Co-compact Gabor systems on locally compact abelian groups

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Abstract: In this work we extend classical structure and duality results in Gabor analysis on the euclidean space to the setting of second countable locally compact abelian (LCA) groups. We formulate the concept of rationally oversampling of Gabor systems in an LCA group and prove corresponding characterization results via the Zak transform. From these results we derive non-existence results for critically sampled continuous Gabor frames. We obtain general characterizations in time and in frequency domain of when two Gabor generators yield dual frames. Moreover, we prove the Walnut and Janssen representation of the Gabor frame operator and consider the Wexler-Raz biorthogonality relations for dual generators. Finally, we prove the duality principle for Gabor frames. Unlike most duality results on Gabor systems, we do not rely on the fact that the translation and modulation groups are discrete and co-compact subgroups. Our results only rely on the assumption that either one of the translation and modulation group (in some cases both) are co-compact subgroups of the time and frequency domain. This presentation offers a unified approach to the study of continuous and the discrete Gabor frames.

1 Introduction

In Gabor analysis structure and duality results, such as the Zibulski-Zeevi, the Walnut and the Janssen representation of the frame operator, the Wexler-Raz biorthogonal relations, and the duality principle, play an important role. These results go back to a series of papers in the 1990s [14, 30–32, 39–43, 45–47] on (discrete) regular Gabor systems in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^n)$ with modulations and translations along full-rank lattices. The results now constitute a fundamental part of the theory. In $L^2(\mathbb{R}^n)$, a regular Gabor system is a discrete family of functions of the form $\{E_\gamma T_\lambda g\}_{\lambda\in A\mathbb{Z}^n,\gamma\in B\mathbb{Z}^n}$, where $g \in L^2(\mathbb{R}^n)$, $E_\gamma T_\lambda g(x) = \mathrm{e}^{2\pi i\gamma\cdot x}\, g(x-\lambda)$, and $A, B \in \mathrm{GL}_n(\mathbb{R})$.

For Gabor systems on locally compact abelian (LCA) groups, the picture is a lot less complete. Rieffel [38] proved in 1988 a weak form of the Janssen representation called the *fundamental identity in Gabor analysis* (FIGA) for Gabor systems in $L^2(G)$ with modulations and translations along a closed subgroup in $G\times\widehat{G}$, where G is a second countable LCA group and $\widehat{G}$ its dual group. Most other structure and duality results assume Gabor systems in $L^2(G)$ with modulations and translations along discrete and co-compact subgroups (also called uniform lattices), e.g., the Wexler-Raz biorthogonal relations for such uniform lattice Gabor systems appear implicitly in

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the work of Gröchenig [23]. Uniform lattices are discrete subgroups whose quotient group is compact, and thus, they are natural generalizations of the concept of full-rank lattices in $\mathbb{R}^n$. However, not all LCA groups possess uniform lattices. This naturally leads to the question to what extent the classical results on Gabor theory mentioned above can be formulated for non-lattice Gabor systems. The current paper gives an answer to this question.

Thus, in this work we set out to extend the theory of structure and duality results to a large class of Gabor systems in $L^2(G)$, where G is a second countable LCA group. We will focus on so-called *co-compact Gabor systems* $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$, where translation and modulation of $g \in L^2(G)$ are along closed, co-compact (i.e., the quotient group is compact) subgroups $\Lambda \subset G$ and $\Gamma \subset \widehat{G}$, respectively. In $L^2(\mathbb{R}^n)$ co-compact Gabor systems are of the form $\big\{\mathrm{e}^{2\pi i\gamma\cdot x}\, g(x-\lambda)\big\}_{\lambda\in A(\mathbb{R}^s\times\mathbb{Z}^{d-s}),\gamma\in B(\mathbb{R}^r\times\mathbb{Z}^{d-r})}$ for some choice of $0 \le r, s \le d$. Depending on the parameters r and s, these Gabor systems range from discrete over semi-continuous to continuous families. If only one of the subsets Λ and Γ is a closed, co-compact subgroup, we will use the terminology *semi* co-compact Gabor system. Clearly, co-compact and semi co-compact Gabor systems need not be discrete. More importantly, such systems exist for all LCA groups, and this setup unifies discrete and continuous Gabor theory.

For co-compact Gabor systems we prove Walnut's representation (Theorem 5.5) and Janssen's representation (Theorem 5.7) of the Gabor frame operator, the Wexler-Raz biorthogonal relations (Theorem 6.5), and the duality principle (Theorem 6.7). As an example, we mention that this generalized duality principle for $L^2(\mathbb{R}^n)$ says that the co-compact Gabor system

$$\Big\{\mathrm{e}^{2\pi i\gamma\cdot x}\, g(x-\lambda) : \lambda \in A(\mathbb{R}^s\times\mathbb{Z}^{d-s}), \gamma \in B(\mathbb{R}^r\times\mathbb{Z}^{d-r})\Big\}$$

is a continuous frame if, and only if, the adjoint system

$$\Big\{\mathrm{e}^{2\pi i\gamma\cdot x}\, g(x-\lambda) : \lambda \in (B^T)^{-1}(\{0\}^r\times\mathbb{Z}^{d-r}), \gamma \in (A^T)^{-1}(\{0\}^s\times\mathbb{Z}^{d-s})\Big\}$$

is a Riesz sequence. We recall that a family of vectors $\{f_k\}_{k\in M}$ in a Hilbert space $\mathcal{H}$ is a continuous frame with respect to a measure μ on the index set M if $\|f\|^2 \asymp \int_M |\langle f, f_k\rangle|^2\, d\mu$ for all $f \in \mathcal{H}$ and that $\{g_k\}_{k\in\mathbb{N}}$ is a Riesz sequence if $\|c\|_{\ell^2}^2 \asymp \|\sum_k c_k g_k\|^2$ for all finite sequences $c = \{c_k\}_{k\in\mathbb{N}}$. Our proof of the duality principle relies on a simple characterization of Riesz sequences in Hilbert spaces (Theorem 6.6).

As we will see, the setting of co-compact Gabor systems is indeed a natural framework for structure and duality results. Closedness of the modulation and translation subgroups is a standard assumption, and one cannot get very far without it, e.g., closedness allows for applications of key identifications between subgroups and their annihilators as well as applications of the Weil and the Poisson formulas. Co-compactness is, on the other hand, non-standard, and to the best of our knowledge this work is the first systematic study of co-compact Gabor systems. Under the second countability assumption on G, co-compactness is the weakest assumption that yields an *adjoint* Gabor system with modulations and translations along discrete and countable subgroups. In this way, co-compactness of the respective subgroups is the most general setting for which the Wexler-Raz biorthogonal relations, the duality conditions for dual generators and the duality principle can be phrased in a way that resembles the classical statements in $L^2(\mathbb{R}^n)$. As an example we mention that in the Wexler-Raz biorthogonal relations, one characterize duality of two Gabor frames by a biorthogonality condition of the corresponding adjoint Gabor systems. Since $L^2(G)$ is separable, such a biorthogonality condition is only possible if the adjoint systems are *countable* sequences (which co-compactness exactly guarantees). Furthermore, co-compact Gabor systems are precisely the setting, where the Walnut and Janssen representation of the continuous frame operator are a *discrete* representation.

However, we begin our work on Gabor systems with a study of *semi* co-compact Gabor systems as special cases of co-compact translation invariant systems, recently introduced in [7, 29]. For translation invariant systems we consider fiberization characterization of frames for translation invariant subspaces (Theorem 3.1), generalizing results from [5, 7, 8, 39]. Using these fiberization techniques we will develop Zak transform methods for Gabor analysis in $L^2(G)$. This leads among other things to a concept of rational oversampling in LCA groups (Theorem 4.3) and a Zibulski-Zeevi representation (Corollary 4.4). Furthermore, we will prove the non-existence of continuous, semi co-compact Gabor frames at "critical density" (Theorem 4.2). We also give characterizations of generators of dual semi co-compact Gabor frames (Theorems 4.7 and 4.9).

There are several advantages of the LCA group approach, one being that the essential ingredient in our arguments often becomes more transparent than in the special cases. The abstract approach also allows us to unify results from the standard settings where G is usually $\mathbb{R}^n$, $\mathbb{Z}^n$, or $\mathbb{Z}_n$. This is not only useful for the sake of generalizations, but, in some instances, it can also simplify the proofs in the special cases. As an example we mention that our proof of the Zak transform characterization of Gabor frames is based on two applications of the same result on fiberizations of $L^2(G)$, but for two different LCA groups G. In the Euclidean setting this would require two different fiberization results, one for $G = \mathbb{R}^n$ and one for $G = A\mathbb{Z}^n$ for $A \in \mathrm{GL}_n(\mathbb{R})$. In the setting of LCA groups we can unify such results into one general result. On the other hand, even for $G = \mathbb{R}^n$ most of our results are new.

For related work on locally compact (abelian) groups we refer to the recent papers [2, 3, 7, 8, 13, 18, 29, 34] as well as the book [20] and the references therein.

The paper is organized as follows. In Section 2 we give a brief introduction to harmonic analysis on LCA groups and frame theory. In Section 3 we study co-compact translation invariant systems, and specialize to *semi* co-compact Gabor systems in Section 4. In Section 5 we study the frame operator of Gabor systems, and in Section 6 we present duality results on co-compact Gabor frames.

2 Preliminaries

In the following sections we set up notation and recall some useful results from Fourier analysis on locally compact abelian groups and continuous frame theory.

2.1 Fourier analysis on locally compact abelian groups

In this paper G will denote a second countable locally compact abelian group. To G we associate its dual group $\widehat{G}$ which consists of all characters, i.e., all continuous homomorphisms from G into the torus $\mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\}$. Under pointwise multiplication $\widehat{G}$ is also a locally compact abelian group. Throughout the paper we use addition and multiplication as group operation in G and $\widehat{G}$, respectively. By the Pontryagin duality theorem, the dual group of $\widehat{G}$ is isomorphic to G as a topological group, i.e., $\widehat{\widehat{G}} \cong G$. Moreover, if G is discrete, then $\widehat{G}$ is compact, and if G is compact, then $\widehat{G}$ is discrete.

We denote the Haar measure on G by μ_G. The (left) Haar measure on any locally compact group is unique up to a positive constant. From μ_G we define $L^1(G)$ and the Hilbert space $L^2(G)$ over the complex field in the usual way. $L^2(G)$ is separable, because G is assumed to be second countable. For functions $f \in L^1(G)$ we define the Fourier transform

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_G f(x)\,\overline{\omega(x)}\,d\mu_G(x), \quad \omega \in \widehat{G}.$$

If $f \in L^1(G), \hat{f} \in L^1(\widehat{G})$, and the measure on G and $\widehat{G}$ are normalized so that the Plancherel theorem holds (see [27, (31.1)]), the function f can be recovered from $\hat{f}$ by the inverse Fourier

transform
$$f(x) = \mathcal{F}^{-1}\hat{f}(x) = \int_{\widehat{G}} \hat{f}(\omega)\,\omega(x)\,d\mu_{\widehat{G}}(\omega), \quad a.e.\ x \in G.$$
We assume that the measure on a group μ_G and its dual group $\mu_{\widehat{G}}$ are normalized this way, and we refer to them as *dual measures*. We will consider $\mathcal{F}$ as an isometric isomorphism between $L^2(G)$ and $L^2(\widehat{G})$.

On any locally compact abelian group G, we define the following three operators. For $a \in G$, the operator T_a, called *translation* by a, is defined by
$$T_a : L^2(G) \to L^2(G),\ (T_a f)(x) = f(x-a), \quad x \in G.$$
For $\chi \in \widehat{G}$, the operator E_χ, called *modulation* by χ, is defined by
$$E_\chi : L^2(G) \to L^2(G),\ (E_\chi f)(x) = \chi(x) f(x), \quad x \in G.$$
For $t \in L^\infty(G)$ the operator M_t, called *multiplication* by t, is defined by
$$M_t : L^2(G) \to L^2(G),\ (M_t f)(x) = t(x) f(x), \quad x \in G.$$
The following commutator relations will be used repeatedly: $T_a E_\chi = \overline{\chi(a)}\, E_\chi T_a,\ \mathcal{F}T_a = E_{a^{-1}}\mathcal{F}$, and $\mathcal{F}E_\chi = T_\chi \mathcal{F}$.

For a subset H of an LCA group G, we define its annihilator as
$$\mathsf{A}(\widehat{G}, H) = \{\omega \in \widehat{G} \,|\, \omega(x) = 1 \text{ for all } x \in H\}.$$
When the group $\widehat{G}$ is understood from the context, we will simply denote the annihilator $\mathsf{A}(\widehat{G}, H) = H^\perp$. The annihilator is a closed subgroup in $\widehat{G}$, and if H is a closed subgroup itself, then $\widehat{H} \cong \widehat{G}/H^\perp$ and $\widehat{G/H} \cong H^\perp$. These relations show that for a closed subgroup H the quotient G/H is compact if and only if $H^\perp$ is discrete.

Lemma 2.1. *Let H be a closed subgroup of G. If G/H is finite, then $H^\perp \cong G/H$.*

Proof. Note that any finite group G is self-dual, that is, $\widehat{G} \cong G$. And so, by application of the isomorphism $H^\perp \cong \widehat{G/H}$ we find that $H^\perp \cong \widehat{G/H} \cong G/H$. □

We also remind the reader of Weil's formula; it relates integrable functions over G with integrable functions on the quotient space G/H when H is a closed normal subgroup of G. For a closed subgroup H of G we Let $\pi_H : G \to G/H,\ \pi_H(x) = x + H$ be the *canonical map* from G onto G/H. If $f \in L^1(G)$, then the function $\dot{x} \mapsto \int_H f(x+h)\,d\mu_H(h)$, $\dot{x} = \pi_H(x)$ defined almost everywhere on G/H, is integrable. Furthermore, when two of the Haar measures on G, H and G/H are given, then the third can be normalized such that
$$\int_G f(x)\,dx = \int_{G/H} \int_H f(x+h)\,d\mu_H(h)\,d\mu_{G/H}(\dot{x}). \tag{2.1}$$
Hence, if two of the measures on $G, H, G/H, \widehat{G}, H^\perp$ and $\widehat{G}/H^\perp$ are given, and these two are not dual measures, then by requiring dual measures and Weil's formula (2.1), all other measures are uniquely determined. To ease notation, we will often write dh in place of $d\mu_H(h)$ and likewise for other measures.

A *Borel section* or a fundamental domain of a closed subgroup H in G is a Borel measurable subset X of G which meets each coset G/H once. Any closed subgroup H in G has a Borel section [35, Lemma 1.1]; however, we shall in the following usually only consider Borel sections of discrete subgroups H. We always equip Borel sections of G with the Haar measure $\mu_G|_X$. Assume that H is a discrete subgroup. It follows that $\mu_G(X)$ is finite if, and only if, H is co-compact, i.e., H is a uniform lattice [7]. From [7], we also have that the mapping $x \mapsto x + H$ from (X, μ_G) to $(G/H, \mu_{G/H})$ is measure-preserving, and the mapping $Q(f) = f'$ defined by

$$f'(x + H) = f(x), \quad x + H \in G/H, x \in X, \tag{2.2}$$

is an isometry from $L^2(X, \mu_G)$ onto $L^2(G/H, \mu_{G/H})$.

For more information on harmonic analysis on locally compact abelian groups, we refer the reader to the classical books [22, 26, 27, 37].

2.2 Frame theory

One of the central concept of this paper is that of a frame. The definition is as follows.

Definition 2.2. Let $\mathcal{H}$ be a complex Hilbert space, and let (M, Σ_M, μ_M) be a measure space, where Σ_M denotes the σ-algebra and μ_M the non-negative measure. A family of vectors $\{f_k\}_{k\in M}$ is called a *frame* for $\mathcal{H}$ with respect to (M, Σ_M, μ_M) if

(a) the mapping $M \to \mathbb{C}, k \mapsto \langle f, f_k\rangle$ is measurable for all $f \in \mathcal{H}$, and

(b) there exists constants $A, B > 0$ such that

$$A\|f\|^2 \le \int_M |\langle f, f_k\rangle|^2 \, d\mu_M(k) \le B\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{2.3}$$

The constants A and B are called *frame bounds.*

If $\{f_k\}_{k\in M}$ is measurable and the upper bound in the above inequality (2.3) holds, then $\{f_k\}_{k\in M}$ is said to be a *Bessel* system or family with constant B. A frame $\{f_k\}_{k\in M}$ is said to be *tight* if we can choose $A = B$; if, furthermore, $A = B = 1$, then $\{f_k\}_{k\in M}$ is said to be a *Parseval frame.*

If μ_M is the counting measure and $\Sigma_M = 2^M$ the discrete σ-algebra, we say that $\{f_k\}_{k\in M}$ is a *discrete frame* whenever (2.3) is satisfied; for this measure space, any family of vectors is obviously measurable. *Because the results of the present paper can be formulated for the discrete and continuous setting, we shall refer to either cases as* frames *and be more specific when necessary.* We mention that in the literature frames and discrete frames are usually called continuous frames and frames, respectively. The concept of continuous frames was introduced by Kaiser [33] and Ali, Antoine, and Gazeau [1]. For an introduction to frame theory, we refer the reader to [11].

To a Bessel family $\{f_k\}_{k\in M}$ for $\mathcal{H}$, we associate the the *synthesis operator* $T : L^2(M, \mu_M) \to \mathcal{H}$ defined weakly by

$$T\{c_k\}_{k\in M} = \int_M c_k f_k \, \mu_M(k). \tag{2.4}$$

This is a bounded linear operator. Its adjoint operator $T^* : \mathcal{H} \to L^2(M, \mu_M)$ is called the *analysis operator*, and it is given by

$$T^* f = \{\langle f, f_k\rangle\}_{k\in M}. \tag{2.5}$$

The *frame operator* $S : \mathcal{H} \to \mathcal{H}$ is then defined as $S = TT^*$. We remark that the frame operator is the unique operator satisfying

$$\langle Sf, g\rangle = \int_M \langle f, f_k\rangle \langle f_k, g\rangle \, d\mu_M(k) \quad \text{for all } f, g \in \mathcal{H}, \tag{2.6}$$

and that it is well-defined, bounded and self-adjoint for any Bessel system $\{f_k\}_{k\in M}$; it is invertible if $\{f_k\}_{k\in M}$ is a frame.

In case the frame inequalities (2.3) only hold for $f \in \mathcal{K} := \overline{\operatorname{span}}\,\{f_k\}_{k\in M} \subset \mathcal{H}$, we say that $\{f_k\}_{k\in M}$ is a *basic frame* or a frame for its closed linear span. For discrete frames such frames are usually called frame sequences; we will not adopt this terminology as basic frames need not be sequences. A frame for $\mathcal{H}$ is clearly a basic frame with $\mathcal{K} = \mathcal{H}$. If we need to stress that a basic frame spans all of $\mathcal{H}$, we use the terminology *total frame*. Now, let us briefly comment on the definition of the subspace $\mathcal{K}$.

From the Bessel property of a (basic) frame $\{f_k\}$, we see that:

$$\overline{\operatorname{im} T} = (\ker T^*)^\perp = \{f \in \mathcal{H} : \langle f, f_k\rangle = 0 \ \forall k \in M\}^\perp = \overline{\operatorname{span}}\,\{f_k\}_{k\in M}\,.$$

The lower frame bound for $f \in \mathcal{K}$ implies that the operator $T^*|_{\mathcal{K}}$ is bounded from below, i.e., $\|T^*|_{\mathcal{K}} f\| \ge \sqrt{A}\,\|f\|$, which is equivalent to $T^*|_{\mathcal{K}}$ being injective with closed range which, in turn, implies that T has closed range. Since $T^*|_{\mathcal{K}}$ is injective, the range of T is dense in $\mathcal{K}$. It follows that $\operatorname{im} T = \mathcal{K}$.

We will only consider measures μ_M that are σ-finite. Assume that $\{f_k\}$ is measurable. It is known that T as in (2.4) defines a bounded linear operator if, and only if, $\{f_k\}_{k\in M}$ is a Bessel family [36]. Hence, the argument in the preceding paragraph shows that $\{f_k\}_{k\in M}$ is a basic frame if, and only if, T as in (2.4) defines a bounded linear operator with $\operatorname{im} T = \mathcal{K}$.

Two Bessel systems $\{f_k\}_{k\in M}$ and $\{g_k\}_{k\in M}$ are said to be *dual frames* for $\mathcal{H}$ if

$$\langle f, g\rangle = \int_M \langle f, g_k\rangle \langle f_k, g\rangle \, d\mu_M(k) \quad \text{for all } f, g \in \mathcal{H}. \tag{2.7}$$

In this case

$$f = \int_M \langle f, g_k\rangle f_k \, d\mu_M(k) \quad \text{for } f \in \mathcal{H}, \tag{2.8}$$

holds in the weak sense. For discrete frames, equation (2.8) holds in the usual strong sense, i.e., with (unconditional) convergence in the $\mathcal{H}$ norm. Two dual frames are indeed frames. We also mention that to a given frame for $\mathcal{H}$ one can always find at least one dual frame, the so-called canonical dual frame $\{S^{-1} f_k\}_{k\in M}$.

Let us end this section with the definition of a Riesz sequence.

Definition 2.3. Let $\{f_k\}_{k=1}^\infty$ be a sequence in a Hilbert space $\mathcal{H}$. If there exists constants $A, B > 0$ such that

$$A \sum_k |c_k|^2 \le \Big\| \sum_k c_k f_k \Big\|_{\mathcal{H}}^2 \le B \sum_k |c_k|^2$$

for all finite sequence $\{c_k\}_{k=1}^\infty$, then we call $\{f_k\}_{k=1}^\infty$ a *Riesz sequence*. If furthermore $\overline{\operatorname{span}}\{f_k\}_{k=1}^\infty = \mathcal{H}$, then $\{f_k\}_{k=1}^\infty$ is a *Riesz basis*.

3 Translation invariant systems

Before we focus on Gabor systems, let us first show some results concerning the class of translation invariant systems, recently introduced in [7, 29], which contains the class of (semi) co-compact Gabor systems.

We define translation invariant systems as follows. Let P be a countable or an uncountable index set, let $g_p \in L^2(G)$ for $p \in P$, and let H be a closed, co-compact subgroup in G. For a compact abelian group, the group is metrizable if, and only if, the character group is countable

[26, (24.15)]. Hence, since G/H is compact and metrizable, the group $\widehat{G/H} \cong H^\perp$ is discrete and countable. *Unless stated otherwise we equip $H^\perp$ with the counting measure and assume a fixed Haar measure μ_G on G.*

The (co-compact) *translation invariant* (TI) system generated by $\{g_p\}_{p\in P}$ with translation along the closed, co-compact subgroup H is the family of functions $\{T_h g_p\}_{h\in H, p\in P}$. We will use the following standing assumptions on the index set P:

(I) (P, Σ_P, μ_P) is a σ-finite measure space,

(II) $p \mapsto g_p, (P, \Sigma_P) \to (L^2(G), B_{L^2(G)})$ is measurable,

(III) $(p, x) \mapsto g_p(x), (P \times G, \Sigma_P \otimes B_G) \to (\mathbb{C}, B_\mathbb{C})$ is measurable.

We say that $\{g_p\}_{p\in P}$ is *admissible* or, when g_p is clear from the context, simply that the measure space P is admissible. The nature of these assumptions are discussed in [29]. Observe that any closed subgroup P of G (or $\widehat{G}$) with the Haar measure is admissible if $p \to g_p$ is continuous, e.g., if $g_p = T_p g$ for some function $g \in L^2(G)$.

If P is countable, we equip it with a weighted counting measure. If the subgroup H is also discrete, hence a uniform lattice, the system $\{T_h g_p\}_{h\in H, p\in P}$ is a *shift* invariant (SI) system.

3.1 Fiberization

TI systems are of interest to us since the Gabor systems we shall study are special instances of these. As the work of Ron and Shen [39] and Bownik [5] show, certain Gramian and so-called dual Gramian matrices as well as a fiberization technique play an important role in the study of TI systems. The fiberization technique is closely related to Zak transform methods in Gabor analysis, as we will see in Section 4.1.

Let $\Omega \subset \widehat{G}$ be a Borel section of $H^\perp$ in $\widehat{G}$ as defined in Section 2.1. Following [7] we define the *fiberization* mapping $\mathcal{T} : L^2(G) \to L^2(\Omega, \ell^2(H^\perp))$ by

$$\mathcal{T} f(\omega) = \{\hat{f}(\omega\alpha)\}_{\alpha \in H^\perp}, \quad \omega \in \Omega; \tag{3.1}$$

the inner product in $L^2(\Omega, \ell^2(H^\perp))$ is defined in the obvious manner. Fiberization is an isometric, isomorphic operation as shown in [7,8].

Our first result characterizes the frame/Bessel property of TI systems in terms of fibers. It extends results from [5,7,8] to the case of uncountable many generators $\{g_p\}_{p\in P}$.

Theorem 3.1. *Let $0 < A \le B < \infty$, let $H \subset G$ be a closed, co-compact subgroup, and let $\{g_p\}_{p\in P} \subset L^2(G)$, where (P, μ_P) is an admissible measure space. The following assertions are equivalent:*

(i) The family $\{T_h g_p\}_{h\in H, p\in P}$ is a frame for $L^2(G)$ with bounds A and B (or a Bessel system with bound B),

(ii) For almost every $\omega \in \Omega$, the family $\{\mathcal{T} g_p(\omega)\}_{p\in P}$ is a frame for $\ell^2(H^\perp)$ with bounds A and B (or a Bessel system with bound B).

Proof. The proof follows from the proofs in [5,7,8]. Indeed, the key computation in [7] shows that

$$\int_P \int_H \left|\langle f, T_h g_p \rangle_{L^2}\right|^2 d\mu_H(h) d\mu_P(p) = \int_P \int_\Omega \left|\langle \mathcal{T} f(\omega), \mathcal{T} g_p(\omega) \rangle_{\ell^2}\right|^2 d\mu_{\widehat{G}}(\omega) d\mu_P(p)$$

for all $f \in L^2(G)$. Let us outline the argument for the frame case; the Bessel case is similar. Assume that (ii) holds. Then for a.e. $\omega \in \Omega$ we have

$$A \|a\|_{\ell^2} \le \int_P \left|\langle a, \mathcal{T} g_p(\omega)\rangle_{\ell^2}\right|^2 d\mu_P(p) \le B \|a\|_{\ell^2} \quad \text{for all } a \in \ell^2(H^\perp).$$

If we integrate these inequalities over Ω and use that $\mathcal{T}$ is an isometric isomorphism, we arrive at (i) using the key computation above. The other implication follows as in [5]. □

Remark 1. Theorem 3.1 can also be formulated for basic frames using the notion of range functions. A very general version of this result was obtained independently and concurrently in [28]. Theorem 3.1 is closely related to the theory of translation invariant subspaces which very recently has been studied in [4, 28] using Zak transform methods (cf. Section 4.1).

Theorem 3.1 shows that the task of verifying that a given TI system $\{T_h g_p\}_{h\in H, p\in P}$ is a frame for $L^2(G)$ can be replaced by the simpler task of proving that the fibers $\{\mathcal{T} g_p(\omega)\}_{p\in P}$ are a frame for the discrete space $\ell^2(H^\perp)$, however, this needs to be done for every $\omega \in \Omega$. For a uniform lattice H, the Borel section Ω of $H^\perp$ is compact, but for non-discrete, co-compact closed subgroups H, this is not the case, in fact, $m_{\widehat{G}}(\Omega) = \infty$.

Let $\omega \in \Omega$ be given. The analysis operator $L_\omega : \ell^2(H^\perp) \to L^2(P)$ for the family of fibers $\{\mathcal{T} g_p(\omega)\}_{p\in P}$ in $\ell^2(H^\perp)$ is given by:

$$L_\omega c = p \mapsto \langle c, \mathcal{T} g_p(\omega)\rangle_{\ell^2(H^\perp)}, \quad D(L_\omega) = c_{00}(H^\perp). \tag{3.2}$$

Note that we have only defined the analysis operator L_ω for finite sequences since we do not, a priori, assume that the family of fibers is a Bessel system, cf. (2.5). If L_ω is bounded, it extends to a bounded, linear operator on all of $\ell^2(H^\perp)$; clearly, L_ω is bounded with bound $\|L_\omega\| \le \sqrt{B}$ if, and only if, $\{\mathcal{T} g_p(\omega)\}_{p\in P}$ is a Bessel system with bound B. In this case the adjoint is the synthesis operator $L_\omega^* : L^2(P) \to \ell^2(H^\perp)$ given by:

$$L_\omega^* f = \left\{ \int_P f(p)\, \hat{g}_p(\omega\alpha) d\mu_P(p) \right\}_{\alpha\in H^\perp}, \qquad \text{where } f \in L^2(P).$$

From results in [10, Chapter 3] and [36] we know that this synthesis operator $L_\omega^* : L^2(P) \to \ell^2(H^\perp)$ is a well-defined, bounded linear operator if, and only if, the fibers $\{\mathcal{T} g_p(\omega)\}_{p\in P}$ is a Bessel system. The frame operator $L_\omega^* L_\omega$ of the family of fibers is called the *dual Gramian* and is denoted by $\tilde{\mathcal{G}}_\omega : \ell^2(H^\perp) \to \ell^2(H^\perp)$. Again, using results from [10, Chapter 3], the frame operator is a bounded, linear operator acting on all of $\ell^2(H^\perp)$ precisely when the fibers form a Bessel system. Paying attention to the operator bounds and Bessel constants, we therefore have the following result, extending results from [7, 8] to the case of uncountably many generators.

Proposition 3.2. *Let $B > 0$, let $H \subset G$ be a closed, co-compact subgroup, and let $\{g_p\}_{p\in P} \subset L^2(G)$, where (P, μ_P) is an admissible measure space. The following assertions are equivalent:*

(i) $\{T_h g_p\}_{h\in H, p\in P}$ *is a Bessel system with bound B,*

(ii) $\operatorname{ess\,sup}_{\omega\in\Omega} \|\tilde{\mathcal{G}}_\omega\| \le B$,

(iii) $\operatorname{ess\,sup}_{\omega\in\Omega} \|L_\omega\| \le \sqrt{B}$.

In a similar fashion, it is possible to generalize [8, Proposition 4.9(2)] and the corresponding result in [7] to the case of uncountably many generators.

4 Semi co-compact Gabor systems and characterizations

In the the rest of this article we will concentrate on Gabor systems. A *Gabor system* in $L^2(G)$ with generator $g \in L^2(G)$ is a family of functions of the form

$$\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}\,, \text{where } \Gamma \subseteq \widehat{G} \text{ and } \Lambda \subseteq G.$$

We will usually assume that at least one of the subsets $\Gamma \subset \widehat{G}$ or $\Lambda \subset G$ is a closed subgroup; if either of these subsets is not a closed subgroup, it will be assumed to be, at least, admissible as an index set (cf. the previous section). We often use that semi co-compact Gabor systems are unitarily equivalent to co-compact translation invariant systems in either time or in frequency domain. If both Γ and Λ are closed and co-compact subgroups, we say that $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ is a *co-compact Gabor system*; if only one of the sets Γ and Λ is a closed and co-compact subgroup, we name the Gabor system *semi co-compact*. If both Γ and Λ are discrete and co-compact, we recover the well-known uniform lattice Gabor systems.

4.1 Characterizations of Gabor frames and the Zak transform

The fiberization technique from Theorem 3.1 will play a crucial role in the characterizations of semi co-compact Gabor frames, presented in this subsection. From Theorem 3.1 for the TI system $\{T_\gamma \mathcal{F}^{-1} T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$, which is unitarily equivalent with $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$, we immediately have a characterization of the frame property of Gabor systems.

Proposition 4.1. *Let $g \in L^2(G)$, and let $0 < A \le B < \infty$. Let Γ be a closed, co-compact subgroup of $\widehat{G}$, and let $(\Lambda, \Sigma_\Lambda, \mu_\Lambda)$ be an admissible measure space in G. The following assertions are equivalent:*

(i) $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ is a frame for $L^2(G)$ with bounds A and B,

(ii) $\left\{\{g(x+\lambda+\alpha)\}_{\alpha\in\Gamma^\perp}\right\}_{\lambda\in\Lambda}$ is a frame for $\ell^2(\Gamma^\perp)$ with bounds A and B for a.e. $x \in X$, where X is a Borel section of $\Gamma^\perp$ in $\widehat{G}$.

We will apply Theorem 3.1 once more to Proposition 4.1 under stronger assumptions on Λ. In the following we will always assume that Λ is a closed subgroup of G. For a moment, let us even assume that $\Lambda = \Gamma^\perp$, where Γ is a closed, co-compact subgroup of $\widehat{G}$. Note that this implies that Λ is discrete and countable. For uniform lattice Gabor systems the condition $\Lambda = \Gamma^\perp$ is called *critical density* by Gröchenig [23] since Borel sections X and Ω of the lattices $\Gamma^\perp$ and $\Lambda^\perp$ in this case satisfy $m_G(X) m_{\widehat{G}}(\Omega) = 1$. Theorem 6.5.2 in [23] states that the uniform lattice Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ only can be frame for $L^2(G)$ if $m_G(X) m_{\widehat{G}}(\Omega) \le 1$. Clearly this is not a necessary condition when either Λ or Γ is non-discrete since, for closed, co-compact subgroups, a Borel section of its annihilator has finite measure if and only if the subgroup itself is discrete.

Now, back to the assumption $\Lambda = \Gamma^\perp$ with Γ being a (not necessarily discrete) closed, co-compact subgroup of $\widehat{G}$. In this case, the system in Proposition 4.1(ii) is a shift invariant system of the form $\{T_\lambda \varphi_x\}_{\lambda\in\Lambda}$ in $\ell^2(\Lambda)$ with countably many generators $\varphi_x := \{g(x+\alpha)\}_{\alpha\in\Lambda}$. We now apply the fiberization techniques from Section 3.1 with $G = \Lambda$ and $H = \Lambda$. Since the annihilator $H^\perp$ in this case is $\mathsf{A}(\widehat{\Lambda}, \Lambda) = \{1\}$, the fiberization map (3.1) is simply $\mathcal{T} f(\omega) = \{\hat{f}(\omega)\}$ for $\omega \in \Omega$, where Ω is a Borel section of $\{1\}$ in $\widehat{\Lambda}$, hence, $\Omega = \widehat{\Lambda}$. The Fourier transform of the generator $\varphi_x \in \ell^2(\Lambda)$ is

$$\hat{\varphi}_x(\omega) = \sum_{\alpha\in\Lambda} g(x+\alpha)\overline{\omega(\alpha)}, \tag{4.1}$$

which is the *Zak transform* $Z_\Lambda g(x, \omega)$ of g with respect to the discrete group $\Lambda \subset G$.

By Theorem 3.1 (or a result in [7], to be more precise), $\{T_\lambda \varphi_x\}_{\lambda\in\Lambda}$ is a basic frame in $\ell^2(\Lambda)$ with bounds A and B if, and only if, $\{\hat{\varphi}_x(\omega)\}$ is a basic frame in $\ell^2(\mathsf{A}(\widehat{\Lambda},\Lambda)) \cong \mathbb{C}$ with bounds A, B for almost all $\omega \in \widehat{\Lambda}$. Now, a scalar $\{\hat{\varphi}_x(\omega)\}$ is a basic frame in $\mathbb{C}$ with bounds A and B if, and only if, its norm squared, whenever non-zero, is bounded between A and B. We conclude that $\{E_\gamma T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ is a Gabor basic frame in $L^2(G)$ with bound A and B if, and only if,

$$A \le \Big|\sum_{\alpha\in\Lambda} g(x+\alpha)\overline{\omega(\alpha)}\Big|^2 \le B \quad \text{for a.e. } x\in X, \omega\in\Omega=\widehat{\Lambda} \text{ for which } \hat{\varphi}_x(\omega)\neq 0. \tag{4.2}$$

In particular, whenever $\Lambda = \Gamma^\perp$ with Γ being a closed, co-compact subgroup, we see that $\{E_\gamma T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ is a total Gabor frame for all of $L^2(G)$ if, and only if, $A \le |Z_\Lambda g(x,\omega)|^2 \le B$ for almost any $x\in X, \omega\in\Omega=\widehat{\Lambda}$. Still assuming $\Gamma=\Lambda^\perp$, this result can be shown to hold for any closed subgroup $\Lambda\subset G$ [2, Theorem 2.6]. However, the next result shows a non-existence phenomenon of such continuous Gabor frames.

Theorem 4.2. *Let $g\in L^2(G)$, let $0<A\le B<\infty$, and let Λ be a closed subgroup of G. Suppose that Λ is either discrete or co-compact. Then the following assertions are equivalent:*

(i) $\{E_\gamma T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ is a frame for $L^2(G)$ with bounds A and B,

(ii) The subgroup Λ is discrete and co-compact, hence a uniform lattice, and $\{E_\gamma T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ is a Riesz basis for $L^2(G)$ with bounds A and B.

Proof. The implication (ii)⇒(i) is trivial so we only have to consider (i)⇒(ii).

Assume first that the subgroup Λ is discrete. Then $\Gamma=\Lambda^\perp$ is co-compact. We use the notation from the paragraphs preceding Theorem 4.2. Then, as shown above, assertion (i) is equivalent to $\{\hat{\varphi}_x(\omega)\}$ being a frame for $\mathbb{C}$ for almost every $x\in X$, $\omega\in\widehat{\Lambda}$. However, a one element set is a frame if, and only if, it is a Riesz basis with the same bounds. Now, we repeat the argument above, but in the reverse direction using a Riesz sequence variant of Theorem 3.1. By [8, Theorem 4.3] the scalar $\{\hat{\varphi}_x(\omega)\}$ is a Riesz basis for $\mathbb{C}$ if, and only if, the SI system $\{T_\lambda\varphi_x\}_{\lambda\in\Lambda}$ is a Riesz basis in $\ell^2(\Lambda)$ with the same bounds. By a result in [7], which generalizes [8, Theorem 4.3], this is equivalent to $\{T_\gamma\mathcal{F}^{-1}T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ being a so-called *continuous* Riesz basis. However, as shown in [7] *continuous* Riesz sequences only exist if $\Lambda^\perp$ is discrete. Hence, $\{T_\gamma\mathcal{F}^{-1}T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ is actually a (discrete) Riesz basis. By unitarily equivalence, this implies that $\{E_\gamma T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ is a Riesz basis.

Assume now that Λ is co-compact. Then $\Gamma=\Lambda^\perp$ is discrete. Note that $\{T_\lambda E_\gamma g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ is unitarily equivalent to $\{E_\gamma T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ and repeat the argument above for the co-compact TI system $\{T_\lambda E_\gamma g\}_{\gamma\in\Lambda^\perp,\lambda\in\Lambda}$ □

Remark 2. In the extreme case $\Lambda=G$, Theorem 4.2 tell us that $\{T_\lambda g\}_{\lambda\in G}$ cannot be a frame for $L^2(G)$ *unless* G is discrete; if G is discrete, then $\widehat{G}$ is compact, and any $g\in L^2(G)$ with $0<A\le|\hat{g}(\omega)|^2\le B$ for a.e. $\omega\in\widehat{G}$ will generate a frame $\{T_\lambda g\}_{\lambda\in G}$ with bounds A, B. For discrete (irregular) Gabor systems in $L^2(\mathbb{R}^n)$ such questions are studied in [12]. On the other hand, totality in $L^2(G)$ of the set $\{T_\lambda g\}_{\lambda\in G}$ is achievable for both discrete and non-discrete LCA groups G; e.g., take any $g\in L^2(G)$ with $\hat{g}(\omega)\neq 0$ for a.e. $\omega\in\widehat{G}$.

Due to Theorem 4.2 we wish to relax the "critical" density condition $\Lambda=\Gamma^\perp$, but in such a way that we still can apply Zak transform methods. For regular Gabor systems

$$\left\{\mathrm{e}^{2\pi i\gamma x}\, g(x-\lambda) : \gamma\in\Gamma=A\mathbb{Z}^n, \lambda\in\Lambda=B\mathbb{Z}^n\right\} \tag{4.3}$$

in $L^2(\mathbb{R}^n)$ with $A, B \in \mathrm{GL}_n(\mathbb{R})$ rational density, where $A\mathbb{Z}^n \cap B\mathbb{Z}^n$ is a full-rank lattice, is such a relaxation; for $n = 1$ rational density simply means $AB = \frac{p}{q} \in \mathbb{Q}$. Our assumptions on the subgroups Λ and Γ in the remainder of this section will mimic the setup of rational density, and the characterization will depend on a vector-valued Zak transform similar to the case of $L^2(\mathbb{R}^n)$ [6,40,47].

For a closed subgroup H of G the Zak transform Z_H as introduced by Weil, albeit not under this name, of a continuous function $f \in C_c(G)$ is:

$$Z_H f(x, \omega) = \int_H f(x+h)\overline{\omega(h)}\, dh \qquad \text{for a.e. } x \in X,\ \omega \in \widehat{H}.$$

The Zak transform extends to a unitary operator from $L^2(G)$ onto $L^2(G/H \times \widehat{G}/H^\perp)$ [2,44]. We will use the Zak transform for discrete subgroups $H = \Gamma^\perp$, where Γ is co-compact, in which case, the convergence of the series $Z_H f(x, \alpha) = \sum_{\alpha \in \Gamma^\perp} f(x+\alpha)\overline{\omega(\alpha)}$ is in the L^2-norm for a.e. x and ω.

The next result shows that the frame property of a Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ in $L^2(G)$ under certain assumptions of Λ and Γ is equivalent with the frame property of a family of associated Zak transformed variants of the Gabor system in $\mathbb{C}^p$.

Theorem 4.3. *Let $g \in L^2(G)$, and let $0 < A \le B < \infty$. Let Γ be a closed, co-compact subgroup of $\widehat{G}$. Suppose that Λ is a closed subgroup of G such that $p := \left|\Gamma^\perp/(\Lambda\cap\Gamma^\perp)\right| < \infty$. Let $\{\chi_1, \dots, \chi_p\} := \mathsf{A}(\widehat{\Gamma^\perp}, \Lambda\cap\Gamma^\perp)$. Equip Λ with some Haar measure μ_Λ, and let $\mu_{\Lambda/(\Lambda\cap\Gamma^\perp)}$ be the unique Haar measure on $\Lambda/(\Lambda\cap\Gamma^\perp)$ such that for all $f \in L^1(\Lambda)$*

$$\int_\Lambda f(x)\, d\mu_\Lambda(x) = p \int_{\Lambda/(\Lambda\cap\Gamma^\perp)} \sum_{\ell\in\Lambda\cap\Gamma^\perp} f(x+\ell)\, d\mu_{\Lambda/(\Lambda\cap\Gamma^\perp)}(\dot{x}).$$

Also, we let $K \subset \Lambda$ denote a Borel section of $\Lambda\cap\Gamma^\perp$ in Λ and μ_K be a measure on K isometric to $\mu_{\Lambda/(\Lambda\cap\Gamma^\perp)}$ in the sense of (2.2)*. Then the following assertions are equivalent:*

(i) $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ *is a frame for $L^2(G)$ with bounds A and B,*

(ii) $A\,\|c\|^2_{\mathbb{C}^p} \le \int_K |\langle c, \{Z_{\Gamma^\perp} g(x+\kappa, \omega\chi_i)\}_{i=1}^p\rangle_{\mathbb{C}^p}|^2\, d\mu_K(\kappa) \le B\,\|c\|^2_{\mathbb{C}^p}$ *for all $c \in \mathbb{C}^p$, a.e. $x \in X$ and $\omega \in \widehat{\Gamma^\perp}$, where X is a Borel section of $\Gamma^\perp$ in G,*

(iii)

$$A \le \operatorname*{ess\,inf}_{(x,\omega)\in X\times\widehat{\Gamma^\perp}} \lambda_p(x,\omega), \quad B \ge \operatorname*{ess\,sup}_{(x,\omega)\in X\times\widehat{\Gamma^\perp}} \lambda_1(x,\omega),$$

where $\lambda_i(x,\omega)$ denotes the i-th largest eigenvalue value of the $p\times p$ matrix $\tilde{\mathcal{G}}(x,\omega)$, whose (i,j)-th entry is

$$\tilde{\mathcal{G}}(x,\omega)_{(i,j)} = \int_K Z_{\Gamma^\perp} g(x+\kappa, \omega\chi_i)\overline{Z_{\Gamma^\perp} g(x+\kappa, \omega\chi_j)} d\mu_K(\kappa).$$

Proof. We first remark that $\mathsf{A}(\widehat{\Gamma^\perp}, \Lambda\cap\Gamma^\perp) \cong \Gamma^\perp/(\Lambda\cap\Gamma^\perp)$ by Lemma 2.1. This shows that $\{\chi_1, \dots, \chi_p\}$ is well-defined due to the assumption $p = \left|\Gamma^\perp/(\Lambda\cap\Gamma^\perp)\right| < \infty$.

By Proposition 4.1, assertion (i) is equivalent to the sequence $\left\{\{g(x+\lambda+\alpha)\}_{\alpha\in\Gamma^\perp}\right\}_{\lambda\in\Lambda}$ being a frame for $\ell^2(\Gamma^\perp)$ with bounds A and B for a.e. $x \in X$. Since $\Lambda\cap\Gamma^\perp$ is a subgroup of Λ, every $\lambda \in \Lambda$ can be written in a unique way as $\lambda = \mu+\kappa$ with $\mu \in \Lambda\cap\Gamma^\perp$ and $\kappa \in \Lambda/(\Lambda\cap\Gamma^\perp)$. Letting

$\varphi_\kappa := \{g(x+\alpha+\kappa)\}_{\alpha\in\Gamma^\perp}$, we can write the above sequence as $\{T_\mu\varphi_\kappa\}_{\mu\in\Lambda\cap\Gamma^\perp,\kappa\in\Lambda/(\Lambda\cap\Gamma^\perp)}$. By assumption, this is a co-compact translation invariant system in $\ell^2(\Gamma^\perp)$. The Fourier transform of $\varphi_\kappa \in \ell^2(\Gamma^\perp)$ is

$$\hat{\varphi}_\kappa(\omega) = \sum_{\alpha\in\Gamma^\perp} g(x+\alpha+\kappa)\overline{\omega(\alpha)} \quad \text{for a.e. } \omega \in \widehat{\Gamma^\perp},$$

hence $\hat{\varphi}_\kappa(\omega) = Z_{\Gamma^\perp} g(x+\kappa,\omega)$. As above, we apply the fiberization techniques from Section 3.1 with $G = \Gamma^\perp$ and $H = \Lambda\cap\Gamma^\perp$. The relationship between the measures via Weil's formula in the assumption guarantees that the subgroups are equipped with the correct measures. Since the annihilator $H^\perp$ in this case is $\mathsf{A}(\widehat{\Gamma^\perp},\Lambda\cap\Gamma^\perp)$, the fiberization map (3.1) is $\mathcal{T}f(\omega) = \{\hat{f}(\omega\chi)\}_{\chi\in\mathsf{A}(\widehat{\Gamma^\perp},\Lambda\cap\Gamma^\perp)}$ for $\omega\in\widehat{\Gamma^\perp}$. By Theorem 3.1, we see that assertion (i) is equivalent to the system

$$\Big\{\{Z_{\Gamma^\perp} g(x+\kappa,\omega\chi)\}_{\chi\in\mathsf{A}(\widehat{\Gamma^\perp},\Lambda\cap\Gamma^\perp)}\Big\}_{\kappa\in\Lambda/(\Lambda\cap\Gamma^\perp)}$$

being a frame in $\ell^2(\mathsf{A}(\widehat{\Gamma^\perp},\Lambda\cap\Gamma^\perp)) \cong \mathbb{C}^p$ with bounds A and B for a.e. $x\in X$ and $\omega\in\widehat{\Gamma^\perp}$. This proves (i)$\Leftrightarrow$(ii).

The dual Gramian matrix $\tilde{\mathcal{G}}(x,\omega)$ is a matrix representation of the frame operator of the system in (ii) which shows the equivalence (ii)$\Leftrightarrow$(iii). □

Under the assumption $p = \left|\Gamma^\perp/(\Lambda\cap\Gamma^\perp)\right| < \infty$, we can view $\{Z_{\Gamma^\perp} g(x+\kappa,\omega\chi)\}_{\chi\in\mathsf{A}(\widehat{\Gamma^\perp},\Lambda\cap\Gamma^\perp)}$ as a column vector in $\mathbb{C}^p$. This vector is sometimes called a vector-valued Zak transform of g. We remark that the quotient group $\Lambda/(\Lambda\cap\Gamma^\perp)$ in Theorem 4.3 can be infinite, even uncountably infinite. If it is finite, however, we have the following simplification.

Corollary 4.4. *In addition to the assertions in Theorem 4.3 assume that Λ is discrete, $q := \left|\Lambda/(\Lambda\cap\Gamma^\perp)\right| < \infty$ and let Λ be equipped with the counting measure. Let κ_i, $i = 1,\dots,q$, be a set of coset representatives of $\Lambda/(\Lambda\cap\Gamma^\perp)$, and let $\{\chi_1,\dots,\chi_p\} := \mathsf{A}(\widehat{\Gamma^\perp},\Lambda\cap\Gamma^\perp)$. Then the following assertions are equivalent.*

- *(i)* $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ *is a frame for $L^2(G)$ with bounds A and B,*
- *(ii)* $\{\{Z_{\Gamma^\perp} g(x+\kappa_i,\omega\chi_j)\}_{j=1}^p\}_{i=1}^q$ *is a frame for $\mathbb{C}^p$ w.r.t. p^{-1} times the counting measure, i.e.,*
  $A\left\|c\right\|_{\mathbb{C}^p}^2 \le \frac{1}{p}\sum_{i=1}^q \left|\langle c, \{Z_{\Gamma^\perp} g(x+\kappa_i,\omega\chi_j)\}_{j=1}^p\rangle_{\mathbb{C}^p}\right|^2 \le B\left\|c\right\|_{\mathbb{C}^p}^2$ *for all $c\in\mathbb{C}^p$,*
  for a.e. $x\in X$ and $\omega\in\widehat{\Gamma^\perp}$, where X is a Borel section of $\Gamma^\perp$ in G,
- *(iii)*
  $$A \le p^{-1} \operatorname*{ess\,inf}_{(x,\omega)\in X\times\widehat{\Gamma^\perp}} \sigma_p(x,\omega)^2, \quad B \ge p^{-1} \operatorname*{ess\,sup}_{(x,\omega)\in X\times\widehat{\Gamma^\perp}} \sigma_1(x,\omega)^2,$$
  where $\sigma_k(x,\omega)$ denotes the k-th largest singular value of the $q\times p$ matrix $\Phi(x,\omega)$, whose (i,j)-th entry is $Z_{\Gamma^\perp} g(x+\kappa_i,\omega\chi_j)$.

The matrix $p^{-1/2}\Phi(x,\omega)$ is called the Zibulski-Zeevi representation; it is the transpose of the matrix representation of the synthesis operator associated with the frame in Corollary 4.4(ii). This shows that the Zibulski-Zeevi representation is possible for Gabor systems with translation along a discrete (but not necessarily co-compact) subgroup $\Lambda\subset G$ and modulation along a co-compact (but not necessarily discrete) subgroup $\Gamma\subset\widehat{G}$.

For lattice Gabor systems (4.3) in $L^2(\mathbb{R}^n)$, Corollary 4.4 reduces to [6, Theorem 4.1]. We remark that, in this case, the roles of p and q are the same as in [6, Theorem 4.1] which can be seen by an application of the second isomorphism theorem

$$p = \left|\Gamma^\perp/(\Lambda \cap \Gamma^\perp)\right|, \qquad q = \left|\Lambda/(\Lambda \cap \Gamma^\perp)\right| = \left|(\Lambda + \Gamma^\perp)/\Gamma^\perp\right|,$$

and by noting that Γ is assumed to be $\mathbb{Z}^n$ in [6]. In particular, for regular Gabor systems in $L^2(\mathbb{R})$ with time and frequency shift parameters a and b, we have $ab = p/q \in \mathbb{Q}$, where p and q are relative prime.

Using range functions, the equivalence of (i) and (ii) in all results in this subsection can be formulated for basic frames. For Corollary 4.4 this simply reads: $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a basic frame in $L^2(G)$ if, and only if, $\{\{Z_{\Gamma^\perp} g(x+\kappa_i, \omega\chi_j)\}_{j=1}^p\}_{i=1}^q$ is a basic frame in $\mathbb{C}^p$. In the following Example 1 we apply this version of Corollary 4.4 to a non-discrete Gabor system and calculate its Zibulski-Zeevi representation.

Example 1. Let $r \in \mathbb{N}$ be prime. We consider Gabor systems $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ in $L^2(\mathbb{Z}(r^\infty))$, where the Prüfer r-group $G = \mathbb{Z}(r^\infty)$, the discrete group of all r^n-roots of unity for all $n \in \mathbb{N}$, is equipped with the discrete topology and multiplication as group operation. Its dual group can be identified with the r-adic integers $\widehat{G} = \mathbb{I}_r$. For $m, n \in \mathbb{N}$ define $\Lambda \subset \mathbb{Z}(r^\infty)$ and $\Gamma^\perp \subset \mathbb{Z}(r^\infty)$ as all r^n and r^m roots of unity, respectively. Then Λ is a discrete, closed subgroup of $\mathbb{Z}(r^\infty)$, and Γ is a co-compact, closed subgroup of $\mathbb{I}_r$. Note that neither Λ nor Γ are uniform lattices. Let X and Ω denote Borel sections of the subgroups $\Gamma^\perp \subset G$ and $\Lambda^\perp \subset \widehat{G}$, respectively. For any $n, m \in \mathbb{N}$, we have $m_G(X) m_{\widehat{G}}(\Omega) = \infty$. Moreover,

$$p = \left|\Gamma^\perp/(\Lambda\cap\Gamma^\perp)\right| = r^{m-\min\{m,n\}}, \qquad q = \left|\Lambda/(\Lambda\cap\Gamma^\perp)\right| = r^{n-\min\{m,n\}}.$$

If $m \geq n$, then $p = r^{m-n}$, $q = 1$, and the Zibulski-Zeevi representation is (up to scaling of $p^{-1/2}$) given as a (row) vector of length p:

$$\Phi(x,\omega) = \{Z_{\Gamma^\perp} g(x, \omega\chi_j)\}_{j=1}^p,$$

where $\{\chi_j\}_{j=1}^p = \mathsf{A}(\widehat{\Gamma^\perp}, \Lambda)$. On the other hand, if $n \geq m$, then $p = 1$, $q = r^{n-m}$, and the Zibulski-Zeevi representation $\Phi(x,\omega) = \{Z_{\Gamma^\perp}(x+\kappa_i, \omega)\}_{i=1}^q$ is a (column) vector of length q, where $\{\kappa_i\}_{i=1}^q$ is a set of coset representatives of $\Lambda/\Gamma^\perp$.

Thus, for any $m, n \in \mathbb{N}$, the system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame for its closed linear span, i.e., a basic frame in $L^2(\mathbb{Z}(r^\infty))$, with bounds A and B if, and only if,

$$A \leq \frac{1}{p}\left\|\Phi(x,\omega)\right\|^2 \leq B$$

for almost every $x \in X$ and $\omega \in \widehat{\Gamma^\perp}$ for which $\|\Phi(x,\omega)\| \neq 0$, where $\Phi(x,\omega)$ is given as above.

Remark 3. As an alternative to the Zak transform decomposition of g used above in part (ii) of Theorem 4.3 and Corollary 4.4, we can use a less time-frequency symmetric variant. The details are as follows. By a unitary transform on $\mathbb{C}^p$ the vector $\left\{1/\sqrt{p} Z_{\Gamma^\perp} g(x+\kappa, \omega\chi_i)\right\}_{i=1}^p$ is mapped to the vector

$$\psi_\kappa(x,\omega) := \left\{ \sum_{\alpha\in\Lambda\cap\Gamma^\perp} g(x+\alpha+\kappa+\ell_i)\overline{\omega(\alpha)} \right\}_{i=1}^p, \tag{4.4}$$

where ℓ_i, $i = 1, \dots, p$, are distinct coset representatives of $\Gamma^\perp/(\Lambda \cap \Gamma^\perp)$, and $\kappa \in K$. The assertions in Theorem 4.3 are, therefore, equivalent with

$$A\,\|c\|^2_{\mathbb{C}^p} \le \int_K |\langle c, \psi_\kappa(x,\omega)\rangle_{\mathbb{C}^p}|^2 \, d\mu_K(\kappa) \le B\,\|c\|^2_{\mathbb{C}^p} \quad \text{for all } \ c \in \mathbb{C}^p,$$

a.e. $x \in X$ and $\omega \in \widehat{\Gamma^\perp}$, where X is a Borel section of $\Gamma^\perp$ in G. Here μ_K is the measure on K isometric to $\mu_{\Lambda/(\Lambda\cap\Gamma^\perp)}$ (in the sense of (2.2)) such that for all $f \in L^1(\Lambda)$

$$\int_\Lambda f(x)\, d\mu_\Lambda(x) = \int_{\Lambda/(\Lambda\cap\Gamma^\perp)} \sum_{\ell \in \Lambda\cap\Gamma^\perp} f(x+\ell)\, d\mu_{\Lambda/(\Lambda\cap\Gamma^\perp)}(\dot{x});$$

note that this is different from the measure μ_K used in Theorem 4.3. Then the assertions in Corollary 4.4 are equivalent to the fact that

$$A\,\|c\|^2_{\mathbb{C}^p} \le \sum_{i=1}^{q} |\langle c, \psi_{\kappa_i}(x,\omega)\rangle_{\mathbb{C}^p}|^2 \le B\,\|c\|^2_{\mathbb{C}^p} \quad \text{for all } \ c \in \mathbb{C}^p,$$

for a.e. $x \in X$ and $\omega \in \widehat{\Gamma^\perp}$, where X is a Borel section of $\Gamma^\perp$ in G.

If we switch the assumptions on Λ and Γ and consider TI systems of the form $\{T_\lambda E_\gamma g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$, we obtain the following variant of Proposition 4.1.

Proposition 4.5. *Let $g \in L^2(G)$, and let $0 < A \le B < \infty$. Let Λ be a closed, co-compact subgroup of G, and let $(\Gamma, \Sigma_\Gamma, \mu_\Gamma)$ be an admissible measure space in $\widehat{G}$. The following assertions are equivalent:*

- *(i) $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ is a frame for $L^2(G)$ with bounds A and B,*
- *(ii) $\left\{\{\hat{g}(\omega\gamma\beta)\}_{\beta\in\Lambda^\perp}\right\}_{\gamma\in\Gamma}$ is a frame for $\ell^2(\Lambda^\perp)$ with bounds A and B for a.e. $\omega \in \Omega$, where Ω is a Borel section of $\Lambda^\perp$ in G.*

From Proposition 4.5 we get the following variant of Theorem 4.3; we leave the corresponding formulation of Corollary 4.4 to the reader.

Theorem 4.6. *Let $g \in L^2(G)$, and let $0 < A \le B < \infty$. Let Λ be a closed, co-compact subgroup of G. Suppose that Γ is a closed subgroup of $\widehat{G}$ such that $p := \left|\Lambda^\perp/(\Gamma\cap\Lambda^\perp)\right| < \infty$. Let $\{\chi_1, \dots, \chi_p\} := \mathsf{A}(\widehat{\Lambda^\perp}, \Gamma\cap\Lambda^\perp)$. Equip Γ with some Haar measure μ_Γ, and let $\mu_{\Gamma/(\Gamma\cap\Lambda^\perp)}$ be the unique Haar measure over $\Gamma/(\Gamma\cap\Lambda^\perp)$ such that for all $f \in L^1(\Gamma)$*

$$\int_\Gamma f(x)\, d\mu_\Gamma(x) = p \int_{\Gamma/(\Gamma\cap\Lambda^\perp)} \sum_{\ell\in\Gamma\cap\Lambda^\perp} f(x+\ell)\, d\mu_{\Gamma/(\Gamma\cap\Lambda^\perp)}(\dot{x}).$$

Also, we let $K \subset \Gamma$ denote a Borel section of $\Gamma\cap\Lambda^\perp$ in Γ and μ_K be a measure on K isometric to $\mu_{\Gamma/(\Gamma\cap\Lambda^\perp)}$ in the sense of (2.2). Then the following assertions are equivalent:

- *(i) $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ is a frame for $L^2(G)$ with bounds A and B,*
- *(ii) $A\,\|c\|^2_{\mathbb{C}^p} \le \int_K |\langle c, \{Z_{\Lambda^\perp}\hat{g}(\omega\kappa, x+\chi_i)\}_{i=1}^p\rangle_{\mathbb{C}^p}|^2\, d\mu_K(\kappa) \le B\,\|c\|^2_{\mathbb{C}^p}$ for all $c \in \mathbb{C}^p$, a.e. $\omega \in \Omega$ and $x \in \widehat{\Lambda^\perp}$, where Ω is a Borel section of $\Lambda^\perp$ in $\widehat{G}$.*

4.2 Characterizations of dual Gabor frames

By a result on so-called characterizing equations from [29], we now characterize when two semi co-compact Gabor systems are dual frames. Using the equivalence of frame properties for systems $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ and $\{T_\gamma \mathcal{F}^{-1} T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ with generator $g \in L^2(G)$ yields the following characterizing equations in the time domain.

Theorem 4.7 ([29])**.** *Let Γ be a closed, co-compact subgroup of $\widehat{G}$, and let $(\Lambda, \Sigma_\Lambda, \mu_\Lambda)$ be an admissible measure space in G. Suppose that the two systems $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ and $\{E_\gamma T_\lambda h\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ are Bessel systems. Then the following statements are equivalent:*

(i) $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ and $\{E_\gamma T_\lambda h\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ are dual frames for $L^2(G)$,

(ii) for each $\alpha \in \Gamma^\perp$ we have

$$s_\alpha(x) := \int_\Lambda \overline{g(x-\lambda-\alpha)} h(x-\lambda)\, d\mu_\Lambda(\lambda) = \delta_{\alpha,0} \quad a.e.\ x \in G, \tag{4.5}$$

If we want to stress the dependence of the generators g and h in (4.5), we use the notation $s_{g,h,\alpha} : G \to \mathbb{C}$.

Corollary 4.8. *Let Γ be a closed, co-compact subgroup of $\widehat{G}$, and let $(\Lambda, \Sigma_\Lambda, \mu_\Lambda)$ be an admissible measure space in G. The family $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ is an A-tight frame for $L^2(G)$ if and only if $s_{g,g,\alpha}(x) = A\,\delta_{\alpha,0}$ a.e. for each $\alpha \in \Gamma^\perp$.*

Example 2. Let $g \in L^2(G)$ and consider $\{E_\gamma T_\lambda g\}_{\gamma\in\widehat{G},\lambda\in\Lambda}$, where $(\Lambda, \Sigma_\Lambda, \mu_\Lambda)$ be an admissible measure space in G. By Corollary 4.8 we see that $\{E_\gamma T_\lambda g\}_{\gamma\in\widehat{G},\lambda\in\Lambda}$ is a Parseval frame for $L^2(G)$ if, and only if, for a.e. $x \in G$

$$\int_\Lambda |g(x-\lambda)|^2\, d\mu_\Lambda(\lambda) = 1. \tag{4.6}$$

If we take $\Lambda = G$ with the Haar measure, then equation (4.6) becomes simply $\|g\| = 1$ which is the well-known inversion formula for the short-time Fourier transform [23, 24].

Suppose now that G contains a uniform lattice. Take Λ as a uniform lattice in G, and let X denote a (relatively compact) Borel section of Λ in G. Equation (4.6) becomes

$$\sum_{\lambda\in\Lambda} |g(x-\lambda)|^2 = |X|^{-1}.$$

Let $g_1, \dots, g_r \in L^2(G)$ be functions positive on X with support $\operatorname{supp} g_i \subset \overline{X}$ so that g_i is constant on X for at least one index i. Following [13], the function on G defined by the r-fold convolution

$$W_r := g_1 \mathbb{1}_X * g_2 \mathbb{1}_X * \dots * g_r \mathbb{1}_X$$

is called a weighted B-spline of order r. As shown in [13], the function W_r is non-negative and satisfies a partition of unity condition up to a constant, say $\sum_{\lambda\in\Lambda} W_r(x-\lambda) = C_r$. Take $g \in L^2(G)$ so that

$$|g(x)|^2 = \frac{1}{C_r\,|X|} W_r(x), \quad \text{e.g.,} \quad g(x) = \frac{1}{(C_r\,|X|)^{1/2}} \sqrt{W_r(x)}.$$

Then $\{E_\gamma T_\lambda g\}_{\gamma\in\widehat{G},\lambda\in\Lambda}$ is a Parseval frame.

Viewing Gabor systems as unitarily equivalent to $\{T_\lambda E_\gamma g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$, we arrive at characterizing equations for duality in the frequency domain.

Theorem 4.9 ([29]). *Let Λ be a closed, co-compact subgroup of G, and let $(\Gamma, \Sigma_\Gamma, \mu_\Gamma)$ be an admissible measure space in $\widehat{G}$. Suppose that the two systems $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ and $\{E_\gamma T_\lambda h\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ are Bessel systems. Then the following statements are equivalent:*

(i) $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ and $\{E_\gamma T_\lambda h\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ are dual frames for $L^2(G)$,

(ii) for each $\beta \in \Lambda^\perp$ we have

$$t_\beta(\omega) := \int_\Gamma \overline{\hat{g}(\omega\gamma^{-1}\beta^{-1})}\hat{h}(\omega\gamma^{-1})\, d\mu_\Gamma(\gamma) = \delta_{\beta,1} \quad a.e.\ \omega \in \widehat{G}. \tag{4.7}$$

As for $s_{g,h,\alpha}$ we write $t_{g,h,\beta} : \widehat{G} \to \mathbb{C}$ for t_β in (4.7) if we want to stress the dependence of the generators g and h.

Corollary 4.10. *Let Λ be a closed, co-compact subgroup of G, and let $(\Gamma, \Sigma_\Gamma, \mu_\Gamma)$ be an admissible measure space in $\widehat{G}$. The family $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ is an A-tight frame for $L^2(G)$ if and only if $t_{g,g,\beta}(x) = A\,\delta_{\beta,1}$ a.e. for each $\beta \in \Lambda^\perp$.*

Let us now consider co-compact Gabor systems, i.e., we take both Λ and Γ to be closed, co-compact subgroups. We first remark that in this case, under the Bessel system assumption, we have equivalence of conditions (4.5) and (4.7). More importantly, $s_{g,h,\alpha}$ and $t_{g,h,\beta}$ can be written as a Fourier series.

Remark 4. (i) For $g, h \in L^2(G)$ assume that two co-compact Gabor systems $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ and $\{E_\gamma T_\lambda h\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ are Bessel systems with bounds B_g and B_h, respectively. By an application of Cauchy-Schwarz' inequality and [29, Proposition 3.3], we see that $s_{g,h,\alpha} \in L^\infty(G)$; to be precise:

$$|s_{g,h,\alpha}(x)| \le B_g^{1/2} B_h^{1/2} \quad \text{for a.e. } x \in G.$$

(ii) Note that $s_{g,h,\alpha} : G \to \mathbb{C}$ is Λ-periodic. Furthermore, G/Λ is compact and s_α is uniformly bounded, we can therefore consider $s_{g,h,\alpha}$ as a function in $L^2(G/\Lambda)$ and its Fourier series is given by

$$s_{g,h,\alpha}(x) = \sum_{\beta\in\Lambda^\perp} c_{\alpha,\beta}\beta(x) \quad \text{with } c_{\alpha,\beta} = \int_{G/\Lambda} s_{g,h,\alpha}(\dot{x})\overline{\beta(\dot{x})}\, d\dot{x}.$$

We can compute the Fourier coefficients $c_{\alpha,\beta}$ directly using Weil's formula:

$$\begin{aligned} c_{\alpha,\beta} &= \int_{G/\Lambda} s_{g,h,\alpha}(\dot{x})\overline{\beta(\dot{x})}\, d\dot{x} = \int_{G/\Lambda}\int_\Lambda \overline{g(x-\lambda-\alpha)}\, h(x-\lambda)\, \overline{\beta(x-\lambda)}\, d\lambda\, d\dot{x} \\ &= \int_G h(x)\, \overline{\beta(x) g(x-\alpha)}\, dx = \langle h, E_\beta T_\alpha g\rangle. \end{aligned} \tag{4.8}$$

(iii) Similarly, we find $t_{g,h,\beta}(\omega) = \sum_{\alpha\in\Gamma^\perp} \langle \hat{h}, E_\alpha T_\beta \hat{g}\rangle\, \omega(\alpha)$.

5 The frame operator of Gabor systems

Let us begin with the definition of the frame operator. Let $g \in L^2(G)$, and let $\Lambda \subset G$, $\Gamma \subset \widehat{G}$ be closed subgroups. If $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a Bessel system, the frame operator introduced in (2.6) reads:

$$S \equiv S_{g,g} : L^2(G) \to L^2(G), \quad S = \int_\Gamma \int_\Lambda \langle \cdot\,, E_\gamma T_\lambda g\rangle E_\gamma T_\lambda g \, d\lambda\, d\gamma,$$

given weakly by

$$\langle S f_1, f_2\rangle = \int_\Gamma \int_\Lambda \langle f_1, E_\gamma T_\lambda g\rangle \langle E_\gamma T_\lambda g, f_2\rangle \, d\lambda\, d\gamma \quad \forall f_1, f_2 \in L^2(G).$$

Similarly, for two Gabor Bessel systems generated by the functions $g, h \in L^2(G)$, we introduce the operator

$$S_{g,h} : L^2(G) \to L^2(G), \quad S_{g,h} = \int_\Gamma \int_\Lambda \langle \cdot\,, E_\gamma T_\lambda g\rangle E_\gamma T_\lambda h \, d\lambda\, d\gamma. \tag{5.1}$$

We follow the Gabor theory tradition, referring to this operator as a (mixed) frame operator. If we want to emphasize the role of Λ and Γ, we denote this operator $S_{g,h,\Lambda,\Gamma}$, where Λ specifies the translation subgroup and Γ the modulation subgroup.

As in Gabor theory on $L^2(\mathbb{R}^n)$, it is straightforward to show that the frame operator commutes with time-frequency shifts with respect to the groups Λ and Γ.

Lemma 5.1. *Suppose that Γ and Λ are closed subgroups. Let $g, h \in L^2(G)$ and let $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$, $\{E_\gamma T_\lambda h\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ be Bessel systems. Then, for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$, the following holds:*

(i) $S_{g,h} E_\gamma T_\lambda = E_\gamma T_\lambda S_{g,h}$,

(ii) If $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame, then

$$S^{-1} E_\gamma T_\lambda = E_\gamma T_\lambda S^{-1}.$$

Lemma 5.1 implies that the canonical dual of a Gabor frame again is a Gabor system of the form $\{E_\gamma T_\lambda h\}_{\lambda\in\Lambda,\gamma\in\Gamma}$, where $h = S^{-1} g$. Finally, we note that by a direct application of the Plancherel theorem, one can show that for all $f_1, f_2 \in L^2(G)$,

$$\langle S_{g,h,\Lambda,\Gamma} f_1, f_2\rangle = \langle S_{\hat{g},\hat{h},\Gamma,\Lambda} \hat{f}_1, \hat{f}_2\rangle,$$

where Λ and Γ are only assumed to be measure spaces.

5.1 Feichtinger's algebra

In applications of our results, one often needs to show that the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ generated by $g \in L^2(G)$ constitutes a Bessel family. This task, however, can be non-trivial, and even if g generates a Bessel system for subgroups Λ_1 and Γ_1, it may not generate a Bessel system for another pair of translation and modulation groups Λ_2 and Γ_2. A solution to this problem is to consider functions in the *Feichtinger algebra* $S_0(G)$. It follows from [17, Theorem 3.3.1] that Gabor systems $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ with respect to any two uniform lattices Λ and Γ in $\mathbb{R}^n$ generated by functions in $S_0(\mathbb{R}^n)$ are Bessel systems. The proof relies on properties of the Wiener-Amalgam spaces. The purpose of this section is to give an alternate proof in the setting of LCA groups that any $g \in S_0(G)$ generates a Bessel system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ for any two closed subgroups $\Lambda \subset G$ and $\Gamma \subset \widehat{G}$.

Let $g \in C_c(G)$ be a non-zero function with $\mathcal{F}g \in L^1(\widehat{G})$. The Feichtinger algebra $S_0(G)$ is then defined as follows:

$$S_0(G) := \big\{ f : G \to \mathbb{C} \,:\, f \in L^1(G) \text{ and } \int_G \int_{\widehat{G}} |\mathcal{V}_g f(x,\omega)| \, d\omega \, dx < \infty \big\},$$

where $\mathcal{V}_g f(x,\omega) := \int_G f(t)\overline{\omega(t) g(t-x)} \, dt$ is the short time Fourier transform of f with the window g. Equipped with the norm $\|f\|_{S_0} := \int_{G\times\widehat{G}} |\mathcal{V}_g f(x,\omega)| d\omega dx$, the function space $S_0(G)$ is a Fourier-invariant Banach space that is dense in $L^2(G)$ and whose members are continuous and integrable functions. Moreover, $S_0(G)$ is continuously embedded in $L^1(G)$, that is, there exists a constant $C > 0$ such that

$$\|f\|_{L^1(G)} \le C \|f\|_{S_0(G)} \quad \text{for all } f \in S_0(G).$$

If $g, h \in S_0(G)$, then $\mathcal{V}_g h \in S_0(G \times \widehat{G})$. Furthermore, for any closed subgroup $H \subset G$ the *restriction mapping*

$$\mathcal{R}_H : S_0(G) \to S_0(H), \; (\mathcal{R}_H f)(x) := f(x), \quad x \in H$$

is a surjective, bounded and linear operator. We refer the reader to [16, 17, 21] for a detailed introduction to $S_0(G)$.

In order to prove Theorem 5.4, we need the following two results. Lemma 5.2 relies on properties (ii) and (iv) from above, whereas Lemma 5.3 is an adaptation of [29, Lemma 2.2].

Lemma 5.2. *Let H be a closed subgroup in G and let $a \in G$, $g \in S_0(G)$. Then there exists some constant $K_H > 0$ which depends on H such that*

$$\int_H |g(x-a)| \, d\mu_H(x) \le K_H \, \|g\|_{S_0(G)} \quad \textit{for all } a \in G.$$

Proof. The result follows from the fact that $S_0(H)$ is continuously embedded in $L^1(H)$ and the boundedness of the restriction mapping:

$$\begin{aligned} \int_H |g(x-a)| \, d\mu_H(x) &= \|\mathcal{R}_H(T_a g)\|_{L^1(H)} \le C \, \|\mathcal{R}_H(T_a g)\|_{S_0(H)} \\ &\le C C_H \, \|T_a g\|_{S_0(G)} = C C_H \, \|g\|_{S_0(G)}. \end{aligned}$$

Here we also used that the S_0-norm is invariant under translation. Now take $K_H = CC_H$. □

Lemma 5.3. *Let $g \in L^2(G)$ and $\Gamma \subset \widehat{G}$ be a closed subgroup. For all $f \in C_c(G)$*

$$\int_\Gamma |\langle f, E_\gamma T_\lambda g\rangle|^2 \, d\mu_\Gamma(\gamma) = \int_G \int_{\Gamma^\perp} f(x)\overline{f(x-\alpha)}\,\overline{T_\lambda g(x)}\, T_\lambda g(x-\alpha) \, d\mu_{\Gamma^\perp}(\alpha) \, d\mu_G(x). \tag{5.2}$$

With these results in hand, we can prove that functions in $S_0(G)$ always generate Gabor Bessel systems.

Theorem 5.4. *Let $g \in S_0(G)$ and let $\Lambda \subset G$ and $\Gamma \subset \widehat{G}$ be closed subgroups. Then $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a Bessel system with bound $B = K_{\Lambda,\Gamma} \, \|g\|^2_{S_0(G)}$, where $K_{\Lambda,\Gamma}$ is a constant that only depends on Λ and Γ.*

Proof. From Lemma 5.3 follows that for all $f \in C_c(G)$:

$$\begin{aligned}
&\int_\Lambda \int_\Gamma |\langle f, E_\gamma T_\lambda g\rangle|^2 \, d\gamma \, d\lambda \\
&= \int_\Lambda \int_G \int_{\Gamma^\perp} f(x) \overline{f(x-\alpha)} \, \overline{T_\lambda g(x)} \, T_\lambda g(x-\alpha) \, d\alpha \, dx \, d\lambda \\
&= \int_\Lambda \int_{G/\Gamma^\perp} \int_{\Gamma^\perp} \int_{\Gamma^\perp} g(x-\lambda-\alpha) \, \overline{f(x-\alpha)} \, \overline{g(x-\lambda-\alpha')} \, f(x-\alpha') \, d\alpha \, d\alpha' \, d\dot{x} \, d\lambda.
\end{aligned}$$

In the latter equality we used Weil's formula and a change of variables $\alpha + \alpha' \mapsto \alpha$. An application of the triangle inequality and the Cauchy-Schwarz inequality now yields the following estimate:

$$\begin{aligned}
&\int_\Lambda \int_\Gamma |\langle f, E_\gamma T_\lambda g\rangle|^2 \, d\mu_\Gamma(\gamma) \, d\mu_\Lambda(\lambda) \\
&\le \int_{G/\Gamma^\perp} \int_{\Gamma^\perp} \int_{\Gamma^\perp} \big| f(x-\alpha) f(x-\alpha') \big| \int_\Lambda \big| g(x-\lambda-\alpha) g(x-\lambda-\alpha') \big| \, d\lambda \, d\alpha \, d\alpha' d\dot{x} \\
&\le \int_{G/\Gamma^\perp} \Big(\int_{\Gamma^\perp} \big| f(x-\alpha) \big|^2 \int_\Lambda \int_{\Gamma^\perp} \big| g(x-\lambda-\alpha) g(x-\lambda-\alpha') \big| \, d\alpha' \, d\lambda \, d\alpha \Big)^{1/2} \\
&\qquad \Big(\int_{\Gamma^\perp} \big| f(x-\alpha') \big|^2 \int_\Lambda \int_{\Gamma^\perp} \big| g(x-\lambda-\alpha) g(x-\lambda-\alpha') \big| \, d\alpha \, d\lambda \, d\alpha' \Big)^{1/2} d\dot{x}. \qquad (5.3)
\end{aligned}$$

The order of integration can be rearranged due to Tonelli's theorem. We now apply Proposition 5.2 to the two innermost integrals and find that there exists a constant $K_{\Lambda,\Gamma} > 0$ such that

$$\int_\Lambda \int_{\Gamma^\perp} \big| g(x-\lambda-\alpha) g(x-\lambda-\alpha') \big| \, d\alpha \, d\lambda = \int_\Lambda \big| g(x-\lambda-\alpha') \big| \int_{\Gamma^\perp} \big| g(x-\lambda-\alpha) \big| \, d\alpha \, d\lambda \le K_{\Lambda,\Gamma} \, \|g\|^2_{S_0(G)},$$

where $\alpha' \in \Gamma^\perp$. Using this inequality in (5.3) yields the Bessel bound:

$$\begin{aligned}
&\int_\Lambda \int_\Gamma |\langle f, E_\gamma T_\lambda g\rangle|^2 \, d\mu_\Gamma(\gamma) \, d\mu_\Lambda(\lambda) \\
&\qquad \le \int_{G/\Gamma^\perp} \Big(\int_{\Gamma^\perp} \big| f(x-\alpha) \big|^2 K_{\Lambda,\Gamma} \, \|g\|^2_{S_0(G)} \Big)^{1/2} \\
&\qquad\qquad \Big(\int_{\Gamma^\perp} \big| f(x-\alpha') \big|^2 K_{\Lambda,\Gamma} \, \|g\|^2_{S_0(G)} \Big)^{1/2} d\mu_{G/\Gamma^\perp}(\dot{x}) \\
&\qquad = K_{\Lambda,\Gamma} \, \|g\|^2_{S_0(G)} \, \|f\|^2_{L^2(G)}.
\end{aligned}$$

Since $C_c(G)$ is dense in $L^2(G)$, the result follows. □

5.2 The Walnut representation of the frame operator

The continuous Gabor frame operator associated with *semi* co-compact Gabor systems defined in (5.1) can be converted into a *discrete* transform called the Walnut representation. The Walnut representation plays an important role the usual discrete (lattice) theory of Gabor analysis. For Gabor theory on $L^2(\mathbb{R})$ the result goes back to [43] and is also presented in [24]. See [9] for a detailed analysis of the convergence properties of the Walnut representation in $L^2(\mathbb{R})$.

In order to state our version of the Walnut representation, we need to introduce two dense subspaces of $L^2(G)$:

$$\mathcal{D}_s := \big\{ f \in L^2(G) \, : \, f \in L^\infty(G) \text{ and } \operatorname{supp} f \text{ is compact in } G \big\} \qquad (5.4)$$

and

$$\mathcal{D}_t := \big\{ f \in L^2(G) \,:\, \hat{f} \in L^\infty(\widehat{G}) \text{ and } \operatorname{supp} \hat{f} \text{ is compact in } \widehat{G} \big\}. \tag{5.5}$$

Recall also the definition of s_α and t_β from (4.5) and (4.7), respectively.

Theorem 5.5. *Let $g,h \in L^2(G)$. Let Γ be a closed, co-compact subgroup of $\widehat{G}$, and let $(\Lambda, \Sigma_\Lambda, \mu_\Lambda)$ be an admissible measure space in G. Suppose that $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ and $\{E_\gamma T_\lambda h\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ are Bessel systems, and let $S_{g,h}$ be the associated mixed frame operator. Then*

$$S_{g,h} f = \sum_{\alpha\in\Gamma^\perp} M_{s_\alpha} T_\alpha f \qquad \text{for all } f \in \mathcal{D}_s, \tag{5.6}$$

with unconditional, norm convergence in $L^2(G)$.

Proof. By the proof of the main result in [29], we have that for all $f_1, f_2 \in \mathcal{D}_s$,

$$\begin{aligned} \langle S_{g,h} f_1, f_2 \rangle &= \int_\Lambda \int_\Gamma \langle f_1, E_\gamma T_\lambda g\rangle \langle E_\gamma T_\lambda h, f_2\rangle \, d\gamma \, d\lambda \\ &= \sum_{\alpha\in\Gamma^\perp} \big\langle M_{s_\alpha} T_\alpha f_1, f_2 \big\rangle. \end{aligned}$$

Moreover, the convergence is absolute and thus unconditionally. Because $\mathcal{D}_s$ is dense in $L^2(G)$ spaces we have that $\langle S_{g,h} f_1, f_2\rangle = \sum_{\alpha\in\Gamma^\perp} \big\langle M_{s_\alpha} T_\alpha f_1, f_2\big\rangle$ holds for all $f_2 \in L^2(G)$. By the Orlicz-Pettis Theorem (see, e.g., [15]), this implies unconditional L^2-norm convergence for (5.6). □

Remark 5. If we assume $g, h \in S_0(G)$, then (5.6) extends to all of $L^2(G)$.

Remark 6. In Theorem 5.5, if we instead assume that Λ is a closed, co-compact subgroup of G and that $(\Gamma, \Sigma_\Gamma, \mu_\Gamma)$ is an admissible measure space in $\widehat{G}$, then

$$\mathcal{F} S_{g,h} f = \sum_{\beta\in\Lambda^\perp} M_{t_\beta} T_\beta \hat{f} \qquad \text{for all } f \in \mathcal{D}_t \tag{5.7}$$

holds.

We can now easily show the following result.

Corollary 5.6. *(i) Under the assumptions of Theorem 5.5 and if $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame with bounds A and B, then*

$$A \le \int_\Lambda |g(x+\lambda)|^2 \, d\lambda \le B \qquad \text{a.e. } x \in G.$$

(ii) Under the assumptions of Remark 6 and if $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame with bounds A and B, then

$$A \le \int_\Gamma |\hat{g}(\omega\gamma)|^2 \, d\gamma \le B \qquad \text{a.e. } \omega \in \widehat{G}.$$

In either case, if $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a Bessel system with bound B, then the upper bound holds.

Proof. If $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame, then, in particular,

$$A\,\|f\|^2 \le \langle S_{g,g}f,f\rangle \le B\,\|f\|^2 \qquad \text{for all } f\in\mathcal{D}_s(G).$$

Pick now a function $f\in\mathcal{D}_s(G)$ so that the support of f lies within a fundamental domain of the discrete group $\Gamma^\perp\subset G$. Then, by (5.6),

$$\begin{aligned}
& A\,\|f\|^2 \le \langle \sum_{\alpha\in\Gamma^\perp} M_{s_\alpha}T_\alpha f, f\rangle \le B\,\|f\|^2 \\
\Leftrightarrow\quad & A\,\|f\|^2 \le \langle s_0 f, f\rangle \le B\,\|f\|^2 \\
\Leftrightarrow\quad & A\int_G |f(x)|^2\,dx \le \int_G \Big(\int_\Lambda |g(x+\lambda)|^2\,d\lambda\Big)\,|f(x)|^2\,dx \le B\int_G |f(x)|^2\,dx.
\end{aligned}$$

From this assertion (i) follows. By use of (5.7), one proves assertion (ii) in the same fashion. □

5.3 The Janssen representations of the frame operator

The Walnut representation was formulated for semi co-compact Gabor systems. In case both Λ and Γ are co-compact, closed subgroups, we can offer a more time-frequency symmetrical representation of the Gabor frame operator; this is the so-called Janssen representation.

Theorem 5.7. *Let $g,h\in L^2(G)$ and let $\Lambda\subset G,\Gamma\subset\widehat{G}$ be closed, co-compact subgroups such that $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ and $\{E_\gamma T_\lambda h\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ are Bessel systems. Suppose that the pair (g,h) satisfies* condition A*:*

$$\sum_{\alpha\in\Gamma^\perp}\sum_{\beta\in\Lambda^\perp}\big|\langle h, E_\beta T_\alpha g\rangle\big| < \infty. \tag{5.8}$$

Then

$$S_{g,h} = \sum_{\alpha\in\Gamma^\perp}\sum_{\beta\in\Lambda^\perp}\langle h, E_\beta T_\alpha g\rangle E_\beta T_\alpha \tag{5.9}$$

with absolute convergence in the operator norm.

Proof. Define the operator $\tilde{S}: L^2(G)\to L^2(G)$ by

$$\tilde{S} = \sum_{\alpha\in\Gamma^\perp}\sum_{\beta\in\Lambda^\perp}\langle h, E_\beta T_\alpha g\rangle\, E_\beta T_\alpha.$$

This series converges absolutely in the operator norm by (5.8). Hence, the convergence is unconditionally. Replacing s_α in the Walnut representation by its Fourier series representation from Remark 4 yields

$$\begin{aligned}
\langle S_{g,h}f_1,f_2\rangle &= \langle \sum_{\alpha\in\Gamma^\perp} M_{s_\alpha}T_\alpha f_1, f_2\rangle = \langle \sum_{\alpha\in\Gamma^\perp}\sum_{\beta\in\Lambda^\perp}\langle h,E_\beta T_\alpha g\rangle \beta(x) T_\alpha f_1, f_2\rangle \\
&= \sum_{\alpha\in\Gamma^\perp}\sum_{\beta\in\Lambda^\perp}\langle h,E_\beta T_\alpha g\rangle\langle E_\beta T_\alpha f_1, f_2\rangle = \langle \tilde{S}f_1,f_2\rangle
\end{aligned}$$

for $f_1,f_2\in\mathcal{D}_s$. Since $\mathcal{D}_s$ is dense in $L^2(G)$, it follows that $S_{g,h}=\tilde{S}$.

□

Note that (5.9) indicates convergence in the uniform operator topology, while Walnut's representation, on the other hand, conveyed convergence in the strong operator topology.

For generators $g, h \in S_0(G)$ in Feichtinger's algebra, the assumptions of the Janssen representation in Theorem 5.7 are automatically satisfied. The Bessel condition follows from Theorem 5.4, while (5.8) follows from the next result.

Proposition 5.8. *Let $g, h \in S_0(G)$, and let Λ and Γ be closed subgroups in G and $\widehat{G}$, respectively. The pair (g, h) satisfies (5.8), that is,*

$$\int_{\Lambda^\perp} \int_{\Gamma^\perp} |\langle g, E_\beta T_\alpha h\rangle| \, d\alpha \, d\beta < \infty.$$

Proof. By [17, Corollary 7.6.6] we have that $g, h \in S_0(G)$ implies $(x, \omega) \mapsto \langle g, E_\omega T_x h\rangle \in S_0(G \times \widehat{G})$. If we restrict this mapping to $\Gamma^\perp \times \Lambda^\perp \subset G \times \widehat{G}$ and use that S_0 is continuously embedded into L^1, we find that (5.8) is satisfied. □

The next version of the Janssen representation holds for arbitrary (not necessarily co-compact) closed subgroups $\Lambda \subset G, \Gamma \subset \widehat{G}$. It is called the *fundamental identity of Gabor analysis* (FIGA). In [19] Feichtinger and Luef give a detailed answer to when (5.10) holds in the setting of $\mathbb{R}^n$, see also [17, 21] for related results. The FIGA was first proved by Rieffel [38] for generators g, h in the Schwartz-Bruhat space $S(G)$. Rieffel's proof uses the Poisson summation formula and also holds for the non-separable case with closed subgroups in $G \times \widehat{G}$; it is also possbile to give an argument based on Janssen's proof for (lattice) Gabor systems in $L^2(\mathbb{R})$ [30, 31].

Theorem 5.9. *Let $f_1, f_2, g, h \in L^2(G)$, and let $\Lambda \subset G, \Gamma \subset \widehat{G}$ be closed subgroups. Assume that $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ and $\{E_\gamma T_\lambda h\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ are Bessel systems. If*

$$(\alpha, \beta) \mapsto \langle E_{\overline{\beta}} T_\alpha f_1, f_2\rangle \langle h, E_{\overline{\beta}} T_\alpha g\rangle \in L^1(\Gamma^\perp \times \Lambda^\perp),$$

then

$$\langle S_{g,h} f_1, f_2\rangle = \int_{\Gamma^\perp} \int_{\Lambda^\perp} \langle h, E_\beta T_\alpha g\rangle \langle E_\beta T_\alpha f_1, f_2\rangle \, d\beta \, d\alpha. \tag{5.10}$$

6 Co-compact Gabor systems and their adjoint systems

The Janssen representation shows that the frame operator of a co-compact Gabor system can be written in terms of the system $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Gamma^\perp}$. In this section we present further results that connect a co-compact Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ with its *adjoint* Gabor system $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Gamma^\perp}$.

The time-frequency shifts in a Gabor system and its adjoint system are characterized by the fact that they commute [21, Section 3.5.3], [24, Lemma 7.4.1]. That is, for $(\lambda, \gamma) \in \Lambda \times \Gamma$ the point $(\alpha, \beta) \subset G \times \widehat{G}$ belongs to $\Gamma^\perp \times \Lambda^\perp$ if and only if

$$(E_\gamma T_\lambda)(E_\beta T_\alpha) = (E_\beta T_\alpha)(E_\gamma T_\lambda).$$

We remind the reader our convention equipping the annihilator of Λ and Γ with the counting measure. The following results will, therefore, only after appropriate modification take the familiar form of the lattice Gabor theory in, e.g., $L^2(\mathbb{R}^n)$.

6.1 Bessel bound duality

Bessel bound duality states that a co-compact Gabor system is a Bessel system with bound B if, and only if, the discrete adjoint Gabor system $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is a Bessel system with bound B. The result is stated in Proposition 6.4, and its proof is divided into two parts, Lemma 6.2 and 6.3.

We begin with the definition of the operator $L_x : D(L_x) \to L^2(\Lambda)$ with $D(L_x) \subset \ell^2(\Gamma^\perp)$. Let $x \in G$, let $g \in L^2(G)$ be given and let $\{c_\alpha\}_{\alpha\in\Gamma^\perp}$ be a finite sequence. Then for almost every $x \in G$ we define the linear operator

$$L_x(\{c_\alpha\}_{\alpha\in\Gamma^\perp}) = \lambda \mapsto \sum_{\alpha\in\Gamma^\perp} g(x-\lambda-\alpha)\, c_\alpha, \quad D(L_x) = c_{00}(\Gamma^\perp). \tag{6.1}$$

Note that L_x essentially (up to complex conjugations, etc.) is the analysis operator, as introduced in (3.2), of the family of fibers associated with the TI system $\left\{T_\gamma \mathcal{F}^{-1} T_\lambda g\right\}_{\gamma\in\Gamma,\lambda\in\Lambda}$. In light of Proposition 3.2, we therefore have the following result.

Lemma 6.1. *If $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a Bessel system with bound B, then for almost every $x \in G$ the operator L_x extends to a linear, bounded operator with domain $\ell^2(\Gamma^\perp)$ and bound $B^{1/2}$.*

Let us now show one direction of the Bessel duality between a co-compact Gabor system and its adjoint.

Lemma 6.2. *Let $\Lambda \subset G$ and $\Gamma \subset \widehat{G}$ be closed, co-compact subgroups. If $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a Bessel system with bound B, then $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is a Bessel system with bound B.*

Proof. We consider the discrete Gabor system $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ and its associated synthesis mapping

$$\mathrm{F} : \ell^2(\Gamma^\perp \times \Lambda^\perp) \to L^2(G), \quad \mathrm{F}c(\alpha,\beta) = \sum_{\alpha\in\Gamma^\perp}\sum_{\beta\in\Lambda^\perp} c(\alpha,\beta) E_\beta T_\alpha g.$$

We will show that F is a well-defined, linear and bounded operator with $\|\mathrm{F}\| \le B^{1/2}$; the result then follows from [10, Theorem 3.2.3]. To this end, let $c \in \ell^2(\Gamma^\perp \times \Lambda^\perp)$ be a finite sequence and for each $x \in G$ consider

$$m_\alpha(x) := \sum_{\beta\in\Lambda^\perp} c(\alpha,\beta)\beta(x), \quad \alpha \in \Gamma^\perp. \tag{6.2}$$

It is clear that $\{m_\alpha(x)\}_{\alpha\in\Gamma^\perp}$ is a finite sequence as well. Note that m_α as a function of $x \in G$ is constant on cosets of Λ. Thus m_α defines a function on G/Λ, which we will denote by $m_\alpha(\dot{x})$. By use of the identification $G/\Lambda \cong \widehat{\Lambda^\perp}$ and the Parseval equality, we find

$$\begin{aligned}\int_{G/\Lambda} |m_\alpha(\dot{x})|^2 \, d\mu_{G/\Lambda}(\dot{x}) &= \int_{\widehat{\Lambda^\perp}} \Big| \sum_{\beta\in\Lambda^\perp} c(\alpha,\beta)\beta(x) \Big|^2 d\mu_{\widehat{\Lambda^\perp}}(x) \\ &= \|c(\alpha,\beta)\|^2_{\ell^2(\Lambda^\perp)} = \sum_{\beta\in\Lambda^\perp} |c(\alpha,\beta)|^2. \end{aligned}\tag{6.3}$$

By definition we have that

$$\mathrm{F}c = \sum_{\alpha\in\Gamma^\perp}\sum_{\beta\in\Lambda^\perp} c(\alpha,\beta) E_\beta T_\alpha g = \sum_{\alpha\in\Gamma^\perp} M_{m_\alpha} T_\alpha g.$$

Using this expression together with Weil's formula we find the following for the norm of $\mathrm{F}c$:

$$\begin{aligned}\|\mathrm{F}c\|^2 &= \int_G |\mathrm{F}c(x)|^2\,d\mu_G(x) = \int_G \sum_{\alpha,\alpha'\in\Gamma^\perp} m_\alpha(x) g(x-\alpha)\,\overline{m_{\alpha'}(x) g(x-\alpha')}\,d\mu_G(x) \\ &= \int_{G/\Lambda}\int_\Lambda \Big(\sum_{\alpha\in\Gamma^\perp} m_\alpha(\dot{x}) g(x-\lambda-\alpha)\Big)\Big(\sum_{\alpha'\in\Gamma^\perp}\overline{m_{\alpha'}(\dot{x}) g(x-\lambda-\alpha')}\Big)\,d\mu_\Lambda(\lambda)\,d\mu_{G/\Lambda}(\dot{x}) \\ &= \int_{G/\Lambda} \|L_x m_\alpha(\dot{x})\|^2_{L^2(\Lambda)}\,d\mu_{G/\Lambda}(\dot{x}). \qquad (6.4)\end{aligned}$$

The rearranging of the summation is possible because the summations over $\Gamma^\perp$ are finite. Since $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a Bessel system with bound B, we know by Lemma 6.1 that L_x is bounded by $B^{1/2}$. We therefore have that

$$\|L_x m_\alpha(\dot{x})\|^2 \le B\,\|m_\alpha(\dot{x})\|^2 = B\sum_{\alpha\in\Gamma^\perp}|m_\alpha(\dot{x})|^2.$$

Using this together with (6.3) and (6.4) yields the following inequality.

$$\|\mathrm{F}c\|^2 \le B\int_{G/\Lambda}\sum_{\alpha\in\Gamma^\perp}|m_\alpha(x)|^2\,d\mu_{G/\Lambda}(\dot{x}) = B\sum_{\alpha\in\Gamma^\perp}\sum_{\beta\in\Lambda^\perp}|c(\alpha,\beta)|^2 = B\,\|c\|^2_{\ell^2(\Gamma^\perp\times\Lambda^\perp)}.$$

We conclude that F is bounded by $B^{1/2}$ and so $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is a Bessel system with bound B. □

Note that in the classical discrete and co-compact setting we simply apply Lemma 6.2 to the adjoint Gabor system, as it would also be discrete and co-compact. However, in our case the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is co-compact and the adjoint system is discrete (and not necessarily co-compact). We thus need another result for the reverse direction.

In order to prove the reverse direction, Lemma 6.3, we will reuse calculations from Lemma 6.2. Furthermore, the proof also relies on Lemma 5.3. Adapted to co-compact $\Gamma\subset\widehat{G}$ it states that for all $f\in C_c(G)$

$$\int_\Gamma |\langle f, E_\gamma T_\lambda g\rangle|^2\,d\mu_\Gamma(\gamma) = \int_G \sum_{\alpha\in\Gamma^\perp} f(x)\overline{f(x-\alpha)}\,\overline{T_\lambda g(x)}\,T_\lambda g(x-\alpha)\,d\mu_G(x). \qquad (6.5)$$

Lemma 6.3. *Let $\Lambda\subset G$ and $\Gamma\subset\widehat{G}$ be closed, co-compact subgroups. If $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is a Bessel system with bound B, then $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a Bessel system with bound B.*

Proof. Note that for finite sequences $c\in\ell^2(\Gamma^\perp\times\Lambda^\perp)$ the calculations in (6.4) still hold. We let $m_\alpha(x)$ be given as in (6.2). By assumption we know that the synthesis mapping F of the adjoint Gabor system $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is bounded by $B^{1/2}$. We therefore have that

$$\|Fc\|^2 = \int_{G/\Lambda}\|L_x m_\alpha(\dot{x})\|^2_{L^2(\Lambda)}\,d\mu_{G/\Lambda}(\dot{x}) \le B\,\|c\|_{\ell^2(\Gamma^\perp\times\Lambda^\perp)} \quad \forall c\in\ell^2(\Gamma^\perp\times\Lambda^\perp).$$

By use of (6.3) we rewrite the norm of c and find

$$\int_{G/\Lambda}\|L_x m_\alpha(\dot{x})\|^2_{L^2(\Lambda)}\,d\mu_{G/\Lambda}(\dot{x}) \le B\int_{G/\Lambda}\|m_\alpha(\dot{x})\|^2_{\ell^2(\Gamma^\perp)}\,d\mu_{G/\Lambda}(\dot{x}). \qquad (6.6)$$

This implies that

$$\|L_x m_\alpha(\dot{x})\|^2_{L^2(\Lambda)} \le B\,\|m_\alpha(\dot{x})\|^2_{\ell^2(\Gamma^\perp)}. \tag{6.7}$$

If $c(\alpha,\beta)=0$ for all $\beta\neq 1$, then $m_\alpha(x)=c(\alpha,1)$. Therefore the mapping from all finite $c\in\ell^2(\Gamma^\perp\times\Lambda^\perp)$ to $m_\alpha(x)$ in (6.2) is a surjection onto all finite sequences indexed by $\Gamma^\perp$. From (6.7) we can therefore conclude that L_x is a bounded operator from all finite sequences to $L^2(\Lambda)$ with $\|L_x\|\le B^{1/2}$. Since L_x is also linear, it uniquely extends to a bounded operator from all of $\ell^2(\Gamma^\perp)$ to $L^2(\Lambda)$.

Let now $f\in C_c(G)$ and consider the finite sequence $c=\{\overline{f(x-\alpha)}\}_{\alpha\in\Gamma^\perp}$. Replacing $\{m_\alpha(\dot{x})\}_{\alpha\in\Gamma^\perp}$ with c in (6.6) yields the following inequality:

$$\int_{G/\Gamma^\perp}\|L_x c\|^2_{L^2(\Lambda)}\,d\mu_{G/\Gamma^\perp}(\dot{x}) \le B\int_{G/\Gamma^\perp}\sum_{\alpha\in\Gamma^\perp}|f(x-\alpha)|^2\,d\mu_{G/\Gamma^\perp}(\dot{x}) = B\,\|f\|^2_{L^2(G)}. \tag{6.8}$$

Concerning the left hand side of (6.8), we find that

$$\begin{aligned}
&\int_{G/\Gamma^\perp}\|L_x c\|^2_{L^2(\Lambda)}\,d\mu_{G/\Gamma^\perp}(\dot{x})\\
&=\int_{G/\Gamma^\perp}\int_\Lambda\sum_{\alpha,\alpha'\in\Gamma^\perp} g(x-\lambda-\alpha)\,\overline{f(x-\alpha)}\,\overline{g(x-\lambda-\alpha')}\,f(x-\alpha')\,d\lambda\,d\mu_{G/\Gamma^\perp}\\
&=\int_{G/\Gamma^\perp}\int_\Lambda\sum_{\alpha,\alpha'\in\Gamma^\perp} g(x-\lambda-\alpha'-\alpha)\,\overline{f(x-\alpha'-\alpha)}\,\overline{g(x-\lambda-\alpha')}\,f(x-\alpha')\,d\lambda\,d\mu_{G/\Gamma^\perp}\\
&=\int_G\int_\Lambda\sum_{\alpha\in\Gamma^\perp} g(x-\lambda-\alpha)\,\overline{f(x-\alpha)}\,\overline{g(x-\lambda)}\,f(x)\,d\lambda\,d\mu_G(x)\\
&=\int_\Lambda\int_\Gamma|\langle f,E_\gamma T_\lambda g\rangle|^2\,d\mu_\Gamma(\gamma)\,d\mu_\Lambda(\lambda).
\end{aligned} \tag{6.9}$$

The last equality follows by (6.5). From (6.8) and (6.9) we conclude that

$$\int_\Lambda\int_\Gamma|\langle f,E_\gamma T_\lambda g\rangle|^2\,d\mu_\Gamma(\gamma)\,d\mu_\Lambda(\lambda)\le B\,\|f\|^2\quad\text{for all } f\in C_c(G).$$

Since this holds for all f in a dense subset of $L^2(G)$ we draw the conclusion that $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a Bessel system with bound B. □

The combination of Lemma 6.2 and 6.3 yields the Bessel bound duality between a co-compact Gabor system and its discrete adjoint system.

Proposition 6.4. *Let $B>0$ and $g,h\in L^2(G)$ be given. Let $\Gamma\subset G$ and $\Lambda\subset\widehat{G}$ be closed, co-compact subgroups. Then $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a Bessel system with bound B if, and only if, $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is a Bessel system with bound B.*

6.2 Wexler-Raz biorthogonality relations

We now turn our attention to a characterization of dual co-compact Gabor frame generators by a biorthogonality condition of the corresponding (discrete) adjoint Gabor systems. Feichtinger and Kozek [17] proved the Wexler-Raz biorthogonality relations for Gabor systems with translation and modulation along *uniform lattices* on elementary LCA groups, i.e., $G=\mathbb{R}^n\times\mathbb{T}^\ell\times\mathbb{Z}^k\times F_m$, where F_m is a finite group. For a proof in the discrete and finite setting and on the real line we refer to the original papers [45] and [31].

Theorem 6.5. *Let $\Lambda \subset G$ and $\Gamma \subset \widehat{G}$ be closed, co-compact subgroups. Let $g, h \in L^2(G)$ and assume that $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ and $\{E_\gamma T_\lambda h\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ are Bessel systems. Then the two Gabor systems are dual frames if, and only if,*

$$\langle h, E_\beta T_\alpha g\rangle = \delta_{\beta,1}\delta_{\alpha,0} \qquad \forall \alpha \in \Gamma^\perp, \beta \in \Lambda^\perp. \tag{6.10}$$

Proof. Assume that the two Gabor systems are dual frames. Then, for each $\alpha \in \Gamma^\perp$, we have $s_\alpha = \delta_{\alpha,0}$ for a.e. $x \in G$. By uniqueness of the Fourier coefficients (4.8), the conclusion in (6.10) follows. The converse direction is immediate. □

Remark 7. (i).

(i) From equation (6.10) with $\alpha' \in \Gamma^\perp, \beta' \in \Lambda^\perp$ we find

$$\delta_{\beta,1}\delta_{\alpha,0} = \langle h, E_\beta T_\alpha g\rangle = \langle E_{\beta'} T_{\alpha'} h, \overline{\beta(\alpha)} E_{\beta'\beta} T_{\alpha'+\alpha} g\rangle.$$

And thus the Wexler-Raz biorthogonality relations (6.10) can equivalently be stated as

$$\langle E_\beta T_\alpha h, E_{\beta'} T_{\alpha'} g\rangle = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'} \ \ \forall\, \alpha, \alpha' \in \Gamma^\perp, \beta, \beta' \in \Lambda^\perp.$$

(ii) For canonical dual frames $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ and $\left\{E_\gamma T_\lambda S^{-1} g\right\}_{\lambda\in\Lambda,\gamma\in\Gamma}$, the biorthogonal sequences $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ and $\left\{E_\beta T_\alpha S^{-1} g\right\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ are actually dual Riesz bases for the subspace $\overline{\operatorname{span}}\,\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$, see [31, Proposition 3.3].

6.3 The duality principle

The duality principle for lattice Gabor systems in $L^2(\mathbb{R}^n)$ was proven simultaneously by three groups of authors, Daubechies, Landau and Landau [14], Janssen [31], and Ron and Shen [40]. Theorem 6.7 below generalizes this principle to co-compact Gabor systems in $L^2(G)$. Our proof of the duality principle relies on the following result on Riesz sequences in abstract Hilbert spaces, cf. Definition 2.3. It is a subspace variant of [11, Theorem 3.4.4] and [25, Theorem 7.13]; its proof is due to Ole Christensen.

Theorem 6.6. *Let $\{f_k\}$ be a sequence in a Hilbert space. Then the following statements are equivalent:*

(a) $\{f_k\}$ is a Riesz sequence with lower bound A and upper bound B,

(b) $\{f_k\}$ is a Bessel system with bound B and possesses a biorthogonal system $\{g_k\}$ that is also a Bessel system with bound A^{-1}.

Proof. Assume that (a) holds. Set $V = \overline{\operatorname{span}}\,\{f_k\}$. Let $\{g_k\}$ be the unique dual Riesz sequence of $\{f_k\}$ in V so that $\overline{\operatorname{span}}\,\{g_k\} = V$. This implies (b).

Assume that (b) holds. Since $\{f_k\}$ and $\{g_k\}$ are biorthogonal, it follows that

$$f_j = \sum_k \langle f_j, g_k\rangle f_k$$

for all j. By linearity, we have, for any $f \in \operatorname{span}\{f_k\}$,

$$f = \sum_k \langle f, g_k\rangle f_k.$$

This formula extends to $\overline{\operatorname{span}}\{f_k\}$ by continuity. Now, for any $f \in \overline{\operatorname{span}}\{f_k\}$, we have

$$\begin{aligned}\|f\|^2 = |\langle f, f\rangle| &= \Big|\sum_k \langle f, g_k\rangle \langle f_k, f\rangle\Big| \\ &\le \Big(\sum_k |\langle f, g_k\rangle|^2 \sum_k |\langle f, f_k\rangle|^2\Big)^{1/2} \\ &\le A^{-1/2}\,\|f\| \Big(\sum_k |\langle f, f_k\rangle|^2\Big)^{1/2}. \end{aligned} \tag{6.11}$$

We see that $\{f_k\}$ is a frame sequence with lower frame bound A; by assumption the upper frame bound is B. By the fact that $\{f_k\}$ possesses a biorthogonal sequence, it follows that $\{f_k\}$ is, in fact, a Riesz sequence with the same bounds. □

Theorem 6.7. *Let $g \in L^2(G)$. Let $\Lambda \subset G$ and $\Gamma \subset \widehat{G}$ be closed, co-compact subgroups. Then $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame for $L^2(G)$ with bounds A and B if, and only if, $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ Riesz sequence with bounds A and B.*

Proof. Let $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ be a frame with bounds A and B. The canonical dual frame $\{E_\gamma T_\lambda S^{-1} g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ has bounds B^{-1} and A^{-1}. By Proposition 6.4, the sequences $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ and $\{E_\beta T_\alpha S^{-1} g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ are Bessel systems with bound B and A^{-1}, respectively. By Wexler-Raz biorthogonal relations, these two families are biorthogonal, hence, by Theorem 6.6, $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is a Riesz sequence with bounds A and B.

Conversely, suppose $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is a Riesz sequence with bounds A and B. The dual Riesz sequence of $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is of the form $\{E_\beta T_\alpha h\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ for some $h \in L^2(G)$ and has bounds B^{-1} and A^{-1}. Using Proposition 6.4 we see that $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ has Bessel bound B. On the other hand, $\{E_\gamma T_\lambda h\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ has Bessel bound A^{-1}. By Wexler-Raz biorthogonal relations, $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ and $\{E_\gamma T_\lambda h\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ are dual frames. By a computation as in (6.11), we see that A is a lower frame bound for $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$. □

The co-compactness assumption on Λ and Γ is a natural framework for the duality principle. Indeed, if the Gabor system is not co-compact, the adjoint system is not discrete. However, we know by a result of Bownik and Ross [7] that continuous Riesz sequences do not exist. Hence, if either Λ or Γ is not co-compact, the adjoint Gabor system cannot be a Riesz "sequence".

Since a Riesz sequence with bounds $A = B$ is an orthogonal sequence, we have the following corollary of Theorem 6.7.

Corollary 6.8. *Let Γ and Λ be closed, co-compact subgroups. A Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a tight frame if, and only if, $\{E_\beta T_\alpha g\}_{\alpha\in\Gamma^\perp,\beta\in\Lambda^\perp}$ is an orthogonal system. In these cases, the frame bound is given by $A = \|g\|^2$.*

We end this paper with the following general remark:

Remark 8. We have stated the results of the current paper for Gabor systems generated by a single function, however, most of the results can be stated for finitely or even infinitely many generators; the non-existence result, Theorem 4.2, is of course an exception to this rule.

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