

Density and duality theorems for regular Gabor frames

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Abstract: We investigate Gabor frames on locally compact abelian groups with time-frequency shifts along non-separable, closed subgroups of the phase space. Density theorems in Gabor analysis state necessary conditions for a Gabor system to be a frame or a Riesz basis, formulated only in terms of the index subgroup. In the classical results the subgroup is assumed to be discrete. We prove density theorems for general closed subgroups of the phase space, where the necessary conditions are given in terms of the “size” of the subgroup. From these density results we are able to extend the classical Wexler-Raz biorthogonal relations and the duality principle in Gabor analysis to Gabor systems with time-frequency shifts along non-separable, closed subgroups of the phase space. Even in the euclidean setting, our results are new.

1 Introduction

Classical harmonic analysis on locally compact abelian (LCA) groups provides a natural framework for many of the topics considered in modern time-frequency analysis. The setup is as follows. Let (G, \cdot) denote a second countable LCA group, and let (\widehat{G}, \cdot) denote its dual group, consisting of all characters. One then defines the translation operator T_λ , $\lambda \in G$, as

$$T_\lambda : L^2(G) \rightarrow L^2(G), (T_\lambda f)(x) = f(x\lambda^{-1}), \quad x \in G,$$

and the modulation operator E_γ , $\gamma \in \widehat{G}$, as

$$E_\gamma : L^2(G) \rightarrow L^2(G), (E_\gamma f)(x) = \gamma(x)f(x), \quad x \in G.$$

The central objects of this work are so-called regular Gabor systems in $L^2(G)$ with modulation and translation along a closed subgroup Δ of $G \times \widehat{G}$ generated by a window function $g \in L^2(G)$; this is a collection of functions of the following form:

$$\mathcal{G}(g, \Delta) := \{\pi(\nu)g\}_{\nu \in \Delta}, \quad \text{where } \pi(\nu) := E_\gamma T_\lambda \text{ for } \nu = (\lambda, \gamma) \in G \times \widehat{G}.$$

The tensor product $G \times \widehat{G}$ is called the phase-space or the time-frequency plane, and $\pi(\nu)g$ is a time-frequency shift of g .

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We are interested in linear operators of the form

$$C_{g,\Delta} : L^2(G) \rightarrow L^2(\Delta), \quad C_{g,\Delta}f = \nu \mapsto \langle f, \pi(\nu)g \rangle$$

as well as their left-inverses (if they exist) and their adjoint operators. The $C_{g,\Delta}$ transform is called an *analysis* operator, while its adjoint is called *synthesis*. In the analysis process $C_{g,\Delta}f$ of a function $f \in L^2(G)$, we obtain information of the time-frequency content in the function f .

If the operator $C_{g,\Delta}$ is bounded below and above, we say that $\mathcal{G}(g, \Delta)$ is a Gabor frame for $L^2(G)$. In case the two constants from these bounds can be taken to be equal, we say that $\mathcal{G}(g, \Delta)$ is a tight frame; if they can be taken to be equal to one, $\mathcal{G}(g, \Delta)$ is said to be a Parseval frame. One can show that the property of being a frame allows for stable reconstruction of any $f \in L^2(G)$ from its time-frequency information given by $C_{g,\Delta}f$. In particular, if $C_{g,\Delta}$ is bounded from below and above, then there exists another function $h \in L^2(G)$ such that $C_{h,\Delta}$ is a bounded operator and such that

$$\langle f_1, f_2 \rangle = \int_{\Delta} C_{g,\Delta}f_1(\nu) \overline{C_{h,\Delta}f_2(\nu)} d\nu$$

for all $f_1, f_2 \in L^2(G)$, where $d\nu$ denotes the Haar measure on Δ . Two such Gabor systems $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(h, \Delta)$ are said to be *dual* Gabor frames. If $\mathcal{G}(g, \Delta)$ is a frame with $C_{g,\Delta}$ being surjective, we say that $\mathcal{G}(g, \Delta)$ is a *Riesz family*.

In case $\Delta = G \times \widehat{G}$ the analysis operator $C_{g,\Delta}$ is the well-known *short-time Fourier transform*, usually written \mathcal{V}_g , which is an isometry for any window function $g \in L^2(G)$ satisfying $\|g\| = 1$. In the language of frame theory, $\mathcal{G}(g, G \times \widehat{G})$ is said to be a Parseval frame. However, for proper subgroups $\Delta \subsetneq G \times \widehat{G}$ window functions $g \in L^2(G)$ leading to isometric transforms $C_{g,\Delta}$, or more generally to Gabor frames $\mathcal{G}(g, \Delta)$ for $L^2(G)$, might not exist.

The density theorems in Gabor analysis are such non-existence results formulated only as *necessary conditions on the subgroup* Δ for a Gabor system to be a frame or a Riesz basis. In particular, the subgroup Δ needs to possess a certain amount of density. The classical density results are stated for *uniform lattices* Δ , i.e., discrete and co-compact subgroups of $G \times \widehat{G}$, where the density is measured by the volume of a fundamental domain of Δ . For $\Delta = P\mathbb{Z}^{2d}$, $P \in \text{GL}_{2d}(\mathbb{R})$, in $G \times \widehat{G} = \mathbb{R}^{2d}$, this volume is exactly $|\det P|$. In this work we introduce a generalization of this density measure for non-lattices so, for closed subgroups Δ of $G \times \widehat{G}$, we set

$$d(\Delta) := \mu_{(G \times \widehat{G})/\Delta}((G \times \widehat{G})/\Delta).$$

If Δ is a uniform lattice equipped with the counting measure, then $d(\Delta)$ is exactly the measure of the fundamental domain. Note that $d(\Delta) < \infty$ precisely when Δ is co-compact, i.e., $(G \times \widehat{G})/\Delta$ is compact. A typical density result says that if Δ is a uniform lattice and $\mathcal{G}(g, \Delta)$ is a frame for $L^2(G)$, then $d(\Delta) \leq 1$. For separable uniform lattices $\Delta = \Lambda \times \Gamma \subset G \times \widehat{G}$ this result was proved by Gröchenig in [20], and for non-separable uniform lattices (in *elementary* LCA groups) it is a consequence of results by Feichtinger and Kozek [14]. Gröchenig's proof is elementary using the Poisson summation formula, while the argument for general lattices relies, as is often the case for results on non-separable lattices, on the theory of pseudo differential operators. We will give alternative proofs using only time-frequency analysis techniques. More importantly, we will generalize density results to arbitrary closed subgroups $\Delta \subset G \times \widehat{G}$. We will show that:

- (a) If $\mathcal{G}(g, \Delta)$ is a frame for $L^2(G)$, then $d(\Delta) < \infty$.
- (b) If Δ is a discrete subgroup and $\mathcal{G}(g, \Delta)$ is a frame, then $d(\Delta) \leq 1$.
- (c) $\{\pi(\nu)g\}_{\nu \in \Delta}$ is a Riesz family for $L^2(G)$ if, and only if, Δ is a uniform lattice with $d(\Delta) = 1$ and $\{\pi(\nu)g\}_{\nu \in \Delta}$ is a frame for $L^2(G)$.

While (b) might be expected, it is rather surprising that density results can be formulated for non-discrete Gabor systems. This extension relies crucially on the fact that the new measure $d(\Delta)$ contains information on both the subgroup Δ and its Haar measure. The forward direction in (c) is also somewhat unexpected. The seemingly weak assumption that $\mathcal{G}(g, \Delta)$ is a Riesz family for some closed subgroup Δ of the phase space has the strong conclusion that Δ is a uniform lattice and that $d(\Delta) = 1$. Moreover, we will see that in statement (a) it is, in general, not possible to be quantitative, that is, if Δ is non-discrete and co-compact, it will be possible to construct a frame $\mathcal{G}(g, \Delta)$ regardless of the value of $d(\Delta) < \infty$. This illustrates that the non-discrete case is rather different from the usual Gabor theory for lattices. We will exhibit several of these differences in Section 5 and 6.

From our generalized density theorems, we are then able to extend the duality theory in Gabor analysis to Gabor systems $\mathcal{G}(g, \Delta)$ with time-frequency shifts along arbitrary closed subgroups $\Delta \subset G \times \widehat{G}$. The most fundamental duality principle says that the Gabor system $\mathcal{G}(g, \Delta)$ is a Parseval frame, i.e., the system is associated with an isometric transform $C_{g, \Delta}$, if and only if $\mathcal{G}(d(\Delta)^{-1/2}g, \Delta^\circ)$ is an orthonormal set, where Δ° denotes the adjoint of Δ .

We will prove two results that can be seen as an extension of this result. Firstly, to dual frames, where one allows for two different window functions $g, h \in L^2(G)$ in the analysis and synthesis transforms; this extension is known as the Wexler-Raz biorthogonality relations. Secondly to non-tight frames; this result is simply known as the duality principle. The Wexler-Raz biorthogonality relations were previously available for non-separable, uniform lattices $\Delta \subset G \times \widehat{G}$ on elementary LCA groups $G = \mathbb{R}^n \times \mathbb{T}^\ell \times \mathbb{Z}^k \times F_m$ by the work of Feichtinger and Kozek [14], while the duality principle was formulated (without bounds) by Feichtinger and Zimmermann [16] for Gabor systems $\mathcal{G}(g, \Delta)$ in $L^2(\mathbb{R}^n)$ with Δ being a non-separable, full-rank lattice in \mathbb{R}^{2n} . The authors proved in [28] both the Wexler-Raz biorthogonality relations and the duality principle on LCA groups for *separable*, co-compact subgroups $\Delta = \Lambda \times \Gamma \subset G \times \widehat{G}$ using the theory of translation invariant systems; an approach that does not generalize to the non-separable case.

Usually, the density/duality theory for *non-separable* lattice Gabor systems relies on the theory of pseudo-differential operators and von Neumann algebra techniques. In particular, the results of Feichtinger and Kozek [14] use concepts of function space Gelfand triples and generalized Kohn-Nirenberg symbols. To cite from Gröchenig's book [21]:

These generalizations [*density and duality results for non-separable time-frequency lattices in the euclidean space*], however, require a completely different approach that involves the analysis of pseudo-differential operators with periodic symbols.

The present paper provides density and duality theorems for Gabor systems $\mathcal{G}(g, \Delta)$ with time-frequency shifts along (possibly non-separable) closed subgroups $\Delta \subset G \times \widehat{G}$ for general second countable LCA groups G . In spite of the above comments, we are able to develop the theory solely within the setting of time-frequency analysis. Indeed, our proofs are based on Weil's formula, the Fourier transform, the short-time Fourier transform, and frame theory.

We mention that duality results in the discrete case have been generalized in other directions; we refer the reader to [4, 10, 11, 22, 35] and the references therein. Generalizations of the density theorem also exist; in particular, Ramanathan and Steger [32] obtained density results for $\mathcal{G}(g, \Delta)$ in $L^2(\mathbb{R}^n)$, where Δ is a discrete set, but not necessarily a subgroup. We refer the reader to the survey paper by Heil [24] for a detailed account of the history and evolution of density results in Gabor analysis. For an introduction to Gabor analysis and frame theory, we refer to [6, 21].

The paper is organized as follows: Section 2 and 3 contain preliminary facts and results on Fourier analysis on LCA groups and frame theory, respectively. Some new results on the non-existence of continuous Riesz families are included in Section 3; these results are essential for our development in the later sections, however, they are also of independent interest. In Section 4 we introduce Gabor systems and show three key lemmas that will be important in the proofs of the main results in Sections 5 and 6. In these sections we show density and duality results for Gabor frames $\mathcal{G}(g, \Delta)$, where Δ is a closed subgroup of the time-frequency domain $G \times \widehat{G}$. Appendix A contains results on the Feichtinger algebra S_0 that are needed for the proofs in Section 6.

2 Harmonic analysis on LCA groups

We let G denote a second countable locally compact abelian group. To G we associate its dual group \widehat{G} which consists of all characters, i.e., all continuous homomorphisms from G into the torus $\mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\}$. Under pointwise multiplication \widehat{G} is also a locally compact abelian group. Throughout the paper we use multiplication as group operation in G , \widehat{G} , and $G \times \widehat{G}$, and we denote the identity element by e . By the Pontryagin duality theorem, the dual group of \widehat{G} is isomorphic to G as a topological group, i.e., $\widehat{\widehat{G}} \cong G$.

We denote the Haar measure on G by μ_G . The (left) Haar measure on any locally compact group is unique up to a positive constant. From μ_G we define $L^1(G)$ and the Hilbert space $L^2(G)$ over the complex field in the usual way. Since G is assumed to be second countable, these function spaces are separable. We define the Fourier transform of $f \in L^1(G)$ by

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_G f(x) \overline{\omega(x)} d\mu_G(x), \quad \omega \in \widehat{G}.$$

If $f \in L^1(G)$, $\hat{f} \in L^1(\widehat{G})$, and the measures on G and \widehat{G} are normalized so that the Plancherel theorem holds (see [26, (31.1)]), then the function f can be recovered from \hat{f} by the inverse Fourier transform

$$f(x) = \mathcal{F}^{-1}\hat{f}(x) = \int_{\widehat{G}} \hat{f}(\omega) \omega(x) d\mu_{\widehat{G}}(\omega), \quad a.e. \ x \in G.$$

If, in addition, f is continuous, the inversion formula holds pointwise. We assume that the measure on a group μ_G and the measure on its dual group $\mu_{\widehat{G}}$ are normalized this way, and we refer to them as *dual measures*. Under this convention, the Fourier transform \mathcal{F} is an isometric isomorphism between $L^2(G)$ and $L^2(\widehat{G})$.

For $\nu = (\lambda, \gamma) \in G \times \widehat{G}$, we let $\pi(\nu)$ denote the time-frequency shift operator $E_\gamma T_\lambda$. It is clear that $\pi(\nu)$ is a unitary operator on $L^2(G)$. The commutator relation

$$T_\lambda E_\gamma = \overline{\gamma(\lambda)} E_\gamma T_\lambda$$

leads to the following useful identities:

$$\pi(\nu)^* = \overline{\gamma(\lambda)} \pi(\nu^{-1}), \tag{2.1}$$

$$\pi(\nu_1)\pi(\nu_2) = \overline{\gamma_2(\lambda_1)}\pi(\nu_1\nu_2), \quad (2.2)$$

$$\pi(\nu_1)\pi(\nu_2) = \gamma_1(\lambda_2)\overline{\gamma_2(\lambda_1)}\pi(\nu_2)\pi(\nu_1), \quad (2.3)$$

where $\nu_i = (\gamma_i, \lambda_i)$, $i = 1, 2$, and $\pi(\nu)^*$ denotes the adjoint operator of $\pi(\nu)$.

We let Δ denote a closed subgroup of $G \times \widehat{G}$ with measure μ_Δ . To ease notation, when the measure is clear from the context, we write $d\nu$ in place of $d\mu_\Delta(\nu)$ and likewise for other measures. In our settings Weil's formula will relate integrable functions over $G \times \widehat{G}$ with integrable functions on the quotient space $(G \times \widehat{G})/\Delta$, where Δ is a closed subgroup of $G \times \widehat{G}$. Let $c_\Delta : G \times \widehat{G} \rightarrow (G \times \widehat{G})/\Delta$, $c_\Delta(\chi) = \chi\Delta$ be the *canonical map* from $G \times \widehat{G}$ onto $(G \times \widehat{G})/\Delta$. If $f \in L^1(G \times \widehat{G})$, then the function $\dot{\chi} \mapsto \int_\Delta f(\chi\nu) d\nu$ with $\dot{\chi} = c_\Delta(\chi)$, defined almost everywhere on $(G \times \widehat{G})/\Delta$, is integrable. Furthermore, when two out of the three Haar measures on $G \times \widehat{G}$, Δ and $(G \times \widehat{G})/\Delta$ are given, the third can be normalized in a unique way so that *Weil's formula*

$$\int_{G \times \widehat{G}} f(\chi) d\chi = \int_{(G \times \widehat{G})/\Delta} \int_\Delta f(\chi\nu) d\nu d\dot{\chi} \quad (2.4)$$

holds.

The annihilator group Δ^\perp of $\Delta \subset G \times \widehat{G}$ is given by

$$\Delta^\perp = \left\{ (\beta, \alpha) \in \widehat{G} \times G : \gamma(\alpha)\beta(\lambda) = 1 \text{ for all } \nu = (\lambda, \gamma) \in \Delta \right\}.$$

The annihilator is a closed subgroup of $\widehat{G} \times G$. Moreover,

$$\widehat{\Delta} \cong (\widehat{G} \times G)/\Delta^\perp \quad \text{and} \quad ((G \times \widehat{G})/\Delta)^\wedge \cong \Delta^\perp.$$

These relations show that for the closed subgroup Δ the quotient $(G \times \widehat{G})/\Delta$ is compact if, and only if, Δ^\perp is discrete. Finally, we define the adjoint Δ° of $\Delta \subset G \times \widehat{G}$ as

$$\Delta^\circ := \{\mu \in G \times \widehat{G} : \pi(\mu)\pi(\nu) = \pi(\nu)\pi(\mu) \quad \forall \nu \in \Delta\}.$$

The annihilator and adjoint of a closed subgroup Δ are identical up to a change of coordinates. To see this, we introduce the mapping

$$\Phi : G \times \widehat{G} \rightarrow \widehat{G} \times G, \quad \Phi(x, \omega) = (\omega^{-1}, x) \quad \text{for } (x, \omega) \in G \times \widehat{G}.$$

It is clear that Φ is a measure preserving, topological group isomorphism. It follows from (2.3) that $\Phi(\Delta^\circ) = \Delta^\perp$, see also [14].

We will use the following **general setup**. We assume a Haar measure on G . On the dual group of any LCA group, we assume the dual measure (such that the Plancherel theorem holds). Furthermore, we assume a Haar measure on the closed subgroup Δ of $G \times \widehat{G}$. By requiring that Weil's formula (2.4) holds, there is a uniquely determined measure $\mu_{(G \times \widehat{G})/\Delta}$ on $(G \times \widehat{G})/\Delta$. From this measure, we define the *size* of the subgroup Δ as

$$d(\Delta) = \mu_{(G \times \widehat{G})/\Delta}((G \times \widehat{G})/\Delta).$$

Intuitively, small values of $d(\Delta)$ suggest that Δ is “dense”, while large values of $d(\Delta)$ suggest that Δ is “sparse”.

Remark 1. (i) In case Δ is co-compact, by the Plancherel identity, we see that the discrete group Δ^\perp is equipped with the Haar measure $d(\Delta)^{-1}\mu_c$, where μ_c is the counting measure. In particular, the canonical choice $d(\Delta) = 1$ comes from the probability measure of $(G \times \widehat{G})/\Delta$ or, equivalently, the counting measure on Δ^\perp .

- (ii) In case Δ is a discrete, co-compact subgroup, i.e., a uniform lattice, then $d(\Delta)$ is closely related to the *lattice size* of Δ . Let $s(\Delta) = \mu_{G \times \widehat{G}}(X)$, where X is a Borel section, also called a fundamental domain, of Δ in $G \times \widehat{G}$ [5, 20]. Now, if we equip Δ with the counting measure, then $s(\Delta) = d(\Delta)$. Especially for \mathbb{R}^n , if $\Delta = P\mathbb{Z}^{2n}$, $P \in \text{GL}_{\mathbb{R}}(2n)$, then $d(\Delta) = |\det(P)|$.

Lemma 2.1. *Let Δ be a closed subgroup of $G \times \widehat{G}$. Then the following holds:*

- (i) $d(\Delta) < \infty$ if, and only if, Δ is co-compact,
- (ii) $d(\Delta^\perp) < \infty$ if, and only if, $d(\Delta^\circ) < \infty$ if, and only if, Δ is discrete.

Furthermore, if Δ is discrete and co-compact, then

- (iii) $\Delta^\perp \subset \widehat{G} \times G$ and $\Delta^\circ \subset G \times \widehat{G}$ are discrete and co-compact subgroups,
- (iv) $d(\Delta)d(\Delta^\perp) = 1$ and $d(\Delta)d(\Delta^\circ) = 1$.

Proof. Statement (i) is just a reformulation of the fact that the Haar measure of an LCA group is finite if, and only if, the group is compact. Since Δ^\perp is discrete if, and only if $(G \times \widehat{G})/\Delta$ is co-compact, statement (ii) follows from (i). Statements (iii) and (iv) for Δ^\perp can be found in [20]. The statements for Δ° follow by the relationship between Δ° and Δ^\perp , see also [14, Lemma 7.7.4]. \square

3 Frame theory

We need a rather general variant of frames, usually called continuous frames, introduced by Ali, Antoine, and Gazeau [1] and Kaiser [31].

Definition 3.1. Let \mathcal{H} be a complex Hilbert space, and let (M, Σ_M, μ_M) be a measure space, where Σ_M denotes the σ -algebra and μ_M the non-negative measure. A family of vectors $\{f_k\}_{k \in M}$ in \mathcal{H} is a *frame* for $\mathcal{K} := \overline{\text{span}}\{f_k\}_{k \in M}$ with respect to (M, Σ_M, μ_M) if

- (a) $k \mapsto f_k$ is weakly measurable, i.e., for all $f \in \mathcal{K}$, the mapping $M \rightarrow \mathbb{C}, k \mapsto \langle f, f_k \rangle$ is measurable, and
- (b) there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \int_M |\langle f, f_k \rangle|^2 d\mu_M(k) \leq B \|f\|^2 \quad \text{for all } f \in \mathcal{K}. \quad (3.1)$$

The constants A and B are called *frame bounds*.

A frame $\{f_k\}_{k \in M}$ is said to be *tight* if we can choose $A = B$; if, furthermore, $A = B = 1$, then $\{f_k\}_{k \in M}$ is a *Parseval frame*. When $\{f_k\}_{k \in M}$ is a frame for its closed linear span \mathcal{K} , we say that $\{f_k\}_{k \in M}$ is a *basic* frame. If $\mathcal{K} = \mathcal{H}$, we say $\{f_k\}_{k \in M}$ is *total*. If $\{f_k\}_{k \in M}$ is weakly measurable and the upper bound in the inequality (3.1) holds, then $\{f_k\}_{k \in M}$ is a *Bessel family* with constant B .

To a Bessel family $F := \{f_k\}_{k \in M}$ for $\mathcal{K} \subset \mathcal{H}$, we associate the *analysis operator* C_F given by

$$C_F : \mathcal{H} \rightarrow L^2(M, \mu_M), \quad f \mapsto (k \mapsto \langle f, f_k \rangle).$$

The frame condition (3.1) simply says that this operator on \mathcal{K} is bounded below and above by \sqrt{A} and \sqrt{B} , respectively, hence $C_F|_{\mathcal{K}}$ is an injective, bounded linear operator with

closed range. The adjoint D_F of C_F is the so-called *synthesis operator*; it is given weakly by

$$D_F : L^2(M, \mu_M) \rightarrow \mathcal{H}, \quad c \mapsto \int_M c(k) f_k d\mu_M(k).$$

For two Bessel families $F = \{f_k\}_{k \in M}$ and $G = \{g_k\}_{k \in M}$ we define the *mixed frame operator* $S_{F,G} = D_F C_G$. If F is a frame, the *frame operator* $S_F := S_{F,F}$ is a bounded, invertible, self-adjoint and positive operator. The Bessel families F and G are said to be *dual frames* for \mathcal{H} if $S_{F,G} = I_{\mathcal{H}}$, i.e.,

$$\langle f, g \rangle = \int_M \langle f, g_k \rangle \langle f_k, g \rangle d\mu_M(k) \quad \text{for all } f, g \in \mathcal{H}. \quad (3.2)$$

In this case we say that the following assignment

$$f = \int_M \langle f, g_k \rangle f_k d\mu_M(k) \quad \text{for } f \in \mathcal{H},$$

holds in the weak sense. Dual frames for subspaces \mathcal{K} of \mathcal{H} are defined similarly. Two dual frames are indeed frames for \mathcal{H} , see, e.g., [27]. On the other hand, given a frame $F = \{f_k\}_{k \in M}$ for \mathcal{H} one can always find at least one dual frame; the canonical choice is $\{S_F^{-1} f_k\}_{k \in M}$. The following result is well-known in frame theory.

Theorem 3.2. *Let \mathcal{H} be a Hilbert space, and let $A, B > 0$. Then the following statements are equivalent:*

- (i) $\{f_k\}_{k \in M}$ is a frame for \mathcal{H} with bounds A and B ;
- (ii) $\{f_k\}_{k \in M}$ is a Bessel family in \mathcal{H} with bound B and there exists another Bessel family $\{g_k\}_{k \in M}$ in \mathcal{H} with bound A^{-1} such that (3.2) holds.

Frames as defined in Definition 3.1 are often called *continuous frames*, with the notion *discrete frames* reserved for the case, where M is countable and μ_M is the counting measure. We will not adapt this terminology. A family of vectors $\{f_k\}_{k \in M}$ will be called *continuous* if μ_M is non-atomic and *discrete* if μ_M is purely atomic *on σ -finite subsets*. Recall that a set $E \in \Sigma$ of positive measure is an *atom* if for any measurable subset F of E either $\mu_M(F) = 0$ or $\mu_M(E \setminus F) = 0$. A measure is called *purely atomic* if every measurable set of positive measure contains an atom and *non-atomic* if there are no atoms. Every measure can be uniquely decomposed as a sum of a purely atomic and a non-atomic measure in the sense of Johnson [30].

Let us explain our terminology of discrete and continuous frames. For $a \in L^2(M, \mu_M)$ the support $K := \text{supp } a$ is σ -finite, hence we can write $K = \cup_{i \in I} M_i \cup N$, where each M_i is an atom of finite measure, I is at most countable, and N is an atomless measurable set. Assume first μ_M is atomic whenever restricted to the subalgebra $\Sigma_K = \{E \cap K : E \in \Sigma\}$, where K is a σ -finite set. Then $\mu_M(N) = 0$. Since we are interested in L^2 -functions, we will either tacitly ignore such null sets or simply say we have equality up to sets of measure zero. Functions, or rather equivalence classes of functions, in $L^2(M, \mu_M)$ are constant on every atom of M , hence any frame $k \mapsto f_k$ will also be constant on atoms (up to sets of measure zero). Moreover, $a \in L^2(M, \mu_M)$ is of the form $\sum_{i \in I} a_i \mathbf{1}_{M_i}$ for some coefficients $a_i \in \mathbb{C}$. Let f_i denote the value of $k \mapsto f_k$ on M_i . Then the synthesis operator on $a \in L^2(M, \mu_M)$ is

$$D_F a = \sum_{i \in I} \mu_M(M_i) a_i f_i \quad \text{for all } f \in \mathcal{H}, \quad (3.3)$$

which is indeed a discrete representation. Assume, on the other hand, that μ_M is non-atomic on each σ -finite subset K of M . In the decomposition $K = \cup_{i \in I} M_i \cup N$, we now have $K = N$ since K is atomless. We can write $K = \cup_{i \in I} K_i$, where K_i is of finite measure. Then, by a classical result of Sierpinski [37, Lemma 52.α], the measure μ_M takes a continuum of values $[0, \mu_M(K_i)]$ on the measurable subsets of K_i , which justifies the name *continuous frame*.

If $F = \{f_k\}_{k \in M}$ is a basic frame with bounds A and B and if C_F has dense range, then $C_F|_{\mathcal{K}}$ is invertible on all of $L^2(M, \mu_M)$, and we say that $\{f_k\}_{k \in M}$ is a *basic Riesz family* with bounds A and B . Equivalently, one can define basic Riesz families with bounds A and B as families of vectors $F = \{f_k\}_{k \in M}$ for which D_F defined on simple, integrable functions is bounded below and above by \sqrt{A} and \sqrt{B} , respectively, i.e.,

$$A \|a\|_{L^2(M)}^2 \leq \left\| \int_M a(k) f_k d\mu_M(k) \right\|^2 \leq B \|a\|_{L^2(M)}^2$$

for all simple functions a on M with finite support. If M is countable and equipped with the counting measure, a basic Riesz family is simply a Riesz basis for its closed linear span, also called a Riesz sequence.

Our notion of discrete frames might appear overly technical compared to the usual definition (i.e., M countable and μ_M the counting measure). However, it allows us to classify the following two pathological “continuous” examples as discrete frames.

Example 1. Consider the following two examples with a “continuous” index set $M = \mathbb{R}$:

- (a) Let $\mathcal{H} = \ell^2(\mathbb{Z})$, let $\{e_k\}_{k \in \mathbb{Z}}$ be its standard orthonormal basis, and equip $M = \mathbb{R}$ with a purely atomic measure, whose atoms are the intervals $[n, n+1)$, $n \in \mathbb{Z}$, each with measure 1. Define $\{f_k\}_{k \in M} \subset \mathcal{H}$ by $f_k = e_{\lfloor k \rfloor}$, $k \in \mathbb{R}$.
- (b) Let $\mathcal{H} = L^2([0, 1])$, and let $M = \mathbb{R}$. Fix $a \in \mathbb{R}$ and define $\mu_M = \sum_{n \in \mathbb{Z}} \delta_{n+a}$, where δ_x denotes the Dirac measure at $x \in \mathbb{R}$. Define $\{f_k\}_{k \in M} \subset \mathcal{H}$ by $f_k(x) = e^{2\pi i k x}$.

It is not difficult to show that both in case (a) and (b) the family $\{f_k\}_{k \in M}$ is a Parseval frame; it is even a Riesz family. Since the measure in both cases is purely atomic, the frame $\{f_k\}_{k \in M}$ is said to be discrete.

There has recently been some interest in the study of (continuous) Riesz families [2, 36], also called Riesz-type frames in [18]. Example 1(b) is a concrete version of [2, Proposition 3.7]. The following result shows, however, that this concept brings little new to the well-studied subject of (discrete) Riesz sequences. To be more concise, Proposition 3.3 shows that norm bounded, basic Riesz families necessarily are discrete.

Proposition 3.3. *Let (M, Σ, μ_M) be a measure space. Suppose that $\{f_k\}_{k \in M}$ is a basic Riesz family in \mathcal{H} that is essentially norm bounded, i.e., $C := \sup_{k \in M} \|f_k\| < \infty$. Then $\{f_k\}_{k \in M}$ is a discrete family. Furthermore, it holds that*

$$\inf_{E \in \Sigma_0} \mu_M(E) > 0, \quad \text{where } \Sigma_0 = \{E \in \Sigma : \mu_M(E) > 0\}. \quad (3.4)$$

Proof. Let a be an integrable simple function on M . From the computation

$$\left\| \int_M a(k) f_k dm(k) \right\| \leq \int_M |a(k)| \|f_k\| dm(k) \leq C \int_M |a(k)| dm(k),$$

we see that the lower Riesz bound implies

$$\sqrt{A} \|a\|_{L^2(M)} \leq C \|a\|_{L^1(M)}.$$

For $a = \mathbf{1}_E$ with $E \in \Sigma$ and $0 < \mu_M(E) < \infty$, this in turn implies that

$$\sqrt{A} \mu_M(E)^{1/2} \leq C \mu_M(E),$$

and hence $\mu_M(E) \geq A/C^2$. This shows the furthermore-part.

Let $K = \cup_{i \in I} K_i$ be any σ -finite set, where each K_i is of finite measure. We need to show that μ_M restricted to the subalgebra $\Sigma_K = \{E \cap K : E \in \Sigma\}$ is purely atomic. Suppose on the contrary that it is not. Then there is an atomless set N of positive measure in Σ_K . For some $i_0 \in I$ the intersection $N \cap K_{i_0}$ has positive measure. Clearly, $N_0 := N \cap K_{i_0}$ is also atomless, hence we can split this set into two sets of positive measure. The smallest in measure of these two sets, say N_1 , is of measure $\mu_M(N_1) \leq \mu_M(N_0)/2$. Continuing this way we obtain sets of arbitrarily small measure, contradicting (3.4). \square

From (3.4) we see that for norm-bounded Riesz families, the atoms M_i in the representation (3.3) are bounded from below in measure. Hence, even if we consider the sum in (3.3) as a Riemann type sum, there is a bound to any refinement.

If we assume that M is a Hausdorff topological group and that μ_M satisfies certain weak regularity assumptions, e.g., M being a locally compact group with the usual left Haar measure, then the existence of a norm-bounded Riesz family forces the group M to be discrete.

Proposition 3.4. *Let M be a Hausdorff topological group with a left Haar measure μ_M (as defined by Fremlin [17, Def. 441D]). If $\{f_k\}_{k \in M}$ is a norm bounded basic Riesz family, then M is a discrete group.*

Proof. From Proposition 443O in [17] we know that μ_M is *not* non-atomic if and only if there is the discrete topology on M . However, by Proposition 3.3, the measure μ_M is clearly not non-atomic, thus the result follows. \square

We end this section with a Riesz family variant of Theorem 3.2.

Theorem 3.5 ([28]). *Let \mathcal{H} be a Hilbert space, let $A, B > 0$, and let M be a countable index set equipped with the counting measure. Then the following statements are equivalent:*

- (i) $\{f_k\}_{k \in M}$ is a basic Riesz family (i.e., a Riesz sequence) in \mathcal{H} with bounds A and B ;
- (ii) $\{f_k\}_{k \in M}$ is a Bessel family \mathcal{H} with bound B and there exists a Bessel family $\{g_k\}_{k \in M}$ in \mathcal{H} with bound A^{-1} such that $\langle f_k, g_\ell \rangle = \delta_{k,\ell}$, $k, \ell \in M$.

4 Gabor systems

The Gabor system $\mathcal{G}(g, \Delta) = \{\pi(\nu)g\}_{\nu \in \Delta}$ is *regular* when Δ is a closed subgroup of $G \times \widehat{G}$. If Δ is not a subgroup, e.g., merely a set of points, the Gabor system is *irregular*. If $\Delta = \Lambda \times \Gamma$ for closed subgroups $\Lambda \subset G$ and $\Gamma \subset \widehat{G}$, we say that $\mathcal{G}(g, \Delta) = \{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ is a *separable* Gabor system. If Δ is not assumed to have this form, $\mathcal{G}(g, \Delta)$ is *non-separable*. In this work we shall consider non-separable, regular Gabor systems.

The analysis, synthesis, and the (mixed) frame operator for Gabor Bessel systems are defined as in Section 3. In particular, the (mixed) frame operator for two Gabor Bessel systems generated by the functions $g, h \in L^2(G)$ takes the form

$$S_{g,h} : L^2(G) \rightarrow L^2(G), \quad S_{g,h} = \int_{\Delta} \langle \cdot, \pi(\nu)g \rangle \pi(\nu)h \, d\nu.$$

If $g = h$, we recover the frame operator $S_g = S_{g,g}$, also simply denoted by S .

It is straightforward to show that the frame operator commutes with time-frequency shifts with respect to the group Δ .

Lemma 4.1. *Let $g, h \in L^2(G)$, $\Delta \subset G \times \widehat{G}$, and let $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(h, \Delta)$ be Bessel systems. If Δ is a closed subgroup of $G \times \widehat{G}$, then the following holds:*

(i) $S_{g,h}\pi(\nu) = \pi(\nu)S_{g,h}$ for all $\nu \in \Delta$,

(ii) If $\{\pi(\nu)g\}_{\nu \in \Delta}$ is a frame, then

$$S^{-1}\pi(\nu) = \pi(\nu)S^{-1} \quad \text{and} \quad S^{-1/2}\pi(\nu) = \pi(\nu)S^{-1/2} \quad \text{for all } \nu \in \Delta.$$

Remark 2. Lemma 4.1 implies that the canonical dual frame of a Gabor frame again is a Gabor system $\mathcal{G}(S^{-1}g, \Delta)$ and that the Gabor system $\mathcal{G}(S^{-1/2}g, \Delta)$ is a Parseval frame. In particular, if $\mathcal{G}(g, \Delta)$ is a Riesz basis, then $\mathcal{G}(S^{-1/2}g, \Delta)$ is an orthonormal basis.

We are interested in those pairs $(g, \Delta) \subset (L^2(G), G \times \widehat{G})$ for which $\mathcal{G}(g, \Delta)$ is a Gabor frame for $L^2(G)$, that is, closed subgroups $\Delta \subset G \times \widehat{G}$ and window functions $g \in L^2(G)$ for which there exists constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \int_{\Delta} |\langle f, \pi(\nu)g \rangle|^2 \, d\nu \leq B \|f\|^2$$

for all $f \in L^2(G)$.

We will need the following well-known Plancherel theorem for the short-time Fourier transform $\mathcal{V}_g := C_{g, G \times \widehat{G}}$.

Lemma 4.2 ([20]). *For $f, f_1, f_2, g, h \in L^2(G)$ the following assertions are true:*

(i) $\mathcal{V}_g f \in L^2(G \times \widehat{G})$ and $\|\mathcal{V}_g f\|_{L^2(G \times \widehat{G})}^2 = \int_{G \times \widehat{G}} |\langle f, \pi(\nu)g \rangle|^2 \, d\nu = \|g\|^2 \|f\|^2$,

(ii)

$$\int_{G \times \widehat{G}} |\langle f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, f_2 \rangle| \, d\nu \leq \|f_1\| \|f_2\| \|g\| \|h\| < \infty, \quad (4.1)$$

(iii)

$$\int_{G \times \widehat{G}} \langle f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, f_2 \rangle \, d\nu = \langle f_1, f_2 \rangle \langle h, g \rangle. \quad (4.2)$$

Proof. Statement (i) and (iii) can be found in [20]. The inequality (4.1) follows directly from the following computation:

$$\begin{aligned} \int_{G \times \widehat{G}} |\langle f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, f_2 \rangle| \, d\nu &\leq \left(\int_{G \times \widehat{G}} |\langle f_1, \pi(\nu)g \rangle|^2 \, d\nu \right)^{1/2} \left(\int_{G \times \widehat{G}} |\langle f_2, \pi(\nu)h \rangle|^2 \, d\nu \right)^{1/2} \\ &= \|f_1\| \|f_2\| \|g\| \|h\|. \end{aligned}$$

□

Lemma 4.2 shows that for any non-zero function $g \in L^2(G)$ the system $\mathcal{G}(g, G \times \widehat{G})$ is a tight frame with bound $A = \|g\|^2$. More generally, if $\langle g, h \rangle \neq 0$, then $\mathcal{G}(g, G \times \widehat{G})$ and $\mathcal{G}(h, G \times \widehat{G})$ are dual frames, and we have a (weak) reproducing formula:

$$f = \frac{1}{\langle h, g \rangle} \int_{G \times \widehat{G}} \langle f, \pi(\nu)g \rangle \pi(\nu)h \, d\nu \quad \text{for all } f \in L^2(G).$$

4.1 Three key lemmas

In this subsection we prove three observations that will be important in the subsequent sections.

Lemma 4.3. *Let $\chi = (x, \omega) \in (G \times \widehat{G})$ and $\mu = (\alpha, \beta) \in \Delta^\circ \subset G \times \widehat{G}$. If Δ is a closed subgroup of $G \times \widehat{G}$, then the equality*

$$\langle h, \pi(\mu)g \rangle \langle \pi(\mu)f_1, f_2 \rangle = \int_{(G \times \widehat{G})/\Delta} \overline{\omega(\alpha)\beta(x)} \int_{\Delta} \langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle \, d\nu \, d\chi$$

holds for all $f_1, f_2, g, h \in L^2(G)$.

Proof. If $\mu \in \Delta^\circ$, then by Lemma 4.2(iii) we have that, for $f_1, f_2, g, h \in L^2(G)$,

$$\langle h, \pi(\mu)g \rangle \langle \pi(\mu)f_1, f_2 \rangle = \int_{G \times \widehat{G}} \langle \pi(\mu)f_1, \pi(\chi)\pi(\mu)g \rangle \langle \pi(\chi)h, f_2 \rangle \, d\chi.$$

By Weil's formula for the closed subgroup Δ , the above equality becomes

$$\langle h, \pi(\mu)g \rangle \langle \pi(\mu)f_1, f_2 \rangle = \int_{(G \times \widehat{G})/\Delta} \int_{\Delta} \langle \pi(\mu)f_1, \pi(\chi\nu)\pi(\mu)g \rangle \langle \pi(\chi\nu)h, f_2 \rangle \, d\nu \, d\chi.$$

For $\chi = (x, \omega) \in G \times \widehat{G}$ and $\mu = (\alpha, \beta) \in \Delta^\circ \subset G \times \widehat{G}$ we have

$$\langle \pi(\mu)f_1, \pi(\chi\nu)\pi(\mu)g \rangle \langle \pi(\chi\nu)h, f_2 \rangle = \overline{\omega(\alpha)\beta(x)} \langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle,$$

which follows from (2.2), (2.3), and $\pi(\nu)\pi(\mu) = \pi(\mu)\pi(\nu)$ for $\nu \in \Delta, \mu \in \Delta^\circ$. This completes the proof. \square

Lemma 4.4. *Let Δ be a closed, co-compact subgroup of $G \times \widehat{G}$, and let $g, h \in L^2(G)$. If $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(h, \Delta)$ are dual frames, then*

$$\langle h, \pi(\mu)g \rangle = d(\Delta) \delta_{\mu,e} \quad \text{for all } \mu \in \Delta^\circ.$$

Proof. Let $\mu = (\alpha, \beta) \in \Delta^\circ$ and take $f \in L^2(G)$. By Lemma 4.3 we have that

$$\langle h, \pi(\mu)g \rangle \langle \pi(\mu)f, f \rangle = \int_{(G \times \widehat{G})/\Delta} \overline{\omega(\alpha)\beta(x)} \int_{\Delta} \langle \pi(\chi)^* f, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f \rangle \, d\nu \, d\chi.$$

Since $\{\pi(\nu)g\}_{\nu \in \Delta}$ and $\{\pi(\nu)h\}_{\nu \in \Delta}$ are dual frames by assumption, this equation simplifies to

$$\begin{aligned} \langle h, \pi(\mu)g \rangle \langle \pi(\mu)f, f \rangle &= \int_{(G \times \widehat{G})/\Delta} \overline{\omega(\alpha)\beta(x)} \langle \pi(\chi)^* f, \pi(\chi)^* f \rangle \, d\chi \\ &= \int_{(G \times \widehat{G})/\Delta} \overline{\omega(\alpha)\beta(x)} \, d\chi \langle f, f \rangle. \end{aligned}$$

The function $\chi \mapsto \mu(\chi) := \overline{\omega(\alpha)}\beta(x)$ is continuous on the compact domain $(G \times \widehat{G})/\Delta$, and it satisfies $\mu(\chi_1\chi_2) = \mu(\chi_1)\mu(\chi_2)$. Therefore, [25, Lemma 23.19] implies that, for all $f \in L^2(G)$,

$$\langle h, \pi(\mu)g \rangle \langle \pi(\mu)f, f \rangle = \begin{cases} d(\Delta) \langle f, f \rangle & \text{if } \mu = e, \\ 0 & \text{if } \mu \neq e. \end{cases}$$

It follows that $\langle h, \pi(\mu)g \rangle = d(\Delta)\delta_{\mu,e}$ for $\mu \in \Delta^\circ$. \square

Lemma 4.5. *Let Δ be a closed subgroup of $G \times \widehat{G}$, and let $f_1, f_2, g, h \in L^2(G)$. If $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(h, \Delta)$ are Bessel systems with Bessel bound B_g and B_h , respectively, then for fixed f_1 and f_2 , the mapping*

$$\varphi: G \times \widehat{G} \rightarrow \mathbb{C}, \quad \chi \mapsto \int_{\Delta} \langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle d\nu$$

is continuous, constant on cosets of Δ (i.e., Δ -periodic), and $\varphi(\chi) \leq B_g^{1/2} B_h^{1/2} \|f_1\| \|f_2\|$ for all $\chi \in G \times \widehat{G}$. Furthermore, the generators g and h satisfy: $|\langle h, g \rangle| \leq d(\Delta) B_g^{1/2} B_h^{1/2}$.

Proof. By the Cauchy-Schwarz inequality, we see that

$$\begin{aligned} & \int_{\Delta} |\langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle| d\nu d\chi \\ & \leq \left(\int_{\Delta} |\langle \pi(\chi)^* f_1, \pi(\nu)g \rangle|^2 d\nu \right)^{1/2} \left(\int_{\Delta} |\langle \pi(\chi)^* f_2, \pi(\nu)h \rangle|^2 d\nu \right)^{1/2} \\ & \leq B_g^{1/2} B_h^{1/2} \|f_1\| \|f_2\|. \end{aligned} \quad (4.3)$$

This computation shows that φ is well-defined and bounded. The continuity of φ can be shown using the Bessel property of $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(h, \Delta)$ and the strong continuity of $\nu \mapsto \pi(\nu)$. The fact that the mapping φ is Δ -periodic is easily verified. We have only left to prove the furthermore-part. By Lemma 4.2 the mapping $\chi \mapsto \langle f_1, \pi(\chi)g \rangle \langle \pi(\chi)h, f_2 \rangle$ lies in $L^1(G \times \widehat{G})$. We can therefore apply Weil's formula for the subgroup Δ to find that

$$\int_{G \times \widehat{G}} \langle f_1, \pi(\chi)g \rangle \langle \pi(\chi)h, f_2 \rangle d\chi = \int_{(G \times \widehat{G})/\Delta} \int_{\Delta} \langle f_1, \pi(\chi\nu)g \rangle \langle \pi(\chi\nu)h, f_2 \rangle d\nu d\chi. \quad (4.4)$$

For any $\chi = (x, \omega) \in G \times \widehat{G}$ and $\nu = (\lambda, \gamma) \in G \times \widehat{G}$ we have, by (2.2),

$$\langle f_1, \pi(\chi\nu)g \rangle \langle \pi(\chi\nu)h, f_2 \rangle = \langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle.$$

With this, equation (4.4) becomes

$$\int_{G \times \widehat{G}} \langle f_1, \pi(\chi)g \rangle \langle \pi(\chi)h, f_2 \rangle d\chi = \int_{(G \times \widehat{G})/\Delta} \int_{\Delta} \langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle d\nu d\chi. \quad (4.5)$$

Lemma 4.2, together with (4.3) and (4.5), yields

$$\begin{aligned} |\langle f_1, f_2 \rangle \langle h, g \rangle| &= \left| \int_{G \times \widehat{G}} \langle f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, f_2 \rangle d\nu \right| \\ &\leq \int_{(G \times \widehat{G})/\Delta} \int_{\Delta} |\langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle| d\nu d\chi \\ &\leq \int_{(G \times \widehat{G})/\Delta} B_g^{1/2} B_h^{1/2} \|f_1\| \|f_2\| d\chi. \end{aligned}$$

The bound on $|\langle h, g \rangle|$ now follows from taking $f_1 = f_2$. \square

5 Density results

Our first density result shows that co-compactness of $\Delta \subset G \times \widehat{G}$ is a necessary condition for the frame property of a Gabor system $\mathcal{G}(g, \Delta)$.

Theorem 5.1. *Let Δ be a closed subgroup of $G \times \widehat{G}$, and let $g \in L^2(G)$. If $\mathcal{G}(g, \Delta)$ is a frame for $L^2(G)$ with bounds $0 < A \leq B < \infty$, then the following holds:*

- (i) *the quotient group $(G \times \widehat{G})/\Delta$ is compact, i.e., $d(\Delta) < \infty$,*
- (ii) *$A d(\Delta) \leq \|g\|^2 \leq B d(\Delta)$.*

Proof. By Lemma 4.2 the mapping $\chi \mapsto |\langle f, \pi(\chi)g \rangle|^2$ lies in $L^1(G \times \widehat{G})$. Weil's formula for the subgroup Δ then gives

$$\begin{aligned} \int_{G \times \widehat{G}} |\langle f, \pi(\chi)g \rangle|^2 d\chi &= \int_{(G \times \widehat{G})/\Delta} \int_{\Delta} |\langle f, \pi(\chi\nu)g \rangle|^2 d\nu d\dot{\chi} \\ &= \int_{(G \times \widehat{G})/\Delta} \int_{\Delta} |\langle \pi(\chi)^* f, \pi(\nu)g \rangle|^2 d\nu d\dot{\chi}, \end{aligned} \quad (5.1)$$

where $|\langle f, \pi(\chi\nu)g \rangle| = |\langle \pi(\chi)^* f, \pi(\nu)g \rangle|$ follows from (2.2). The frame assumption of $\{\pi(\nu)g\}_{\nu \in \Delta}$ states that

$$A \|f\|^2 \leq \int_{\Delta} |\langle f, \pi(\nu)g \rangle|^2 d\nu \leq B \|f\|^2 \quad \text{for all } f \in L^2(G).$$

Integrating the lower frame inequality for $\pi(\chi)^* f$ over $(G \times \widehat{G})/\Delta$ yields the following:

$$A \|f\|^2 \int_{(G \times \widehat{G})/\Delta} d\dot{\chi} = A \int_{(G \times \widehat{G})/\Delta} \|\pi(\chi)^* f\|^2 d\dot{\chi} \leq \int_{(G \times \widehat{G})/\Delta} \int_{\Delta} |\langle \pi(\chi)^* f, \pi(\nu)g \rangle|^2 d\nu d\dot{\chi}.$$

By Lemma 4.2(i) and (5.1) the term on the far right equals $\|f\|^2 \|g\|^2$. We conclude that

$$A \int_{(G \times \widehat{G})/\Delta} d\dot{\chi} \leq \|g\|^2 < \infty.$$

The measure of the quotient group $(G \times \widehat{G})/\Delta$ is finite if, and only if $(G \times \widehat{G})/\Delta$ is compact. This proves (i) and the lower inequality in (ii). To get the upper bound in (ii), we look at the upper frame inequality and proceed as above to find:

$$\|g\|^2 \|f\|^2 = \int_{G \times \widehat{G}} |\langle f, \pi(\nu)g \rangle|^2 d\nu \leq B \|f\|^2 \int_{(G \times \widehat{G})/\Delta} d\dot{\chi}.$$

□

The assertions of Theorem 5.1 are not true in general if we assume that $\mathcal{G}(g, \Delta)$ is a basic frame, i.e., a frame for its closed linear span, instead of assuming that $\mathcal{G}(g, \Delta)$ is a total frame. Density results for basic frames in the case of lattice Gabor systems in $L^2(\mathbb{R}^n)$ have recently been obtained in [19]; we will not consider such extensions here.

Let us consider some implications of the density result in Theorem 5.1 for a couple of specific locally compact abelian groups. The first result shows an extreme behavior for the p -adic numbers. From Lemma 4.2 we know that for the short-time Fourier transform any nonzero window will generate a Gabor frames. However, for the p -adic numbers no other time-frequency subgroup will have a frame generator.

Corollary 5.2. *For a prime number p , consider the p -adic numbers \mathbb{Q}_p . Let Δ be a closed subgroup of $\mathbb{Q}_p \times \widehat{\mathbb{Q}}_p$. If $\mathcal{G}(g, \Delta)$ is a frame for some $g \in L^2(\mathbb{Q}_p)$, then $\Delta = \mathbb{Q}_p \times \widehat{\mathbb{Q}}_p$.*

Proof. The result follows from Theorem 5.1 together with the fact that the only co-compact subgroup of $\mathbb{Q}_p \times \widehat{\mathbb{Q}}_p$ is the entire group itself. \square

Corollary 5.3. *Let $g \in L^2(\mathbb{R}^n)$. If the system $\mathcal{G}(g, \Delta)$ is a regular Gabor frame for $L^2(\mathbb{R}^n)$, then the closed subgroup Δ is of the form $A(\mathbb{Z}^k \times \mathbb{R}^{2n-k})$ for some $A \in \text{GL}_{\mathbb{R}}(2n)$ and $0 \leq k \leq 2n$.*

Proof. Any closed subgroup Δ of \mathbb{R}^{2n} is isomorphic to $\{0\}^\ell \times \mathbb{Z}^k \times \mathbb{R}^{2n-k-\ell}$ for $0 \leq k + \ell \leq 2n$. The subgroup $\{0\}^\ell \times \mathbb{Z}^k \times \mathbb{R}^{2n-k-\ell}$ is co-compact exactly when $\ell = 0$. Hence, by Theorem 5.1, the subgroup Δ is of the form $A(\mathbb{Z}^k \times \mathbb{R}^{2n-k})$ for some $A \in \text{GL}_{\mathbb{R}}(2n)$ and $0 \leq k \leq 2n$. \square

The next results relate the norm of a Gabor frame generator to the subgroup size $d(\Delta)$.

Corollary 5.4. *Let Δ be a closed subgroup of $G \times \widehat{G}$, and let $g \in L^2(G)$. If $\mathcal{G}(g, \Delta)$ is a tight frame with bound A , then $\mathcal{G}(g, \Delta^\circ)$ is an orthogonal system with $\|g\|^2 = d(\Delta)A$.*

Proof. The canonical dual frame of $\mathcal{G}(g, \Delta)$ is $\mathcal{G}(\frac{1}{A}g, \Delta)$. From Theorem 5.1 we know that Δ is co-compact and $\|g\|^2 = d(\Delta)$. By Lemma 4.4 it follows that $\mathcal{G}(g, \Delta)$ is an orthogonal system with $\langle \frac{1}{A}g, g \rangle = d(\Delta)$. \square

Corollary 5.5. *Let Δ be a closed subgroup of $G \times \widehat{G}$, and let $g \in L^2(G)$. If $\mathcal{G}(g, \Delta)$ is a frame, then $\|S^{-1/2}g\|^2 = d(\Delta)$.*

Proof. Lemma 4.3 and Remark 2 show that $\mathcal{G}(S^{-1/2}g, \Delta)$ is a Parseval frame. The result now follows from Corollary 5.4. \square

If we in addition to co-compactness in Theorem 5.1 assume that Δ is discrete, i.e., a uniform lattice, we have a quantitative density theorem.

Theorem 5.6. *Let Δ be a discrete subgroup of $G \times \widehat{G}$ equipped with the counting measure, and let $g \in L^2(G)$. If $\mathcal{G}(g, \Delta)$ is a frame for $L^2(G)$, then Δ is a uniform lattice with $d(\Delta) \leq 1$.*

Proof. By Theorem 5.1 it follows that Δ is a uniform lattice. From Corollary 5.5 we have that $\|S^{-1/2}g\|^2 = d(\Delta)$. Taking $f = S^{-1/2}g$ in the upper frame inequality for $\mathcal{G}(S^{-1/2}g, \Delta)$ yields, using that Δ is discrete, that $\|S^{-1/2}g\|^2 \leq 1$. We conclude that $d(\Delta) \leq 1$. \square

In Theorem 5.6 the assumption that Δ is discrete is essential for the bound $d(\Delta) \leq 1$. Indeed, in Example 2 in Section 6.5 we will show that in $L^2(\mathbb{R}^n)$ for $\Delta \subset \mathbb{R}^{2n}$ separable and co-compact, but non-discrete, it will always be possible to construct a frame $\mathcal{G}(g, \Delta)$ regardless of the value of $d(\Delta)$. The construction relies on the duality principle, which is why the example is relegated to Section 6.5.

Theorem 5.7. *Let Δ be a closed subgroup of $G \times \widehat{G}$, and let $g \in L^2(G)$. Then $\mathcal{G}(g, \Delta)$ is a total Riesz family for $L^2(G)$ if, and only if, Δ is a uniform lattice, $d(\Delta) = 1$, and $\mathcal{G}(g, \Delta)$ is a frame.*

Proof. Assume that $\mathcal{G}(g, \Delta)$ is a total Riesz family. By Proposition 3.4 and Theorem 5.1, the subgroup Δ is discrete and co-compact. Hence, $\mathcal{G}(g, \Delta)$ is a Riesz basis, and $\mathcal{G}(S^{-1/2}g, \Delta)$ is therefore an orthonormal basis for $L^2(G)$, see Remark 2. It follows that $\|S^{-1/2}g\|^2 = 1$. Furthermore, by Corollary 5.5, we have that $\|S^{-1/2}g\|^2 = d(\Delta)$. Hence $d(\Delta) = 1$.

For the converse implication note that $\|S^{-1/2}g\|^2 = d(\Delta) = 1$ by Corollary 5.5. By isometry of the time-frequency shifts we see that $\langle \pi(\nu)g, \pi(\nu)S^{-1}g \rangle = 1$ for all $\nu \in \Delta$. By Theorem 5.4.7 and Proposition 5.4.8 in [6], it follows that $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(S^{-1}g, \Delta)$ are dual Riesz bases, and we conclude that $\mathcal{G}(g, \Delta)$ is a Riesz basis. Alternatively, we can arrive at this conclusion as follows. Again by isometry of $\pi(\nu)$, we see that $\|\pi(\nu)S^{-1/2}g\|^2 = 1$ for all $\nu \in \Delta$. Hence $\mathcal{G}(S^{-1/2}g, \Delta)$ is a discrete Parseval frame whose elements have norm 1, and thus it is actually an orthonormal basis. As $\mathcal{G}(g, \Delta)$ is the image of the orthonormal basis $\mathcal{G}(S^{-1/2}g, \Delta)$ under the bounded, invertible operator $S^{1/2}$, it follows that $\mathcal{G}(g, \Delta)$ is a Riesz basis for $L^2(G)$. Here, we tacitly used Lemma 4.1. \square

Owing to Theorem 5.6 discrete Gabor frames $\mathcal{G}(g, \Delta)$, for which $d(\Delta) = 1$, are called *critically sampled*. Let us for a moment consider critically sampled *separable* Gabor systems that are systems of the form $\mathcal{G}(g, \Lambda \times \Lambda^\perp) = \{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Lambda^\perp}$ for some closed subgroup Λ of G . The following density result is a slightly stronger variant of Theorem 5.7 for the special case of separable critical sampling.

Corollary 5.8. *Let Λ be a closed subgroup of G , and let $g \in L^2(G)$. If $\{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Lambda^\perp}$ is a frame for $L^2(G)$, then Λ is a uniform lattice of G and $\{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Lambda^\perp}$ is a Riesz basis.*

Proof. By Theorem 5.1 the quotient group $(G \times \widehat{G})/(\Lambda \times \Lambda^\perp) \cong (G/\Lambda \times \widehat{G}/\Lambda^\perp)$ has to be compact. This only happens if both G/Λ and $\widehat{G}/\Lambda^\perp$ are compact. Hence Λ is a co-compact subgroup of G and $\widehat{G}/\Lambda^\perp \cong \widehat{\Lambda}$ is compact in \widehat{G} . The latter conclusion implies that Λ is discrete. Thus Λ is a uniform lattice. The fact that $\{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Lambda^\perp}$ is a Riesz basis now follows from Theorem 5.7. \square

From Corollary 5.8 we see that if there are no uniform lattices in G , then there do not exist any separable, critically sampled Gabor frames for $L^2(G)$. For the Prüfer p -group $G = \mathbb{Z}(p^\infty)$ the only uniform lattice is the Prüfer p -group itself, therefore there is only one type of critically sampled Gabor system, namely $\{T_\lambda g\}_{\lambda \in \mathbb{Z}(p^\infty)}$.

Corollary 5.8 has the following direct implications.

Corollary 5.9. *Let $g \in L^2(G)$.*

- (i) *If $\{T_\lambda g\}_{\lambda \in G}$ is a frame for $L^2(G)$, then G is discrete.*
- (ii) *If $\{E_\gamma g\}_{\gamma \in \widehat{G}}$ is a frame for $L^2(G)$, then G is compact.*

Let us end this section with commenting on yet another difference between discrete and non-discrete Gabor systems. For a full-rank lattice Δ in \mathbb{R}^{2n} , Bekka [3] proved (using von Neumann algebra techniques) that there exists $g \in L^2(\mathbb{R}^n)$ so that $\mathcal{G}(g, \Delta)$ is a frame if, and only if, there exists $g \in L^2(\mathbb{R}^n)$ so that $\mathcal{G}(g, \Delta)$ is total, i.e., the linear span of the functions in $\mathcal{G}(g, \Delta)$ is dense in $L^2(\mathbb{R}^n)$. This equivalence is not true for non-discrete Gabor systems, e.g., take $\Delta = \mathbb{R}^n \times \{0\}^n$. Then $\{\pi(\nu)g\}_{\nu \in \Delta} = \{T_\lambda g\}_{\lambda \in \mathbb{R}^n}$, and it follows from Corollary 5.9 that no $g \in L^2(\mathbb{R})$ can generate a Gabor frame since \mathbb{R}^n is not discrete, see also [7]. However, for any function g such that $\hat{g}(\omega) \neq 0$ for a.e. $\omega \in \mathbb{R}^n$, we see that

$$0 = \langle T_\lambda g, f \rangle = \langle E_{-\lambda} \hat{g}, \hat{f} \rangle = \mathcal{F}^{-1}(\hat{g} \bar{\hat{f}})(-\lambda) \quad \text{for all } \lambda \in \mathbb{R}$$

implies that $f = 0$, hence $\{T_\lambda g\}_{\lambda \in \mathbb{R}^n}$ is total. This argument obviously also works for $\Delta = G \times \{0\} \subset G \times \widehat{G}$. In general, co-compactness of $\Delta \subset G \times \widehat{G}$ is not necessary for $\mathcal{G}(g, \Delta)$ to be total.

6 Duality results

To simplify the formulation of the duality result and to avoid working with infinite subgroup sizes, we introduce the following variant of $d(\Delta)$:

$$\tilde{d}(\Delta) = \begin{cases} d(\Delta) = \int_{(G \times \widehat{G})/\Delta} 1 d\chi & \text{if } (G \times \widehat{G})/\Delta \text{ is compact,} \\ 1 & \text{otherwise.} \end{cases}$$

The precise value of $\tilde{d}(\Delta)$ for non-co-compact subgroups Δ is not important as we just need that $\tilde{d}(\Delta)$ is finite for all closed subgroups.

6.1 The Wexler-Raz biorthogonality relations

Theorem 6.1. *Let Δ be a closed subgroup of $G \times \widehat{G}$, and let $g, h \in L^2(G)$. Suppose that $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(h, \Delta)$ are Bessel systems. Then the following statements are equivalent:*

- (i) $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(h, \Delta)$ are dual frames,
- (ii) $\langle h, \pi(\mu)g \rangle = \tilde{d}(\Delta)\delta_{\mu,e}$ for all $\mu \in \Delta^\circ$.

If either and hence both of the assertions hold, then $(G \times \widehat{G})/\Delta$ is compact.

Proof. Assume that $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(h, \Delta)$ are dual frames for $L^2(G)$. By Theorem 5.1 this implies that Δ is co-compact. It follows by Lemma 4.4 that

$$\langle h, \pi(\mu)g \rangle = d(\Delta)\delta_{\mu,e} \quad \text{for all } \mu \in \Delta^\circ.$$

Assume now that $\langle h, \pi(\mu)g \rangle = \tilde{d}(\Delta)\delta_{\mu,e}$ for $\mu \in \Delta^\circ$. Suppose Δ is not co-compact. Then the cardinality of Δ° is uncountable. However, this contradicts the assumption that h and $\pi(\mu)g$ are biorthogonal for each $\mu \in \Delta^\circ$ since $L^2(G)$ is separable. Thus, the subgroup Δ is co-compact. By Lemma 4.3 we have

$$d(\Delta)\delta_{\mu,e}\langle \pi(\mu)f_1, f_2 \rangle = \int_{(G \times \widehat{G})/\Delta} \overline{\omega(\alpha)}\beta(x)\varphi(\chi)d\chi, \quad (6.1)$$

where $\varphi(\chi) = \int_{\Delta} \langle \pi(\chi)^*f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^*f_2 \rangle d\nu$ and $f_1, f_2 \in L^2(G)$ are arbitrary. By Lemma 4.5 the mapping φ is a bounded function on the compact domain $(G \times \widehat{G})/\Delta$. It therefore has a Fourier series indexed by the dual group of $(G \times \widehat{G})/\Delta$, which is topologically isomorphic to the discrete group Δ° . The right hand side of equation (6.1) are the Fourier coefficients of φ . Indeed, by assumption, all but one are identically zero. We thus have the Fourier series expansion

$$\varphi(\chi) = d(\Delta)^{-1} \sum_{\mu \in \Delta^\circ} d(\Delta)\delta_{\mu,e}\langle \pi(\mu)f_1, f_2 \rangle = \langle f_1, f_2 \rangle, \quad (6.2)$$

which holds for almost all $\chi \in G \times \widehat{G}$. Moreover, by Lemma 4.5 the function φ is continuous. Hence, equality (6.2) holds pointwise; in particular, for $\chi = e$, it yields

$$\varphi(e) = \int_{\Delta} \langle f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, f_2 \rangle d\nu = \langle f_1, f_2 \rangle.$$

Thus $\{\pi(\nu)g\}_{\nu \in \Delta}$ and $\{\pi(\nu)h\}_{\nu \in \Delta}$ are dual frames for $L^2(G)$. □

6.2 The Janssen representation

For any closed subgroup Δ in $G \times \widehat{G}$, Lemma 4.3 states that the Fourier transform of the Δ -periodic function

$$\varphi : G \times \widehat{G} \rightarrow \mathbb{C}, \quad \chi \mapsto \int_{\Delta} \langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle d\nu$$

is given by

$$\hat{\varphi}(\mu) = \langle h, \pi(\mu)g \rangle \langle \pi(\mu)f_1, f_2 \rangle, \quad \mu \in \Delta^\circ.$$

Indeed, $\varphi \in L^1((G \times \widehat{G})/\Delta)$ since

$$\int_{(G \times \widehat{G})/\Delta} \left| \int_{\Delta} \langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle d\nu \right| d\chi \leq \|f_1\| \|f_2\| \|g\| \|h\|.$$

Using the Fourier inversion formula, we then recover the *fundamental identity in Gabor analysis* (6.4) for Gabor systems in $L^2(G)$ by Rieffel [33].

Theorem 6.2. *Let Δ be any closed subgroup of $G \times \widehat{G}$, and define $\varphi \in L^1((G \times \widehat{G})/\Delta)$ as above. If $\hat{\varphi} \in L^1(\Delta^\circ)$, then for almost all $\chi = (x, \omega) \in G \times \widehat{G}$:*

$$\int_{\Delta} \langle \pi(\chi)^* f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, \pi(\chi)^* f_2 \rangle d\nu = \int_{\Delta^\circ} \omega(\alpha) \overline{\beta(x)} \langle h, \pi(\mu)g \rangle \langle \pi(\mu)f_1, f_2 \rangle d\mu, \quad (6.3)$$

where $\mu = (\alpha, \beta) \in \Delta^\circ$. If, furthermore, φ is continuous, then the inversion formula (6.3) holds pointwise, and for $\chi = (e_G, e_{\widehat{G}})$ we find that

$$\int_{\Delta} \langle f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, f_2 \rangle d\nu = \int_{\Delta^\circ} \langle h, \pi(\mu)g \rangle \langle \pi(\mu)f_1, f_2 \rangle d\mu. \quad (6.4)$$

Corollary 6.3. *Let Δ be a closed subgroup of $G \times \widehat{G}$, and let $g, h \in L^2(G)$. Suppose that $\mathcal{G}(g, \Delta)$ and $\mathcal{G}(h, \Delta)$ are Bessel systems. If the functions g, h and $f_1, f_2 \in L^2(G)$ satisfy*

$$\int_{\Delta^\circ} |\langle h, \pi(\mu)g \rangle \langle \pi(\mu)f_1, f_2 \rangle| d\mu < \infty, \quad (6.5)$$

then

$$\langle S_{g,h} f_1, f_2 \rangle = \int_{\Delta} \langle f_1, \pi(\nu)g \rangle \langle \pi(\nu)h, f_2 \rangle d\nu = \int_{\Delta^\circ} \langle h, \pi(\mu)g \rangle \langle \pi(\mu)f_1, f_2 \rangle d\mu. \quad (6.6)$$

Proof. The Bessel assumption, Lemma 4.5 and (6.5) ensure that the conclusion (6.4) of Theorem 6.2 holds. \square

In [15] the authors determine sufficient conditions on the functions g, h, f_1, f_2 under which (6.4) holds. In particular, we mention that (6.4) holds if $g, h \in L^2(G)$ and f_1, f_2 belong to the Feichtinger algebra $S_0(G)$, cf. [15] and Theorem A.4 and Corollary A.5 in the appendix.

Assume that Δ is a closed, co-compact subgroup of $G \times \widehat{G}$. The measure on Δ° in the right hand side of (6.6) is then given by $d(\Delta)^{-1} \sum_{\mu \in \Delta^\circ}$. The pair (g, Δ) satisfies *condition A* if $\sum_{\mu \in \Delta^\circ} |\langle g, \pi(\mu)g \rangle| < \infty$. Now, if the Gabor system $\{\pi(\nu)g\}_{\nu \in \Delta}$ is a Bessel family and condition A holds, then (6.6) yields the *Janssen representation* of the frame operator:

$$S_g = d(\Delta)^{-1} \sum_{\mu \in \Delta^\circ} \langle g, \pi(\mu)g \rangle \pi(\mu)$$

with absolute convergence in the (uniform) operator norm. It follows from Proposition A.3 and the comments preceding Corollary A.5 that any $g \in S_0(G)$ satisfies condition A. The mixed frame operator $S_{g,h}$, $g, h \in L^2(G)$, has a similar Janssen representation.

6.3 The duality principle

In this section we proof an extended version of the duality theorem for Gabor frames. The original result on separable lattice Gabor systems on $L^2(\mathbb{R}^d)$ goes back to Daubechies, Landau and Landau [9], Janssen [29], and Ron and Shen [34]. Our proof is inspired by one direction of Janssen's proof; the important fact is that Janssen's computations carry over from the setting of discrete, separable Gabor systems in $L^2(\mathbb{R})$ to regular, non-separable Gabor systems in $L^2(G)$. From this idea, we prove that for any closed subgroup Δ in $G \times \widehat{G}$ the Gabor system $\{\pi(\nu)g\}_{\nu \in \Delta}$ is a Bessel system with bound B , if, and only if, $\{\pi(\mu)g\}_{\mu \in \Delta^\circ}$ is a Bessel system with bound B . We remind the reader that a Gabor system is a Bessel system with bound B *with respect to* the measure on the associated time-frequency subgroup. In case Δ is co-compact, the measure on Δ° is $d(\Delta)^{-1} \sum_{\mu \in \Delta^\circ}$, see Remark 1, and the Bessel duality principle in Theorem 6.4 states that for $g \in L^2(G)$ and $B > 0$ we have:

$$\int_{\Delta} |\langle f, \pi(\nu)g \rangle|^2 d\nu \leq B \|f\|^2 \quad \text{if, and only if} \quad \sum_{\mu \in \Delta^\circ} |\langle f, \pi(\mu)g \rangle|^2 \leq d(\Delta)B \|f\|^2$$

for all $f \in L^2(G)$. When Δ is a full-rank lattice in \mathbb{R}^{2n} , the Bessel duality result is well-known, and the result is stated in [16, Proposition 3.5.10], albeit without bounds. Note, however, that the Bessel duality principle is true for any closed subgroup of $G \times \widehat{G}$ and that neither co-compactness nor discreteness is needed. The generalized duality principle will then follow from the Bessel duality principle and the Wexler-Raz biorthogonality relations using general frame theory.

Theorem 6.4. *Let Δ be a closed subgroup of $G \times \widehat{G}$, and let $g \in L^2(G)$. Then $\mathcal{G}(g, \Delta)$ is a Bessel system with bound B if, and only if, $\mathcal{G}(g, \Delta^\circ)$ is a Bessel system with bound B .*

Proof. By symmetry of the Bessel duality principle, we only have to prove one of the two implications. We assume that $g \in L^2(G)$ and that $\{\pi(\nu)g\}_{\nu \in \Delta}$ is a Bessel system with bound B . In other words, we assume that the analysis operator $C_{g,\Delta} : L^2(G) \rightarrow L^2(\Delta)$ is bounded with operator bound \sqrt{B} . Therefore, its adjoint, the synthesis operator $D_{g,\Delta}$ given weakly by

$$\langle D_{g,\Delta}a, h \rangle = \int_{\Delta} a(\nu) \langle \pi(\nu)g, h \rangle d\nu \quad \text{for all } h \in L^2(G)$$

is also bounded by \sqrt{B} :

$$\|D_{g,\Delta}\varphi\|_{L^2(G)}^2 \leq B \|\varphi\|_{L^2(\Delta)}^2 \quad \text{for all } \varphi \in L^2(\Delta). \quad (6.7)$$

For each $\chi \in G \times \widehat{G}$, we define $\varphi(\nu) = \langle \pi(\chi)h, \pi(\nu)f \rangle$, $\nu \in \Delta$, where $f, h \in S_0(G)$, normalized so that $\|h\|_{L^2(G)} = 1$. It follows from Proposition A.3 together with the comments preceding Corollary A.5 that $\varphi \in S_0(\Delta)$.

By a change of variables and using the properties of the time-frequency shift operator we find that

$$\begin{aligned} \|D_{g,\Delta}\varphi\|_{L^2(G)}^2 &= \langle D_{g,\Delta}\varphi, D_{g,\Delta}\varphi \rangle = \int_{\Delta} \varphi(\nu) \int_{\Delta} \langle \pi(\nu)g, \pi(\nu')g \rangle \overline{\varphi(\nu')} d\nu' d\nu \\ &= \int_{\Delta} \varphi(\nu) \int_{\Delta} \langle \pi(\nu)g, \pi(\nu')g \rangle \langle \pi(\nu')f, \pi(\chi)h \rangle d\nu' d\nu \end{aligned}$$

$$\begin{aligned}
&= \int_{\Delta} \varphi(\nu) \int_{\Delta} \langle \pi(\nu)g, \pi(\nu\nu')g \rangle \langle \pi(\nu\nu')f, \pi(\chi)h \rangle d\nu' d\nu \\
&= \int_{\Delta} \varphi(\nu) \int_{\Delta} \langle \pi(\nu)g, \pi(\nu)\pi(\nu')g \rangle \langle \pi(\nu)\pi(\nu')f, \pi(\chi)h \rangle d\nu' d\nu \\
&= \int_{\Delta} \varphi(\nu) \int_{\Delta} \langle g, \pi(\nu')g \rangle \langle \pi(\nu')f, \pi(\nu)^* \pi(\chi)h \rangle d\nu' d\nu.
\end{aligned}$$

Note that the order of integration can be interchanged by Fubini's theorem since $\varphi \in S_0(\Delta) \subset L^1(\Delta)$.

Since f and h belong to $S_0(G)$ and $g \in L^2(G)$, the fundamental identity in Gabor analysis (6.4) holds (see the comments following Corollary 6.3). Hence,

$$\|D_{g,\Delta}\varphi\|_{L^2(G)}^2 = \int_{\Delta} \varphi(\nu) \int_{\Delta^\circ} \langle f, \pi(\mu)g \rangle \langle \pi(\mu)g, \pi(\nu)^* \pi(\chi)h \rangle d\mu d\nu. \quad (6.8)$$

For the adjoint system $\{\pi(\mu)g\}_{\mu \in \Delta^\circ}$ it follows from Corollary A.5 that the frame operator S_{g,Δ° is well-defined on the subspace $S_0(G)$ of $L^2(G)$. Thus, since $f \in S_0(G)$, we have by definition that

$$\langle S_{g,\Delta^\circ} f, \pi(\nu)^* \pi(\chi)h \rangle = \int_{\Delta^\circ} \langle f, \pi(\mu)g \rangle \langle \pi(\mu)g, \pi(\nu)^* \pi(\chi)h \rangle d\mu.$$

Hence, we can continue (6.8):

$$\begin{aligned}
\|D_{g,\Delta}\varphi\|_{L^2(G)}^2 &= \int_{\Delta} \langle S_{g,\Delta^\circ} f, \pi(\nu)^* \pi(\chi)h \rangle \langle \pi(\nu)^* \pi(\chi)h, f \rangle d\nu \\
&= \int_{\Delta} \langle S_{g,\Delta^\circ} f, \pi(\nu\chi)h \rangle \langle \pi(\nu\chi)h, f \rangle d\nu
\end{aligned}$$

where we have also used (2.1), (2.2), and a change of variables from ν^{-1} to ν . Thus, with our choice of φ , the inequality (6.7) becomes

$$\int_{\Delta} \langle S_{g,\Delta^\circ} f, \pi(\nu\chi)h \rangle \langle \pi(\nu\chi)h, f \rangle d\nu \leq B \int_{\Delta} |\langle \pi(\nu\chi)h, f \rangle|^2 d\nu.$$

Integrating over the quotient $(G \times \widehat{G})/\Delta$ yields that

$$\int_{(G \times \widehat{G})/\Delta} \int_{\Delta} \langle S_{g,\Delta^\circ} f, \pi(\nu\chi)h \rangle \langle \pi(\nu\chi)h, f \rangle d\nu d\chi \leq B \int_{(G \times \widehat{G})/\Delta} \int_{\Delta} |\langle \pi(\nu\chi)h, f \rangle|^2 d\nu d\chi,$$

and further by Weil's formula we find that

$$\int_{G \times \widehat{G}} \langle S_{g,\Delta^\circ} f, \pi(\chi)h \rangle \langle \pi(\chi)h, f \rangle d\chi \leq B \int_{G \times \widehat{G}} |\langle \pi(\chi)h, f \rangle|^2 d\chi.$$

Using the orthogonality relations of the short-time Fourier transform in Lemma 4.2(iii), we arrive at

$$\int_{\Delta^\circ} |\langle f, \pi(\mu)g \rangle|^2 d\mu = \langle S_{g,\Delta^\circ} f, f \rangle \leq B \|f\|^2$$

for any $f \in S_0(G)$. Since $S_0(G)$ is dense in $L^2(G)$, we conclude that $\{\pi(\mu)g\}_{\mu \in \Delta^\circ}$ is a Bessel system with bound B . \square

The duality principle can now be proven using general frame theory; the proof strategy is similar to the proof of the duality principle for separable, co-compact subgroups in [28].

Theorem 6.5. *Let Δ be a closed subgroup of $G \times \widehat{G}$ and let $g \in L^2(G)$. Then the following statements are equivalent:*

- (i) $\mathcal{G}(g, \Delta)$ is a Gabor frame with bounds A and B ,
- (ii) $\mathcal{G}(g, \Delta^\circ)$ is a basic Riesz family with bounds $\tilde{d}(\Delta) A$ and $\tilde{d}(\Delta) B$.

If either and hence both of the assertions hold, then $(G \times \widehat{G})/\Delta$ is compact.

Proof. Suppose either (i) or (ii) holds. Then by Theorem 5.1 and Proposition 3.4 the group $(G \times \widehat{G})/\Delta$ is compact, and equivalently Δ° is discrete. Assume now that (i) holds. Then $\{\pi(\nu)g\}_{\nu \in \Delta}$ has the canonical dual frame $\{\pi(\nu)S^{-1}g\}_{\nu \in \Delta}$ with frame bounds B^{-1} and A^{-1} . Therefore in particular $\{\pi(\nu)g\}_{\nu \in \Delta}$ and $\{\pi(\nu)S^{-1}g\}_{\nu \in \Delta}$ are Bessel systems with bounds B and A^{-1} , respectively. By Theorem 6.4 then also $\{\pi(\mu)(d(\Delta))^{-1/2}g\}_{\mu \in \Delta^\circ}$ and $\{\pi(\mu)(d(\Delta))^{-1/2}S^{-1}g\}_{\mu \in \Delta^\circ}$ are Bessel systems with respect to the counting measure on Δ° with bound B and A^{-1} , respectively. By Theorem 6.1 we also have that the duality of the frames $\{\pi(\nu)g\}_{\nu \in \Delta}$ and $\{\pi(\nu)S^{-1}g\}_{\nu \in \Delta}$ imply that $\{\pi(\mu)(d(\Delta))^{-1/2}g\}_{\mu \in \Delta^\circ}$ and $\{\pi(\mu)(d(\Delta))^{-1/2}S^{-1}g\}_{\mu \in \Delta^\circ}$ are bi-orthogonal. By Theorem 3.5 it now follows that $\{\pi(\mu)(d(\Delta))^{-1/2}g\}_{\mu \in \Delta^\circ}$ is a basic Riesz family with bounds A and B . The converse implication is similar where Theorem 3.2 instead of Theorem 3.5 is used. \square

Let us comment on a difference between the Bessel duality and the duality principle. We have proven both results for any closed subgroup Δ of $G \times \widehat{G}$. However, for non-co-compact subgroups, the duality principle is vacuously true, in the sense that Theorem 5.1 and Proposition 3.4 imply that both statements in Theorem 6.5 are false. This is not so for the Bessel duality. In fact, Corollary A.5 shows that for any closed subgroup Δ in $G \times \widehat{G}$ any function g in the Feichtinger algebra $S_0(G)$ will generate a Bessel system $\mathcal{G}(g, \Delta)$. It is thus the additional lower frame inequality and lower Riesz family condition that restrict the interesting (non-empty) statements of Theorem 6.5 to the case where $(G \times \widehat{G})/\Delta$ is compact. It is, however, remarkable that both of the assumptions limit the admissible subgroups Δ to exactly those that have a compact quotient $(G \times \widehat{G})/\Delta$. A similar comment holds for the Wexler-Raz biorthogonality relations in Theorem 6.1.

Corollary 6.6. *Let Δ be a closed subgroup of $G \times \widehat{G}$, and let $g \in L^2(G)$. Then $\mathcal{G}(g, \Delta)$ is a tight frame with bound A if, and only if, $\mathcal{G}(g, \Delta^\circ)$ is an orthogonal system with $\|g\|^2 = d(\Delta)A$.*

Proof. One implication follows by Corollary 5.4. For the other note that an orthogonal system $\{\pi(\mu)g\}_{\mu \in \Delta^\circ}$ with $\|g\|^2 = d(\Delta)A$ is a basic Riesz family, where both bounds are $d(\Delta)A$. By Theorem 6.5 the Gabor system $\{\pi(\nu)g\}_{\nu \in \Delta}$ is a tight frame with bound A . \square

We are now ready to show existence of tight Gabor frames in $L^2(\mathbb{R}^n)$ for very “sparse”, but non-discrete, subgroups Δ ; here we mean sparse (or thin) in the sense that $d(\Delta)$ can be arbitrarily large. On the other hand, if Δ is discrete, we saw in Theorem 5.6 that $d(\Delta) \leq 1$ is necessary for the existence of Gabor frames.

Example 2. Let $G = \mathbb{R}^n$ and let $\Delta = \Lambda \times \Gamma$, where Λ and Γ are closed, co-compact subgroups of \mathbb{R}^n . Then $\Gamma = P(\mathbb{Z}^r \times \mathbb{R}^{n-r})$ and $\Lambda = Q(\mathbb{Z}^s \times \mathbb{R}^{n-s})$ for some $P, Q \in \text{GL}_{\mathbb{R}}(n)$ and $0 \leq r, s \leq n$. If we consider P and Q as $n \times n$ matrices and the columns of P and Q as vectors in \mathbb{R}^n , we can take the last $n - r$ and the last $n - s$ columns of P and Q , respectively, to be orthonormal vectors. We then equip $\Gamma^\perp = (P^T)^{-1}(\mathbb{Z}^r \times \{0\}^{n-r})$ and $\Lambda^\perp = (Q^T)^{-1}(\mathbb{Z}^s \times \{0\}^{n-s})$ with the counting measure times $|\det P|^{-1}$ and $|\det Q|^{-1}$,

respectively. It follows that $d(\Delta) = |\det(PQ)|$. We split the construction of tight Gabor frames $\{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ in three cases:

- (a) $r < n$, any $P, Q \in \text{GL}_\mathbb{R}(n)$,
- (b) $s < n$, any $P, Q \in \text{GL}_\mathbb{R}(n)$,
- (c) $r = n$ and $s = n$, and $d(\Delta) = |\det(PQ)| \leq 1$ for $P, Q \in \text{GL}_\mathbb{R}(n)$.

Case (a) and (b) correspond to non-discrete subgroups, while case (c) is the well-known setup of discrete Gabor systems.

- (a) Applying the dilation operator $D_{Q^{-1}}$, defined on $L^2(\mathbb{R}^n)$ by $D_A f(x) = \det A^{1/2} f(Ax)$, to the functions in $\{E_\gamma T_\lambda g\}_{\lambda \in Q(\mathbb{Z}^s \times \mathbb{R}^{n-s}), \gamma \in P(\mathbb{Z}^r \times \mathbb{R}^{n-r})}$ we obtain:

$$\{E_\gamma T_\lambda \tilde{g}\}_{\lambda \in \mathbb{Z}^s \times \mathbb{R}^{n-s}, \gamma \in QP(\mathbb{Z}^r \times \mathbb{R}^{n-r})}, \quad \text{with } \tilde{g} := D_{Q^{-1}} g.$$

The adjoint Gabor system is

$$\{E_\beta T_\alpha \tilde{g}\}_{\alpha \in A(\mathbb{Z}^r \times \{0\}^{n-r}), \beta \in \mathbb{Z}^s \times \{0\}^{n-s}}, \quad (6.9)$$

where $A = ((QP)^T)^{-1}$. We will choose \tilde{g} so that this adjoint system is an orthonormal system. By Corollary 6.6, the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ generated by $g = D_Q \tilde{g}$ will then be a Parseval frame for $L^2(\mathbb{R}^n)$.

Obviously, the system $\{E_\beta \mathbb{1}_{[0,1]^n}\}_{\beta \in \mathbb{Z}^s \times \{0\}^{n-s}}$ is orthonormal. We consider the columns of A as vectors in \mathbb{R}^n and redefine the last $n - r$ columns of A to be an orthonormal basis of the orthogonal complement of the first r column vectors of A . For each $z \in \mathbb{Z}^r$ define

$$K_z = [0, 1]^n \cap A([z, z + 1]^r \times \mathbb{R}^{n-r}).$$

Let $Z = \{z \in \mathbb{Z}^r : K_z \neq \emptyset\}$; as usual, our set relations should be understood only up to sets of measure zero. Since Z is finite and the subspace $A(\mathbb{R}^r \times \{0\}^{n-r})$ of co-dimension $n - r > 0$, we can find points $\{y_z \in \mathbb{Z}^n : z \in Z\}$ that satisfy

$$(K_z + y_z) \cap (K_{z'} + y_{z'} + \alpha) = \emptyset \quad \forall \alpha \in A(\mathbb{Z}^r \times \{0\}^{n-r}) \setminus \{0\}^n \quad (6.10)$$

for all $z, z' \in Z$. The choice of y_z is illustrated in Figure 1. Define $\tilde{g} \in L^2(\mathbb{R})$ by

$$\tilde{g} = \sum_{z \in Z} \mathbb{1}_{K_z + y_z}.$$

By \mathbb{Z}^n -periodicity of $e^{2\pi i \beta x}$ for $\beta \in \mathbb{Z}^n$ and the fact that $y_z \in \mathbb{Z}^n$, we see that $\{E_\beta \tilde{g}\}_{\beta \in \mathbb{Z}^s \times \{0\}^{n-s}}$ is an orthonormal set. By (6.10), the translates $T_\alpha \tilde{g}$ and $T_{\alpha'} \tilde{g}$ have disjoint support for $\alpha, \alpha' \in A(\mathbb{Z}^r \times \{0\}^{n-r})$ whenever $\alpha \neq \alpha'$. Combining these two facts, we see that the adjoint Gabor system (6.9) is orthonormal. As a conclusion we have that tight Gabor frames $\{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ exist for any value of $d(\Delta) = |\det(PQ)| > 0$ with generators g having compact support.

- (b) An application of the Fourier transform switches the role of Λ and Γ . Hence, we can construct a tight Gabor frame generator g in the frequency domain by directly referring to case (a). This approach, however, leads to bandlimited generators. If compactly supported generators are desired, slight modifications of the procedure in (a) are necessary.

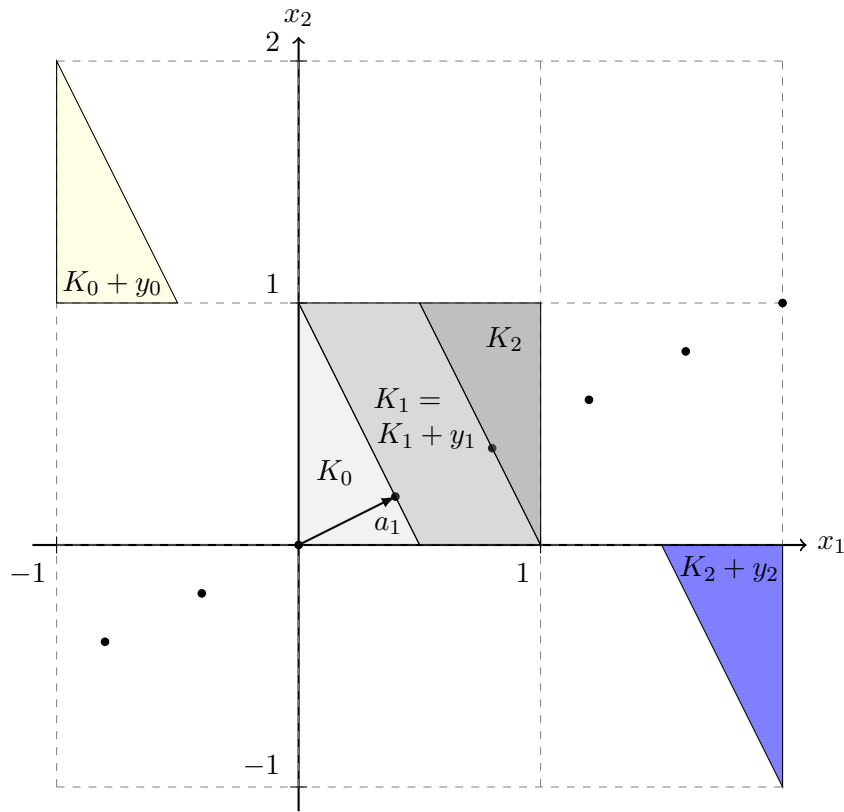


Figure 1: An example for $n = 2, r = 1$ showing the choice of the integer vectors y_z in (6.10). The dots show $A(\mathbb{Z} \times \{0\})$, where $A = [a_1 \ a_2]$ is a 2×2 matrix, and the column vector a_1 is illustrated as a geometric vector on the plot, and a_2 is orthogonal to a_1 . Then $Z = \{0, 1, 2\}$, and we can take $y_0 = (-1, 1), y_1 = (0, 0), y_2 = (1, -1)$. With this choice the set $\cup_{z \in Z} (K_z + y_z)$, and its translates along the “dots” $A(\mathbb{Z} \times \{0\}) \setminus \{(0, 0)\}$ are disjoint.

- (c) When $r = n$ and $s = n$, both $\Lambda = P\mathbb{Z}^n$ and $\Gamma = Q\mathbb{Z}^n$ are full-rank lattice in \mathbb{R}^n , and we equip these discrete subgroups with the counting measure. Under this setup, Han and Wang [23] prove that $d(\Delta) = |\det(PQ)| \leq 1$ is equivalent with the existence of a tight Gabor frames $\{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$. Their proof is constructive and the constructed Gabor windows g is, as above, a characteristic function of a given set. However, in this case the set might be unbounded, in which case the generator will not have compact support.

For uniform lattices Δ in $G \times \widehat{G}$ it follows from the duality principle in Theorem 6.5 applied to the density result in Theorem 5.6 that $\mathcal{G}(g, \Delta)$ being a (basic) Riesz family implies that $d(\Delta) \geq 1$. For non-discrete Gabor systems this conclusion is false, in fact, by the duality principle, Example 2 shows that $d(\Delta)$ can take any value in \mathbb{R}_+ .

A The Feichtinger algebra S_0

For functions $f, g \in L^1(G)$ involution and convolution are defined by

$$f^\dagger(x) = \overline{f(x^{-1})} \quad \text{and} \quad (f * g)(x) = \int_G f(s)g(xs^{-1}) ds,$$

respectively. The function space S_0 is a Fourier-invariant Banach space that is dense in L^2 and whose members are continuous and integrable functions. It can be defined as follows.

Definition A.1. For an LCA group G we define

$$S_0(G) = \left\{ f \in L^1(G) : \int_{\widehat{G}} \|E_\omega f * f^\dagger\|_{L^1} d\omega < \infty \right\}.$$

We endow $S_0(G)$ with the norm $\|f\|_{S_0, g} := \|\mathcal{V}_g f\|_{L^1(G \times \widehat{G})}$ for a fixed $g \in S_0(G)$.

The space $S_0(G)$ is a Banach algebra under convolution and pointwise multiplication, also known as the Feichtinger algebra [12]. It is a special instance of both a modulation space and a Wiener amalgam space, namely, M^1 and $W(\mathcal{FL}^1, L^1)$. Note that, for $f, g \in S_0(G)$,

$$\int_{\widehat{G}} \|E_\omega f * g^\dagger\|_{L^1(G)} d\omega = \int_G \|\mathcal{F}(f \cdot \overline{T_x g})\|_{L^1(\widehat{G})} dx = \|\mathcal{V}_g f\|_{L^1(G \times \widehat{G})},$$

where we have used that

$$\mathcal{V}_g f(x, \omega) = \mathcal{F}(f \cdot \overline{T_x g})(\omega), \quad (x, \omega) \in G \times \widehat{G}, \quad (\text{A.1})$$

for all $f, g \in L^2(G)$.

For the proof of Theorem 6.4 we need the following two properties of S_0 :

- The product of two short-time Fourier transforms of L^2 -functions with windows in $S_0(G)$ is a function in $S_0(G \times \widehat{G})$ (Theorem A.4).
- For any window $g \in L^2(G)$ and any closed subgroup Δ of $G \times \widehat{G}$, the frame operator

$$S_{g, \Delta} : L^2(G) \rightarrow L^2(G), \quad S_{g, \Delta} f = \int_{\Delta} |\langle f, \pi(\nu)g \rangle|^2 d\nu$$

with domain $D(S_{g, \Delta}) = S_0(G) \subset L^2(G)$ is well-defined (Corollary A.5).

The aim of this appendix is to give a proof of these statements. The material presented here is known for the lattice case in $L^2(\mathbb{R}^n)$ [8, 15, 16, 21], and the generalization to $L^2(G)$ is routine using standard harmonic analysis. We have included the proofs for completeness. Along the way, we obtain a direct proof of the Hölder inequalities for certain Wiener amalgam spaces.

We need the following further properties of short-time Fourier transform.

Lemma A.2. Let $g, g_i, f, f_i \in L^2(G)$, $i = 1, 2$ and $x, \alpha \in G$ and $\omega, \beta \in \widehat{G}$. Then the short-time Fourier transform

$$\mathcal{V}_g : L^2(G) \rightarrow L^2(G \times \widehat{G}), \quad \mathcal{V}_g f(x, \omega) = \langle f, E_\omega T_x g \rangle$$

satisfies the following relations:

- $\mathcal{V}_g E_\beta T_\alpha f = \beta(\alpha) E_{(e_{\widehat{G}}, \alpha^{-1})} T_{(\alpha, \beta)} \mathcal{V}_g f$, where $e_{\widehat{G}}$ denotes the identity element in \widehat{G} ,
- $\mathcal{V}_{E_\beta T_\alpha g} E_\beta T_\alpha f = \beta(x) \overline{\omega(\alpha)} \mathcal{V}_g f$,
- $\mathcal{F}(\mathcal{V}_{g_1} f_1 \cdot \overline{\mathcal{V}_{g_2} f_2})(\beta, \alpha) = \langle f_1, E_\beta T_{\alpha^{-1}} f_2 \rangle \langle E_\beta T_{\alpha^{-1}} g_2, g_1 \rangle$, where \mathcal{F} is the Fourier transform on $G \times \widehat{G}$.

Proof. Assertion (a) follows from:

$$\begin{aligned} (\mathcal{V}_g E_\beta T_\alpha f)(x, \omega) &= \langle E_\beta T_\alpha f, E_\omega T_x g \rangle = \langle f, T_{\alpha^{-1}} E_{\omega\beta^{-1}} T_x g \rangle = \overline{\omega(\alpha)} \beta(\alpha) \langle f, E_{\omega\beta^{-1}} T_{x\alpha^{-1}} g \rangle \\ &= \overline{\omega(\alpha)} \beta(\alpha) \mathcal{V}_g f(x\alpha^{-1}, \omega\beta^{-1}) = \beta(\alpha) (E_{(e_{\widehat{G}}, \alpha^{-1})} T_{(\alpha, \beta)} \mathcal{V}_g f)(x, \omega). \end{aligned}$$

Assertion (b) follows by similar manipulations, using the unitarity of $E_\beta T_\alpha$:

$$\begin{aligned} \mathcal{V}_{E_\beta T_\alpha g} E_\beta T_\alpha f(x, \omega) &= \langle E_\beta T_\alpha f, E_\omega T_x E_\beta T_\alpha g \rangle = \beta(x) \overline{\omega(\alpha)} \langle E_\beta T_\alpha f, E_\beta T_\alpha E_\omega T_x g \rangle \\ &= \beta(x) \overline{\omega(\alpha)} \langle f, E_\omega T_x g \rangle = \beta(x) \overline{\omega(\alpha)} \mathcal{V}_g f(x, \omega). \end{aligned}$$

For (c) we do the following:

$$\begin{aligned} \mathcal{F}(\mathcal{V}_{g_1} f_1 \cdot \overline{\mathcal{V}_{g_2} f_2})(\beta, \alpha) &= \int_{G \times \widehat{G}} \mathcal{V}_{g_1} f_1(x, \omega) \cdot \overline{\mathcal{V}_{g_2} f_2(x, \omega)} \beta(x) \overline{\omega(\alpha)} d(x, \omega) \\ &\stackrel{(b)}{=} \int_{G \times \widehat{G}} \mathcal{V}_{g_1} f_1(x, \omega) \cdot \overline{\mathcal{V}_{E_\beta T_{\alpha^{-1} g_2}} E_\beta T_{\alpha^{-1}} f_2(x, \omega)} d(x, \omega) = \langle \mathcal{V}_{g_1} f_1, \mathcal{V}_{E_\beta T_{\alpha^{-1} g_2}} E_\beta T_{\alpha^{-1}} f_2 \rangle \\ &\stackrel{(4.2)}{=} \langle f_1, E_\beta T_{\alpha^{-1}} f_2 \rangle \langle E_\beta T_{\alpha^{-1}} g_2, g_1 \rangle. \end{aligned}$$

□

The norm $\|\cdot\|_{S_0, g}$ on $S_0(G)$ depends on $g \in S_0(G)$. However, any function g induces an equivalent norm. Indeed, for $f, g_1, g_2 \in S_0(G)$ one can show that

$$\|g_1\|_{L^2}^2 \|g_2\|_{S_0, g_1}^{-1} \|f\|_{S_0, g_2} \leq \|f\|_{S_0, g_1} \leq \|g_2\|_{L^2}^{-2} \|g_2\|_{S_0, g_1} \|f\|_{S_0, g_2}.$$

As a consequence, a function $f \in L^2(G)$ belongs to $S_0(G)$ if, and only if, $\mathcal{V}_g f \in L^1(G \times \widehat{G})$ for any and thus all $g \in S_0(G)$.

Proposition A.3. *If $f, g \in S_0(G)$, then $\mathcal{V}_g f \in S_0(G \times \widehat{G})$ and*

$$\|\mathcal{V}_g f\|_{S_0, \mathcal{V}_g f} = \|f\|_{S_0, f} \|g\|_{S_0, g}.$$

Proof. Let $f, g \in S_0(G)$. By the argument preceding the proposition, we have that $\mathcal{V}_g f \in L^1(G \times \widehat{G})$. Using Lemma A.2 we find:

$$\begin{aligned} &\int_{G \times \widehat{G}} \int_{\widehat{G} \times G} |\langle \mathcal{V}_g f, E_{(\beta, \alpha)} T_{(x, \omega)} \mathcal{V}_g f \rangle| d(\beta, \alpha) d(x, \omega) \\ &\stackrel{(a)}{=} \int_{G \times \widehat{G}} \int_{\widehat{G} \times G} |\langle \mathcal{V}_g f, E_{(\beta, \alpha x)} \mathcal{V}_g E_\omega T_x f \rangle| d(\beta, \alpha) d(x, \omega) \\ &\stackrel{(A.1)}{=} \int_{G \times \widehat{G}} \int_{\widehat{G} \times G} |\mathcal{F}(\mathcal{V}_g f \cdot \overline{\mathcal{V}_g E_\omega T_x f})(\beta, \alpha x)| d(\beta, \alpha) d(x, \omega) \\ &\stackrel{(c)}{=} \int_{G \times \widehat{G}} \int_{\widehat{G} \times G} |\langle f, E_{\beta\omega} T_{\alpha^{-1}} f \rangle \langle E_\beta T_{x^{-1}\alpha^{-1}} g, g \rangle| d(\beta, \alpha) d(x, \omega) \\ (\alpha \mapsto \alpha^{-1}, \beta \mapsto \beta\omega^{-1}) &= \int_{G \times \widehat{G}} \int_{\widehat{G} \times G} |\langle f, E_\beta T_\alpha f \rangle \langle E_{\beta\omega^{-1}} T_{\alpha x^{-1}} g, g \rangle| d(\beta, \alpha) d(x, \omega) \\ &= \int_{G \times \widehat{G}} |\mathcal{V}_f f(\alpha, \beta)| \int_{G \times \widehat{G}} |\mathcal{V}_g g(\alpha x^{-1}, \beta\omega^{-1})| d(x, \omega) d(\alpha, \beta) \\ &= \|f\|_{S_0, f} \|g\|_{S_0, g}. \end{aligned}$$

□

Theorem A.4. Let $f_i \in L^2(G)$ and $g_i \in S_0(G)$, $i = 1, 2$. Then the mapping

$$\varphi : G \times \widehat{G} \rightarrow \mathbb{C}, (x, \omega) \mapsto (\mathcal{V}_{g_1} f_1 \cdot \overline{\mathcal{V}_{g_2} f_2})(x, \omega)$$

belongs to $S_0(G \times \widehat{G})$.

Proof. It is clear that $\varphi \in L^1(G \times \widehat{G})$. Now, let $g_0 \in S_0(G)$ and define $\varphi_0 := \mathcal{V}_{g_0} g_0$. By Proposition A.3 the function $\varphi_0 \in S_0(G \times \widehat{G})$, and thus $\varphi_0^2 \in S_0(G \times \widehat{G})$. To finish the proof, it suffices to show that $\|\varphi\|_{S_0, \varphi_0^2} = \|\mathcal{V}_{\varphi_0^2} \varphi\|_{L^1(G \times \widehat{G} \times \widehat{G} \times G)} < \infty$. We show this in two steps.

Step 1: Using the definition of the short-time Fourier transform and (A.1) we find that

$$\begin{aligned} \|\mathcal{V}_{\varphi_0^2} \varphi\|_{L^1} &= \int_{G \times \widehat{G}} \int_{\widehat{G} \times G} |\langle \varphi, E_{(\beta, \alpha)} T_{(x, \omega)} \varphi_0^2 \rangle| d(\beta, \alpha) d(x, \omega) \\ &= \int_{G \times \widehat{G}} \int_{\widehat{G} \times G} |\mathcal{F}(\mathcal{V}_{g_1} f_1 \cdot \overline{\mathcal{V}_{g_2} f_2} \cdot \overline{T_{(x, \omega)} \varphi_0^2})(\beta, \alpha)| d(\beta, \alpha) d(x, \omega) \\ &= \int_{G \times \widehat{G}} \|\mathcal{F}(\mathcal{V}_{g_1} f_1 \overline{T_{(x, \omega)} \varphi_0}) * \mathcal{F}(\overline{\mathcal{V}_{g_2} f_2} T_{(x, \omega)} \varphi_0)\|_{L^1(\widehat{G} \times G)} d(x, \omega) \\ &\leq \int_{G \times \widehat{G}} \|\mathcal{F}(\mathcal{V}_{g_1} f_1 \overline{T_{(x, \omega)} \varphi_0})\|_{L^1(\widehat{G} \times G)} \|\mathcal{F}(\overline{\mathcal{V}_{g_2} f_2} T_{(x, \omega)} \varphi_0)\|_{L^1(\widehat{G} \times G)} d(x, \omega) \\ &\leq \left(\int_{G \times \widehat{G}} \|\mathcal{F}(\mathcal{V}_{g_1} f_1 \overline{T_{(x, \omega)} \varphi_0})\|_{L^1}^2 d(x, \omega) \right)^{1/2} \left(\int_{G \times \widehat{G}} \|\mathcal{F}(\overline{\mathcal{V}_{g_2} f_2} T_{(x, \omega)} \varphi_0)\|_{L^1}^2 d(x, \omega) \right)^{1/2} \end{aligned} \quad (\text{A.2})$$

Step 2: We now show that both factors in (A.2) are finite. Consider the first of the factors. By use of Lemma A.2, we find the following:

$$\begin{aligned} &\int_{G \times \widehat{G}} \|\mathcal{F}(\mathcal{V}_{g_1} f_1 \overline{T_{(x, \omega)} \mathcal{V}_{g_0} g_0})\|_{L^1}^2 d(x, \omega) \\ &\stackrel{(a)}{=} \int_{G \times \widehat{G}} \left(\int_{\widehat{G} \times G} |\mathcal{F}(\mathcal{V}_{g_1} f_1 \overline{\mathcal{V}_{g_0} E_\omega T_x g_0})(\beta, \alpha x)| d(\beta, \alpha) \right)^2 d(x, \omega) \\ &\stackrel{(c)}{=} \int_{G \times \widehat{G}} \left(\int_{\widehat{G} \times G} |\langle f_1, E_\beta T_{\alpha^{-1} x^{-1}} E_\omega T_x g_0 \rangle \langle E_\beta T_{\alpha^{-1} x^{-1}} g_0, g_1 \rangle| d(\beta, \alpha) \right)^2 d(x, \omega) \end{aligned} \quad (\text{A.3})$$

By use of the short-time Fourier transform, expansion of the square term and a change of variables $\alpha \mapsto \alpha^{-1} x^{-1}$ we rewrite (A.3) to yield the following:

$$\begin{aligned} &\int_{G \times \widehat{G}} \|\mathcal{F}(\mathcal{V}_{g_1} f_1 \overline{T_{(x, \omega)} \mathcal{V}_{g_0} g_0})\|_{L^1}^2 d(x, \omega) \\ &= \iiint |\mathcal{V}_{g_0} f_1(x\alpha, \omega\beta)| |\mathcal{V}_{g_0} f_1(x\tilde{\alpha}, \omega\tilde{\beta})| |\mathcal{V}_{g_0} g_1(\alpha, \beta)| |\mathcal{V}_{g_0} g_1(\tilde{\alpha}, \tilde{\beta})| d(\alpha, \beta) d(\tilde{\alpha}, \tilde{\beta}) d(x, \omega) \\ &= \int |\mathcal{V}_{g_0} g_1(\alpha, \beta)| \int |\mathcal{V}_{g_0} g_1(\tilde{\alpha}, \tilde{\beta})| \int |\mathcal{V}_{g_0} f_1(x\alpha, \omega\beta)| |\mathcal{V}_{g_0} f_1(x\tilde{\alpha}, \omega\tilde{\beta})| d(x, \omega) d(\tilde{\alpha}, \tilde{\beta}) d(\alpha, \beta) \\ &\leq \|\mathcal{V}_{g_0} g_1\|_{L^1}^2 \|\mathcal{V}_{g_0} f_1\|_{L^2}^2 = \|g_1\|_{S_0, g_0}^2 \|g_0\|_{L^2}^2 \|f_1\|_{L^2}^2, \end{aligned}$$

where all integrals are over $G \times \widehat{G}$. The bound for the other term in (A.2) is obtained similarly. Combining step 1 and 2 yields that

$$\|\varphi\|_{S_0, \varphi_0^2} = \|\mathcal{V}_{\varphi_0^2} \varphi\|_{L^1} \leq \|g_0\|_{L^2}^2 \|g_1\|_{S_0, g_0} \|g_2\|_{S_0, g_0} \|f_1\|_{L^2} \|f_2\|_{L^2},$$

where $\varphi_0 = \mathcal{V}_{g_0} g_0$, $g_0 \in S_0(G)$. □

Step 1 in the proof of Theorem A.4 shows that

$$\|f \cdot g\|_{W(\mathcal{FL}^1, L^1)} \leq \|f\|_{W(\mathcal{FL}^1, L^2)} \|g\|_{W(\mathcal{FL}^1, L^2)}.$$

Using the Hölder inequality rather than the Cauchy-Schwarz inequality in (A.2) yields a Hölder inequality for Wiener amalgam spaces:

$$\|f \cdot g\|_{W(\mathcal{FL}^1, L^1)} \leq \|f\|_{W(\mathcal{FL}^1, L^p)} \|g\|_{W(\mathcal{FL}^1, L^q)}, \quad 1 = 1/p + 1/q, \quad 1 \leq p, q \leq \infty \quad (\text{A.4})$$

for $f \in W(\mathcal{FL}^1, L^p)$ and $g \in W(\mathcal{FL}^1, L^q)$. In the special case of $G = \mathbb{R}^n$ the inequality (A.4) plays an important role in [8, 15]. On the other hand, (A.4) holds for more general Wiener amalgam spaces [13].

From [12, Theorem 7] we have the following important property of S_0 . For any closed subgroup H of G the restriction mapping

$$R_H : S_0(G) \rightarrow S_0(H), \quad R_H f(x) = f(x), \quad x \in H$$

is onto and bounded.

Corollary A.5. *Let Δ be a closed subgroup of $G \times \widehat{G}$. If $g \in S_0(G)$, then there exists a constant $K > 0$ (which only depends on Δ) such that*

$$\int_{\Delta} |\langle f, \pi(\nu)g \rangle|^2 d\nu \leq K \|g\|_{S_{0,g}}^2 \|g\|_{L^2}^2 \|f\|_{L^2}^2 \quad \text{for all } f \in L^2(G).$$

Proof. From Theorem A.4 we have that the mapping

$$\varphi : G \times \widehat{G} \rightarrow \mathbb{C}, \quad (x, \omega) \mapsto |\langle f, E_{\omega} T_x g \rangle|^2$$

belongs to $S_0(G \times \widehat{G})$ for any $f \in L^2(G)$. By the comment preceding Corollary A.5, we have that for a closed subgroup Δ of $G \times \widehat{G}$, the mapping $\nu \mapsto |\langle f, \pi(\nu)g \rangle|^2$ also belongs to $S_0(\Delta)$. Hence, it belongs, in particular, to $L^1(\Delta)$. Therefore

$$\int_{\Delta} |\langle f, \pi(\nu)g \rangle|^2 d\nu = \left\| |\langle f, \pi(\cdot)g \rangle|^2 \right\|_{L^1(\Delta)} \leq C \|R_{\Delta} \varphi\|_{S_0} \leq C \|R_{\Delta}\|_{op} \|\varphi\|_{S_0}.$$

The result now follows by the proof of Theorem A.4. \square

Corollary A.5 shows that the frame operator S_g with $g \in S_0(G)$ is well-defined and bounded on $L^2(G)$. However, it also shows, and this is what we used in Section 6.4, that the operator S_g , $g \in L^2(G)$, is well-defined when the domain is restricted to the subspace $S_0(G)$ of $L^2(G)$. We refer to [16] for further results of this nature for $G = \mathbb{R}^n$.

References

- [1] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau. Continuous frames in Hilbert space. *Ann. Physics*, 222(1):1–37, 1993.
- [2] A. A. Arefijamaal, R. A. K. Gol, R. R. Tousi, and N. Tavallaei. A new approach to continuous Riesz bases. *Journal of Sciences-Islamic Republic of Iran*, 24:63–69, 2014.
- [3] B. Bekka. Square integrable representations, von Neumann algebras and an application to Gabor analysis. *J. Fourier Anal. Appl.*, 10(4):325–349, 2004.

- [4] P. G. Casazza, G. Kutyniok, and M. C. Lammers. Duality principles in frame theory. *J. Fourier Anal. Appl.*, 10(4):383–408, 2004.
- [5] J. W. S. Cassels. *An introduction to the geometry of numbers*. Classics in Mathematics. Springer-Verlag, Berlin, 1997. Corrected reprint of the 1971 edition.
- [6] O. Christensen. *An introduction to frames and Riesz bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2003.
- [7] O. Christensen, B. Deng, and C. Heil. Density of Gabor frames. *Appl. Comput. Harmon. Anal.*, 7(3):292–304, 1999.
- [8] E. Cordero and K. Gröchenig. Time-frequency analysis of localization operators. *J. Funct. Anal.*, 205(1):107–131, 2003.
- [9] I. Daubechies, H. J. Landau, and Z. Landau. Gabor time-frequency lattices and the Wexler-Raz identity. *J. Fourier Anal. Appl.*, 1(4):437–478, 1995.
- [10] D. Dutkay, D. Han, and D. Larson. A duality principle for groups. *J. Funct. Anal.*, 257(4):1133–1143, 2009.
- [11] Z. Fan, A. Heinecke, and Z. Shen. Duality for frames. *J. Fourier Anal. Appl.*, to appear.
- [12] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92(4):269–289, 1981.
- [13] H. G. Feichtinger. Banach convolution algebras of Wiener type. In *Functions, series, operators, Vol. I, II (Budapest, 1980)*, volume 35 of *Colloq. Math. Soc. János Bolyai*, pages 509–524. North-Holland, Amsterdam, 1983.
- [14] H. G. Feichtinger and W. Kozek. Quantization of TF lattice-invariant operators on elementary LCA groups. In *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser Boston, Boston, MA, 1998.
- [15] H. G. Feichtinger and F. Luef. Wiener amalgam spaces for the fundamental identity of Gabor analysis. *Collect. Math.*, Vol. Extra:233–253, 2006.
- [16] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis. In *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 123–170. Birkhäuser Boston, Boston, MA, 1998.
- [17] D. H. Fremlin. *Measure theory. Vol. 4: Topological Measure Spaces*. Torres Fremlin, Colchester, 2006.
- [18] J.-P. Gabardo and D. Han. Frames associated with measurable spaces. *Adv. Comput. Math.*, 18(2-4):127–147, 2003.
- [19] J.-P. Gabardo, D. Han, and Y.-Z. Li. Lattice tiling and density conditions for subspace Gabor frames. *J. Funct. Anal.*, 265(7):1170–1189, 2013.
- [20] K. Gröchenig. Aspects of Gabor analysis on locally compact abelian groups. In *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 211–231. Birkhäuser Boston, Boston, MA, 1998.

- [21] K. Gröchenig. *Foundations of time-frequency analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.
- [22] D. Han and D. Larson. Frame duality properties for projective unitary representations. *Bull. Lond. Math. Soc.*, 40(4):685–695, 2008.
- [23] D. Han and Y. Wang. Lattice tiling and the Weyl-Heisenberg frames. *Geom. Funct. Anal.*, 11(4):742–758, 2001.
- [24] C. Heil. History and evolution of the density theorem for Gabor frames. *J. Fourier Anal. Appl.*, 13(2):113–166, 2007.
- [25] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations*. Die Grundlehren der mathematischen Wissenschaften 115. Springer-Verlag, Berlin-New York, second edition, 1963.
- [26] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups*. Die Grundlehren der mathematischen Wissenschaften 152. Springer-Verlag, New York, 1970.
- [27] M. S. Jakobsen and J. Lemvig. Reproducing formulas for generalized translation invariant systems on locally compact abelian groups. *Trans. Amer. Math. Soc.*, to appear, (arXiv: 1405:4948).
- [28] M. S. Jakobsen and J. Lemvig. Co-compact gabor systems on locally compact abelian groups. *J. Fourier Anal. Appl.*, to appear, (arXiv:1411:4948).
- [29] A. J. E. M. Janssen. Duality and biorthogonality for Weyl-Heisenberg frames. *J. Fourier Anal. Appl.*, 1(4):403–436, 1995.
- [30] R. A. Johnson. Atomic and nonatomic measures. *Proc. Amer. Math. Soc.*, 25:650–655, 1970.
- [31] G. Kaiser. *A friendly guide to wavelets*. Birkhäuser Boston Inc., Boston, MA, 1994.
- [32] J. Ramanathan and T. Steger. Incompleteness of sparse coherent states. *Appl. Comput. Harmon. Anal.*, 2(2):148–153, 1995.
- [33] M. A. Rieffel. Projective modules over higher-dimensional noncommutative tori. *Canad. J. Math.*, 40(2):257–338, 1988.
- [34] A. Ron and Z. Shen. Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbf{R}^d)$. *Duke Math. J.*, 89(2):237–282, 1997.
- [35] D. T. Stoeva and O. Christensen. On R-Duals and the Duality Principle in Gabor Analysis. *J. Fourier Anal. Appl.*, 21(2):383–400, 2015.
- [36] Z.-Q. Xiang. New characterizations of Riesz-type frames and stability of alternate duals of continuous frames. *Adv. Math. Phys.*, pages 1–11, 2013.
- [37] A. C. Zaanen. *Integration*. North-Holland Publishing Co., Amsterdam; Interscience Publishers John Wiley & Sons, Inc., New York, 1967. Completely revised edition of An introduction to the theory of integration.