Reproducing formulas for generalized translation invariant systems on locally compact abelian groups

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Abstract: In this paper we connect the well established discrete frame theory of generalized shift invariant systems to a continuous frame theory. To do so, we let Γ_j, $j \in J$, be a countable family of closed, co-compact subgroups of a second countable locally compact abelian group G and study systems of the form $\cup_{j \in J} \{g_{j,p}(\cdot - \gamma)\}_{\gamma \in \Gamma_j, p \in P_j}$ with generators $g_{j,p}$ in $L^2(G)$ and with each P_j being a countable or an uncountable index set. We refer to systems of this form as generalized translation invariant (GTI) systems. Many of the familiar transforms, e.g., the wavelet, shearlet and Gabor transform, both their discrete and continuous variants, are GTI systems. Under a technical α local integrability condition (α-LIC) we characterize when GTI systems constitute tight and dual frames that yield reproducing formulas for $L^2(G)$. This generalizes results on generalized shift invariant systems, where each P_j assumed to be countable and each Γ_j is a uniform lattice in G, to the case of uncountably many generators and (not necessarily discrete) closed, co-compact subgroups. Furthermore, even in the case of uniform lattices Γ_j, our characterizations improve known results since the class of GTI systems satisfying the α-LIC is strictly larger than the class of GTI systems satisfying the previously used local integrability condition. As an application of our characterization results, we obtain new characterizations of translation invariant continuous frames and Gabor frames for $L^2(G)$. In addition, we will see that the admissibility conditions for the continuous and discrete wavelet and Gabor transform in $L^2(\mathbb{R}^n)$ are special cases of the same general characterizing equations.

1 Introduction

In harmonic analysis one is often interested in determining conditions on generators of function systems, e.g., Gabor and wavelet systems, that allow for reconstruction of any function in a given class of functions from its associated transform via a reproducing formula. The work of Hernández, Labate, and Weiss [30] and of Ron and Shen [46] on generalized shift invariant systems in $L^2(\mathbb{R}^n)$ presented a unified theory for many of the familiar *discrete* transforms, most notably the Gabor and the wavelet transform. The generalized shift invariant systems are collections of functions of the form $\cup_{j \in J} \{T_\gamma g_j\}_{\gamma \in \Gamma_j}$, where J is a countable index set, T_γ denotes translation by γ, Γ_j a full-rank lattice in $\mathbb{R}^n$, and $\{g_j\}_{j \in J}$ a subset of $L^2(\mathbb{R}^n)$. Here, the word "shift" is

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used since the translations are discrete and the word "generalized" since the shift lattices Γ_j are allowed to change with the parameter $j \in J$. The main result of Hernández, Labate, and Weiss [30] is a characterization, by so-called t_α-equations, of all functions g_j that give rise to isometric transforms, called Parseval frames in frame theory.

The goal of this work is to connect the discrete transform theory of generalized shift invariant systems to a continuous/integral transform theory. In doing so, the scope of the "unified approach" started in [30, 46] will be vastly extended. What more is, this new theory will cover "intermediate" steps, the semi-continuous transforms, and we will do so in a very general setting of square integrable functions on locally compact abelian groups. In particular, we recover the usual characterization results for discrete and continuous Gabor and wavelet systems as special cases. For discrete wavelets in $L^2(\mathbb{R})$ with dyadic dilation, this result was obtained in 1995, independently by Gripenberg [23] and Wang [48], and it can be stated as follows. Define the translation operator $T_b f(x) = f(x - b)$ and dilation operator $D_a f(x) = |a|^{-1/2} f(x/a)$ for $b \in \mathbb{R}, a \neq 0$. The discrete wavelet system $\{T_{2^j k} D_{2^j} \psi\}_{j,k \in \mathbb{Z}}$ generated by $\psi \in L^2(\mathbb{R})$ is indeed a generalized shift invariant system with $J = \mathbb{Z}$, $\Gamma_j = 2^j \mathbb{Z}$, and $g_j = D_{2^j} \psi$. Now, the linear operator W_d defined by

$$W_d : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2), \quad W_d f(j, k) = \langle f, T_{2^j k} D_{2^j} \psi \rangle$$

is isometric if, and only if, for all $\alpha \in \bigcup_{j \in \mathbb{Z}} 2^{-j} \mathbb{Z}$, the following t_α-equations hold:

$$t_\alpha := \sum_{j \in \mathbb{Z} \,:\, \alpha \in 2^{-j} \mathbb{Z}} \widehat{\psi}(2^j \xi) \overline{\widehat{\psi}(2^j (\xi + \alpha))} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \widehat{\mathbb{R}}, \tag{1.1}$$

where $\widehat{\mathbb{R}}$ denotes the Fourier domain. In the language of frame theory, we say that generators $\psi \in L^2(\mathbb{R})$ of discrete Parseval wavelet frames have been characterized by t_α-equations.

Calderón [6] discovered in 1964 that any function $\psi \in L^2(\mathbb{R})$ satisfying the Calderón admissibility condition

$$\int_{\mathbb{R} \setminus \{0\}} \frac{|\widehat{\psi}(a\xi)|^2}{|a|} \, da = 1 \quad \text{for a.e. } \xi \in \widehat{\mathbb{R}} \tag{1.2}$$

leads to reproducing formulas for the *continuous* wavelet transform. To be precise, the linear operator W_c defined by

$$W_c : L^2(\mathbb{R}) \to L^2\big(\mathbb{R} \setminus \{0\} \times \mathbb{R}, \tfrac{da\,db}{a^2}\big), \quad W_c f(a, b) = \langle f, T_b D_a \psi \rangle$$

is isometric if, and only if, the Calderón admissibility condition holds. We will see that the Calderón admissibility condition is nothing but the t_α-equation (there is only one!) for the continuous wavelet system. Similar results hold for the Gabor case; here the continuous transform is usually called the short-time Fourier transform. Actually, the theory is not only applicable to the Gabor and wavelet setting, but to a very large class of systems of functions including shearlet and wave packet systems, which we shall call *generalized translation invariant* systems. We refer the reader to the classical texts [12, 14, 27] and the recent book [38] for introductions to the specific cases of Gabor, wavelet, shearlet and wave packet analysis.

In [36], Kutyniok and Labate generalized the results of Hernández, Labate, and Weiss to generalized shift invariant systems $\cup_{j \in J} \{T_\gamma g_j\}_{\gamma \in \Gamma_j}$ in $L^2(G)$, where G is a second countable locally compact abelian group and Γ_j is a family of uniform lattices (i.e., Γ_j is a discrete subgroup and the quotient group G/Γ_j is compact) indexed by a countable set J. The main goal of the present paper is to develop the corresponding theory for *semi-continuous* and *continuous* frames in $L^2(G)$. In order to achieve this, we will allow non-discrete translation groups Γ_j, and we

will allow for each translation group to have uncountable many generators, indexed by some index set P_j, $j \in J$. We say that the corresponding family $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ in $L^2(G)$ is a *generalized translation invariant system*. To be precise, we will, for each $j \in J$, take P_j to be a σ-finite measure space with measure μ_{P_j} and Γ_j to be closed, co-compact (i.e., the quotient group G/Γ_j is compact) subgroups. We mention that *any* locally compact abelian group has a co-compact subgroup, namely the group itself. On the other hand, there exist groups that do not contain uniform lattices, e.g., the p-adic numbers. Thus, the theory of generalized translation invariant systems is applicable to a larger class of locally compact abelian groups than the theory of generalized shift invariant systems.

The two wavelet cases described above fit our framework. The discrete wavelet system can be written as $\cup_{j \in \mathbb{Z}} \{T_\gamma (D_{2^j} \psi)\}_{\gamma \in 2^j \mathbb{Z}}$, so we see that P_j is a singleton and μ_{P_j} a weighted counting measure for each $j \in J = \mathbb{Z}$, and that there are countably many different (discrete) Γ_j. For the continuous wavelet system on the form $\{T_\gamma (D_p \psi)\}_{\gamma \in \mathbb{R}, p \in \mathbb{R} \setminus \{0\}}$, we have that J is a singleton, e.g., $\{j_0\}$ since there is only one translation subgroup $\Gamma_{j_0} = \mathbb{R}$. On the other hand, here P_{j_0} is uncountable and $\mu_{P_{j_0}}$ a weighted Lebesgue measure. We stress that our setup can handle countable many (distinct) Γ_j and countable many P_j, each being uncountable.

The characterization results in [30, 36] rely on a technical condition on the generators and the translation lattices, the so-called *local integrability condition*. This condition is straightforward to formulate for generalized translation invariant systems, however, we will replace it by a strictly weaker condition, termed α *local integrability condition*. Therefore, even for generalized *shift* invariant systems in the euclidean setting, our work extends the characterization results by Hernández, Labate, and Weiss [30]. Under the α local integrability condition, we show in Theorem 3.5 that $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a Parseval frame for $L^2(G)$, that is, the associated transform is isometric if, and only if,

$$t_\alpha := \sum_{j \in J \,:\, \alpha \in \Gamma_j^\perp} \int_{P_j} \overline{\hat{g}_{j,p}(\omega)} \hat{g}_{j,p}(\omega + \alpha) \, d\mu_{P_j}(p) = \delta_{\alpha,0} \quad \text{a.e. } \omega \in \widehat{G}$$

for every $\alpha \in \cup_{j \in J} \Gamma_j^\perp$, where $\Gamma_j^\perp = \left\{ \omega \in \widehat{G} : \omega(x) = 0 \text{ for all } x \in \Gamma_j \right\}$ denotes the annihilator of Γ_j. Now, returning to the two main examples of this introduction, the discrete and continuous wavelet transform, we see why the number of the t_α-equations in (1.1) and (1.2) are so different. In the discrete case the corresponding union of the annihilators of the translation groups is $\cup_{j \in \mathbb{Z}} 2^{-j} \mathbb{Z}$, while in the continuous case the annihilator of $\mathbb{R}$ is simply $\{0\}$, which corresponds to only one t_α-equation ($\alpha = 0$).

Finally, as Kutyniok and Labate [36] restrict their attention to Parseval frames, there are currently no characterization results available for *dual* (discrete) frames in the setting of locally compact abelian groups. Hence, one additional objective of this paper is to prove characterizing equations for *dual* generalized translation invariant frames to remedy this situation.

For a related study of reproducing formulas from a purely group representation theoretical point of view, we refer to the work of Führ [20], and De Mari, De Vito [15], and the references therein.

The paper is organized as follows. We recall some basic theory about locally compact abelian groups and introduce the generalized translation invariant systems in Section 2.1 and 2.2, respectively. Additionally, in Section 2.3 we give a short introduction to the theory of continuous frames and g-frames. In Section 3 we present our main characterization result for dual generalized translation invariant frames (Theorem 3.4) and, as corollary, then for Parseval frames (Theorem 3.5). In Section 3.2 and 3.3 we relate several conditions used in our main results.

Finally, we consider the special case of translation invariant systems and apply our characterization results on concrete groups and to concrete examples in Sections 3.4 and 4. Specifically, we consider discrete and continuous wavelet systems in $L^2(\mathbb{R}^n)$, shearlets in $L^2(\mathbb{R}^2)$, discrete, semi-continuous and continuous Gabor frames on LCA groups and GTI systems over the p-adic integers and numbers.

During the final stages of this project, we realized that Bownik and Ross [4] have completed a related investigation. As they consider and characterize the structure of translation invariant subspaces on locally compact abelian groups, their results do not overlap with our results in any way. However, they do consider translations along a closed, co-compact subgroup. We adopt their terminology of *translation* invariance, in place of shift invariance, to emphasize the fact that Γ_j need not be discrete.

2 Preliminaries

In the following sections we set up notation and recall some useful results from Fourier analysis on locally compact abelian (LCA) groups and continuous frame theory. Furthermore, we will prove two important lemmas, Lemma 2.2 and 2.4.

2.1 Fourier analysis on locally compact abelian groups

Throughout this paper G will denote a second countable locally compact abelian group. We note that the following statements are equivalent: (i) G is second countable, (ii) $L^2(G)$ is separable, (iii) G is metrizable and σ-compact. Note that the metric on G can be chosen to be translation invariant.

To G we associate its dual group $\widehat{G}$ consisting of all characters, i.e., all continuous homomorphisms from G into the torus $\mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\}$. Under pointwise multiplication $\widehat{G}$ is also a locally compact abelian group. We will use addition and multiplication as group operation in G and $\widehat{G}$, respectively. Note that in the introduction we used addition as group operation in $\widehat{G}$. By the Pontryagin duality theorem, the dual group of $\widehat{G}$ is isomorphic to G as a topological group, i.e., $\widehat{\widehat{G}} \cong G$. We recall the well-known facts that if G is discrete, then $\widehat{G}$ is compact, and vice versa.

We denote the Haar measure on G by μ_G. The (left) Haar measure on any locally compact group is unique up to a positive constant. From μ_G we define $L^1(G)$ and the Hilbert space $L^2(G)$ over the complex field in the usual way.

For functions $f \in L^1(G)$ we define the Fourier transform

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_G f(x)\overline{\omega(x)} \, d\mu_G(x), \quad \omega \in \widehat{G}.$$

If $f \in L^1(G), \hat{f} \in L^1(\widehat{G})$, and the measure on G and $\widehat{G}$ are normalized so that the Plancherel theorem holds (see [32, (31.1)]), the function f can be recovered from $\hat{f}$ by the inverse Fourier transform

$$f(x) = \mathcal{F}^{-1}\hat{f}(x) = \int_{\widehat{G}} \hat{f}(\omega)\omega(x) \, d\mu_{\widehat{G}}(\omega), \quad x \in G.$$

From now on we always assume that the measure on a group μ_G and its dual group $\mu_{\widehat{G}}$ are normalized this way, and we refer to them as *dual measures*. As in the classical Fourier analysis $\mathcal{F}$ can be extended from $L^1(G) \cap L^2(G)$ to an isometric isomorphism between $L^2(G)$ and $L^2(\widehat{G})$.

On any locally compact abelian group G, we define the following two linear operators. For $a \in G$, the operator T_a, called *translation* by a, is defined by

$$T_a : L^2(G) \to L^2(G), \ (T_a f)(x) = f(x - a), \quad x \in G.$$

For $\chi \in \widehat{G}$, the operator E_χ, called *modulation* by χ, is defined by

$$E_\chi : L^2(G) \to L^2(G), \quad (E_\chi f)(x) = \chi(x) f(x), \quad x \in G.$$

Together with the Fourier transform $\mathcal{F}$, the two operators E_χ and T_a share the following commutator relations: $T_a E_\chi = \overline{\chi(a)} E_\chi T_a$, $\mathcal{F} T_a = E_{a^{-1}} \mathcal{F}$, and $\mathcal{F} E_\chi = T_\chi \mathcal{F}$.

For a subgroup H of an LCA group G, we define its annihilator as

$$H^\perp = \{\omega \in \widehat{G} : \omega(x) = 1 \text{ for all } x \in H\}.$$

The annihilator $H^\perp$ is a closed subgroup in $\widehat{G}$, and if H is closed, then $\widehat{H} \cong \widehat{G}/H^\perp$ and $\widehat{G/H} \cong H^\perp$.

We will repeatedly use Weil's formula; it relates integrable functions over G with integrable functions on the quotient space G/H when H is a closed subgroup of G. We mention the following results concerning Weil's formula [44].

Theorem 2.1. *Let H be a closed subgroup of G. Let $\pi_H : G \to G/H$, $\pi_H(x) = x + H$ be the canonical map from G onto G/H. If $f \in L^1(G)$, then the following holds:*

(i) *The function $\dot{x} \mapsto \int_H f(x+h)\, d\mu_H(h)$, $\dot{x} = \pi_H(x)$ defined almost everywhere on G/H, is integrable.*

(ii) *(Weil's formula) Let two of the Haar measures on G, H and G/H be given, then the third can be normalized such that*

$$\int_G f(x)\, d\mu_G(x) = \int_{G/H} \int_H f(x+h)\, d\mu_H(h)\, d\mu_{G/H}(\dot{x}). \tag{2.1}$$

(iii) *If (2.1) holds, then the respective dual measures on $\widehat{G}, H^\perp \cong \widehat{G/H}, \widehat{G}/H^\perp \cong \widehat{H}$ satisfy*

$$\int_{\widehat{G}} \hat{f}(\omega)\, d\mu_{\widehat{G}}(\omega) = \int_{\widehat{G}/H^\perp} \int_{H^\perp} \hat{f}(\omega\gamma)\, d\mu_{H^\perp}(\gamma)\, d\mu_{\widehat{G}/H^\perp}(\dot{\omega}). \tag{2.2}$$

Remark 1. Since a Haar measure and its dual are chosen so that the Plancherel theorem holds we have the following uniqueness result: If two of the measures on $G, H, G/H, \widehat{G}, H^\perp$ and $\widehat{G}/H^\perp$ are given, and these two are not dual measures, by requiring Weil's formulas (2.1) and (2.2), all other measures are uniquely determined.

For more information on harmonic analysis on locally compact abelian groups, we refer the reader to the classical books [17, 31, 32, 44].

For a Borel set $E \subset \widehat{G}$ with $\mu_{\widehat{G}}(\overline{E}) = 0$, we define:

$$\mathcal{D} = \{f \in L^2(G) : \hat{f} \in L^\infty(\widehat{G}) \text{ and } \operatorname{supp} \hat{f} \text{ is compact in } \widehat{G} \setminus E\}. \tag{2.3}$$

It is not difficult to show that $\mathcal{D}$ is dense in $L^2(G)$ exactly when $\mu_{\widehat{G}}(\overline{E}) = 0$. We will frequently prove our results on $\mathcal{D}$ and extend by a density argument. The role of the set E is to allow for "blind spots" of transforms – a term coined by Führ [21]. We will let E be an unspecified set satisfying $\mu_{\widehat{G}}(\overline{E}) = 0$; the specific choice of E depends on the application, e.g., in the Gabor and wavelet case [30] one would usually take $E = \emptyset$ and $E = \{0\}$, respectively.

The following result relies on Weil's formula and will play an important part of the proofs in Section 3.

Lemma 2.2. *Let H be a closed subgroup of an LCA group G with Haar measure μ_H. Suppose that $f_1, f_2 \in \mathcal{D}$ and $\varphi, \psi \in L^2(G)$. Then*

$$\int_H \langle f_1, T_h \varphi \rangle \langle T_h \psi, f_2 \rangle \, d\mu_H(h) = \int_{\widehat{G}} \int_{H^\perp} \hat{f}_1(\omega) \overline{\hat{f}_2(\omega \alpha)} \overline{\hat{\varphi}(\omega)} \hat{\psi}(\omega \alpha) \, d\mu_{H^\perp}(\alpha) \, d\mu_{\widehat{G}}(\omega).$$

Proof. Let $h \in H$. An application of the Plancherel theorem together with Weil's formula yields

$$\langle f_1, T_h \varphi \rangle = \langle \hat{f}_1, \widehat{T_h \varphi} \rangle = \langle \hat{f}_1, E_{-h} \hat{\varphi} \rangle = \int_{\widehat{G}} \hat{f}_1(\omega) \overline{\hat{\varphi}(\omega)} \omega(h) \, d\mu_{\widehat{G}}(\omega)$$

$$= \int_{\widehat{G}/H^\perp} \int_{H^\perp} \hat{f}_1(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \omega(h) \gamma(h) \, d\mu_{H^\perp}(\gamma) \, d\mu_{\widehat{G}/H^\perp}(\dot{\omega})$$

$$= \int_{\widehat{H}} \left(\omega(h) \int_{H^\perp} \hat{f}_1(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \, d\mu_{H^\perp}(\gamma) \right) d\mu_{\widehat{H}}(\omega),$$

where we tacitly used that $\widehat{G}/H^\perp \cong \widehat{H}$. A similar calculation can be done for $\langle T_h \psi, f_2 \rangle$. To ease notation, we define $[\hat{f}, \hat{\varphi}](\omega, H^\perp) = \int_{H^\perp} \hat{f}(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \, d\mu_{H^\perp}(\gamma)$ for $f \in \mathcal{D}$. Again, by the Plancherel theorem and Weil's formula we have

$$\int_H \langle f_1, T_h \varphi \rangle \langle T_h \psi, f_2 \rangle \, d\mu_H(h)$$

$$= \int_H \left(\int_{\widehat{H}} \omega(h) [\hat{f}_1, \hat{\varphi}](\omega, H^\perp) \, d\mu_{\widehat{H}}(\omega) \right) \overline{\left(\int_{\widehat{H}} \omega(h) [\hat{f}_2, \hat{\psi}](\omega, H^\perp) \, d\mu_{\widehat{H}}(\omega) \right)} d\mu_H(h)$$

$$= \left\langle \mathcal{F}^{-1}[\hat{f}_1, \hat{\varphi}](\,\cdot\,, H^\perp), \mathcal{F}^{-1}[\hat{f}_2, \hat{\psi}](\,\cdot\,, H^\perp) \right\rangle_{L^2(H)}$$

$$= \left\langle [\hat{f}_1, \hat{\varphi}](\,\cdot\,, H^\perp), [\hat{f}_2, \hat{\psi}](\,\cdot\,, H^\perp) \right\rangle_{L^2(\widehat{H})}$$

$$= \int_{\widehat{G}/H^\perp} \left(\int_{H^\perp} \hat{f}_1(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \, d\mu_{H^\perp}(\gamma) \right) \overline{\left(\int_{H^\perp} \hat{f}_2(\omega \gamma) \overline{\hat{\psi}(\omega \gamma)} \, d\mu_{H^\perp}(\gamma) \right)} d\mu_{\widehat{G}/H^\perp}(\dot{\omega})$$

$$= \int_{\widehat{G}/H^\perp} \left[\int_{H^\perp} \hat{f}_1(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \left(\int_{H^\perp} \overline{\hat{f}_2(\omega \beta)} \hat{\psi}(\omega \beta) \, d\mu_{H^\perp}(\beta) \right) d\mu_{H^\perp}(\gamma) \right] d\mu_{\widehat{G}/H^\perp}(\dot{\omega})$$

$$= \int_{\widehat{G}/H^\perp} \left[\int_{H^\perp} \hat{f}_1(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \left(\int_{H^\perp} \overline{\hat{f}_2(\omega \gamma \alpha)} \hat{\psi}(\omega \gamma \alpha) \, d\mu_{H^\perp}(\alpha) \right) d\mu_{H^\perp}(\gamma) \right] d\mu_{\widehat{G}/H^\perp}(\dot{\omega})$$

$$= \int_{\widehat{G}} \hat{f}_1(\omega) \overline{\hat{\varphi}(\omega)} \left(\int_{H^\perp} \overline{\hat{f}_2(\omega \alpha)} \hat{\psi}(\omega \alpha) \, d\mu_{H^\perp}(\alpha) \right) d\mu_{\widehat{G}}(\omega)$$

$$= \int_{\widehat{G}} \int_{H^\perp} \hat{f}_1(\omega) \overline{\hat{f}_2(\omega \alpha)} \overline{\hat{\varphi}(\omega)} \hat{\psi}(\omega \alpha) \, d\mu_{H^\perp}(\alpha) \, d\mu_{\widehat{G}}(\omega).$$

Here $\mathcal{F}$ denotes the Fourier transform on H. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

2.2 Definition of generalized translation invariant systems

Let $J \subset \mathbb{Z}$ be a countable index set. For each $j \in J$, let P_j be a countable or an uncountable index set, let $g_{j,p} \in L^2(G)$ for $p \in P_j$, and let Γ_j be a closed, co-compact subgroup in G. Recall that co-compact subgroups are subgroups of G for which G/Γ_j is compact. For a compact abelian group, the group is metrizable if, and only if, the character group is countable [31, (24.15)]. Hence, since G/Γ_j is compact and metrizable, the group $\widehat{G/\Gamma_j} \cong \Gamma_j^\perp$ is discrete and countable. Unless stated otherwise we equip $\Gamma_j^\perp$ with the counting measure and assume a fixed Haar measure μ_G on G. By Remark 1 this uniquely determines the measures on $\Gamma_j, G/\Gamma_j, \widehat{G}$, and $\widehat{G}/\Gamma_j^\perp$.

The *generalized translation invariant* (GTI) system generated by $\{g_{j,p}\}_{p \in P_j, j \in J}$ with translation along closed, co-compact subgroups $\{\Gamma_j\}_{j \in J}$ is the family of functions $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$. To ease notation, we will suppress the dependence of j in $g_{j,p}$ and write the GTI system as $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$.

If we take $\Gamma = \Gamma_j$ for each $j \in J$, we obtain a *translation invariant* (TI) system in the sense that $f \in \cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma, p \in P_j}$ implies $T_\gamma f \in \cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma, p \in P_j}$ for all $\gamma \in \Gamma$. However, generalized translation invariant systems are more general than translation invariant systems since we allow for a different subgroup for each set of generators $\{g_{j,p}\}_{p \in P_j}$.

When each P_j is countable and each Γ_j is a uniform lattice, i.e., a *discrete*, co-compact subgroup, we recover the generalized shift invariant (GSI) systems considered in [36]. However, we note that there exist locally compact abelian groups that do not contain any uniform lattices. As an example we mention the p-adic numbers, whose only discrete subgroup is the neutral element which is not a uniform lattice. In other cases, such as the p-adic integers, the LCA group will have only trivial examples of uniform lattices, e.g., the neutral element, but have plenty non-trivial co-compact subgroups, see Example 10 in Section 4.

Finally, as an alternative generalization of uniform lattices, we mention the idea of so-called quasi-lattices, see [28,29]. In contrast to closed, co-compact subgroups, quasi-lattices are discrete subsets in G that are not necessarily groups.

2.3   Frame theory

The central concept of this section is that of a continuous frame. The definition is as follows.

Definition 2.3. Let $\mathcal{H}$ be a complex Hilbert space, and let (M, Σ_M, μ_M) be a measure space, where Σ_M denotes the σ-algebra and μ_M the non-negative measure. A family of vectors $\{f_k\}_{k \in M}$ is called a *continuous frame* for $\mathcal{H}$ with respect to (M, Σ_M, μ_M) if

(a) $k \mapsto f_k$ is weakly measurable, i.e., for all $f \in \mathcal{H}$, the mapping $M \to \mathbb{C}, k \mapsto \langle f, f_k \rangle$ is measurable, and

(b) there exist constants $A, B > 0$ such that

$$A \|f\|^2 \le \int_M |\langle f, f_k \rangle|^2 \, d\mu_M(k) \le B \|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{2.4}$$

The constants A and B are called *frame bounds*.

Remark 2. As we will only consider separable Hilbert spaces in this paper, we can replace weak measurability of $k \mapsto f_k$ with (strong) measurability with respect to the Borel algebra in $\mathcal{H}$ by Pettis' theorem.

In cases where it will cause no confusion, we will simply say that $\{f_k\}_{k \in M}$ is a frame for $\mathcal{H}$. If $\{f_k\}_{k \in M}$ is weakly measurable and the upper bound in the above inequality (2.4) holds, then $\{f_k\}_{k \in M}$ is said to be a *Bessel family* with constant B. A frame $\{f_k\}_{k \in M}$ is said to be *tight* if we can choose $A = B$; if, furthermore, $A = B = 1$, then $\{f_k\}_{k \in M}$ is said to be a *Parseval frame*.

Two Bessel families $\{f_k\}_{k \in M}$ and $\{g_k\}_{k \in M}$ are said to be *dual frames* if

$$\langle f, g \rangle = \int_M \langle f, g_k \rangle \langle f_k, g \rangle \, d\mu_M(k) \quad \text{for all } f, g \in \mathcal{H}. \tag{2.5}$$

In this case we say that the following assignment

$$f = \int_M \langle f, g_k \rangle f_k \, d\mu_M(k) \quad \text{for } f \in \mathcal{H}, \tag{2.6}$$

holds in the weak sense. Equation (2.6) is often called a *reproducing formula* for $f \in \mathcal{H}$. The following argument shows that two such dual frames indeed are frames, and we shall say that the frame $\{f_k\}_{k \in M}$ is dual to $\{g_k\}_{k \in M}$, and vice versa. We need to show that both Bessel families $\{f_k\}_{k \in M}$ and $\{g_k\}_{k \in M}$ satisfy the lower frame bound. By taking $f = g$ in (2.5) and using the Cauchy-Schwarz inequality, we have

$$\|f\|^2 = \int_M \langle f, f_k \rangle \langle g_k, f \rangle \, d\mu_M(k) \leq \Big(\int_M |\langle f, f_k \rangle|^2 \, d\mu_M(k) \Big)^{1/2} \Big(\int_M |\langle f, g_k \rangle|^2 \, d\mu_M(k) \Big)^{1/2}$$
$$\leq \Big(\int_M |\langle f, f_k \rangle|^2 \, d\mu_M(k) \Big)^{1/2} \sqrt{B_g} \, \|f\|.$$

In the last step we used that $\{g_k\}_{k \in M}$ has an upper frame bound B_g. Rearranging the terms in the above inequality gives

$$\frac{1}{B_g} \|f\|^2 \leq \int_M |\langle f, f_k \rangle|^2 \, d\mu_M(k).$$

Hence, the Bessel family $\{f_k\}_{k \in M}$ satisfies the lower frame condition and is a frame. A similar argument shows that $\{g_k\}_{k \in M}$ satisfies the lower frame condition. This completes the argument. Moreover, by a polarization argument, it follows that two Bessel families $\{f_k\}_{k \in M}$ and $\{g_k\}_{k \in M}$ are dual frames if, and only if,

$$\langle f, f \rangle = \int_M \langle f, g_k \rangle \langle f_k, f \rangle \, d\mu_M(k) \quad \text{for all } f \in \mathcal{H}.$$

We mention that to a given frame for $\mathcal{H}$ one can always find at least one dual frame. For more information on (continuous) frames, we refer to [1, 2, 8, 18, 22, 34].

To a frame $\{f_k\}_{k \in M}$ for $\mathcal{H}$, we associate the *frame transform* given by

$$\mathcal{H} \to L^2(M, \mu_M), \quad f \mapsto (k \mapsto \langle f, f_k \rangle).$$

As mentioned in the introduction, this transform is isometric if, and only if, the family $\{f_k\}_{k \in M}$ is a Parseval frame. A similar conclusion holds for a pair of dual frames.

Let (M_1, Σ_1, μ_1) and (M_2, Σ_2, μ_2) be measure spaces. We say that a family $\{f_k\}_{k \in M_1}$ in the Hilbert space $\mathcal{H}$ is *unitarily equivalent* to a family $\{g_k\}_{k \in M_2}$ in the Hilbert space $\mathcal{K}$ if there is a point isomorphism $\iota \colon M_1 \to M_2$, i.e., ι is a (measurable) bijection such that $\iota(\Sigma_1) = \Sigma_2$ and $\mu_1 \circ \iota^{-1} = \mu_2$, a unitary mapping $U \colon \mathcal{K} \to \mathcal{H}$, and measurable mapping $M_1 \to \mathbb{C}, k \mapsto c_k$ with $|c_k| = 1$ such that $f_k = c_k U g_{\iota(k)}$ for all $k \in M_1$. This notion of unitarily equivalence generalizes a similar concept from [1]. Unitarily equivalence is important to us since it preserves many of the properties we are interested in, e.g., the frame property, including the frame bounds. The following lemma tells us that "pairwise" unitarily equivalence preserves the property of being dual frames.

Lemma 2.4. *Let $\{f_k\}_{k \in M_1}$ and $\{\tilde{f}_k\}_{k \in M_1}$ be families in $\mathcal{H}$, and let $\{g_k\}_{k \in M_2}$ and $\{\tilde{g}_k\}_{k \in M_2}$ be families in $\mathcal{K}$. Suppose that*

$$f_k = c_k U g_{\iota(k)} \quad \text{and} \quad \tilde{f}_k = c_k U \tilde{g}_{\iota(k)}$$

for some point isomorphism $\iota \colon M_1 \to M_2$, a unitary mapping $U \colon \mathcal{K} \to \mathcal{H}$, and a measurable mapping $M_1 \to \mathbb{C}, k \mapsto c_k$ with $|c_k| = 1$ for $k \in M_1$. Then $\{f_k\}_{k \in M_1}$ and $\{\tilde{f}_k\}_{k \in M_1}$ are dual frames with respect to (M_1, Σ_1, μ_1) if, and only if, $\{g_k\}_{k \in M_2}$ and $\{\tilde{g}_k\}_{k \in M_2}$ are dual frames with respect to (M_2, Σ_2, μ_2).

Proof. Assume that $\{f_k\}_{k \in M_1}$ and $\{\tilde{f}_k\}_{k \in M_1}$ are a pair of dual frames. Since the composition of measurable functions is again measurable, then by our assumptions it follows that $\{g_k\}_{k \in M_2}$ and $\{\tilde{g}_k\}_{k \in M_2}$ are weakly measurable. They are obviously Bessel families. For $f \in \mathcal{K}$ and $g \in \mathcal{H}$ we compute:

$$\langle f, U^*g \rangle = \langle Uf, g \rangle = \int_{M_1} \langle Uf, \tilde{f}_k \rangle \langle f_k, g \rangle \, d\mu_1(k) = \int_{M_1} \langle Uf, c_k U \tilde{g}_{\iota(k)} \rangle \langle c_k U g_{\iota(k)}, g \rangle \, d\mu_1(k)$$

$$= \int_{M_1} \langle f, \tilde{g}_{\iota(k)} \rangle \langle g_{\iota(k)}, U^*g \rangle \, d\mu_1(k) = \int_{M_2} \langle f, \tilde{g}_k \rangle \langle g_k, U^*g \rangle \, d\mu_2(k),$$

where the last equality follows from the properties of the point isomorphism. Since U^* is invertible on all of $\mathcal{K}$, this implies that $\{g_k\}_{k \in M_2}$ and $\{\tilde{g}_k\}_{k \in M_2}$ are dual frames. The opposite implication follows by symmetry. $\qquad \square$

If μ_M is the counting measure and $\Sigma_M = 2^M$ the discrete σ-algebra, we say that $\{f_k\}_{k \in M}$ is a *discrete frame* whenever (2.4) is satisfied; for this measure space, any family of vectors is obviously weakly measurable. For discrete frames, equation (2.6) holds in the usual strong sense, i.e., with (unconditional) convergence in the $\mathcal{H}$ norm.

Lastly, we combine the notion of continuous frames with that of generalized frames, also known as g-frames. Let (M_j, Σ_j, μ_j) be a measure space for each $j \in J$, where $J \subset \mathbb{Z}$ is a countable index set. We will say that a union $\cup_{j \in J} \{f_{j,k}\}_{k \in M_j}$ is a g-frame for $\mathcal{H}$, or simply a frame, with respect to $\{L^2(M_j, \mu_j) : j \in J\}$ if

(a) $k \mapsto f_{j,k}, M_j \to \mathcal{H}$ is measurable for each $j \in J$, and

(b) there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{j \in J} \int_{M_j} |\langle f, f_{j,k} \rangle|^2 \, d\mu_{M_j}(k) \leq B \|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{2.7}$$

The above definition and statements about continuous frames carry over to continuous g-frames; we refer to the original paper by Sun [47] for a detailed account of g-frames. Lemma 2.4 is also easily transferred to this new setup. We will repeatedly use that it is sufficient to verify the various frame properties on a dense subset of $\mathcal{H}$. The precise statement is as follows.

Lemma 2.5. *Let $\mathcal{D}$ be a dense subset of $\mathcal{H}$, and let (M_j, μ_j) be a measure space for each $j \in J$.*

(i) *Suppose that $\cup_{j \in J} \{f_{j,k}\}_{k \in M_j}$ and $\cup_{j \in J} \{g_{j,k}\}_{k \in M_j}$ are Bessel families in $\mathcal{H}$. If, for $f \in \mathcal{D}$,*

$$\langle f, f \rangle = \sum_{j \in J} \int_{M_j} \langle f, f_{j,k} \rangle \langle g_{j,k}, f \rangle \, d\mu_{M_j}(k), \tag{2.8}$$

then equation (2.8) holds for all $f \in \mathcal{H}$, i.e., $\cup_{j \in J} \{f_{j,k}\}_{k \in M_j}$ and $\cup_{j \in J} \{g_{j,k}\}_{k \in M_j}$ are dual frames.

(ii) *Suppose that (M_j, μ_{M_j}) are σ-finite and $\cup_{j \in J} \{f_{j,k}\}_{k \in M_j}$ weakly measurable. If, for $f \in \mathcal{D}$,*

$$\langle f, f \rangle = \sum_{j \in J} \int_{M_j} \langle f, f_{j,k} \rangle \langle f_{j,k}, f \rangle \, d\mu_{M_j}(k), \tag{2.9}$$

then equation (2.9) holds for all $f \in \mathcal{H}$, i.e., $\cup_{j \in J} \{f_{j,k}\}_{k \in M_j}$ is a Parseval frame.

Proof. (i): The first statement follows by a straightforward generalization of the proof of the same result for discrete frames [19, Lemma 7]. The duality of $\cup_{j \in J} \{f_{j,k}\}_{k \in M_j}$ and $\cup_{j \in J} \{g_{j,k}\}_{k \in M_j}$ follows then by polarization.

(ii): Without loss of generality we can assume that the measure space (M_j, μ_{M_j}) is bounded for each $j \in J$. By use of Lebesgue's bounded convergence theorem, equation (2.9) for $f \in \mathcal{D}$ implies that $\cup_{j \in J} \{f_{j,k}\}_{k \in M_j}$ is a Bessel family on all of $\mathcal{H}$; a similar argument can be found in the proof of [43, Proposition 2.5]. The result now follows from (i). $\qquad \square$

3   Generalized translation invariant systems

In this section we will work with generalized translation invariant systems $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$, introduced in Section 2.2, in the setting of continuous g-frames. In order to do this, we let $(P_j, \Sigma_{P_j}, \mu_{P_j})$ be a σ-finite measure space for each $j \in J$, where $J \subset \mathbb{Z}$ is a countable index set. For a topological space T, we let B_T denote the Borel algebra of T. We now consider $M_j := P_j \times \Gamma_j$, and let $\Sigma_{M_j} := \Sigma_{P_j} \otimes B_{\Gamma_j}$ and $\mu_{M_j} := \mu_{P_j} \otimes \mu_{\Gamma_j}$ denote the product algebra and the product measure on $P_j \times \Gamma_j$, respectively.

We will work under the following **standing hypotheses** on the generalized translation invariant system $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$. For each $j \in J$:

(I) $(P_j, \Sigma_{P_j}, \mu_{P_j})$ is a σ-finite measure space,

(II) the mapping $p \mapsto g_p, (P_j, \Sigma_{P_j}) \to (L^2(G), B_{L^2(G)})$ is measurable,

(III) the mapping $(p, x) \mapsto g_p(x), (P_j \times G, \Sigma_{P_j} \otimes B_G) \to (\mathbb{C}, B_{\mathbb{C}})$ is measurable.

Consider $T_\gamma g_p$ as a function of $(p, \gamma) \in P_j \times \Gamma_j$ into $L^2(G)$. This function is continuous in γ and measurable in p. Such functions are sometimes called Carathéodory functions, and since $\Gamma_j \subset G$ is a second countable metric space, it follows that any Carathéodory function, in particular $T_\gamma g_p$, is jointly measurable on $(M_j, \Sigma_{M_j}) = (P_j \times \Gamma_j, \Sigma_{P_j} \otimes B_{\Gamma_j})$. Thus, the family of functions $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ is automatically weakly measurable. A generalized translation invariant system is therefore a frame for $L^2(G)$ if (2.7) is satisfied with respect to the measure spaces $(M_j, \Sigma_{M_j}, \mu_{M_j})$. Similar conclusions are valid with respect to generalized translation invariant systems being Bessel families, Parseval frames, etc. Let us here just observe that for dual frames $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_p\}_{\gamma \in \Gamma_j, p \in P_j}$, we have the reproducing formula

$$f = \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} \langle f, T_\gamma g_p \rangle T_\gamma h_p \, d\mu_{\Gamma_j}(\gamma) \, d\mu_{P_j}(p) \quad \text{for } f \in L^2(G),$$

where the measure on Γ_j is chosen so that the measure on $\Gamma_j^\perp$ is the counting measure.

Remark 3. In Section 3 we always assume the three standing hypotheses. However, in many special cases these assumptions are automatically satisfied:

(a) When P_j is countable for all $j \in J$, we will equip it with a scaled counting measure $k\mu_c$, $k > 0$, and the discrete σ-algebra 2^{P_j}. If all P_j, $j \in J$, are countable, all three standing hypotheses therefore trivially hold.

(b) If P_j is a second countable metric space for all $j \in J$ and if $p \mapsto g_p$ is continuous, then the standing hypotheses (II) and (III) are satisfied. Hence, if P_j is also a subset of G or $\widehat{G}$ equipped with their respective Haar measure, then all three standing hypotheses hold.

The main characterization results are stated in Theorem 3.4 and 3.5. These results rely on the following technical assumption.

Definition 3.1. We say that two generalized translation invariant systems $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_p\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfy the *dual α local integrability condition* (dual α-LIC) if, for all $f \in \mathcal{D}$,

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\widehat{G}} \left| \hat{f}(\omega) \hat{f}(\omega\alpha) \hat{g}_p(\omega) \hat{h}_p(\omega\alpha) \right| d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p) < \infty. \tag{3.1}$$

In case $g_p = h_p$ we refer to (3.1) as the *α local integrability condition* (α-LIC) for the generalized translation invariant system $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$.

The α-LIC should be compared to the local integrability condition for generalized shift invariant systems introduced in [30] for $L^2(\mathbb{R}^n)$ and in [36] for $L^2(G)$. For generalized translation invariant systems $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ the *local integrability conditions* (LIC) becomes

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\operatorname{supp} \hat{f}} \left| \hat{f}(\omega\alpha) \hat{g}_p(\omega) \right|^2 d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p) < \infty \quad \text{for all } f \in \mathcal{D}. \tag{3.2}$$

Since the integrands in (3.1) and (3.2) are measurable on $\mathcal{P}_j \times \widehat{G}$, we are allowed to reorder sums and integrals in the local integrability conditions.

We will see (Lemma 3.9 and Example 1) that the LIC implies the α-LIC, but not vice versa. Moreover, we mention that *dual* local integrability conditions have not been considered in the literature before. The following simple observation will often be used.

Lemma 3.2. *The following assertions are equivalent:*

(i) *The systems $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_p\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfy the dual α-LIC,*

(ii) *for each compact subset $K \subseteq \widehat{G} \setminus E$*

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{K \cap \alpha^{-1} K} \left| \hat{g}_p(\omega) \hat{h}_p(\omega\alpha) \right| d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p) < \infty.$$

Proof. To show that (i) implies (ii), let K be any compact subset in $\widehat{G}$ and define $\hat{f} = \mathbb{1}_K$. Then, by assumption,

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\widehat{G}} \left| \hat{f}(\omega) \hat{f}(\omega\alpha) \hat{g}_p(\omega) \hat{h}_p(\omega\alpha) \right| d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p)$$

$$= \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{K \cap \alpha^{-1} K} \left| \hat{g}_p(\omega) \hat{h}_p(\omega\alpha) \right| d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p) < \infty.$$

To show that (ii) implies (i), take $f \in \mathcal{D}$ and denote supp $\hat{f}$ by K. Note that $\hat{f} \in L^\infty(\widehat{G})$. Hence, we find that

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\widehat{G}} \left| \hat{f}(\omega) \hat{f}(\omega\alpha) \hat{g}_p(\omega) \hat{h}_p(\omega\alpha) \right| d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p)$$

$$\leq \|\hat{f}\|_\infty^2 \sum_{j\in J} \int_{P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_{K\cap\alpha^{-1}K} \left|\hat{g}_p(\omega)\hat{h}_p(\omega\alpha)\right| d\mu_{\widehat{G}}(\omega)\, d\mu_{P_j}(p) < \infty.$$

$\square$

In a similar way, we see that $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j, p\in P_j}$ satisfies the local integrability condition if, and only if, for each compact subset $K\subseteq \widehat{G}\setminus E$

$$\sum_{j\in J} \int_{P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_{K\cap\alpha^{-1}K} \left|\hat{g}_p(\omega)\right|^2 d\mu_{\widehat{G}}(\omega)\, d\mu_{P_j}(p) < \infty. \tag{3.3}$$

Inspired by the definition of the Calderón sum in wavelet theory, we will say that the term $\sum_{j\in J}\int_{P_j}|\hat{g}_p(\omega)|^2\, d\mu_{P_j}(p)$ is the *Calderón integral*. The next result shows that the Calderón integral is bounded if the generalized translation invariant system is a Bessel family. From this it follows that the t_α-equations (3.6) are well-defined. We remark that Proposition 3.3 generalizes [36, Proposition 3.6] and [30, Proposition 4.1] from the uniform lattice setting where each P_j is countable to the setting of generalized translation invariant systems.

Proposition 3.3. *If the generalized translation invariant system $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j, p\in P_j}$ is a Bessel family with bound B, then*

$$\sum_{j\in J} \int_{P_j} |\hat{g}_p(\omega)|^2\, d\mu_{P_j}(p) \leq B \quad \text{for a.e. } \omega\in\widehat{G}. \tag{3.4}$$

Proof. We begin by noting that the Calderón integral in (3.4) is well-defined by our standing hypothesis (III). We assume without loss of generality that $J=\mathbb{Z}$. From the Bessel assumption on $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j, p\in P_j}$, we have

$$\sum_{|j|\leq M} \int_{P_j} \int_{\Gamma_j} \left|\langle f, T_\gamma g_p\rangle\right|^2 d\mu_{\Gamma_j}(\gamma)\, d\mu_{P_j}(p) \leq B\,\|f\|^2$$

for every $M\in\mathbb{N}$ and all $f\in L^2(G)$. By Lemma 2.2 we then get

$$\sum_{|j|\leq M} \int_{P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_{\widehat{G}} \hat{f}(\omega)\overline{\hat{f}(\omega\alpha)}\,\overline{\hat{g}_p(\omega)}\,\hat{g}_p(\omega\alpha)\, d\mu_{\widehat{G}}(\omega)\, d\mu_{P_j}(p) \leq B\,\|f\|^2 \tag{3.5}$$

for every $M\in\mathbb{N}$ and all $f\in\mathcal{D}$. Assume towards a contradiction that there exists a Borel subset $N\subset\widehat{G}$ of positive measure $\mu_{\widehat{G}}(N) > 0$ for which

$$\sum_{j\in J} \int_{P_j} |\hat{g}_p(\omega)|^2\, d\mu_{P_j}(p) > B \quad \text{for a.e. } \omega\in N.$$

In [36] it is assumed that N contains an open ball, but this needs not be the case. However, since $\widehat{G}$ is σ-compact, there exists a compact set K so that $\mu_{\widehat{G}}(K\cap N) > 0$. Set $\delta_M := \inf\{d(\alpha,1) : \alpha\in\Gamma_j^\perp\setminus\{1\}, |j|\leq M\}$. For any discrete subgroup Γ there exists a $\delta > 0$ such that $B(x,\delta)\cap\Gamma = \{x\}$ for $x\in\Gamma$, where $B(x,\delta)$ denotes the open ball of radius δ and center x. It follows that $\delta_M > 0$ since δ_M is the smallest of such radii about $x=1$ from a *finite* union of discrete subgroups $\Gamma_j^\perp$. Let $\mathcal{O}$ be an open covering of K of sets with diameter strictly less than $\delta_M/2$.

Since a finite subset of $\mathcal{O}$ covers K, there is an open set $B \in \mathcal{O}$ so that $\mu_{\widehat{G}}(B \cap K \cap N) > 0$. Define $f \in L^2(G)$ by

$$\hat{f} = \mathbb{1}_{B \cap K \cap N}.$$

By Remark 4 below, we can assume that $\overline{E}$ does not intersect the closure of $B \cap K \cap N$. Therefore, $f \in \mathcal{D}$ and by our assumption we have

$$\sum_{|j| \leq M} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\widehat{G}} \hat{f}(\omega)\overline{\hat{f}(\omega\alpha)}\overline{\hat{g}_p(\omega)}\hat{g}_p(\omega\alpha)\, d\mu_{\widehat{G}}(\omega)\, d\mu_{P_j}(p)$$

$$= \int_{\widehat{G}} |\hat{f}(\omega)|^2 \sum_{|j| \leq M} \int_{P_j} |\hat{g}_p(\omega)|^2\, d\mu_{P_j}(p)\, d\mu_{\widehat{G}}(\omega),$$

where the change of the order of integration above is justified by an application of the Fubini-Tonelli theorem together with the Bessel assumption (3.4) and our standing hypotheses (I) and (III). By letting M tend to infinity, we see that

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\widehat{G}} \hat{f}(\omega)\overline{\hat{f}(\omega\alpha)}\overline{\hat{g}_p(\omega)}\hat{g}_p(\omega\alpha)\, d\mu_{\widehat{G}}(\omega)\, d\mu_{P_j}(p) > B\, \|f\|^2,$$

which contradicts (3.5). $\qquad\square$

Remark 4. In case $\overline{E}$ intersects the closure of $A := B \cap K \cap N$ in the proof of Proposition 3.3, one needs to approximate the function f with functions from $\mathcal{D}$ as defined in (2.3). As we will use such arguments several times in the remainder of this paper, let us consider how to do such a modification in this specific case. Define $E_A = \overline{E} \cap \overline{A}$ and

$$F_n = \left\{ \omega \in A : \inf\{ d(\omega, a) : a \in E_A \} < \tfrac{1}{n} \right\}, \quad \text{for each } n \in \mathbb{N}.$$

Define $\hat{f}_n = \mathbb{1}_{A \setminus F_n} \in \mathcal{D}$. Since $F_{n+1} \subset F_n$ and $\mu_{\widehat{G}}(F_1) < \infty$, we have

$$\|\hat{f} - \hat{f}_n\| = \mu_{\widehat{G}}(F_n) \to \mu_{\widehat{G}}(\cap_{n \in \mathbb{N}} F_n) = \mu_{\widehat{G}}(E_A) = 0 \qquad \text{as } n \to \infty,$$

where $\hat{f} = \mathbb{1}_{B \cap K \cap N}$. Finally, we use $\hat{f}_n$ in place of $\hat{f}$ in the final argument of the proof above, and let $n \to \infty$.

3.1   Characterization results for dual and Parseval frames

We are ready to prove the first of our main results, Theorem 3.4. Under the technical dual α-LIC assumption we characterize dual generalized translation invariant frames in terms of t_α-equations. We stress that these GTI systems are dual frames *with respect to* $\{L^2(M_j, \mu_j) : j \in J\}$ defined in the previous section. Recall that we assume a Haar measure on G to be given, and that we equip every $\Gamma_j^\perp \subset \widehat{G}$ with the counting measure.

Theorem 3.4. *Suppose that* $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ *and* $\cup_{j \in J} \{T_\gamma h_p\}_{\gamma \in \Gamma_j, p \in P_j}$ *are Bessel families satisfying the dual α-LIC. Then the following statements are equivalent:*

(i) $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ *and* $\cup_{j \in J} \{T_\gamma h_p\}_{\gamma \in \Gamma_j, p \in P_j}$ *are dual frames for* $L^2(G)$,

(ii) for each $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$ *we have*

$$t_\alpha(\omega) := \sum_{j \in J : \alpha \in \Gamma_j^\perp} \int_{P_j} \overline{\hat{g}_p(\omega)}\hat{h}_p(\omega\alpha)\, d\mu_{P_j}(p) = \delta_{\alpha,1} \quad a.e.\ \omega \in \widehat{G}. \qquad (3.6)$$

Proof. Let us first show that the t_α-equations are well-defined. Take B to be a common Bessel bound for the two GTI families. By two applications of the Cauchy-Schwarz inequality and Proposition 3.3, we find that

$$\sum_{j\in J\,:\,\alpha\in\Gamma_j^\perp}\int_{P_j}|\hat{g}_p(\omega)||\hat{h}_p(\omega\alpha)|\,d\mu_{P_j}(p) \le \sum_{j\in J}\int_{P_j}|\hat{g}_p(\omega)||\hat{h}_p(\omega\alpha)|\,d\mu_{P_j}(p)$$

$$\le \sum_{j\in J}\Big(\int_{P_j}|\hat{g}_p(\omega)|^2\,d\mu_{P_j}(p)\Big)^{1/2}\Big(\int_{P_j}|\hat{h}_p(\omega\alpha)|^2\,d\mu_{P_j}(p)\Big)^{1/2}$$

$$\le \Big(\sum_{j\in J}\int_{P_j}|\hat{g}_p(\omega)|^2\,d\mu_{P_j}(p)\Big)^{1/2}\Big(\sum_{j\in J}\int_{P_j}|\hat{h}_p(\omega\alpha)|^2\,d\mu_{P_j}(p)\Big)^{1/2} \le B,$$

for a.e. $\omega\in\widehat{G}$. This shows that the t_α-equations are well-defined and converge absolutely.

For $f\in\mathcal{D}$, define the function

$$w_f : G\to\mathbb{C},\ \ w_f(x) := \sum_{j\in J}\int_{P_j}\int_{\Gamma_j}\langle T_xf,T_\gamma g_p\rangle\langle T_\gamma h_p,T_xf\rangle\,d\mu_{\Gamma_j}(\gamma)\,d\mu_{P_j}(p). \tag{3.7}$$

By Lemma 2.2 and the calculation $\widehat{T_xf}(\omega)\overline{\widehat{T_xf}(\omega\alpha)}=\alpha(x)\hat{f}(\omega)\overline{\hat{f}(\omega\alpha)}$, we have

$$w_f(x) = \sum_{j\in J}\int_{P_j}\int_{\widehat{G}}\sum_{\alpha\in\Gamma_j^\perp}\alpha(x)\hat{f}(\omega)\overline{\hat{f}(\omega\alpha)}\overline{\hat{g}_p(\omega)}\hat{h}_p(\omega\alpha)\,d\mu_{\widehat{G}}(\omega)\,d\mu_{P_j}(p).$$

Let $\varphi_{\alpha,j}(p,\omega)$ denote the innermost summand in the right hand side expression above. By our standing hypothesis (III), the function $\varphi_{\alpha,j}$ is $(\Sigma_{P_j}\otimes B_G)$-measurable for each α. Applying Beppo Levi's theorem to the dual α local integrability condition yields that the function $\sum_\alpha\varphi_{\alpha,j}$ belongs to $L^1(P_j\times\widehat{G})$ for each $j\in J$. An application of Fubini's theorem now gives:

$$w_f(x) = \sum_{j\in J}\int_{\widehat{G}}\int_{P_j}\sum_{\alpha\in\bigcup_{j\in J}\Gamma_j^\perp}\mathbb{1}_{\Gamma_j^\perp}(\alpha)\varphi_{\alpha,j}(p,\omega)\,d\mu_{P_j}(p)\,d\mu_{\widehat{G}}(\omega).$$

Lebesgue's dominated convergence theorem then yields:

$$w_f(x) = \sum_{j\in J}\sum_{\alpha\in\bigcup_{j\in J}\Gamma_j^\perp}\alpha(x)\int_{\widehat{G}}\int_{P_j}\mathbb{1}_{\Gamma_j^\perp}(\alpha)\hat{f}(\omega)\overline{\hat{f}(\omega\alpha)}\overline{\hat{g}_p(\omega)}\hat{h}_p(\omega\alpha)\,d\mu_{P_j}(p)\,d\mu_{\widehat{G}}(\omega).$$

By the dual α local integrability condition the summand belongs to $\ell^1(J\times\cup_{j\in J}\Gamma_j^\perp)$ and we can therefore interchange the order of summations. Further, by Lebesgue's bounded convergence theorem, we can interchange the sum over $j\in J$ and the integral over $\mathrm{supp}\,\hat{f}\subset\widehat{G}$. Hence,

$$w_f(x) = \sum_{\alpha\in\bigcup_{j\in J}\Gamma_j^\perp}\alpha(x)\int_{\widehat{G}}\hat{f}(\omega)\overline{\hat{f}(\omega\alpha)}\sum_{j\in J\,:\,\alpha\in\Gamma_j^\perp}\int_{P_j}\overline{\hat{g}_p(\omega)}\hat{h}_p(\omega\alpha)\,d\mu_{P_j}(p)\,d\mu_{\widehat{G}}(\omega).$$

Finally, we arrive at:

$$w_f(x) = \sum_{\alpha\in\bigcup_{j\in J}\Gamma_j^\perp}\alpha(x)\hat{w}(\alpha),\quad\text{where}\quad \hat{w}(\alpha) := \int_{\widehat{G}}\hat{f}(\omega)\overline{\hat{f}(\omega\alpha)}t_\alpha(\omega)\,d\mu_{\widehat{G}}(\omega). \tag{3.8}$$

From the previous calculations and the dual α-LIC, it follows that the convergence in (3.8) is absolute. By the Weierstrass M-test, we see that w_f is the uniform limit of a generalized Fourier series and thus an almost periodic, continuous function.

We start by showing the implication (ii)$\Rightarrow$(i). Inserting (3.6) into (3.8) for $x = 0$ yields

$$w_f(0) = \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} \langle f, T_\gamma g_p \rangle \langle T_\gamma h_p, f \rangle \, d\mu_{\Gamma_j}(\gamma) \, d\mu_{P_j}(p)$$

$$= \sum_{\alpha \in \bigcup\limits_{j \in J} \Gamma_j^\perp} \alpha(0) \int_{\widehat{G}} \hat{f}(\omega) \overline{\hat{f}(\omega\alpha)} \delta_{\alpha,1} \, d\mu_{\widehat{G}}(\omega) = \langle f, f \rangle,$$

and (i) follows by Lemma 2.5(i).

For the converse implication (i)$\Rightarrow$(ii), we have

$$w_f(x) = \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} \langle T_x f, T_\gamma g_p \rangle \langle T_\gamma h_p, T_x f \rangle \, d\mu_{\Gamma_j}(\gamma) \, d\mu_{P_j}(p) = \|f\|^2$$

for each $f \in \mathcal{D}$. Consider now the function $z(x) := w_f(x) - \|f\|^2$. We have shown that w_f is continuous and by construction z is identical to the zero function. Additionally, since w_f equals an absolute convergent, generalized Fourier series, also z can be expressed as an absolute convergent generalized Fourier series $z(x) = \sum\limits_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} \alpha(x)\hat{z}(\alpha)$, with

$$\hat{z}(\alpha) = \begin{cases} \int_{\widehat{G}} |\hat{f}(\omega)|^2 t_1(\omega) \, d\mu_{\widehat{G}}(\omega) - \|f\|^2 & \text{for } \alpha = 1, \\[2mm] \int_{\widehat{G}} \hat{f}(\omega) \overline{\hat{f}(\omega\alpha)} t_\alpha(\omega) \, d\mu_{\widehat{G}}(\omega) & \text{for } \alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{1\}. \end{cases}$$

By the uniqueness theorem for generalized Fourier series [13, Theorem 7.12], the function $z(x)$ is identical to zero if, and only if, $\hat{z}(\alpha) = 0$ for all $\alpha \in \cup_{j \in J} \Gamma_j^\perp$.

In case $\alpha = 1$ we have $\int_{\widehat{G}} |\hat{f}(\omega)|^2 (t_1(\omega) - 1) d\mu_{\widehat{G}}(\omega) = 0$ for $f \in \mathcal{D}$. Hence, since $\mathcal{D}$ is dense in $L^2(G)$, we conclude that $t_1(\omega) = 1$ for a.e. $\omega \in \widehat{G}$. For $\alpha \in \cup_{j \in J} \Gamma_j^\perp \setminus \{1\}$, we have

$$\int_{\widehat{G}} \hat{f}(\omega) \overline{\hat{f}(\omega\alpha)} t_\alpha(\omega) \, d\mu_{\widehat{G}}(\omega) = 0. \tag{3.9}$$

Define the multiplication operator $M_{\overline{t_\alpha}} : L^2(\widehat{G}) \to L^2(\widehat{G})$ by $M_{\overline{t_\alpha}} \hat{f}(\omega) = \overline{t_\alpha(\omega)} \hat{f}(\omega)$. This linear operator is bounded since by Proposition 3.3 $t_\alpha(\omega) \in L^\infty(\widehat{G})$. We can now rewrite the left hand side of (3.9) as an inner-product:

$$\langle \hat{f}, M_{\overline{t_\alpha}} T_{\alpha^{-1}} \hat{f} \rangle_{L^2(\widehat{G})} = 0,$$

where $f \in \mathcal{D}$. Since $\mathcal{D}$ is dense in the complex Hilbert space $L^2(G)$, this implies that $M_{\overline{t_\alpha}} T_{\alpha^{-1}} = 0$. After multiplication with T_α from the right, we have $M_{\overline{t_\alpha}} = 0$ and therefore $t_\alpha = 0$. $\qquad\square$

From Theorem 3.4 we easily obtain the corresponding characterization for tight frames. We state it for Parseval frames only as it is just a matter of scaling.

Theorem 3.5. *Suppose that the generalized translation invariant system $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the α local integrability condition. Then the following assertions are equivalent:*

(i) $\cup_{j\in J}\left\{T_\gamma g_p\right\}_{\gamma\in\Gamma_j,p\in P_j}$ *is a Parseval frame for* $L^2(G)$,

(ii) for each $\alpha\in\bigcup_{j\in J}\Gamma_j^\perp$ *we have*

$$t_\alpha := \sum_{j\in J\,:\,\alpha\in\Gamma_j^\perp}\int_{P_j}\overline{\hat{g}_p(\omega)}\hat{g}_p(\omega\alpha)\,d\mu_{P_j}(p) = \delta_{\alpha,1}\quad a.e.\ \omega\in\widehat{G}.$$

Proof. We first remark that the integrals in (ii) indeed converge absolutely. This follows from two applications of the Cauchy-Schwarz' inequality (as in the proof of Theorem 3.4), which gives:

$$\sum_{j\in J\,:\,\alpha\in\Gamma_j^\perp}\int_{P_j}|\hat{g}_p(\omega)|\,|\hat{g}_p(\omega\alpha)|\,d\mu_{P_j}(p) \le 1.$$

In view of Theorem 3.4 we only have to argue that the assumption on the Bessel family can be omitted. If we assume (i), then clearly $\cup_{j\in J}\left\{T_\gamma g_p\right\}_{\gamma\in\Gamma_j,p\in P_j}$ is a Bessel family and (ii) follows from Theorem 3.4.

Suppose that (ii) holds. Formula (3.8) is still valid, where w_f is defined as in (3.7) with $h_p = g_p$. Setting $x = 0$ in (3.8) yields

$$\|f\|^2 = \sum_{j\in J}\int_{P_j}\int_{\Gamma_j}|\langle f,T_\gamma g_p\rangle|^2\,d\mu_{\Gamma_j}(\gamma)\,d\mu_{P_j}(p)\quad\text{for all }f\in\mathcal{D}.$$

Finally, we conclude by Lemma 2.5(ii) that $\cup_{j\in J}\left\{T_\gamma g_p\right\}_{\gamma\in\Gamma_j,p\in P_j}$ is a Parseval frame for $L^2(G)$.
$\square$

By virtue of Lemma 2.4, we have the following extension of Theorem 3.4 and 3.5.

Corollary 3.6. *The characterization results in Theorem 3.4 and 3.5 extend to systems that are unitarily equivalent to generalized translation invariant systems.*

3.2   On sufficient conditions and the local integrability conditions

Let us now turn to sufficient conditions for a generalized translation invariant system to be a Bessel family or a frame. Proposition 3.7 is a generalization of the results in, e.g., [10] and [9], which state the corresponding result for GSI systems in the euclidean space and locally compact abelian groups, respectively. The result is as follows.

Proposition 3.7. *Consider the generalized translation invariant system* $\cup_{j\in J}\left\{T_\gamma g_p\right\}_{\gamma\in\Gamma_j,p\in P_j}$.

(i) If

$$B := \operatorname*{ess\,sup}_{\omega\in\widehat{G}}\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\left|\hat{g}_p(\omega)\hat{g}_p(\omega\alpha)\right|d\mu_{P_j}(p) < \infty,\tag{3.10}$$

then $\cup_{j\in J}\left\{T_\gamma g_p\right\}_{\gamma\in\Gamma_j,p\in P_j}$ *is a Bessel family with bound* B.

(ii) Furthermore, if also

$$A := \operatorname*{ess\,inf}_{\omega\in\widehat{G}}\left(\sum_{j\in J}\int_{P_j}|\hat{g}_p(\omega)|^2\,d\mu_{P_j}(p) - \sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp\setminus\{0\}}|\hat{g}_p(\omega)\hat{g}_p(\omega\alpha)|\,d\mu_{P_j}(p)\right) > 0,$$

then $\cup_{j\in J}\left\{T_\gamma g_p\right\}_{\gamma\in\Gamma_j,p\in P_j}$ *is a frame for* $L^2(G)$ *with bounds* A *and* B.

Proof. With a few adaptations the result follows from the corresponding proofs in [9] and [10]. $\quad\square$

We refer to (3.10) as the absolute CC-condition, or for short, CC-condition [7]. Proposition 3.7 is useful in applications as a mean to verify that a given family indeed is Bessel, or even a frame. Moreover, in relation to the characterizing results in Theorem 3.4 and 3.5, the CC-condition (3.10) is sufficient for the α-LIC to hold. In contrast, we remark that, by Example 1 in Section 3.3, the CC-condition does not imply the LIC.

Lemma 3.8. *If $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j,p\in P_j}$ and $\cup_{j\in J}\{T_\gamma h_p\}_{\gamma\in\Gamma_j,p\in P_j}$ satisfy*

$$\operatorname*{ess\,sup}_{\omega\in\widehat{G}}\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\left|\hat{g}_p(\omega)\hat{h}_p(\omega\alpha)\right|d\mu_{P_j}(p)<\infty$$

and

$$\operatorname*{ess\,sup}_{\omega\in\widehat{G}}\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\left|\hat{g}_p(\omega\alpha)\hat{h}_p(\omega)\right|d\mu_{P_j}(p)<\infty,$$

then the dual α local integrability condition is satisfied. Furthermore, if $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j,p\in P_j}$ satisfies the CC-condition (3.10), then the α local integrability condition is satisfied.

Proof. By applications of Cauchy-Schwarz' inequality, we find

$$\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\int_{\widehat{G}}|\hat{f}(\omega)\hat{f}(\omega\alpha)\hat{g}_p(\omega)\hat{h}_p(\omega\alpha)|\,d\mu_{\widehat{G}}(\omega)\,d\mu_{P_j}(p)$$

$$\leq\left[\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\int_{\widehat{G}}|\hat{f}(\omega)|^2|\hat{g}_p(\omega)\hat{h}_p(\omega\alpha)|\,d\mu_{\widehat{G}}(\omega)\,d\mu_{P_j}(p)\right]^{1/2}$$

$$\times\left[\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\int_{\widehat{G}}|\hat{f}(\omega\alpha)|^2|\hat{g}_p(\omega)\hat{h}_p(\omega\alpha)|\,d\mu_{\widehat{G}}(\omega)\,d\mu_{P_j}(p)\right]^{1/2}$$

$$=\left[\int_{\widehat{G}}|\hat{f}(\omega)|^2\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}|\hat{g}_p(\omega)\hat{h}_p(\omega\alpha)|\,d\mu_{P_j}(p)\,d\mu_{\widehat{G}}(\omega)\right]^{1/2}$$

$$\times\left[\int_{\widehat{G}}|\hat{f}(\omega)|^2\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}|\hat{g}_p(\omega\alpha)\hat{h}_p(\omega)|\,d\mu_{P_j}(p)\,d\mu_{\widehat{G}}(\omega)\right]^{1/2}<\infty.$$

$\square$

Finally, we show that the LIC implies the (dual) α-LIC. The precise statement is as follows.

Lemma 3.9. *If both $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j,p\in P_j}$ and $\cup_{j\in J}\{T_\gamma h_p\}_{\gamma\in\Gamma_j,p\in P_j}$ satisfy the local integrability condition (3.2), then $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j,p\in P_j}$ and $\cup_{j\in J}\{T_\gamma h_p\}_{\gamma\in\Gamma_j,p\in P_j}$ satisfy the dual α local integrability condition. In particular, if $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j,p\in P_j}$ satisfies the local integrability condition, then it also satisfies the α local integrability condition.*

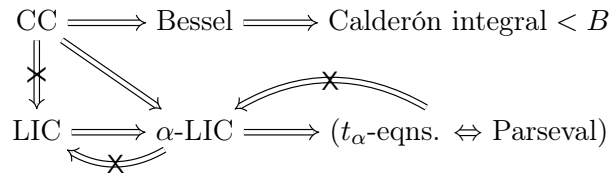*Proof.* By use of Cauchy-Schwarz' inequality and $2\,|cd|\leq|c|^2+|d|^2$, we have

$$\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\int_{\widehat{G}}\left|\hat{f}(\omega)\hat{f}(\omega\alpha)\hat{g}_p(\omega)\hat{h}_p(\omega\alpha)\right|d\mu_{\widehat{G}}(\omega)\,d\mu_{P_j}(p)$$

$$\leq \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^{\perp}} \left(\int_{\alpha^{-1} \operatorname{supp} \hat{f}} \left| \hat{f}(\omega) \hat{h}_p(\omega\alpha) \right|^2 d\mu_{\widehat{G}}(\omega) \right)^{1/2} \left(\int_{\operatorname{supp} \hat{f}} \left| \hat{f}(\omega\alpha) \hat{g}_p(\omega) \right|^2 d\mu_{\widehat{G}}(\omega) \right)^{1/2} d\mu_{P_j}(p)$$

$$= \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^{\perp}} \left(\int_{\operatorname{supp} \hat{f}} \left| \hat{f}(\omega\alpha^{-1}) \hat{h}_p(\omega) \right|^2 d\mu_{\widehat{G}}(\omega) \right)^{1/2} \left(\int_{\operatorname{supp} \hat{f}} \left| \hat{f}(\omega\alpha) \hat{g}_p(\omega) \right|^2 d\mu_{\widehat{G}}(\omega) \right)^{1/2} d\mu_{P_j}(p)$$

$$\leq \tfrac{1}{2} \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^{\perp}} \int_{\operatorname{supp} \hat{f}} \left| \hat{f}(\omega\alpha^{-1}) \hat{h}_p(\omega) \right|^2 d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p)$$

$$+ \tfrac{1}{2} \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^{\perp}} \int_{\operatorname{supp} \hat{f}} \left| \hat{f}(\omega\alpha) \hat{g}_p(\omega) \right|^2 d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p) < \infty,$$

and the statements follow. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The relationships between the various conditions considered above are summarized in the diagram below. To simplify the presentation we do not consider dual frames. An arrow means that the assumption at the tail of the arrow implies the assumption at the head. A crossed out arrow means that one can find a counter example for that implication; clearly, implications to the left in the top line are also not true in general.

$$
\begin{array}{ccc}
\text{CC} \Longrightarrow & \text{Bessel} \Longrightarrow & \text{Calderón integral} < B \\
\Big\Updownarrow\!\!\!\times & & \times \\
\text{LIC} \Longrightarrow & \alpha\text{-LIC} \Longrightarrow & (t_\alpha\text{-eqns.} \Leftrightarrow \text{Parseval})
\end{array}
$$

The crossed out arrows are shown by Example 1 and Example 2 in the next section.

3.3   Two examples on the role of the local integrability conditions

In this section we consider two key examples. Both examples take place in $\ell^2(\mathbb{Z})$; however, they can be extended to $L^2(\mathbb{R})$, see [5]. The first example, Example 1, shows that for a GTI system the α local integrability condition is strictly weaker than the local integrability condition.

Example 1. Let $G = \mathbb{Z}$, $N \in \mathbb{N}$, $N \geq 2$ and consider the co-compact subgroups $\Gamma_j = N^j\mathbb{Z}$, $j \in \mathbb{N}$. Note that $\widehat{G}$ can be identified with the half-open unit interval $[0,1)$ under addition modulo one. To each Γ_j we associate N^j functions $g_{j,p}$, for $p = 0, 1, \ldots, N^j - 1$. Each function $g_{j,p}$ is defined by its Fourier transform

$$\hat{g}_{j,p} = (N-1)^{1/2} N^{-j/2} \mathbb{1}_{[p/N^j,(p+1)/N^j)}.$$

The factor $(N-1)^{1/2}$ is for normalization purposes and does not play a role in the calculations. The annihilator of each Γ_j is given by $\Gamma_j^{\perp} = N^{-j}\mathbb{Z} \cap [0,1)$. Note that the number of elements in $\Gamma_j^{\perp}$ is N^j. We equip both G and $\Gamma_j^{\perp}$ with the counting measure, this implies that the measure on Γ_j is the counting measure multiplied by N^j. For the generalized translation invariant system $\cup_{j \in \mathbb{N}} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p=0,1,\ldots,N^j-1}$ we show the following: (i) the LIC is violated, (ii) the α-LIC holds, (iii) the system is a Parseval frame for $\ell^2(\mathbb{Z})$. It then follows from Theorem 3.5 that the t_α-equations are satisfied.

Ad (i). In order for the LIC to hold we need

$$\sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{\alpha \in \Gamma_j^{\perp}} \int_{K \cap (K-\alpha)} |\hat{g}_{j,p}(\omega)|^2 \, d\omega \; < \; \infty$$

for all compact $K \subseteq [0, 1)$, see Lemma 3.2. In particular for $K = \widehat{G}$, we find

$$\sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{\alpha \in \Gamma_j^{\perp}} \int_0^1 |\hat{g}_{j,p}(\omega)|^2 \, d\omega = (N-1) \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{\alpha \in \Gamma_j^{\perp}} N^{-2j}$$

$$= (N-1) \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} N^{-j} = (N-1) \sum_{j=1}^{\infty} 1 = \infty.$$

Therefore, the local integrability condition is not satisfied.

Ad (ii). By Lemma 3.2 it suffices to show that

$$\sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{\alpha \in \Gamma_j^{\perp}} \int_{\widehat{G} \cap (\widehat{G}-\alpha)} \left| \hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega+\alpha) \right| d\omega < \infty.$$

Due to the support of $\hat{g}_{j,p}$ we have $|\overline{\hat{g}_{j,p}(\omega)}\hat{g}_{j,p}(\omega+\alpha)| = 0$ for $\alpha \in \Gamma_j^{\perp} \setminus \{0\}$. We thus find that

$$\sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{\alpha \in \Gamma_j^{\perp}} \int_0^1 |\hat{g}_{j,p}(\omega)\hat{g}_{j,p}(\omega+\alpha)| \, d\omega = \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \int_0^1 |\hat{g}_{j,p}(\omega)|^2 \, d\omega$$

$$= (N-1) \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} N^{-2j} = (N-1) \sum_{j=1}^{\infty} N^{-j} = 1.$$

Ad (iii). Note that $\sum_{p=0}^{N^j-1} |\hat{g}_{j,p}(\omega)|^2 = (N-1)N^{-j} \math1_{[0,1)}(\omega)$ for $\omega \in [0,1)$ and for all $j \in \mathbb{N}$. Using the frame bound estimates from Proposition 3.7, we have

$$B = \operatorname*{ess\,sup}_{\omega \in [0,1)} \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{\alpha \in \Gamma_j^{\perp}} |\hat{g}_{j,p}(\omega)\hat{g}_{j,p}(\omega+\alpha)|$$

$$= \operatorname*{ess\,sup}_{\omega \in [0,1)} \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} |\hat{g}_{j,p}(\omega)|^2 = \operatorname*{ess\,sup}_{\omega \in [0,1)} (N-1) \sum_{j=1}^{\infty} N^{-j} \math1_{[0,1)}(\omega) = 1.$$

In the same way, for the lower frame bound, we find

$$A = \operatorname*{ess\,inf}_{\omega \in [0,1)} \left(\sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} |\hat{g}_{j,p}(\omega)|^2 - \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{\alpha \in \Gamma_j^{\perp} \setminus \{0\}} |\hat{g}_{j,p}(\omega)\hat{g}_{j,p}(\omega+\alpha)| \right) = 1.$$

These calculations also show that $\cup_{j \in \mathbb{N}} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p=0,1,\dots,N^j-1}$ is actually a union over $j \in \mathbb{N}$ of tight frames $\{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p=0,1,\dots,N^j-1}$ each with frame bound N^{-j}. Furthermore, we see that the CC-condition is satisfied, even though the LIC fails. Hence, the CC-condition does not imply LIC (however, by Lemma 3.8 it does imply the α-LIC).

The following example is inspired by similar constructions in [5] and [36]. It shows two points. Firstly, the α local integrability condition cannot be removed in Theorem 3.5. Secondly, it is possible for a GTI Parseval frame to satisfy the t_α-equations even though the α local integrability condition fails. We show these observations in the reversed order.

Example 2. Let $G = \mathbb{Z}$ and for each $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, let $[m]_k$ denote the residue class of m modulo k. Then, for $\tau_j = 2^{j-1} - 1$, $j \in \mathbb{N}$,

$$\mathbb{Z} = \bigcup_{j \in \mathbb{N}} [\tau_j]_{2^j} = [0]_2 \cup [1]_4 \cup [3]_8 \cup [7]_{16} \cup [15]_{32} \ldots,$$

where the union is disjoint. Now set $g_j = N^{-j/2} \mathbb{1}_{\tau_j}$ and $\Gamma_j = N^j \mathbb{Z}$ for $N = 2$. The GTI system $\cup_{j \in \mathbb{N}} \{T_\gamma g_j\}_{\gamma \in \Gamma_j}$ is essentially a reordering of the standard orthonormal basis $\{e_k\}_{k \in \mathbb{Z}}$ for $\ell^2(\mathbb{Z})$. The factor $N^{-j/2}$ in the definition of g_j is due to the fact that we equip $\Gamma_j^\perp$ with the counting measure. This implies that the measure on Γ_j becomes N^j times the counting measure. One can now show that this GTI system does not satisfy the α-LIC. However, the system does indeed satisfy the t_α-equations. For $\alpha = 0$:

$$\sum_{j=1}^\infty |\hat{g}_j(\omega)|^2 = \sum_{j=1}^\infty 2^{-j} \left| e^{2\pi i \tau_j \omega} \right|^2 = \frac{1}{2-1} = 1,$$

and for $\alpha = k/2^{j^*} \in 2^{-j^*} \mathbb{Z} = \Gamma_{j^*}^\perp$, where k is odd,

$$\sum_{j \in J : \alpha \in \Gamma_j^\perp} \hat{g}_j(\omega) \overline{\hat{g}_j(\omega + \alpha)} = \sum_{j=j^*}^\infty 2^{-j} e^{-2\pi i \frac{k}{2^{j^*}} (2^{j-1}-1)} = e^{2\pi i k 2^{-j^*}} \sum_{j=j^*}^\infty 2^{-j} e^{-2\pi i \frac{2^{j-1}}{2^{j^*}}}$$

$$= e^{2\pi i k 2^{-j^*}} \left(-2^{-j^*} + \sum_{j=j^*+1}^\infty 2^{-j} \right) = 0.$$

If one uses $N \geq 3$, $N \in \mathbb{N}$ in place of $N = 2$, then the α-LIC is still not satisfied. However, even though for suitably chosen τ_j (the formula is more complicated than for $N = 2$, see [5]) $\cup_{j \in \mathbb{N}} \{T_\gamma N^{-j/2} \mathbb{1}_{\tau_j}\}_{\gamma \in N^j \mathbb{Z}}$ is still essentially a reordering of the standard orthonormal basis, every t_α-equation is false. The case $\alpha = 0$ gives $t_\alpha = \frac{1}{N-1} \neq 1$, while the cases $\alpha \neq 0$ give $t_\alpha \neq 0$. We stress that these examples show the existence of generalized translation invariant Parseval frames for $\ell^2(\mathbb{Z})$ which do not satisfy the t_α-equations.

3.4 Characterization results for special groups

Under special circumstances the local integrability condition will be satisfied automatically. In this section we will see that this is indeed the case for TI systems, i.e., $\Gamma_j = \Gamma$ for all $j \in J$, and for GTI systems on compact abelian groups G. For brevity, we will only state the corresponding characterization results for dual frames, but remark here that the results hold equally for Parseval frames, in which case, the Bessel family assumption can be omitted.

Let us begin with a lemma concerning general GTI systems for LCA groups showing that the LIC holds if the annihilators of Γ_j possess a sufficient amount of separation.

Lemma 3.10. *If $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ has a uniformly bounded Calderón integral and if there exists a constant $C > 0$ such that for all compact $K \subseteq \widehat{G}$*

$$\sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} \mu_{\widehat{G}}(K \cap \alpha^{-1} K) \leq C,$$

then $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the local integrability condition.

Proof. By assumption there exists a constant $B > 0$ such that $\sum_{j \in J} \int_{P_j} |\hat{g}_p(\omega)|^2 \, d\mu_{P_j}(p) < B$ for a.e. $\omega \in \widehat{G}$, and we therefore have

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{K \cap \alpha^{-1}K} |\hat{g}_p(\omega)|^2 \, d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p)$$

$$= \sum_{\alpha \in \bigcup\limits_{j \in J} \Gamma_j^\perp} \int_{K \cap \alpha^{-1}K} \sum_{j \in J} \int_{P_j} |\hat{g}_p(\omega)|^2 \, d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p) \leq BC < \infty.$$

$\square$

Now, let us consider the case where all subgroups Γ_j coincide. In other words, we consider *translation invariant* systems. Note that this setting includes the continuous wavelet and Gabor transform as well as the shift invariant systems considered in [30, 36].

Theorem 3.11. *Let Γ be a co-compact subgroup in G. Suppose that $\cup_{j \in J}\{T_\gamma g_p\}_{\gamma \in \Gamma, p \in P_j}$ and $\cup_{j \in J}\{T_\gamma h_p\}_{\gamma \in \Gamma, p \in P_j}$ are Bessel families. Then the following statements are equivalent:*

(i) $\cup_{j \in J}\{T_\gamma g_p\}_{\gamma \in \Gamma, p \in P_j}$ *and* $\cup_{j \in J}\{T_\gamma h_p\}_{\gamma \in \Gamma, p \in P_j}$ *are dual frames for $L^2(G)$,*

(ii) For each $\alpha \in \Gamma^\perp$ we have

$$t_\alpha(\omega) := \sum_{j \in J} \int_{P_j} \overline{\hat{g}_p(\omega)} \hat{h}_p(\omega\alpha) \, d\mu_{P_j}(p) = \delta_{\alpha,1} \quad a.e. \ \omega \in \widehat{G}. \tag{3.11}$$

Proof. Since $\Gamma^\perp$ is a discrete subgroup in $\widehat{G}$ and since the metric on $\widehat{G}$ is translation invariant, there exists a $\delta > 0$ so that the distance between two distinct points from $\Gamma^\perp$ is larger than δ. Thus, for any compact $K \subset \widehat{G}$, the set $\Gamma^\perp \cap (K^{-1}K)$ has finite cardinality because, if not, then $\Gamma^\perp \cap (K^{-1}K)$ would contain a sequence (take one without repetitions) with no convergent subsequence which contradicts the compactness of K. Since $\{\alpha \in \Gamma^\perp : K\alpha \cap K \neq \emptyset\}$ is a subset of $\Gamma^\perp \cap (K^{-1}K)$, it is also of finite cardinality. From this together with the Bessel assumption and Proposition 3.3 we conclude that the assumptions of Lemma 3.10 are satisfied and hence the LIC holds. By Lemma 3.9 the dual α-LIC is satisfied and the result now readily follows from Theorem 3.4. $\square$

For TI systems with translation along the entire group $\Gamma = G$ there is only one t_α-equation in (3.11) since $G^\perp = \{1\}$. To be precise:

Lemma 3.12. *Suppose that $\Gamma = G$. Then assertion (ii) in Theorem 3.11 reduces to*

$$\sum_{j \in J} \int_{P_j} \overline{\hat{g}_p(\omega)} \hat{h}_p(\omega) \, d\mu_{P_j}(p) = 1 \quad a.e. \ \omega \in \widehat{G}.$$

Let us now turn to the familiar setting of [30, 36], where Γ is a uniform lattice, i.e., a discrete, co-compact subgroup. Then there is a compact fundamental domain $F \subset G$ for Γ, such that $G = F\Gamma$, and moreover for any $x \in G$ we have $x = \varphi\gamma$, where $\varphi \in F, \gamma \in \Gamma$ are unique. For a uniform lattice we introduce the *lattice size* $s(\Gamma) := \mu_G(F)$, which is, in fact, independent of the choice of F.

Corollary 3.13. *Let Γ be a uniform lattice in G. Suppose that the two generalized translation invariant systems $\cup_{j \in J}\{T_\gamma g_p\}_{\gamma \in \Gamma, p \in P_j}$ and $\cup_{j \in J}\{T_\gamma h_p\}_{\gamma \in \Gamma, p \in P_j}$ are Bessel families. Then the following statements are equivalent:*

(i) $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma,p\in P_j}$ and $\cup_{j\in J}\{T_\gamma h_p\}_{\gamma\in\Gamma,p\in P_j}$ are dual frames for $L^2(G)$, i.e.,

$$\langle f_1,f_2\rangle = \sum_{j\in J}\int_{P_j} s(\Gamma)\sum_{\gamma\in\Gamma}\langle f_1,T_\gamma g_p\rangle\langle T_\gamma h_p,f_2\rangle\,d\mu_{P_j}(p),\quad \text{for all } f_1,f_2\in L^2(G). \quad (3.12)$$

(ii) For each $\alpha\in\Gamma^\perp$ we have $t_\alpha(\omega)=\delta_{\alpha,1}$ for a.e. $\omega\in\widehat{G}$, where t_α is defined in (3.11)

Remark 5. In the same way, we can state the characterization results for *generalized* shift-invariant systems. In this case we have countable many uniform lattices Γ_j, so we replace $s(\Gamma)$ in Corollary 3.13 with $s(\Gamma_j)$, sum over $\{j\in J:\alpha\in\Gamma_j^\perp\}$ in (3.12), and add the dual α local integrability condition as assumption. We obtain a statement equivalent to the main characterization result in [36]. In contrast to the result in [36], the lattice size $s(\Gamma)$ is contained in the reproducing formula rather than in the t_α-equations.

For compact abelian groups all generalized translation invariant systems satisfy the local integrability condition. The characterization result is as follows.

Theorem 3.14. *Let G be a compact abelian group. Suppose that $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j,p\in P_j}$ and $\cup_{j\in J}\{T_\gamma h_p\}_{\gamma\in\Gamma_j,p\in P_j}$ are Bessel families. Then the following statements are equivalent:*

(i) $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j,p\in P_j}$ and $\cup_{j\in J}\{T_\gamma h_p\}_{\gamma\in\Gamma_j,p\in P_j}$ are dual frames for $L^2(G)$,

(ii) for each $\alpha\in\bigcup_{j\in J}\Gamma_j^\perp$ we have

$$t_\alpha(\omega):=\sum_{j\in J\,:\,\alpha\in\Gamma_j^\perp}\int_{P_j}\overline{\hat{g}_p(\omega)}\hat{h}_p(\omega\alpha)\,d\mu_{P_j}(p)=\delta_{\alpha,1}\quad a.e.\ \omega\in\widehat{G}.$$

Proof. Because G is compact, the dual group $\widehat{G}$ is discrete. All compact $K\subset\widehat{G}$ are therefore finite. Let $\#K$ denote the number of elements in K. From the LIC we then find

$$\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\sum_{\omega\in K\cap\alpha^{-1}K}|\hat{g}_p(\omega)|^2\,d\mu_{P_j}(p) \le \sum_{j\in J}\int_{P_j}\#K\sum_{\omega\in K}|\hat{g}_p(\omega)|^2\,d\mu_{P_j}(p)$$

$$\le (\#K)^2\max_{\omega\in K}\sum_{j\in J}\int_{P_j}|\hat{g}_p(\omega)|^2\,d\mu_{P_j}.$$

By the Bessel assumption and Proposition 3.3, the Calderón integral is bounded. The far right hand side in the above calculation is therefore finite, and the LIC is satisfied. The result now follows from Theorem 3.4 and Lemma 3.9. $\qquad\square$

Finally, let us turn to discrete groups G. In this case, the local integrability condition is *not* automatically satisfied (as we saw in the examples in the previous section), but it has a simple reformulation:

Lemma 3.15. *Suppose G is a discrete abelian group. Then the following statements are equivalent:*

(i) The system $\cup_{j\in J}\{T_\gamma g_p\}_{\gamma\in\Gamma_j,p\in P_j}$ satisfies the local integrability condition,

(ii) $\sum_{j\in J}\int_{P_j}\mu_c(\Gamma_j^\perp)\|g_p\|_{L^2(G)}^2\,d\mu_{P_j}(p)<\infty$, where μ_c denotes the counting measure.

Proof. Note that if G is discrete, then $\widehat{G}$ is compact. Hence the discrete groups $\Gamma_j^\perp$ are also compact and therefore finite. By this observation we can easily show the result. If (i) holds, then

$$\sum_{j \in J} \int_{P_j} \mu_c(\Gamma_j^\perp) \, \|g_p\|_{L^2(G)}^2 \, d\mu_{P_j}(p) \leq \sum_{j \in J} \int_{P_j} \int_{\widehat{G}} \mu_c(\Gamma_j^\perp) \, |\hat{g}_p(\omega)|^2 \, d\mu_{\widehat{G}}(\omega) \, d\mu_{P_j}(p).$$

By (3.3) with $K = \widehat{G}$ the right hand side is finite, and (ii) follows. If (ii) holds, then

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{K \cap \alpha^{-1}K} |\hat{g}_p(\omega)|^2 d\mu_{\widehat{G}}(\omega) d\mu_{P_j}(p) \leq \sum_{j \in J} \int_{P_j} \mu_c(\Gamma_j^\perp) \int_{\widehat{G}} |\hat{g}_p(\omega)|^2 d\mu_{\widehat{G}}(\omega) d\mu_{P_j}(p) < \infty.$$

$\square$

4   Applications and discussions of the characterization results

In this section we study applications of Theorem 3.4 leading to new characterization results. Moreover, we will easily recover known results as special cases of our theory. We consider Gabor and wavelet-like systems for general locally compact abelian groups as well as for specific locally compact abelian groups, e.g., $\mathbb{R}^n, \mathbb{Z}^n, \mathbb{Z}_n$. We also give an example of characterization results for the locally compact abelian group of p-adic numbers, where the theory of generalized *shift* invariant systems is not applicable.

We will focus on verifying the local integrability conditions and on the deriving the characterizing equations, but not on the related question of how to construct generators satisfying these equations. The recent work of Christensen and Goh [9] takes this more constructive approach for generalized *shift* invariant systems on locally compact abelian groups. Under certain assumptions, they explicitly construct dual GSI frames using variants of t_α-equations, which are proved to be sufficient.

4.1   Gabor systems

A *Gabor system* in $L^2(G)$ with generator $g \in L^2(G)$ is a family of functions of the form

$$\{E_\gamma T_\lambda g\}_{\gamma \in \Gamma, \lambda \in \Lambda}, \text{ where } \Gamma \subseteq \widehat{G} \text{ and } \Lambda \subseteq G.$$

Note that a Gabor system $\{E_\gamma T_\lambda g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is not a generalized translation invariant system because $E_\gamma T_\lambda g = T_\lambda(\gamma(\lambda) E_\gamma g)$ cannot be written as $T_\gamma g_{j,p}$ for $j \in J$ and $p \in P_j$ for any $\{g_{j,p}\}$. However, by use of Lemma 2.4, we can establish the following two possibilities to relate Gabor and translation invariant systems.

Firstly, by Lemma 2.4 with $\iota = \mathrm{id}$, $U = \mathcal{F}$ and $c_{\gamma,\lambda} = 1$, we see that the Gabor system $\{E_\gamma T_\lambda g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is a frame if, and only if, the translation invariant system $\{T_\gamma \mathcal{F}^{-1} T_\lambda g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is a frame. By this observation all results for translation invariant systems naturally carry over to Gabor systems. In order to apply the theory established in this paper, we need Γ to be a closed, co-compact subgroup of $\widehat{G}$ and Λ to be equipped with a measure μ_Λ satisfying the standing hypotheses (I)–(III). This approach together with Theorem 3.4 yield t_α-equations in the *time domain* G: for each $\alpha \in \Gamma^\perp$ we have

$$\int_\Lambda \overline{g(x - \lambda)} h(x - \lambda + \alpha) \, d\mu_\Lambda(\lambda) = \delta_{\alpha,0} \quad \text{a.e. } x \in G.$$

Secondly, by Lemma 2.4 with $\iota = \mathrm{id}$, $U = \mathrm{id}$ and $c_{\gamma,\lambda} = \gamma(\lambda)$, we see that the Gabor system $\{E_\gamma T_\lambda g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is a frame if, and only if, the translation invariant system $\{T_\lambda E_\gamma g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is a

frame. This time we need Λ to be a closed, co-compact subgroup of G and Γ to be equipped with a measure satisfying standing hypotheses (I)–(III). In contrast to the first approach, Theorem 3.4 now yields t_α-equations in the *frequency* domain $\widehat{G}$: for each $\beta \in \Lambda^\perp$ we have

$$\int_\Gamma \overline{\hat{g}(\omega\gamma)}\hat{h}(\omega\gamma\beta)\,d\mu_\Gamma(\gamma) = \delta_{\beta,1} \quad \text{a.e. } \omega \in \widehat{G}.$$

Gabor systems play a major role in time-frequency analysis [27] and it is common to require similar properties on Γ and Λ. In the following theorem we characterize dual Gabor frames, where we combine both of the above approaches and require that Λ and Γ are closed, co-compact subgroups. If we consider Parseval frames, then the Bessel assumption in Theorem 4.1 can be omitted.

Theorem 4.1. *Let Λ and Γ be closed, co-compact subgroups of G and $\widehat{G}$ respectively and equip $\Lambda^\perp$ and $\Gamma^\perp$ with the counting measure. Suppose that the two systems $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ and $\{E_\gamma T_\lambda h\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ are Bessel families. Then the following statements are equivalent:*

(i) $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ and $\{E_\gamma T_\lambda h\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ are dual frames for $L^2(G)$,

(ii) for each $\alpha \in \Gamma^\perp$ we have

$$\int_\Lambda \overline{g(x-\lambda)}h(x-\lambda+\alpha)\,d\mu_\Lambda(\lambda) = \delta_{\alpha,0} \quad \text{a.e. } x \in G,$$

(iii) for each $\beta \in \Lambda^\perp$ we have

$$\int_\Gamma \overline{\hat{g}(\omega\gamma)}\hat{h}(\omega\gamma\beta)\,d\mu_\Gamma(\gamma) = \delta_{\beta,1} \quad \text{a.e. } \omega \in \widehat{G}.$$

Proof. By Remark 3 the standing hypotheses are satisfied by the Gabor system. The result now follows from Theorem 3.11 together with Lemma 2.4 and the comments preceding Theorem 4.1. $\qquad\square$

From Theorem 4.1 we can derive numerous results about Gabor systems. We begin with an example concerning the inversion of the short-time Fourier transform.

Example 3. Let $g, h \in L^2(G)$ and consider $\{E_\gamma T_\lambda g\}_{\gamma\in\widehat{G},\lambda\in G}$ and $\{E_\gamma T_\lambda h\}_{\gamma\in\widehat{G},\lambda\in G}$. We equip G and $\widehat{G}$ with their respective Haar measures μ_G and $\mu_{\widehat{G}}$. For $f \in L^2(G)$ we calculate

$$\langle f, E_\gamma T_\lambda g\rangle = \int_G f(x)\overline{g(x-\lambda)}\gamma(x)\,d\mu_G(x) = \mathcal{F}(f(\cdot)\overline{g(\cdot-\lambda)})(\gamma). \tag{4.1}$$

With equation (4.1) and since $\|f\| = \|\mathcal{F}f\|$, we find

$$\int_G \int_{\widehat{G}} |\langle f, E_\gamma T_\lambda g\rangle|^2\,d\mu_{\widehat{G}}(\gamma)\,d\mu_G(\lambda) = \int_G \int_{\widehat{G}} \left|\mathcal{F}(f(\cdot)\overline{g(\cdot-\lambda)})(\gamma)\right|^2\,d\mu_{\widehat{G}}(\gamma)\,d\mu_G(\lambda)$$

$$= \int_G \int_G \left|f(x)\overline{g(x-\lambda)}\right|^2\,d\mu_G(x)\,d\mu_G(\lambda)$$

$$= \int_G \left|f(x)\right|^2 \int_G \left|g(x-\lambda)\right|^2\,d\mu_G(\lambda)\,d\mu_G(x) = \|f\|^2\|g\|^2.$$

The same calculation holds for the Gabor system generated by h. We conclude that both Gabor systems are Bessel families. By Theorem 4.1 the two Gabor systems $\{E_\gamma T_\lambda g\}_{\gamma \in \widehat{G}, \lambda \in G}$ and $\{E_\gamma T_\lambda h\}_{\gamma \in \widehat{G}, \lambda \in G}$ are dual frames for $L^2(G)$ if, and only if, for a.e. $x \in G$

$$\int_G \overline{g(x-\lambda)} h(x-\lambda) \, d\mu_G(\lambda) = \int_G \overline{g(\lambda)} h(\lambda) \, d\mu_G(\lambda) = \overline{\langle g, h \rangle} = 1,$$

that is, $\langle g, h \rangle = 1$. This result is the well-known inversion formula for the short-time Fourier transform [26, 27].

Example 4. Let $G = \Gamma = \mathbb{R}^n, \Lambda = \mathbb{Z}^n$ and $g \in L^2(\mathbb{R}^n)$. We equip G and Γ with the Lebesgue measure and Λ with the counting measure. Then

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^n} \sum_{\lambda \in \mathbb{Z}^n} \langle f_1, E_\gamma T_\lambda g \rangle \langle E_\gamma T_\lambda g, f_2 \rangle \, d\gamma, \quad \text{for all } f_1, f_2 \in L^2(\mathbb{R}^n)$$

if, and only if,

$$\sum_{\lambda \in \mathbb{Z}^n} |g(x-\lambda)|^2 = 1, \text{ a.e. } x \in \mathbb{R}^n.$$

Equivalently in the frequency domain, for all $\beta \in \mathbb{Z}^n$

$$\int_{\mathbb{R}^n} \overline{\hat{g}(\omega+\gamma)} \hat{g}(\omega+\gamma+\beta) \, d\gamma = \delta_{\beta,0} \quad \text{a.e. } \omega \in \mathbb{R}^n.$$

From the time domain characterization, it is clear that the square-root of any uniform B-splines can be used to construct such functions g. The Gabor system with $\Lambda = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$ has similar characterizing equations, see [39, Example 2.1(b)].

Example 5. Let $g, h \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Take $\Lambda = a\mathbb{Z}$ and $\Gamma = b\mathbb{Z}$. We equip $\mathbb{R}$ with the Lebesgue measure and $\Lambda^\perp \cong \frac{1}{a}\mathbb{Z}$, $\Gamma^\perp \cong \frac{1}{b}\mathbb{Z}$ with the counting measure. From this follows that the measure on Λ and Γ is the counting measure multiplied with a and b respectively. Theorem 4.1 now yields the following characterizing equation for dual Gabor systems in $L^2(\mathbb{R})$: If $\{E_\gamma T_\lambda g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ and $\{E_\gamma T_\lambda h\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ are Bessel sequences, then

$$f = ab \sum_{\lambda \in a\mathbb{Z}} \sum_{\gamma \in b\mathbb{Z}} \langle f, E_\gamma T_\lambda g \rangle E_\gamma T_\lambda h, \quad \text{for all } f \in L^2(\mathbb{R})$$

if, and only if, for all $\alpha \in \frac{1}{b}\mathbb{Z}$

$$\sum_{\lambda \in a\mathbb{Z}} \overline{g(x-\lambda)} h(x-\lambda+\alpha) = \frac{1}{a}\delta_{\alpha,0} \quad \text{for a.e. } x \in [0,a].$$

This result is equivalent to the characterization result by Janssen [33]. Higher dimensional versions can be derived similarly; see Ron and Shen [45] for alternative proofs.

One can easily deduce characterization results for Gabor systems in $\ell^2(\mathbb{Z}^d)$ following the approach of the preceding example. We refer to the work of Janssen [16] and Lopez and Han [42] for direct proofs. Finally, we mention the following characterization for finite and discrete Gabor frames.

Example 6. Let $g, h \in \mathbb{C}^d$ and $a, b, d, N, M \in \mathbb{N}$ be such that $aN = bM = d$. Then

$$f = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}h, \quad \text{for all } f \in \mathbb{C}^d$$

if, and only if,

$$\sum_{k=0}^{N-1} \overline{g(x-nM-ka)}h(x-ka) = \frac{1}{M}\delta_{n,0}, \quad \forall x \in \{0, 1, \dots, a-1\}, n \in \{0, 1, \dots, b-1\}.$$

This result appears first in [49] and has been rediscovered in, e.g., [41].

4.2 Wavelet and shearlet systems

Following [4], we let $\operatorname{Epick}(G)$ denote the semigroup of continuous group homomorphisms a of G onto G with compact kernel. This semigroup can be viewed as an extension of the group of topological automorphisms on G; we define the extended *modular function* Δ in $\operatorname{Epick}(G)$ as in [4, Section 6]. The isometric dilation operator $D_a : L^2(G) \to L^2(G)$ is then defined by

$$D_a f(x) = \Delta(a)^{-1/2} f(a(x)).$$

Let $\mathcal{A}$ be a subset of $\operatorname{Epick}(G)$, let Γ be a co-compact subgroup of G, and let Ψ be a subset of $L^2(G)$. The wavelet system generated by Ψ is:

$$W(\Psi, \mathcal{A}, \Gamma) := \{D_a T_\gamma \psi : a \in \mathcal{A}, \gamma \in \Gamma, \psi \in \Psi\}.$$

Depending on the choice of $\mathcal{A}$ and the structure of $\operatorname{Epick}(G)$, it might be desirable to extend the wavelet system with translates of "scaling" functions, that is, $\{T_\gamma \phi : \gamma \in \Gamma, \phi \in \Phi\}$ for some $\Phi \subset L^2(G)$. We denote this extension to a "non-homogeneous" wavelet system by $W_h(\Psi, \Phi, \mathcal{A}, \Gamma)$. If $\operatorname{Epick}(G)$ only contains trivial group homomorphisms, e.g., as in the case of $G = \mathbb{Z}$, it is possible to define the dilation operator on the dual group $\widehat{G}$ via the Fourier transform.

The two wavelet systems introduced above offer a very general setup that include most of the usual wavelet-type systems in $L^2(\mathbb{R}^n)$, e.g., discrete and continuous wavelet and shearlet systems [14, 38] as well as composite wavelet systems.

Example 7. Let us consider the general setup as above, where we make the specific choice $\Gamma = G$ and $\Psi = \{\psi_j\}_{j \in J}$ for some index set $J \subset \mathbb{Z}$. For $a \in \mathcal{A}$ and $\gamma \in \Gamma = G$, we have

$$D_a T_\gamma \psi_j(x) = \Delta(a)^{-1/2} \psi_j(a(x) - \gamma)) = T_{\bar{\gamma}} D_a \psi_j(x)$$

for some $\bar{\gamma} \in a^{-1}\Gamma$ so that $a(\bar{\gamma}) = \gamma$. It follows that $W(\Psi, \mathcal{A}, \Gamma)$ is a (generalized) translation invariant system for $\Gamma_j = G$ with $j \in J$ and $g_{j,p} = g_{j,a} = D_a \psi_j$ for $(j, p) = (j, a) \in J \times \mathcal{A}$. For simplicity we equip each measure space $P_j = \mathcal{A}$, $j \in J$, with the same measure; as usual we require that this measure $\mu_{\mathcal{A}}$ satisfies our standing hypotheses. Further, we define the adjoint of a by $\hat{a}(\omega) = \omega \circ a$ for $\omega \in \widehat{G}$. Using results from [4], it follows that $\hat{a}$ is an isomorphism from $\widehat{G}$ onto $(\ker a)^\perp$ and that

$$\widehat{D_a f}(\omega) = \begin{cases} \Delta(a)^{1/2} \hat{f}(\hat{a}^{-1}(\omega)) & \omega \in (\ker a)^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

As translation invariant systems always satisfy the local integrability condition, we immediately have that $W(\Psi, \mathcal{A}, G)$ is a Parseval frame, that is,

$$f = \sum_{j \in J} \int_{\mathcal{A}} \int_G \langle f, D_a T_\gamma \psi_j \rangle \, D_a T_\gamma \psi_j \, d\mu_G(\gamma) \, d\mu_{\mathcal{A}}(a) \quad \text{for all } f \in L^2(G),$$

if, and only if, for a.e. $\omega \in \widehat{G}$,

$$t_0 = \sum_{j \in J} \int_{\mathcal{A}} \big| \widehat{D_a \psi_j}(\omega) \big|^2 d\mu_{\mathcal{A}}(a) = \sum_{j \in J} \int_{\{a \in \mathcal{A} : \omega \in (\ker a)^\perp\}} \Delta(a) \big| \hat{\psi}_j(\hat{a}^{-1}(\omega)) \big|^2 d\mu_{\mathcal{A}}(a) = 1. \quad (4.2)$$

In particular, it follows that $W(\Psi, \mathcal{A}, G)$ cannot be a Parseval frame for $L^2(G)$ regardless of the measure $\mu_{\mathcal{A}}$ if $\widehat{G} \setminus \cup_{a \in \mathcal{A}}(\ker a)^\perp$ has non-zero measure.

The Calderón admissibility condition (1.2) is a special case of (4.2). To see this, take $G = \mathbb{R}$ and consider the dilation group $\mathcal{A} = \{x \mapsto a^{-1} x : a \in \mathbb{R} \setminus \{0\}\}$ with measure $\mu_{\mathcal{A}}$ defined on the Borel algebra on $\mathbb{R} \setminus \{0\}$ by $d\mu_{\mathcal{A}}(a) = da/a^2$, where $da = d\lambda(a)$ denotes the Lebesgue measure. Higher dimensional versions of Calderón's admissibility condition are obtained similarly, see also [20, 40].

Example 8. We consider wavelet systems in $L^2(\mathbb{R}^n)$ with discrete dilations and semi-continuous translations. Let $A \in \mathrm{GL}(n, \mathbb{R})$ be a matrix whose eigenvalues are strictly larger than one in modulus, set $\mathcal{A} = \{x \mapsto A^j x : j \in \mathbb{Z}\}$, and let Γ be a co-compact subgroup of $\mathbb{R}^n$. The wavelet system generated by $\Psi = \{\psi_\ell\}_{\ell=1}^L \subset L^2(G)$ is given by

$$W(\Psi, A, \Gamma) := \left\{ D_{A^j} T_\gamma \psi_\ell = |\det A|^{-j/2} \psi_\ell(A^{-j} \cdot -\gamma) : \ell = 1, \ldots, L, j \in \mathbb{Z}, \gamma \in \Gamma \right\}.$$

Any co-compact subgroup of $\mathbb{R}^n$ is of the form $\Gamma = P(\mathbb{Z}^k \times \mathbb{R}^{n-k})$ for some $k \in \{0, 1, \ldots, n\}$ and $P \in \mathrm{GL}(n, \mathbb{R})$. Since $W(\{\psi\}, A, \Gamma)$ is unitarily equivalent to $W(\{D_{P^{-1}}\psi\}, P^{-1}AP, \mathbb{Z}^k \times \mathbb{R}^{n-k})$ we can without loss of generality assume that $P = I_n$, i.e., $\Gamma = \mathbb{Z}^k \times \mathbb{R}^{n-k}$.

Clearly, $W(\Psi, A, \Gamma)$ is a generalized translation invariant system for $\Gamma_j = A^j \Gamma$ with $j \in J := \mathbb{Z}$ and $g_{j,\ell} = D_{A^j} \psi_\ell$, where $P_j = \{1, \ldots, L\}$. To get rid of a scaling factor in the representation formula, we will use $\mu_{P_j} = \frac{1}{|\det A|^j} \mu_c$ as measure on $P_j = \{1, \ldots, L\}$, where μ_c denotes the counting measure. The standing assumptions are clearly satisfied. Moreover, the local integrability condition is known to be equivalent to local integrability on $\mathbb{R}^n \setminus \{0\}$ of the Calderón sum [3, Proposition 2.7] and can, therefore, be omitted from the characterization results. It follows that two Bessel families $W(\Psi, A, \Gamma)$ and $W(\Phi, A, \Gamma)$ are dual frames if, and only if, with $B = A^T$,

$$t_\alpha(\omega) = \sum_{l=1}^L \sum_{j \in \mathbb{Z}: \alpha \in B^j(\mathbb{Z}^k \times \{0\}^{n-k})} \hat{\psi}_l(B^{-j} \omega) \overline{\hat{\phi}_l(B^{-j}(\omega + \alpha))} = \delta_{\alpha, 0} \quad \text{for a.e. } \omega \in \mathbb{R}^n,$$

for all $\alpha \in \mathbb{Z}^k \times \{0\}^{n-k}$. For $k = n$ this result was obtained in [11], extending the work of Gripenberg [23] and Wang [48].

Example 9. Let us finally consider the cone-adapted shearlet systems. For brevity we restrict our findings to the non-homogeneous, continuous shearlet transform in dimension two. Let

$$A_a = \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix} \quad \text{and} \quad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

for $a \neq 0$ and $s \in \mathbb{R}$. For $\psi \in L^2(\mathbb{R}^2)$ define

$$\psi_{ast}(x) := a^{-3/4} \psi(A_a^{-1} S_s^{-1}(x-t)) = T_t D_{S_s A_a} \psi.$$

The cone-adapted continuous shearlet system $S_h(\phi, \psi, \tilde{\psi})$ is then defined as the collection:

$$S_h(\phi, \psi_1, \psi_2) = \left\{ T_t \phi : t \in \mathbb{R}^2 \right\} \cup \left\{ T_t D_{S_s A_a} \psi_1 : a \in (0,1], |s| \leq 1 + a^{1/2}, t \in \mathbb{R}^2 \right\}$$

$$\cup \left\{ T_t D_{\tilde{S}_s \tilde{A}_a} \psi_2 : a \in (0,1], |s| \leq 1 + a^{1/2}, t \in \mathbb{R}^2 \right\},$$

where $\tilde{S}_s = S_s^T$ and $\tilde{A}_a = \mathrm{diag}\,(a^{1/2}, a)$. This is a special case of the system W_h introduced above. More importantly, this is a GTI system. To see this claim, take $J = \{0, 1\}$ and $\Gamma = \Gamma_j = \mathbb{R}^2$ for $j \in J$. Define $P_0 = \{0\}$ and let μ_{P_0} be the counting measure on P_0. Define

$$P_1 = \left\{ (a,s) \in \mathbb{R}^2 : a \in (0,1], |s| \leq 1 + a^{1/2} \right\},$$

and let μ_{P_1} be some measure on P_1 so that our standing hypotheses are satisfied. The generators are $g_{0,p} = g_{0,0} = \phi$ for $p = 0 \in P_0$ and $g_{1,p} = g_{1,(a,s)} = D_{\tilde{S}_s \tilde{A}_a} \psi$ for $p = (a,s) \in P_1$. This proves our claim. By Theorem 3.11 and Lemma 3.12 we immediately have that, if $S_h(\phi, \psi_1, \psi_2)$ and $S_h(\phi, \tilde{\psi}_1, \tilde{\psi}_2)$ are Bessel families, then they are dual frames if, and only if,

$$\overline{\hat{\phi}(\omega)} \hat{\phi}(\omega) + \int_{P_1} a^{3/2} \overline{\hat{\psi}_1(A_a S_s^T \omega)} \hat{\tilde{\psi}}_1(A_a S_s^T \omega)\, d\mu_{P_1}(a,s)$$

$$+ \int_{P_1} a^{3/2} \overline{\hat{\psi}_2(\tilde{A}_a \tilde{S}_s^T \omega)} \hat{\tilde{\psi}}_2(\tilde{A}_a \tilde{S}_s^T \omega)\, d\mu_{P_1}(a,s) = 1 \qquad \text{for a.e. } \omega \in \mathbb{R}^2. \quad (4.3)$$

A standard choice for the measure μ_{P_1} in (4.3) is $d\mu_{P_1}(a,s) = \frac{da\,ds}{a^3}$, which comes from the left-invariant Haar measure on the shearlet group. The above characterization result generalizes results from [24, 25, 37].

4.3 Other examples

Example 10. In this example we consider the additive group of p-adic integers $\mathbb{I}_p$. To introduce this group, we first consider the p-adic numbers $\mathbb{Q}_p$. Here p is some fixed prime-number. The p-adic numbers are the completion of the rationals $\mathbb{Q}$ under the p-adic norm, defined as follows. Every non-zero rational x can be uniquely factored into $x = \frac{r}{s} p^n$, where $r, s, n \in \mathbb{Z}$ and p does not divide r nor s. We then define the p-adic norm of x as $\|x\|_p = p^{-n}$, additionally $\|0\|_p := 0$. The p-adic numbers $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ under $\|\cdot\|_p$. It can be shown that all p-adic numbers x can be written uniquely as

$$x = \sum_{j=k}^{\infty} x_j p^j, \tag{4.4}$$

where $x_k \in \{0, 1, \ldots, p-1\}$ and $k \in \mathbb{Z}$, $x_k \neq 0$. The set of all numbers $x \in \mathbb{Q}_p$ for which $x_j = 0$ for $j < 0$ in (4.4) are the p-adic integer $\mathbb{I}_p$. Equivalently, $\mathbb{I}_p = \{x \in \mathbb{Q}_p : \|x\|_p \leq 1\}$. In fact, $\mathbb{I}_p$ is a compact, closed and open subgroup of $\mathbb{Q}_p$. Its dual group $\widehat{\mathbb{I}_p}$ can be identified with the Prüfer p-group $\mathbb{Z}(p^\infty)$, which consists of the union of the p^n-roots of unity for all $n \in \mathbb{N}$. That is,

$$\widehat{\mathbb{I}_p} \cong \mathbb{Z}(p^\infty) := \{ e^{2\pi i m/p^n} \ : \ n \in \mathbb{N}, m \in \{0, 1, \ldots, p^n - 1\} \} \subset \mathbb{C}.$$

We equip $\mathbb{Z}(p^\infty)$ with the discrete topology and multiplication as group operation. For more information on p-adic numbers and their dual group we refer to, e.g., [31, §10, §25]. For $n \in \mathbb{N}$ consider now the subgroups $\Gamma_n^\perp = \{e^{2\pi im/p^n} : m = 0, 1, \ldots, p^n - 1\} \subset \mathbb{Z}(p^\infty)$. Note that all $\Gamma_n^\perp$ are finite groups of order p^n and generated by $e^{2\pi i/p^n}$. Moreover, all $\Gamma_n^\perp$ are nested so that

$$1 \subset \Gamma_1^\perp \subset \Gamma_2^\perp \subset \cdots \subset \mathbb{Z}(p^\infty).$$

Let now $\{g_n\}_{n\in\mathbb{N}} \subset L^2(\mathbb{I}_p)$. By Theorem 3.14 the generalized translation invariant system $\{T_\gamma g_n\}_{\gamma\in\Gamma_n, n\in\mathbb{N}}$ is a Parseval frame for $L^2(\mathbb{I}_p)$ if, and only if, for each $\alpha \in \bigcup_{n\in\mathbb{N}} \Gamma_n^\perp = \mathbb{Z}(p^\infty)$

$$\sum_{k=n^*}^{\infty} \overline{\hat{g}_n(\omega)} \hat{g}_n(\omega\alpha) = \delta_{\alpha,1} \quad \text{for all } \omega \in \mathbb{Z}(p^\infty),$$

where $n^* \in \mathbb{N}$ is the smallest natural number such that $\alpha \in \Gamma_{n^*}^\perp$. Because we consider a GTI system with countably many generators, the standing hypotheses are trivially satisfied, see Section 3.

Returning to the p-adic numbers $\mathbb{Q}_p$, we note that the only co-compact subgroup of $\mathbb{Q}_p$ is $\mathbb{Q}_p$ itself [4]. Therefore any GTI system in $L^2(\mathbb{Q}_p)$ is, in fact, a translation invariant system of the form $\cup_{j\in J} \{T_\gamma g_p\}_{\gamma\in\mathbb{Q}_p, p\in P_j}$. The equations characterizing the dual frame property of such systems are immediate from Theorem 3.11 and Lemma 3.12.

Finally, in the product group $\mathbb{Q}_p \times \mathbb{I}_p$ there are no discrete, co-compact subgroups [4], and thus no generalized shift invariant systems for $L^2(\mathbb{Q}_p \times \mathbb{I}_p)$ can be constructed. However, any subgroup of the form $\mathbb{Q}_p \times \Gamma_n$, where Γ_n is a co-compact subgroup of $\mathbb{I}_p$ as before, is a co-compact subgroup in $\mathbb{Q}_p \times \mathbb{I}_p$, indicating that a large number of generalized translation invariant systems do exist in $L^2(\mathbb{Q}_p \times \mathbb{I}_p)$.

In order to apply Theorem 3.4 to a given GTI system, one needs to verify that the (dual) α-LIC or the stronger LIC holds. By Theorem 3.11 we get this for free for translation invariant systems. For regular wavelet systems as in Example 8 the LIC has an easy characterization [3, Proposition 2.7]. For certain irregular wavelet systems over the real line a detailed analysis of the LIC has been carried out in [35] using Beurling densities. However, for general GTI systems there is no simple interpretation of the local integrability conditions.

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