$ be a function such that $\hat{\psi}$ is compactly supported and the functions $\xi \mapsto \hat{\psi}(a^j\xi)$, $j \in \mathbb{Z}$, form a partition of unity for some $a > 1$. We prove that for sufficiently small translation parameter b the function ψ generates a wavelet frame $\{a^{j/2}\psi(a^jx - bk) : j, k \in \mathbb{Z}\}$ with a dual wavelet frame generated by a finite linear combination of dilations of ψ. The result allows a construction procedure for pairs of dual wavelet frames generated by bandlimited functions with fast decay in the time domain where both generators are explicitly given.

The principal idea used in the proof of Theorem 3 comes from Christensen's construction of dual Gabor frames in [6]. Our construction is similar, but it takes place in the Fourier domain. The proof of Theorem 3 and the construction procedure provided by this theorem are based on the well-known characterizing equations for dual wavelet frames by Chui and Shi [8].

Our aim is to provide a construction of a pair of dual frame generators ψ and ϕ for which the functions ψ and ϕ are explicitly given in the sense that the functions or their Fourier transform are given as finite linear combinations of elementary functions. To be precise, the construction uses ψ as a starting point and defines the dual generator ϕ as a finite linear combination of dilations of ψ with explicitly

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given coefficients. This gives us control of the properties of both generators as opposed to using canonical duals.

The construction of redundant wavelet representations is often restricted to tight frames in order to avoid the cumbersome inversion of the frame operator. However, in this paper we consider general non-tight, non-canonical, non-dyadic dual wavelet frames. The construction of wavelet frames is usually based on the (mixed) unitary or oblique extension principle [7, 9, 12, 13]. These principles lead to dual or tight frame wavelets with many desirable features: compact support, high order of vanishing moments, high smoothness, and symmetry/antisymmetry; in particular, explicitly given spline generators are constructed from B-spline multiresolution analysis in [7,9]. In these and similar constructions one cannot do with fewer than two generators (see [7, Theorem 9] and [9, Theorem 3.8] including the succeeding remark); in addition, higher smoothness leads to more generators or larger support of the generators. Our construction leads to frame wavelet with similar properties, the most notable difference is that the generators have compact support in the Fourier domain, not in the time domain.

Wavelet frames constructed by the unitary extension principle from a B-spline multiresolution analysis will always have one generator with only one vanishing moment yielding a wavelet system with approximation order of at most 2; this problem is circumvented in the oblique extension principle. When multiple generators are needed in our construction, all of these will share the same properties. In Examples 2 and 3 the constructed wavelet frames are generated by only *one* function, and in these cases the smoothness of the generator does not affect the size of the support (that is, in the Fourier domain).

Our construction is explicit, and it works for arbitrary real dilations, but as a drawback the wavelet frame generators will not have compact support in the time domain leading to infinite impulse response filters. In the dyadic case an efficient algorithm can be implemented by using the fast Fourier transform, see for example the fractional spline wavelet software for Matlab by Unser and Blu [3]. The idea is to perform the calculation in the Fourier domain using multiplication and periodization in place of convolution and down-sampling. For this to work, we need the frequency response of the filter coefficients (sometimes simply called filters or masks and often denoted by τ_i, m_i, or H_i), but we get this almost directly from our construction; the frequency response of *both* high pass filters (decomposition and reconstruction) can be obtained from dilations of $\hat{\psi}$. Note that this relies crucially on the fact that the dual generator ϕ is defined as a finite linear combination of dilations of ψ with explicitly given coefficients.

The paper is organized as follows. In Section 2 we prove the main result of this article, Theorem 3. The theorem contains a technical condition on partition of unity, and we address this problem in Example 1 where we explicitly construct functions that satisfy the condition. A note on the terminology: the functions in the "partition of unity" are not assumed to be non-negative, but can take any real value. In Example 2 we give an example of a pair of smooth, fast decaying,

symmetric generators with the translation parameter being 1. The construction of dual wavelet frames using Theorem 3 often imposes the translation parameter to be small, *e.g.* smaller than 1. Consequently, we want methods to expand the range of the translation parameter, and this is the topic of Section 2.2. In Section 3 we show that the representation of functions provided by Theorem 3 with the explicitly given dual frame is advantageous over similar representations using tight frames or canonical dual frames. In Section 4 we present another application of Theorem 3 with generators in the Schwartz space. However, the construction in this example is less explicit than in the first example. We end this paper with some remarks on constructions of pairs of dual wavelet frames for the Hardy space.

We end this introduction by reviewing some basic definitions and with an observation on the canonical dual frame. A frame for a separable Hilbert space $\mathcal{H}$ is a collection of vectors $\{f_j\}_{j\in\mathbb{J}}$ with a countable index set $\mathbb{J}$ if there are constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1 \|f\|^2 \le \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \le C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If the upper bound holds in the above inequality, then $\{f_j\}$ is said to be a Bessel sequence with Bessel constant C_2. For a Bessel sequence $\{f_j\}$ we define the frame operator by

$$S\colon \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle f_j.$$

This operator is bounded, invertible, and positive. A frame $\{f_j\}$ is said to be *tight* if we can choose $C_1 = C_2$; this is equivalent to $S = C_1 \mathrm{I}$ where I is the identity operator. Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be *dual* frames if

$$f = \sum_{j\in\mathbb{J}} \langle f, g_j\rangle f_j \quad \forall f \in \mathcal{H}.$$

It can be shown that two such Bessel sequences are indeed frames. Given a frame $\{f_j\}$, at least one dual always exists; it is called the canonical dual and is given by $\{S^{-1} f_j\}$. Only redundant frames have several duals.

For $f \in L^2(\mathbb{R})$, we define the dilation operator by $D_a f(x) = a^{1/2} f(ax)$ and the translation operator by $T_b f(x) = f(x-b)$ where $1 < a < \infty$ and $b \in \mathbb{R}$. We say that $\{D_a^j T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ is the wavelet system generated by ψ where $a > 1$ and $b > 0$. In the following we use the index set $(j,k) \in \mathbb{Z}\times\mathbb{Z}$ whenever a sequence is stated without index set. If $\{D_a^j T_{bk}\psi\}$ is a frame for $L^2(\mathbb{R})$, the generator ψ is termed a framelet or frame wavelet. For $f \in L^1(\mathbb{R})$ the Fourier transform is defined by $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx$ with the usual extension to $L^2(\mathbb{R})$. Given a measurable set $K \subset \mathbb{R}$ we define the Paley-Wiener space $\check{L}^2(K)$, which is invariant under all translations, by $\check{L}^2(K) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset K\}$.

2. Construction of dual wavelet frames

Our main result, Theorem 3, is obtained from the following result by Chui and Shi [8]. The result is stated in the last two lines of Section 4 on page 263 in their article.

Theorem 1. *Let $a > 1$, $b > 0$, and $\psi, \tilde{\psi} \in L^2(\mathbb{R})$. Suppose the two wavelet systems $\{D_a^j T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_a^j T_{bk}\tilde{\psi}\}_{j,k\in\mathbb{Z}}$ form Bessel families. Then $\{D_a^j T_{bk}\psi\}$ and $\{D_a^j T_{bk}\tilde{\psi}\}$ will be dual frames if the following conditions hold*

$$\sum_{j\in\mathbb{Z}} \overline{\hat{\psi}(a^j\xi)}\hat{\tilde{\psi}}(a^j\xi) = b \qquad \text{a.e. } \xi \in \mathbb{R}, \tag{1}$$

$$\hat{\tilde{\psi}}(\xi)\overline{\hat{\psi}(\xi+q)} = 0 \qquad \text{a.e. } \xi \in \mathbb{R} \text{ for } 0 \neq q \in b^{-1}\mathbb{Z}. \tag{2}$$

The conditions (1) and (2) are also necessary when $a > 1$ is such that a^j is irrational for all positive integers j, see [8, p. 263]. For this reason the above conditions are often refereed to as *characterizing equations* for such irrational dilations. The result in Theorem 1 follows from the general result of characterizing equations for dual wavelet frames [8, Theorem 2].

The next result, Lemma 2, gives a sufficient condition for a wavelet system to be a Bessel sequence. Its proof can be found in [5, Theorem 11.2.3].

Lemma 2. *Let $a > 1$, $b > 0$, and $f \in L^2(\mathbb{R})$. Suppose that*

$$C_2 = \frac{1}{b} \sup_{|\xi|\in[1,a]} \sum_{j,k\in\mathbb{Z}} \left|\hat{f}(a^j\xi)\hat{f}(a^j\xi + k/b)\right| < \infty.$$

Then the affine system $\{D_a^j T_{bk} f\}$ is a Bessel sequence with bound C_2.

Theorem 1 and Lemma 2 are all we need to prove our main result, Theorem 3. The main result contains the technical condition (3) on ψ. In the example following the proof of the main result, Example 1, we explicitly construct functions satisfying this condition.

Theorem 3. *Let $n \in \mathbb{N}$, $a > 1$, and $\psi \in L^2(\mathbb{R})$. Suppose that $\hat{\psi} \in L^\infty(\mathbb{R})$ is a real-valued function with $\operatorname{supp}\hat{\psi} \subset [-a^c, -a^{c-n}] \cup [a^{c-n}, a^c]$ for some $c \in \mathbb{Z}$, and that*

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(a^j\xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{3}$$

Let $b \in \left(0, 2^{-1}a^{-c}\right]$. Then the function ψ and the function ϕ defined by

$$\phi(x) = b\psi(x) + 2b\sum_{j=1}^{n-1} a^{-j}\psi(a^{-j}x) \quad \text{for } x \in \mathbb{R}, \tag{4}$$

generate dual frames $\{D_a^j T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_a^j T_{bk}\phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof. By assumption the function $\hat{\psi}$ is compactly supported in $\mathbb{R}\setminus\{0\}$; the same holds for $\hat{\phi}$ since, by the definition in (4) and the linearity of the Fourier transform,

$$\hat{\phi}(\xi) = b\hat{\psi}(\xi) + 2b\sum_{j=1}^{n-1}\hat{\psi}(a^j\xi).$$

An application of Lemma 2 shows that the functions ψ and ϕ generate wavelet Bessel sequences.

To conclude that ψ and ϕ generate dual wavelet frames we will show that conditions (1) and (2) in Theorem 1 hold. By a^j-dilation periodicity of the sum in condition (1) it is sufficient to verify this condition on the intervals $[-a,-1]$ and $[1,a]$. On these two intervals, only finitely many terms in the sum (3) are nonzero since $\hat{\psi}$ has compact support; in particular, only the terms $j = c-n, c-n+1, \dots, c-1$ contribute which follows from the support of the dilations of $\hat{\psi}$:

$$\begin{aligned}&\operatorname{supp}\hat{\psi}(a^{c-n}\cdot) \subset [-a^n,-1]\cup[1,a^n]\,,\\ &\operatorname{supp}\hat{\psi}(a^{c-n+1}\cdot) \subset [-a^{n-1},-1/a]\cup[1/a,a^{n-1}],\end{aligned}$$

and continuing to

$$\operatorname{supp}\hat{\psi}(a^{c-1}\cdot) \subset [-a,-a^{-n+1}]\cup[a^{-n+1},a].$$

For $|\xi|\in[1,a]$, by the assumption, we have

$$\begin{aligned}1 &= \Big(\sum_{j\in\mathbb{Z}}\hat{\psi}(a^j\xi)\Big)^2 = \Big(\sum_{j=c-n}^{c-1}\hat{\psi}(a^j\xi)\Big)^2\\ &= \Big[\hat{\psi}(a^{c-n}\xi)+\hat{\psi}(a^{c-n+1}\xi)+\cdots+\hat{\psi}(a^{c-1}\xi)\Big]^2\\ &= \hat{\psi}(a^{c-n}\xi)\Big[\hat{\psi}(a^{c-n}\xi)+2\hat{\psi}(a^{c-n+1}\xi)+\cdots+2\hat{\psi}(a^{c-1}\xi)\Big]\\ &\quad+\hat{\psi}(a^{c-n+1}\xi)\Big[\hat{\psi}(a^{c-n+1}\xi)+2\hat{\psi}(a^{c-n+2}\xi)+\cdots+2\hat{\psi}(a^{c-1}\xi)\Big]\\ &\quad+\cdots+\hat{\psi}(a^{c-1}\xi)\Big[\hat{\psi}(a^{c-1}\xi)\Big]\\ &= \frac{1}{b}\sum_{j=c-n}^{c-1}\hat{\psi}(a^j\xi)\hat{\phi}(a^j\xi) = \frac{1}{b}\sum_{j\in\mathbb{Z}}\overline{\hat{\psi}(a^j\xi)}\hat{\phi}(a^j\xi),\end{aligned} \tag{5}$$

hence ψ and ϕ satisfy condition (1).

To realize that ψ and ϕ satisfy equation (2) as well, we note $\operatorname{supp}\hat{\psi}(\cdot\pm q)\subset \bar{B}(\mp q,a^c)$ and $\operatorname{supp}\hat{\phi}\subset[-a^c,-a^{c-2n+1}]\cup[a^{c-2n+1},a^c]\subset\bar{B}(0,a^c)$ where $\bar{B}(x,r)=[x-r,x+r]$ denotes the closed ball with center at x and radius r. The two functions above will have disjoint support modulo null sets whenever $|q|\geq 2a^c$. Consequently, by choosing the translation parameter $b\leq 2^{-1}a^{-c}$, the two functions in condition (2) will have disjoint support for all $q\in b^{-1}\mathbb{Z}\setminus\{0\}$ since $\min|b^{-1}\mathbb{Z}\setminus\{0\}| = 1/b\geq 2a^c$, and the condition will be trivially satisfied. □

Whenever $n = 1$ in Theorem 3 above, we have $\phi = b\psi$ by equation (4), thus ψ generates a tight frame with bound b. In this case, *i.e.*, $n = 1$, the choices of ψ are very limited since functions ψ satisfying the conditions in Theorem 3 with $n = 1$ must be of the form $\hat{\psi} = \chi_{a^c S}$, where $S = [-1, -1/a] \cup [1/a, 1]$. As a consequence, interesting constructions using Theorem 3 are restricted to $n > 1$. For $n > 1$, the dual frames generated by ψ and ϕ will be non-canonical.

The important thing to note about the definition of ϕ in (4) is that ϕ will inherit properties from ψ that are preserved by linearity and dilation, *e.g.* $\hat{\phi}$ will have compact support because $\hat{\psi}$ has this property. This holds also for properties such as smoothness, symmetry, fast decay, and vanishing moments up to some order. If ψ (or $\hat{\psi}$) can be written in terms of elementary functions, the same will hold for ϕ (or $\hat{\phi}$). These observations naturally lead to a review of the properties generally possessed by the dual generators we construct. As mentioned above, all non-trivial applications of Theorem 3 involve $n > 1$, $n \in \mathbb{N}$. We will furthermore assume that $\hat{\psi} \in L^2(\mathbb{R})$ is even, explicitly given, and, when mentioned, a C^r-function for some $r \in \mathbb{N} \cup \{0\}$. In this situation the resulting pair of dual generators has the following properties:

- Explicit and similar form: $\hat{\psi}$ and $\hat{\phi}$ are of similar form, *e.g.* piecewise polynomial of the same order (see Example 2) unlike the situation for the canonical dual (see Section 3). A similar construction procedure for tight frames gives "less" explicitly given generators (see Section 3).
- Compact support in Fourier domain of both ψ and ϕ.
- Fast decay in time domain. For $\hat{\psi} \in C_0^r(\mathbb{R})$ the generator ψ will satisfy $\lim_{|x|\to\infty} x^r \psi(x) = 0$, that is, $\psi(x) = \mathrm{o}(x^{-r})$ as $|x| \to \infty$. The dual generator ϕ has the same properties.
- High order of vanishing moments. In general for $\hat{\psi} \in C_0^r(\mathbb{R})$ the generator ψ will have vanishing moments up to order $r \in \mathbb{N} \cup \{0\}$ since
$$0 = \frac{d^m \hat{\psi}}{d\xi^m}(0) = (-2\pi i)^m \int_{\mathbb{R}} x^m \psi(x) dx \quad \text{for } m = 0, \ldots, r.$$
And again, the same holds for the dual generator ϕ.
- Symmetry: $\hat{\psi}$ and $\hat{\phi}$ are even and real functions and so are ψ and ϕ.
- Frequency overlap between scales for increased stability and non-semiorthogonality: For all $j, k \in \mathbb{Z}$ there is a $j' \neq j$ and a $k' \in \mathbb{Z}$ so that $\langle D_a^j T_{bk} \psi, D_a^{j'} T_{bk'} \psi \rangle \neq 0$. The same holds for the dual generator ϕ.
- Generalized multiresolution structure [1] (see also Section 2.3). The two generators can be associated with the same GMRA with identical core subspace, the Paley-Wiener space $\check{L}^2(K)$ with $K = \cup_{j<0}(a^j \operatorname{supp} \hat{\psi}) \subset [-a^{c-1}, a^{c-1}]$, hence both generators can be associated with the same scaling function. These types of dual wavelet frames are called sibling frames in [7]. Furthermore, the GMRA provides arbitrarily large approximation order [10].

To make Theorem 3 applicable, we need to show how to construct functions that satisfy the technical condition (3) in the theorem. It is important that this

construction is explicit because one of the key features of the theorem is that the dual generator is *explicitly* given in terms of dilations of ψ. In Example 1 we construct a dyadic partition of unity, that is, we construct a function $g \in L^2(\mathbb{R})$ satisfying

$$\sum_{j\in\mathbb{Z}} g(2^j x) = 1 \quad \text{for } a.e.\ x \in \mathbb{R}. \tag{6}$$

This corresponds to condition (3) for dyadic dilation $a = 2$; a generalization of the construction to arbitrary real dilation parameter $a > 1$ is straightforward (replace every occurrence of "2" with "a"). As we shall see a very general class of functions satisfy the condition (see also Example 3).

Example 1. For any $m \in \mathbb{Z}$, any $\delta > 0$ smaller than or equal to $2^m/3$, and a bounded function f on $[2^m - \delta, 2^m + \delta]$ satisfying $f(2^m - \delta) = 0$ and $f(2^m + \delta) = 1$, we define

$$h_1(x) = \begin{cases} f(x) & x \in \bar{B}(2^m, \delta), \\ 1 & x \in \left(2^m + \delta, 2^{m+1} - 2\delta\right), \\ 1 - f(x/2) & x \in \bar{B}(2^{m+1}, 2\delta), \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

Any such $h_1 \in L^2(\mathbb{R})$ will be continuous if f is continuous, and it will satisfy:

$$\sum_{j\in\mathbb{Z}} h_1(2^j x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

We use the same approach to construct $h_2 \in L^2(\mathbb{R})$ satisfying:

$$\sum_{j\in\mathbb{Z}} h_2(2^j x) = \begin{cases} 0 & \text{for } x \ge 0, \\ 1 & \text{for } x < 0, \end{cases}$$

and define $g = h_1 + h_2$. This gives us the dyadic partition of unity almost everywhere.

The function f above could be chosen as any polynomial satisfying $f(2^m - \delta) = 0$ and $f(2^m + \delta) = 1$; this will make g continuous. If we also let the polynomial f satisfy $f'(2^m - \delta) = f'(2^m + \delta) = 0$, then $g \in C^1(\mathbb{R})$. Continuing this way, we can make g as smooth as desired while still keeping g piecewise polynomial.

In the next example we apply the ideas from the above example to Theorem 3 and construct dual wavelet frames with dyadic dilation and translation parameter $b = 1$; actually, any $b \in (0, 1]$ can be used, but we take $b = 1$ for simplicity.

Example 2. Let f be a continuous function on the interval $[1/4, 1/2]$ satisfying $f(1/4) = 1$ and $f(1/2) = 0$. For example f can be any of the functions below:

$$f(x) = 2 - 4x, \tag{8a}$$

$$f(x) = 8(24x^2 - 8x + 1)(2x - 1)^2, \tag{8b}$$

$$f(x) = -16(320x^3 - 192x^2 + 42x - 3)(2x - 1)^3, \tag{8c}$$

$$f(x) = 32(4480x^4 - 3840x^3 + 1280x^2 - 192x + 11)(2x - 1)^4, \tag{8d}$$

$$f(x) = \tfrac{1}{2} + \tfrac{1}{2}\cos \pi(4x - 1). \tag{8e}$$

In definitions (8b) and (8e) the function f satisfy $f'(1/4) = f'(1/2) = 0$, in definition (8c) this also holds for the second derivative, and in (8d) even for the third derivative. As in Example 1 define $\psi \in L^2(\mathbb{R})$ by:

$$\hat{\psi}(\xi) = \begin{cases} 1 - f(2\,|\xi|) & \text{for } |\xi| \in [1/8, 1/4]\,, \\ f(|\xi|) & \text{for } |\xi| \in (1/4, 1/2]\,, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

This way $\hat{\psi}$ becomes a dyadic partition of unity with $\operatorname{supp}\hat{\psi} \subset [-1/2, -1/8] \cup [1/8, 1/2]$, so we can apply Theorem 3 with $c = -1$, $n = 2$, and $b = 1$. Following Theorem 3 we define the dual generator $\phi \in L^2(\mathbb{R})$ by:

$$\hat{\phi}(\xi) = \begin{cases} 2[1 - f(4\,|\xi|)] & \text{for } |\xi| \in [1/16, 1/8]\,, \\ 1 + f(2\,|\xi|) & \text{for } |\xi| \in (1/8, 1/4]\,, \\ f(|\xi|) & \text{for } |\xi| \in (1/4, 1/2]\,, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

whereby ψ and ϕ generate dual frames $\{D_2^j T_k \psi\}_{j,k\in\mathbb{Z}}$ and $\{D_2^j T_k \phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. The translation parameter in these wavelet systems is set to $b = 1$, and each wavelet frame is generated by only one function.

Suppose we let $\hat{\psi} \in L^2(\mathbb{R})$ be piecewise polynomial as defined by equations (8a) to (8d). Then $\hat{\psi} \in C^r(\mathbb{R})$ with $r = 0, 1, 2, 3$, respectively. Further, the generators ψ and ϕ will be real and even, and $\hat{\psi}$ and $\hat{\phi}$ will be piecewise polynomial and have compact support with $\operatorname{supp}\hat{\psi} \subset [-1/2, -1/8] \cup [1/8, 1/2]$ and $\operatorname{supp}\hat{\phi} \subset [-1/2, -1/16] \cup [1/16, 1/2]$. We have a greater number of vanishing moments and faster decay than indicated by the review of properties above: ψ and ϕ will have $r + 1$ vanishing moments and decay as $\mathrm{O}(x^{-r-2})$ as $|x| \to \infty$, *e.g.* using (8b) we have $\hat{\psi}, \hat{\phi} \in C^1(\mathbb{R})$, and ψ and ϕ with vanishing moments up to order 2, and $\psi(x) = \mathrm{O}(x^{-3})$ and $\phi(x) = \mathrm{O}(x^{-3})$, see Figures 1 and 2. The explicit form of ψ and hence ϕ are easily found; in general, they are finite linear combination of sine and cosine of the form $\sin(2\pi\alpha x)/(\pi x)^n$ and $\cos(2\pi\alpha x)/(\pi x)^n$ for integer $n \geq 2 + r$ and $\alpha \in \mathbb{Q}$.

We end the example with some notes on the numerical aspects and the multiresolution structure. We claim that $C_1 = 1/2$ and $C_2 = 1$ are frame bounds

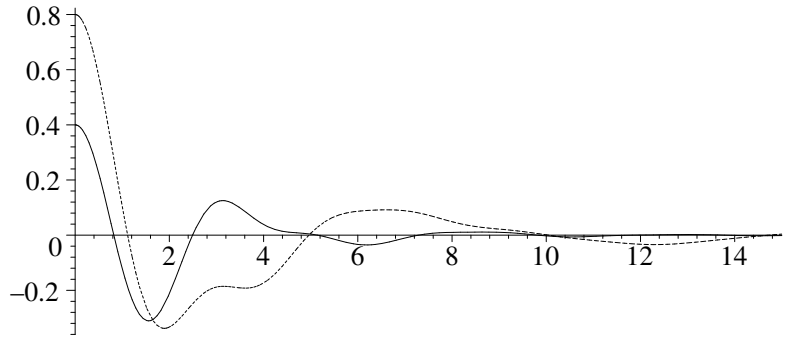

FIGURE 1. A pair of dual generators ψ (solid line) and ϕ (dashed line) in the time domain with f as in (8b).

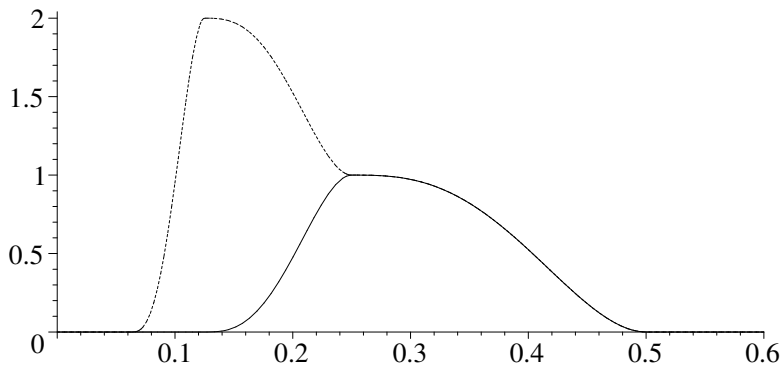

FIGURE 2. A pair of dual generators $\hat{\psi}$ (solid line) and $\hat{\phi}$ (dashed line) in the Fourier domain with f as in (8b).

for $\{D_2^j T_k \psi\}$, that $C_1 = 7/2$ and $C_2 = 5$ are frame bounds for the dual frame $\{D_2^j T_k \phi\}$, and that this holds for any f from equations (8); even more, the frame bounds hold for *any* f satisfying $0 \le f(x) \le 1$ for $x \in [1/4, 1/2]$. To prove the claim observe that

$$\sum_{k \neq 0} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}(2^j \xi) \hat{\psi}(2^j \xi + k) \right| = 0, \qquad \text{for } \xi \in \mathbb{R},$$

by the support of $\hat{\psi}$. This reduces the frame bound estimates in [5, Theorem 11.2.3] to

$$C_1 = \inf_{|\xi| \in [1/4, 1/2]} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}(2^j \xi) \right|^2, \quad C_2 = \sup_{|\xi| \in [1/4, 1/2]} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}(2^j \xi) \right|^2,$$

where C_1 and C_2 are a lower and upper frame bound of $\{D_2^j T_k \psi\}$, respectively. For $|\xi| \in [1/4, 1/2]$ we have, by the definition (9),

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = f(|\xi|)^2 + (1 - f(|\xi|))^2 = 1 - 2f(|\xi|) + 2f(|\xi|)^2,$$

and thus,

$$C_1 = \min_{x\in[\alpha,\beta]} 1 - 2x + 2x^2 = 1/2, \quad C_2 = \max_{x\in[\alpha,\beta]} 1 - 2x + 2x^2,$$

with $\alpha := \min_{1/4\le x\le 1/2} f(x)$ and $\beta := \max_{1/4\le x\le 1/2} f(x)$. Since $0 \le f(x) \le 1$ for $x \in [1/4, 1/2]$, we have $\alpha = 0$ and $\beta = 1$, hence $C_2 = 1$, and this proves the claim for $\{D_2^j T_k \psi\}$; similar calculations will show the claim for the dual frame. In particular, we see that the condition number C_2/C_1 does not depend on the smoothness of the generators, and that the condition number of the dual frame $\{D_2^j T_k \phi\}$ is smaller than the condition number of $\{D_2^j T_k \psi\}$ and the condition number of the canonical dual frame.

The core subspace of the GMRA is the Paley-Wiener space $V_0 = \check{L}^2([-1/4, 1/4])$. The function $\eta \in L^2(\mathbb{R})$ defined by $\hat{\eta} = \chi_{[-1/4,1/4]}$ is a generator for V_0, that is, $\overline{\text{span}}\{T_k\eta\}_{k\in\mathbb{Z}} = V_0$, and $\{T_k\eta\}_{k\in\mathbb{Z}}$ is a tight frame with frame bound 1 for V_0. We note that this frame contains twice as many elements as "necessary" in the sense that $\{T_{2k}\eta\}_{k\in\mathbb{Z}}$ and $\{T_{2k+1}\eta\}_{k\in\mathbb{Z}}$ are orthogonal bases for V_0. Obviously, we can take the refinable symbol $H_0 \in L^2(\mathbb{T})$ to be the 1-periodic extension of $H_0 = \chi_{[-1/8,1/8]}$ so that $\hat{\eta}(2\xi) = H_0(\xi)\hat{\eta}(\xi)$ for $\xi \in \mathbb{R}$; note that the choice of H_0 is not unique, and by letting $H_0 = \chi_{[-3/8,1/4)\cup[-1/8,1/8)\cup[1/4,3/8)}$ we obtain a quadrature mirror filter since $H_0(0) = 1$ and $|H_0(\xi)|^2 + |H_0(\xi + 1/2)|^2 = 1$. The refinable symbol H_0 is sometimes called a low pass filter or mask. As wavelet symbol (high pass filter) for the decomposition H_d and reconstruction H_r we can take $H_d = \hat{\psi}(2\cdot)$ and $H_r = \hat{\phi}(2\cdot)$ extending them to 1-periodic functions; these symbols obviously satisfy $\hat{\psi}(2\xi) = H_d(\xi)\hat{\eta}(\xi)$ and $\hat{\phi}(2\xi) = H_r(\xi)\hat{\eta}(\xi)$.

2.1. **An alternative definition of the dual generator.** The following result resembles Theorem 3, but it gives an alternative way of defining ϕ; note the change from $\psi(a^{-j}x)$ in (4) to $\psi(a^j x)$ in (11). The result follows from the symmetry of the calculations in (5).

Proposition 4. *Let $n \in \mathbb{N}$ and $a > 1$. Suppose $\psi \in L^2(\mathbb{R})$ is as in Theorem 3. Let $b \in \big(0, a^{-c}(1 + a^{n-1})^{-1}\big]$. Then the function ψ and the function ϕ defined by*

$$\phi(x) = b\psi(x) + 2b\sum_{j=1}^{n-1} a^j \psi(a^j x) \quad \text{for } x \in \mathbb{R} \tag{11}$$

generate dual frames $\{D_a^j T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_a^j T_{bk}\phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof. The functions $\hat{\psi}$ and $\hat{\phi}$ satisfy condition (1). This follows from calculations similar to those in (5): We start by factoring out $\hat{\psi}(a^{c-1}\xi)$ instead of $\hat{\psi}(a^{c-n}\xi)$, then $\psi(a^{c-2}\xi)$ and continue in a similar way. To see that condition (2) is satisfied,

we note that $\operatorname{supp}\hat{\phi} \subset [-a^{c+n-1}, -a^{c-n}] \cup [a^{c-n}, a^{c+n-1}]$ since $\operatorname{supp}\hat{\phi}(a^{-n+1}\cdot) \subset [-a^{c+n-1}, -a^{c-1}] \cup [a^{c-1}, a^{c+n-1}]$. The two functions in (2) will have disjoint support modulo null sets whenever $|q| \geq a^c + a^{c+n-1} = a^c(1 + a^{n-1})$. □

The choice of the translation parameter b is more restrictive in Proposition 4 than in Theorem 3 since the support of $\hat{\phi}$ defined by (11) is larger than when defined by (4). Note that $b \in (0, a^{-c}(1 + a^{n-1})^{-1}]$ can be replaced by the simpler, but more restrictive, $b \in (0, a^{-c-n}]$ in the case $a \geq 2$.

2.2. **Expanding the range of the translation parameter.** The construction of dual wavelet frames from Theorem 3 often imposes the translation parameter b to be small, *e.g.* $b < 1$. Hence, it would be interesting to know in which cases we can take $b = 1$. For the sake of simplicity let $a = 2$ for a moment, and assume that ψ satisfies the assumptions of Theorem 3. Obviously, we can take $b = 1$ if the support of $\hat{\psi}$ is contained in $[-1/2, 1/2]$, that is, if $c \leq -1$; this is exactly what we used in Example 2. If $c \geq 0$, we need, in order to achieve $b = 1$, to apply Theorem 3 to $\hat{\psi}(2^{c+1}\cdot)$ in place of $\hat{\psi}$. This dilated version of ψ will still be a dyadic partition of unity and $\operatorname{supp}\hat{\psi}(2^{c+1}\cdot) \subset [-1/2, 1/2]$. Moreover, we have the following result.

Corollary 5. *Let $n \in \mathbb{N}$ and $a > 1$. Suppose $\psi \in L^2(\mathbb{R})$ is as in Theorem 3. Let $b \in (0, 2^{-1}a^{-c}]$. Then the function $\tilde{\psi} := D_b\psi$ and the function $\tilde{\phi} := D_b\phi$, where ϕ is defined as in* (4)*, generate dual frames $\{D_a^j T_k \tilde{\psi}\}_{j,k\in\mathbb{Z}}$ and $\{D_a^j T_k \tilde{\phi}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.*

Proof. The result basically follows from an application of the identity

$$D_b T_{bk} = T_k D_b, \tag{12}$$

and the fact that dilation preserves the frame property and the duality of (wavelet) frames since it is a unitary operator on $L^2(\mathbb{R})$. By assumption $\{D_a^j T_{bk}\psi\}$ and $\{D_a^j T_{bk}\phi\}$ are dual frames for $b \in (0, 2^{-1}a^{-c}]$. The identity (12) yields,

$$D_b D_a^j T_{bk}\psi = D_a^j T_k(D_b\psi),$$

hence $\{D_a^j T_k \tilde{\psi}\}$ is a frame as a unitary image of a wavelet frame where $\tilde{\psi} = D_b\psi$. The same conclusion holds for $\{D_a^j T_k \tilde{\phi}\}$. For all $f \in L^2(\mathbb{R})$, we have

$$f = D_b(D_b^* f) = \sum_{j,k\in\mathbb{Z}} \left\langle f, D_b D_a^j T_{bk}\phi \right\rangle D_b D_a^j T_{bk}\psi = \sum_{j,k\in\mathbb{Z}} \left\langle f, D_a^j T_k \tilde{\phi} \right\rangle D_a^j T_k \tilde{\psi},$$

and conclude that duality is preserved. □

Another approach (for obtaining $b = 1$) makes use of multigenerated wavelet systems. In the following result the constructed dual wavelet frames are generated by m functions again sharing the properties of the starting point function ψ; in particular, if ψ has vanishing moments up to some order, then so will every function in the generator sets Ψ and Φ.

Corollary 6. *Let $n \in \mathbb{N}$ and $a > 1$. Suppose $\psi \in L^2(\mathbb{R})$ is as in Theorem 3. Let $m \in \mathbb{N}$ and $b \in \left(0, 2^{-1}a^{-c}m\right]$. Then the functions $\Psi = \{\psi, T_{b/m}\psi, \dots, T_{(m-1)b/m}\psi\}$ and the functions $\Phi = \{\phi, T_{b/m}\phi, \dots, T_{(m-1)b/m}\phi\}$, where ϕ is defined as in (4), generate dual frames $\{D_a^j T_{bk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$ and $\{D_a^j T_{bk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ for $L^2(\mathbb{R})$.*

Proof. Let $m \in \mathbb{N}$. For b so that $0 < b/m \le 2^{-1}a^{-c}$, the functions ψ and ϕ, where ϕ is defined as in (4), generate dual frames $\{D_a^j T_{bk/m}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_a^j T_{bk/m}\phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. Note that $(m^{-1}\mathbb{Z}) / \mathbb{Z} = \{0, 1, \dots, m-1\}$, and define:

$$\Psi = \left\{\psi, T_{b/m}\psi, T_{2b/m}\psi, \dots, T_{(m-1)b/m}\psi\right\}.$$

It follows immediately that $\{D_a^j T_{b/mk}\psi\}_{j,k\in\mathbb{Z}} = \{D_a^j T_{bk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$. Similarly, we have for ϕ that $\{D_a^j T_{b/mk}\phi\}_{j,k\in\mathbb{Z}} = \{D_a^j T_{bk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$, where

$$\Phi := \left\{\phi, T_{b/m}\phi, T_{2b/m}\phi, \dots, T_{(m-1)b/m}\phi\right\}.$$

We conclude $\{D_a^j T_{bk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$ and $\{D_a^j T_{bk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ are dual frames for $L^2(\mathbb{R})$ for $b/m \le 2^{-1}a^{-c}$, that is, for $b \le 2^{-1}a^{-c}m$. □

It follows from the corollary that, in the dyadic case, we can always obtain $b = 1$ by using 2^{c+1} generators.

2.3. **On the generalized multiresolution structure.** We end this section with a closer study of the GMRA structure of ψ and ϕ. To this end, let $\psi \in L^2(\mathbb{R})$ satisfy the assumptions in Theorem 3. We consider the subspaces $W_j^b(\psi) := \overline{\operatorname{span}}\{D_a^j T_{bk}\psi : k \in \mathbb{Z}\}$. Let $\tilde{\psi} = D_b\psi$ be the generator of frame $\{D_a^j T_k\tilde{\psi}\}$, see Corollary 5. From the identity $T_{bk} = D_b^{-1}T_k D_b$ we have $W_0^b(\psi) = D_b^{-1}W_0^1(\tilde{\psi})$ where $W_j^1(\tilde{\psi}) = \overline{\operatorname{span}}\left\{D_a^j T_k\tilde{\psi} : k \in \mathbb{Z}\right\}$. By [10, Theorem 2.14],

$$W_0^1(\tilde{\psi}) = \left\{f \in L^2(\mathbb{R}) : \hat{f} = m\hat{\tilde{\psi}} \text{ for some measurable, 1-periodic } m\right\}$$

and further, using $\operatorname{supp}\hat{\tilde{\psi}} \subset [-1/2, 1/2]$,

$$W_0^1(\tilde{\psi}) = \left\{f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subset \operatorname{supp}\hat{\tilde{\psi}}\right\} = \check{L}^2(\operatorname{supp}\hat{\tilde{\psi}}),$$

hence $W_0^b(\psi) = \check{L}^2(\operatorname{supp}\hat{\psi})$ by the above, and by dilation, $W_j^b(\psi) = \check{L}^2(a^j \operatorname{supp}\hat{\psi})$. We conclude that the space of negative dilates, also called the core subspace, associated with ψ is given by

$$V_0(\psi) = \overline{\operatorname{span}}\Big(\bigcup_{j<0} W_j^b(\psi)\Big) = \check{L}^2(K), \qquad K = \bigcup_{j<0}(a^j \operatorname{supp}\hat{\psi}) \subset \left[-a^{c-1}, a^{c-1}\right],$$

which is a subspace invariant under all translations. It is straightforward to see $V_0(\psi) = V_0(\phi)$; we will denote this space by V_0. A function $\eta \in L^2(\mathbb{R})$ is said to generate V_0 if $\overline{\operatorname{span}}\{T_{bk}\eta\}_{k\in\mathbb{Z}} = V_0$, and we have that η generates V_0 if, and only if, $\operatorname{supp}\hat{\eta} = K$ (see [10]). If we further require $\{T_{bk}\eta\}_{k\in\mathbb{Z}}$ to be a frame for

V_0, then $\hat{\eta}$ cannot be continuous hence η will be poorly localized in time. This drawback follows from a result in [2]; indeed, the sum $\sum_{k\in\mathbb{Z}} |\hat{\eta}((\xi+k)/b)|^2$ reduces to $|\hat{\eta}(\xi/b)|^2$ for $\xi \in [-1/2, 1/2]$ since $b \le 2^{-1}a^{-c}$ implies $ba^{c-1} \le 1/(2a) \le 1/2-\varepsilon$ for some $\varepsilon > 0$ hence $\operatorname{supp}\hat{\eta}(\cdot/b) = bK \subset [-ba^{c-1}, ba^{c-1}] \subset [-1/2+\varepsilon, 1/2-\varepsilon]$. Now, the conclusion follows from [2, Theorem 3.4]. We note that the constructed wavelet frame will not necessarily be a frame for a fixed dilation level subspace $W_j(\psi)$ of $L^2(\mathbb{R})$. This situation is similar to that of the unitary and oblique extension principles, but in contrast to frame multiresolution analysis.

3. Dual frames versus tight frames

In Theorem 3 we explicitly *construct* the dual frame. One might ask why we do not use the canonical dual frame, or why we do not use the characterizing equations for tight frames to formulate a similar construction procedure of tight frames. In the following we will show that these approaches have some disadvantages compared to Theorem 3.

For a wavelet frame $\{D_a^j T_{bk}\psi\}_{j,k\in\mathbb{Z}}$, the canonical dual frame is given by

$$\left\{S^{-1}D_a^j T_{bk}\psi : j,k \in \mathbb{Z}\right\} = \left\{D_a^j S^{-1} T_{bk}\psi : j,k \in \mathbb{Z}\right\},$$

where S is frame operator of $\{D_a^j T_{bk}\psi\}_{j,k\in\mathbb{Z}}$. In general the canonical dual need *not* have the structure of a wavelet system, and this is one reason to avoid working with canonical dual frames. However, as we show below, the canonical dual of all wavelet frames considered in this paper will be of wavelet structure, hence the canonical dual could be used in the synthesis process in the frame wavelet transform. The problem with this approach is that it is difficult to control which properties the canonical dual frame inherits from the frame since the application of the inverse frame operator can destroy desirable properties. We give an example of this issue in the following.

Let $\psi \in L^2(\mathbb{R})$ be as in the assumptions of Theorem 3. Then $\hat{\psi}(\xi)\overline{\hat{\psi}(\xi+b^{-1}k)} = 0$ for $k \in \mathbb{Z}\setminus\{0\}$, and consequently, by [11, Proposition 7.1.19] in the dyadic case and a simple generalization of parts of the proof of the proposition in the general case, the associated frame operator is the Fourier multiplier given by

$$\widehat{Sf}(\xi) = \left(\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(a^j\xi)\right|^2\right)\hat{f}(\xi) \quad \text{for } a.e.\ \xi \in \mathbb{R}, \tag{13}$$

for all $f \in L^2(\mathbb{R})$ with $C_1 \le \sum_{j\in\mathbb{Z}} |\hat{\psi}(a^j\xi)|^2 \le C_2$ and C_1, C_2 as frame bounds for $\{D_a^j T_{bk}\psi\}$. Since S is a Fourier multiplier, it commutes with all translations, that is, $ST_r = T_r S$ for all $r \in \mathbb{R}$, and the same holds for the inverse frame operator, hence the canonical dual takes the form

$$\left\{D_a^j T_{bk}(S^{-1}\psi) : j,k \in \mathbb{Z}\right\},$$

which is a wavelet frame generated by $S^{-1}\psi$. Moreover, the canonical dual generator is given by

$$\widehat{S^{-1}\psi}(\xi) = \frac{\hat{\psi}(\xi)}{\sum_{j\in\mathbb{Z}}|\hat{\psi}(a^j\xi)|^2} \quad \text{for } a.e.\ \xi \in \mathbb{R}, \tag{14}$$

Since $\operatorname{supp}\hat{\psi} \subset [-a^c, -a^{c-n}] \cup [a^{c-n}, a^c]$ for some $c \in \mathbb{Z}$ and $n \in N$, we conclude, by equation (14), $\operatorname{supp}\widehat{S^{-1}\psi} = \operatorname{supp}\hat{\psi}$ and

$$\widehat{S^{-1}\psi}(\xi) = \frac{\hat{\psi}(\xi)}{\sum_{|j|<n}(\hat{\psi}(a^j\xi))^2} \quad \text{for } a.e.\ \xi \in \mathbb{R}. \tag{15}$$

This implies, among other things, that $\hat{\psi}$ and $\widehat{S^{-1}\psi}$ will have the same regularity. But it also implies that choosing $\hat{\psi}$ to be piecewise linear will not make the canonical dual generator $S^{-1}\psi$ piecewise linear (in the Fourier domain, that is) owing to the denominator in (15). This is unlike the situation in Example 2 where a piecewise polynomial $\hat{\psi}$ by Theorem 3 gave a dual generator $\hat{\phi}$ piecewise polynomial of the same order, *e.g.* a piecewise linear $\hat{\psi}$ gave a piecewise linear $\hat{\phi}$. In general the denominator in (15) makes the expression for the canonical dual generator "less" explicit. The price we pay for using the non-canonical dual is a slightly larger support (in the Fourier domain) of the dual generator.

Since the construction of wavelet frames by Theorem 3 is based on characterizing equations for dual wavelet frames, it would be natural to look for a similar way of constructing tight frames from their characterizing equations. In a naive approach to such a construction one would need to choose $\psi \in L^2(\mathbb{R})$ so that $\hat{\psi}$ is real and the family $\xi \mapsto (\hat{\psi}(a^j\xi))^2$, $j \in \mathbb{Z}$, form a partition of unity and to choose a sufficiently small translation parameter (so that all terms in the series in the so-called "t_q-equations" become zero owing to disjoint support). Following the ideas from Example 1 we take $\psi \in L^2(\mathbb{R})$ as (extending $\hat{\psi}$ to an even function):

$$\hat{\psi}(\xi) = \begin{cases} f(\xi) & \xi \in \bar{B}(a^m, \delta), \\ 1 & \xi \in (a^m + \delta, a^{m+1} - a\delta), \\ \sqrt{1 - (f(\xi/a))^2} & \xi \in \bar{B}(a^{m+1}, a\delta), \\ 0 & \xi \in [0, \infty) \setminus [a^m - \delta, a^{m+1} + a\delta]. \end{cases}$$

for any $m \in \mathbb{Z}$, any $\delta > 0$ smaller than or equal to $a^m/3$, and a bounded function f on $[a^m - \delta, a^m + \delta]$ satisfying $f(a^m - \delta) = 0$, $f(a^m + \delta) = 1$, and $|f| \leq 1$. The important thing to note with this approach is that $\hat{\psi}$ does not inherit properties from f in opposition to the situation in Example 1, *e.g.* taking f to be linear does not make $\hat{\psi}$ piecewise linear because of the square root in the expression above; moreover, it is well known that the property of being a smooth (non-negative) function need not be preserved when taking square roots.

4. Another application of Theorem 3

In Examples 1 and 2 we constructed dual wavelet frames in a rather explicit way. The following construction is less explicit. In the first part of the example below we construct a C^∞ function on $\mathbb{R}$ with compact support satisfying the technical condition (6), and in the second part we apply Theorem 3 to the constructed function.

Example 3 (Part I)**.** Let $f \in C^\infty(\mathbb{R})$ be defined as

$$f(x) = \begin{cases} \mathrm{e}^{-1/x} & x > 0, \\ 0 & x \le 0, \end{cases}$$

and choose positive constants $R > r > 0$ so that

$$\exists \delta > 0 : \bigcup_{j\in\mathbb{Z}} 2^j [r+\delta, R-\delta] = [0,\infty)\,, \tag{16}$$

holds, *e.g.* take $r = 1/8$ and $R = 1/2$. We define $f_1(x) = f(x-r)f(R-x)$ for $x \in \mathbb{R}$, hence $\operatorname{supp} f_1 \subset [r,R]$ and $f_1 \in C_0^\infty(\mathbb{R})$, and we introduce a symmetric version of f_1, denoted f_2, in order to get a dyadic partition of unity of the negative as well as the positive real line.

$$f_2(x) = \begin{cases} f_1(x) & \text{for } x > 0, \\ f_1(-x) & \text{for } x \le 0. \end{cases} \tag{17}$$

The function w will be used to normalize f_2:

$$w(x) = \sum_{j\in\mathbb{Z}} f_2(2^j x).$$

For a fixed $x \in \mathbb{R}$ this sum only has finitely many nonzero terms. Obviously, w is a 2^j-dyadic periodic function and, by (16) and the definition of f_1, it is also bounded away from 0 and ∞:

$$\exists c, C > 0 : c < w(x) < C \quad \text{for all } x \in \mathbb{R}\setminus\{0\},$$

hence we can define a function $g \in C_0^\infty(\mathbb{R})$ by

$$g(x) = \frac{f_2(x)}{w(x)} \quad \text{for } x \in \mathbb{R}\setminus\{0\}, \quad \text{and,} \quad g(0) = 0. \tag{18}$$

This g will be a dyadic partition of unity; the calculations are straightforward:

$$\sum_{j\in\mathbb{Z}} g(2^j x) = \sum_{j\in\mathbb{Z}} \frac{f_2(2^j x)}{w(2^j x)} = \sum_{j\in\mathbb{Z}} \frac{f_2(2^j x)}{w(x)} = \frac{\sum_{j\in\mathbb{Z}} f_2(2^j x)}{\sum_{k\in\mathbb{Z}} f_2(2^k x)} = 1.$$

The construction of g looks indeed less explicit than the piecewise polynomial partition of unity in Example 1 primarily because g is normalized by an infinite series w. This situation improves by noticing that, in practice, the series w reduce to a finite sum since $\operatorname{supp} g = \operatorname{supp} f_2 \subset [-R,-r] \cup [r,R]$. For example, if we let $r = 1/8$ and $R = 1/2$, we can do with three terms $g(x) = f_2(x)/\sum_{j=-1}^{1} f_2(2^j x)$ for all $x \in \mathbb{R}\setminus\{0\}$.

Remark 1. 1. Note that the mirroring step (17) introducing f_2 also makes g symmetric. But it is obvious from the example that we can carry out the construction for the positive part of the real line only to get a dyadic partition of the unity on the positive real line, and, then, by the same approach (but with different choices of r and R), for the negative real line. This way g will not be symmetric.

2. In place of f one could choose any function in $C_0^\infty(\mathbb{R})$ having the same support as f. In place of f_1 one could take any characteristic function $f_1 = \chi_{[2^n,2^{n+1}]}$ for some $n \in \mathbb{N}$ convolved with a smooth $h_\delta \in C_0^\infty(\mathbb{R})$ for a sufficiently small $\delta > 0$, where $h_\delta(x) = \delta^{-1}h(\delta^{-1}x)$, and $\operatorname{supp} h \subset [-1,1]$, $h \geq 0$, $\int h \,\mathrm{d}\mu = 1$, and $h \in C_0^\infty(\mathbb{R})$. Then $\operatorname{supp} h_\delta \subset [-\delta,\delta]$ and $\operatorname{supp} h_\delta * f_1 \subset \big[2^n - \delta, 2^{n+1} + \delta\big]$.

Example 3 (Part II)**.** We take $r = 1/8$ and $R = 1/2$ in Example 3 and set $\hat{\psi} = f_2/\sum_{j=-1}^{1} f_2(2^j\cdot)$ where f_2 is given by (17), hence

$$\hat{\psi}(\xi) = \begin{cases} \frac{\mathrm{e}^{(1/8-\xi)^{-1}}\mathrm{e}^{(\xi-1/2)^{-1}}}{\mathrm{e}^{(1/8-\xi)^{-1}}\mathrm{e}^{(\xi-1/2)^{-1}}+\mathrm{e}^{(1/8-2\xi)^{-1}}\mathrm{e}^{(2\xi-1/2)^{-1}}} & \xi \in (1/8,1/4)\,, \\ 1 & \xi = 1/4, \\ \frac{\mathrm{e}^{(1/8-\xi)^{-1}}\mathrm{e}^{(\xi-1/2)^{-1}}}{\mathrm{e}^{(1/8-\xi/2)^{-1}}\mathrm{e}^{(\xi/2-1/2)^{-1}}+\mathrm{e}^{(1/8-\xi)^{-1}}\mathrm{e}^{(\xi-1/2)^{-1}}} & \xi \in (1/4,1/2)\,, \\ 0 & \xi \in \mathbb{R}_+ \setminus (1/8,1/2)\,, \end{cases}$$

and symmetrically for the negative real line. Applying this to Theorem 3 with $n = 2$, $c = -1$, and $b = 1$ yields a pair of dual wavelet generators with $\hat{\psi}, \hat{\phi} \in C^\infty(\mathbb{R})$, where $\hat{\phi}$ is defined as in (4), and $\operatorname{supp}\hat{\psi} \subset [-1/2,-1/8] \cup [1/8,1/2]$ and $\operatorname{supp}\hat{\phi} \subset [-1/2,-1/16] \cup [1/16,1/2]$. The generators are smooth, rapidly decaying, symmetric dual framelets with vanishing moments of infinite order. It is clear that both belong to the Schwartz space, but it is also clear, from the equation above, that ψ and ϕ are not explicitly given in the time domain.

5. The Hardy space

A similar construction procedure for dual wavelet frames holds for the Hardy space $H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subset [0,\infty)\}$. The result in Corollary 1 can easily be transformed from $L^2(\mathbb{R})$ settings to the Hardy space $H^2(\mathbb{R})$. Indeed, we only need to replace the right hand side b in equation (1) by $b\chi_{[0,\infty)}(\xi)$. In [4, Theorem 1.3] such a transformation is carried out for a similar result on tight wavelet frames [8, Theorem 1]. The analogue version of Theorem 3 for the Hardy space is as follows. Let $n \in \mathbb{N}$ and $a > 1$. Suppose for $\psi \in H^2(\mathbb{R})$ that $\hat{\psi}$ is a real-valued function with $\operatorname{supp}\hat{\psi} \subset [a^{c-n}, a^c]$ for some $c \in \mathbb{Z}$ and that

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(a^j\xi) = \chi_{[0,\infty)}(\xi) \quad \text{for } a.e.\ \xi \in \mathbb{R}.$$

Let $b \in (0, a^{-c}]$; actually, we could even let $b \in \big(0, a^{-c}(1-a^{-2n+1})^{-1}\big]$. Then ψ and ϕ defined by (4) generate dual frames for $H^2(\mathbb{R})$. We note that, in the Hardy

space, the choice of translation parameter becomes less restrictive than for $L^2(\mathbb{R})$. This owes to the fact that $\hat{\psi}$ and $\hat{\phi}$ have smaller support since they are zero on the negative real line.

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