The arc-length and energy of rational Bézier curves

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Abstract

Extending earlier work on polynomial Bézier curves, we define approximations to the length and energy of rational Bézier curves. The approximations are defined directly in terms of the control points and we investigates how the approximations behave under subdivision. The approximation to the arc-length form the basis for a fast recursive and adaptive algorithm which calculates the arc-length of a rational Bézier curve.

1 Introduction

Suppose we want to find the arc-length $\mathcal{L}$ of a Bézier curve γ of degree n as in Figure 1. We can easily find the length of the chord, L_c, i.e., the distance between the end points, and length of the control polygon L_p, i.e., the sum of the distances between consecutive control points. We may consider L_c and L_p as approximations of $\mathcal{L}$ and as $L_c \leq \mathcal{L} \leq L_p$, we might expect that a suitable convex combination of L_c and L_p approximate $\mathcal{L}$ better than either of them. This is indeed the case, if we put $L = \frac{2}{n+1} L_c + \frac{n-1}{n+1} L_p$, then we get the best approximation, in the sense that it under subdivision converges faster than any other affine combination of L_c and L_p. If we subdivide the curve k times, we get 2^k segments γ_j and we can find the chord length $L_c(\gamma_j)$ and the polygon length $L_p(\gamma_j)$ of each segment. We can add these numbers and get the total length of the chords L_c^k and the total length of the control polygons L_p^k. As a Bézier curve is rectifiable we have $L_c^k \nearrow \mathcal{L}$, and it is easily seen that we have $L_p^k \searrow \mathcal{L}$. In an earlier paper [2] it is shown that the error in both cases goes to zero as 2^{-2k}, and that the error when using the convex

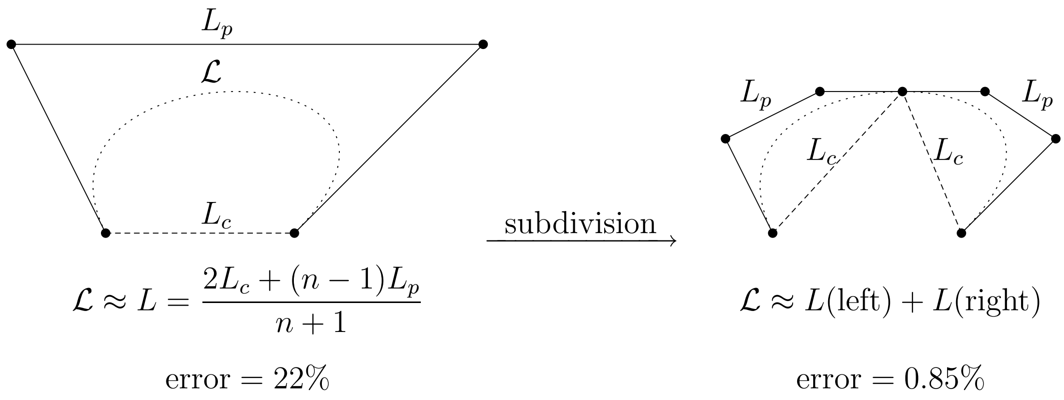

Figure 1: Calculating the length of a Bézier curve

combination L goes to zero as 2^{-4k}:

$$\mathcal{L}(\gamma) = \sum_{j=1}^{2^k} \frac{2L_c(\gamma_j) + (n-1)L_p(\gamma_j)}{n+1} + O\big(2^{-4k}\big), \tag{1}$$

It was furthermore shown that the energy $\mathcal{E} = \frac{1}{2}\int \kappa^2 \,\mathrm{ds}$ of the curve can be approximated by another combination of L_c and L_p:

$$\mathcal{E}(\gamma) = \sum_{j=1}^{2^k} 12 \frac{n-1}{n+1} \frac{L_p(\gamma_j) - L_c(\gamma_j)}{L_c^2(\gamma_j)} + O\big(2^{-2k}\big), \tag{2}$$

In [2] these results was shown for polynomial Bézier curves and in this paper we show that the exact same formulae holds for rational Bézier curves.

At first it might seems surprising that the weights of the rational Bézier curve does not enter in to the formulae. But it is the combination of subdivision and the above formulae which gives the good approximation, and effect of the weights is hidden in the subdivision.

Throughout the paper we consider curves in an affine space E modeled on a vector space V equiped with some inner product $\langle\cdot,\cdot\rangle$, and we equip the space $C^k\big([0,1],V\big)$ of C^k-maps $[0,1] \to V$ with the C^k-norm.

2 Expansion of rational Bézier curves

A key ingredients in the proof of (1) and (2) was theorem 2.2 and theorem 2.3 in [2] which describes an expansion of the curve and its control points in terms of the length of a subinterval. We can reformulate the results as:

Theorem 2.1. *Let $\gamma(t)$ be a C^k-curve, with $k \geq 2$, defined on an interval I. Then the restriction of γ to the interval $[a, a+h] \subset I$ can be written uniquely as*

$$\gamma(a + th) = P + th\mathbf{v} + B_1^2(t)h^2\mathbf{w} + h^3\mathbf{u}(t), \quad t \in [0, 1], \tag{3}$$

where the map

$$\{(a, h) \mid a, a + h \in I\} \to V \times V \times C^k\big([0, 1], V\big) : (a, h) \mapsto (\mathbf{v}, \mathbf{w}, \mathbf{u})$$

is bounded, and

$$\mathbf{u}(0) = \mathbf{u}(1) = \int_0^1 \mathbf{u}(t)\,\mathrm{dt} = \mathbf{0}. \tag{4}$$

Furthermore, if $\gamma'(t) \neq 0$ for all $t \in I$, then $\mathbf{v}$ is bounded away from $\mathbf{0}$. If $\gamma(t)$ is a Bézier curve of degree n, then the restriction of γ to the interval $[a, a + h]$ has control points

$$P_i = P + \frac{i}{n}h\mathbf{v} + 2\frac{i(n-i)}{n(n-1)}h^2\mathbf{w} + h^3\mathbf{u}_i, \tag{5}$$

where the vectors $\mathbf{u}_i$ are bounded functions of (a, h), and satisfies

$$\mathbf{u}_0 = \mathbf{u}_n = \sum_{i=0}^{n} \mathbf{u}_i = \mathbf{0}. \tag{6}$$

Remark 2.2. The points $\mathbf{u}_i$ are the control points for the Bézier curve $\mathbf{u}(t)$.

Remark 2.3. We can determine P, $\mathbf{v}$ and $\mathbf{w}$ by

$$\begin{aligned}
P &= \gamma(a + h) & &\text{or} & P &= P_0 \\
h\mathbf{v} &= \gamma(a + h) - \gamma(a) & &\text{or} & h\mathbf{v} &= P_n - P_0, \\
h^2\mathbf{w} &= 3\int_0^1 (\gamma(a + th) - \gamma(a) - th\mathbf{v})\,\mathrm{dt} & &\text{or} & h^2\mathbf{w} &= \frac{3}{n+1}\sum_{i=0}^{n}\left(P_n - P - \frac{i}{n}\mathbf{v}\right).
\end{aligned}$$

Remark 2.4. In [2] it was not stated explicitly that $\mathbf{u}(t)$ is bounded in the C^k-topology, but this is trivial as soon as we notice that $\gamma(a + th)$ and its first k derivatives with respect to t is bounded as functions of a and h.

In the case of a rational Bézier curves the relation between the expansion of the curve and the expansion of the control points is slightly more complicated:

Theorem 2.5. *Let $\gamma(t)$, $t \in I$, be a rational Bézier curve with positive weights, then the restriction of γ to the interval $[a, a+h] \subset I$ has control points and normalized weights*

$$\begin{aligned} P_i &= P + \frac{i}{n} h\mathbf{v} + 2\frac{i(n-i)}{n(n-1)} h^2 \mathbf{w} + h^3 \mathbf{u}_i \\ \omega_i &= 1 + 2\frac{i(n-i)}{n(n-1)} h^2 \nu + h^3 \mu_i \end{aligned} \tag{7}$$

where $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}_i$, ν, and μ_i are bounded functions of (a, h), and

$$\mathbf{u}_0 = \mathbf{u}_n = \sum_{i=0}^{n} \mathbf{u}_i = \mathbf{0}, \quad \text{and} \quad \mu_0 = \mu_n = \sum_{i=0}^{n} \mu_i = 0. \tag{8}$$

The restriction of the curve can be written as

$$\gamma(a + th) = P + th\mathbf{v} + B_1^2(t) h^2 \left(\mathbf{w} + h^2 \overline{\mathbf{w}}\right) + h^3 \mathbf{u}(t), \quad t \in [0, 1], \tag{9}$$

where $\overline{\mathbf{w}}$ is a bounded function of (a, h), and $(a, h) \mapsto \mathbf{u}$ is a bounded function into $C^k([0, 1], V)$ (for any k), which satisfies

$$\mathbf{u}(0) = \mathbf{u}(1) = \int_0^1 \mathbf{u}(t)\,\mathrm{dt} = \mathbf{0}. \tag{10}$$

Furthermore, if $\gamma'(t) \neq 0$ for all $t \in I$, then $\mathbf{v}$ is bounded away from $\mathbf{0}$.

Proof. We can consider $P_0, \ldots, P_n$ and $\omega_0, \ldots, \omega_n$ as control points for two ordinary Bézier curves, so we do have an expansion (7) such that (8) is satisfied. But when a and h varies we can not consider the points as control points for the restriction of a *fixed* Bézier curve, so we need to show that $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}_i$, ν, and μ_i are bounded functions of (a, h).

A rational Bézier curve is the projection of an ordinary Bézier curve in a space of one higher dimension. We can write the control points of the restriction of this curve as $[P_i \widetilde{\omega}_i, \widetilde{\omega}_i]$, where we according to Theorem 2.1 have

$$\begin{aligned} P_i \widetilde{\omega}_i &= P_0 \widetilde{\omega}_0 + \frac{i}{n} h\widetilde{\mathbf{v}} + 2\frac{i(n-i)}{n(n-1)} h^2 \widetilde{\mathbf{w}} + h^3 \widetilde{\mathbf{u}}_i, \\ \widetilde{\omega}_i &= \widetilde{\omega}_0 + \frac{i}{n} h\widetilde{\sigma} + 2\frac{i(n-i)}{n(n-1)} h^2 \widetilde{\nu} + h^3 \widetilde{\mu}_i, \end{aligned}$$

and as $\gamma(t)$ have positive weights, we know that $\widetilde{\omega}_0$ is bounded away from zero. Due to affine invariance we may, with out loss of generality, assume that $P_0 = 0$, and then we have

$$\mathbf{v} = \frac{P_n - P_0}{h} = \frac{P_n \widetilde{\omega}_n}{h \widetilde{\omega}_n} = \frac{h\widetilde{\mathbf{v}}}{h(\widetilde{\omega}_0 + h\widetilde{\sigma})} = \frac{\widetilde{\mathbf{v}}}{\widetilde{\omega}_0 + h\widetilde{\sigma}} = \frac{\widetilde{\mathbf{v}}}{\widetilde{\omega}_0} + h\widetilde{\widetilde{\sigma}}.$$

As $\widetilde{\sigma}$ is a bounded function of (a, h), so is $\widetilde{\widetilde{\sigma}}$, and as $\widetilde{\omega}_0$ is bounded away from zero, the vector $\mathbf{v}$ is bounded. In order to determine the vector $\mathbf{w}$ we need the control points:

$$\begin{aligned}
P_i = \frac{P_i \widetilde{\omega}_i}{\widetilde{\omega}_i} &= \frac{\frac{i}{n} h \widetilde{\mathbf{v}} + 2\frac{i(n-i)}{n(n-1)} h^2 \widetilde{\mathbf{w}} + h^3 \widetilde{\mathbf{u}}_i}{\widetilde{\omega}_0 + \frac{i}{n} h \widetilde{\sigma} + 2\frac{i(n-i)}{n(n-1)} h^2 \widetilde{\nu} + h^3 \widetilde{\mu}_i} \\
&= \left(\frac{i}{n} h \frac{\widetilde{\mathbf{v}}}{\widetilde{\omega}_0} + 2\frac{i(n-i)}{n(n-1)} h^2 \frac{\widetilde{\mathbf{w}}}{\widetilde{\omega}_0} + h^3 \frac{\widetilde{\mathbf{u}}_i}{\widetilde{\omega}_0} \right) \left(1 - \frac{i}{n} h \frac{\widetilde{\sigma}}{\widetilde{\omega}_0} + O(h^2) \right) \\
&= \frac{i}{n} h \frac{\widetilde{\mathbf{v}}}{\widetilde{\omega}_0} + O(h^2) = \frac{i}{n} h \mathbf{v} + O(h^2)
\end{aligned}$$

From this we obtain

$$h^2 \mathbf{w} = \frac{3}{n+1} \sum_{i=0}^{n} \left(P_i - \frac{i}{n} h \mathbf{v} \right) = O(h^2),$$

and we can see that $\mathbf{w}$ is bounded. To determine ν we need the normalized weights which are defined by

$$\begin{aligned}
\omega_i = \frac{\widetilde{\omega}_i}{\sqrt[n]{\widetilde{\omega}_n^i \widetilde{\omega}_0^{n-i}}} &= \frac{\widetilde{\omega}_0 + \frac{i}{n} h \widetilde{\sigma} + O(h^2)}{\sqrt[n]{(\widetilde{\omega}_0 + h \widetilde{\sigma})^i \widetilde{\omega}_0^{n-i}}} = \frac{\widetilde{\omega}_0 + \frac{i}{n} h \widetilde{\sigma} + O(h^2)}{\widetilde{\omega}_0 \sqrt[n]{(1 + i h \frac{\widetilde{\sigma}}{\widetilde{\omega}_0} + O(h^2))}} \\
&= \left(1 + \frac{i}{n} h \frac{\widetilde{\sigma}}{\widetilde{\omega}_0} + O(h^2) \right) \left(1 - \frac{i}{n} h \frac{\widetilde{\sigma}}{\widetilde{\omega}_0} + O(h^2) \right) = 1 + O(h^2),
\end{aligned}$$

see eg. [1]. Hence

$$h^2 \nu = \frac{3}{n+1} \sum_{i=0}^{n} (\omega_i - 1) = O(h^2),$$

and we can see that ν is bounded. By affine invariance we may assume that $\gamma(0) = 0$, and then we can write the restriction of the curve as

$$\gamma(a + th) = th \widehat{\mathbf{v}} + B_1^2(t) h^2 \widehat{\mathbf{w}} + h^3 \mathbf{u}(t),$$

where $\mathbf{u}(0) = \mathbf{u}(1) = \int_0^1 \mathbf{u}(t)\, \mathrm{d}t = 0$. To prove (9), we only need to show that

$$\widehat{\mathbf{v}} = \mathbf{v}, \tag{*}$$

$$\widehat{\mathbf{w}} - \mathbf{w} = h^2 \overline{\mathbf{w}}, \tag{**}$$

where $\overline{\mathbf{w}}$ is bounded. We can write the restriction of the curve as

$$\gamma(a+th) = \frac{p(t)}{q(t)},$$

where the denominator $q(t)$ is a Bézier curve with control points ω_i and the numerator $p(t)$ is a Bézier curve with control points $P_i\omega_i$. Hence

$$q(t) = 1 + B_1^2(t)h^2\nu + h^3\mu(t),$$

where $\mu(0) = \mu(1) = \int_0^1 \mu(t)\,\mathrm{dt} = 0$. The curve $P(t)$ can be written as

$$p(t) = t\widetilde{\mathbf{v}} + B_1^2(t)h^2\widetilde{\mathbf{w}} + h^3\widetilde{\mathbf{u}}(t),$$

where $\widetilde{\mathbf{u}}(0) = \widetilde{\mathbf{u}}(1) = \int_0^1 \widetilde{\mathbf{u}}(t)\,\mathrm{dt} = \mathbf{0}$. As

$$\gamma(a+h) - \gamma(a) = p(a+h) - p(a) = P_n - P_0,$$

we have

$$\widehat{\mathbf{v}} = \widetilde{\mathbf{v}} = \mathbf{v}.$$

In particular, we have established (*). To determine the vector $\widetilde{\mathbf{w}}$ we need the control points for $p(t)$:

$$\begin{aligned} P_i\omega_i &= \left(\frac{i}{n}h\mathbf{v} + 2\frac{i(n-i)}{n(n-1)}h^2\mathbf{w} + h^3\mathbf{u}_i\right)\left(1 + 2\frac{i(n-i)}{n(n-1)}h^2\nu + h^3\mu_i\right) \\ &= \frac{i}{n}h\mathbf{v} + 2\frac{i(n-i)}{n(n-1)}h^2\mathbf{w} + h^3\mathbf{u}_i \\ &\quad + 2\frac{i^2(n-i)}{n^2(n-1)}h^3\nu\mathbf{v} + 4\frac{i^2(n-i)^2}{n^2(n-1)^2}h^4\nu\mathbf{w} + \frac{i}{n}h^4\mu_i\mathbf{v} + O(h^5). \end{aligned}$$

Thus

$$\begin{aligned} \widetilde{\mathbf{w}} &= \frac{3}{n+1}\sum_{i=0}^{n}\left(P_i\omega_i - \frac{i}{n}h\mathbf{v}\right) \\ &= h^2\mathbf{w} + \frac{1}{2}h^3\nu\mathbf{v} + \frac{2}{5}\frac{n^2+1}{n^2-n}h^4\nu\mathbf{w} + \frac{3}{n^2+n}\sum_{i=0}^{n} i\mu_i h^4\mathbf{v} + O(h^5), \end{aligned}$$

so if we put

$$\widehat{\widehat{\mathbf{w}}} = \frac{2}{5}\frac{n^2+1}{n^2-n}\nu\mathbf{w} + \frac{3}{n^2+n}\sum_{i=0}^{n} i\mu_i\mathbf{v}$$

then $\widetilde{\mathbf{w}} = \mathbf{w} + \frac{1}{2}h\nu\mathbf{v} + h^2\widehat{\widehat{\mathbf{w}}} + O(h^3)$, and we can write

$$p(t) = th\mathbf{v} + B_1^2(t)h^2\left(\mathbf{w} + \frac{1}{2}h\nu\mathbf{v} + h^2\widehat{\widehat{\mathbf{w}}}\right) + h^3\widetilde{\mathbf{u}}(t) + O(h^5),$$

where $\widetilde{\mathbf{u}}(0) = \widetilde{\mathbf{u}}(1) = \int_0^1 \widetilde{\mathbf{u}}(t)\,\mathrm{dt} = \mathbf{0}$. We now have an expression for the restriction of the curve:

$$\begin{aligned}
\gamma(a+th) = \frac{p(t)}{q(t)} &= \frac{th\mathbf{v} + B_1^2(t)h^2\left(\mathbf{w} + \frac{1}{2}h\nu\mathbf{v} + h^2\widehat{\widehat{\mathbf{w}}}\right) + h^3\widehat{\mathbf{u}}(t) + O(h^5)}{q(t) = 1 + B_1^2(t)h^2\nu + h^3\mu(t)} \\
&= \left(th\mathbf{v} + B_1^2(t)h^2\left(\mathbf{w} + \frac{1}{2}h\nu\mathbf{v} + h^2\widehat{\widehat{\mathbf{w}}}\right) + h^3\widehat{\mathbf{u}}(t) + O(h^5)\right) \\
&\qquad \times \left(1 - B_1^2(t)h^2\nu - h^3\mu(t) + O(h^4)\right) \\
&= th\mathbf{v} + B_1^2(t)h^2\left(\mathbf{w} + \frac{1}{2}h\nu\mathbf{v} + h^2\widehat{\widehat{\mathbf{w}}}\right) + h^3\widehat{\mathbf{u}}(t) \\
&\qquad - tB_1^2(t)h^3\nu\mathbf{v} - B_1^2(t)^2h^4\nu\mathbf{w} - t\mu(t)h^4\mathbf{v} + O(h^5),
\end{aligned}$$

and we can determine the vector $\widehat{\mathbf{w}}$:

$$\begin{aligned}
\widehat{\mathbf{w}} &= \frac{3}{h^2}\int_0^1 (\gamma(a+th) - thv)\,\mathrm{dt} \\
&= \mathbf{w} + \frac{1}{2}h\nu\mathbf{v} + h^2\widehat{\widehat{\mathbf{w}}} - \frac{1}{2}h\nu\mathbf{v} - \frac{2}{5}h^2\nu\mathbf{w} - 3h^2\mathbf{v}\int_0^1 t\mu(t)\,\mathrm{dt} + O(h^3) \\
&= \mathbf{w} + h^2\left(\widehat{\widehat{\mathbf{w}}} - \frac{2}{5}\nu\mathbf{w} - 3\mathbf{v}\int_0^1 t\mu(t)\,\mathrm{dt}\right) + O(h^3).
\end{aligned}$$

I.e, $\widehat{\mathbf{w}} - \mathbf{w} = h^2\overline{\mathbf{w}}$, where $\overline{\mathbf{w}}$ is bounded, and (**) is established.

Finally, $h\mathbf{v} = \gamma(a+th) - \gamma(a)$, hence $\mathbf{v} \to \gamma'(a)$ for $h \to 0$, and $\mathbf{v}$ is bounded away from zero if $\gamma'(t) \neq 0$ for all t. □

3 The arc-length and energy

The results (1) and (2) for polynomial curves was proved by using the expansion (5) to calculate the length of the control polygon, and the expansion (3) to calculate the length and energy of the curve $\gamma(a+th)$, see Lemma 3.1, Lemma 3.2 and Lemma 4.1 in [2]. Without the term $\overline{\mathbf{w}}$ in (9) we would of course obtain the same results, but it turns out that $\overline{\mathbf{w}}$ have no effect on the result, (to the appropriate order in h).

Theorem 3.1. *Let $\gamma(t)$ be a rational Bézier curve of degree n, and let the restriction of γ to the interval $[a, a+h]$ have control points $P_0, \ldots, P_n$. If $\mathbf{v}$ and $\mathbf{w}$ are the vectors from the expansions* (7) *and* (9)*, then*

$$L_c\big(\gamma_{|[a,a+h]}\big) = |P_n - P_0| = |\mathbf{v}|h, \tag{11}$$

$$L_p\big(\gamma_{|[a,a+h]}\big) = \sum_{i=1}^{n} |P_i - P_{i-1}| = |\mathbf{v}|h\left(1 + h^2 \frac{2}{3}\frac{n+1}{n-1}\frac{|\mathbf{v}|^2|\mathbf{w}|^2 - \langle \mathbf{v}, \mathbf{w}\rangle^2}{|\mathbf{v}|^4}\right) + O(h^5), \tag{12}$$

$$\mathcal{L}\big(\gamma_{|[a,a+h]}\big) = \int_0^1 \left|\frac{d}{dt}\gamma(a+th)\right| \,\mathrm{dt} = |\mathbf{v}|h\left(1 + h^2 \frac{2}{3}\frac{|\mathbf{v}|^2|\mathbf{w}|^2 - \langle \mathbf{v}, \mathbf{w}\rangle^2}{|\mathbf{v}|^4}\right) + O(h^5), \tag{13}$$

$$\mathcal{E}\big(\gamma_{|[a,a+h]}\big) = \frac{1}{2}\int_0^1 \kappa^2 \left|\frac{d}{dt}\gamma(a+th)\right| \,\mathrm{dt} = 8h\frac{|\mathbf{v}|^2|\mathbf{w}|^2 - \langle \mathbf{v}, \mathbf{w}\rangle^2}{|\mathbf{v}|^5} + O(h^3). \tag{14}$$

Remark 3.2. There are some misprints in Lemma 3.1 and 3.2 in [2]. In both cases an exponent 2 is missing on the term $\left\langle \frac{\mathbf{v}_{|[a,b]}}{|\mathbf{v}_{|[a,b]}|}, \frac{\mathbf{w}_{|[a,b]}}{|\mathbf{v}_{|[a,b]}|}\right\rangle$, and in Lemma 3.2 $\mathcal{L}(p_{|[a,b]})$ should read $L_p(p_{|[a,b]})$.

Proof. The formula for L_c is trivial and the formula for L_p follows directly from the expansion (7) which is same as in the polynomial case. So we need only consider the arc-length $\mathcal{L}$ and the energy $\mathcal{E}$. We put

$$\mathbf{p}(t) = \gamma(a+th) = P + th\mathbf{v} + \cdots + B_1^2(t)h^4\overline{\mathbf{w}} + O(h^5),$$

where "…" means: "the same expressions as in [2]". Then

$$\begin{aligned}\mathbf{p}'(t) &= h\mathbf{v} + \cdots + B_1^{2\prime}(t)h^4\overline{\mathbf{w}} + O(h^5),\\ \mathbf{p}''(t) &= 4h^2\mathbf{w} + 4h^4\overline{\mathbf{w}} + O(h^5),\end{aligned}$$

and

$$\begin{aligned}|\mathbf{p}'(t)|^2 &= h^2|\mathbf{v}|^2 + \cdots + 2B_1^{2\prime}(t)h^5\langle \mathbf{v}, \overline{\mathbf{w}}\rangle + O(h^6)\\ &= h^2|\mathbf{v}|^2\left(1 + \cdots + 2B_1^{2\prime}(t)h^3\frac{\langle \mathbf{v}, \overline{\mathbf{w}}\rangle}{|\mathbf{v}|^2} + O(h^4)\right),\\ |\mathbf{p}'(t)| &= h|\mathbf{v}|\sqrt{1 + \cdots + 2B_1^{2\prime}(t)h^3\frac{\langle \mathbf{v}, \overline{\mathbf{w}}\rangle}{|\mathbf{v}|^2} + O(h^4)}\\ &= h|\mathbf{v}|\left(1 + \cdots + B_1^{2\prime}(t)h^3\frac{\langle \mathbf{v}, \overline{\mathbf{w}}\rangle}{|\mathbf{v}|^2} + O(h^4)\right).\end{aligned}$$

Hence

$$\mathcal{L} = \int_0^1 |\mathbf{p}'(t)|\,\mathrm{dt} = h|\mathbf{v}| \int_0^1 \left(1 + \cdots + B_1^{2\prime}(t) h^3 \frac{\langle \mathbf{v}, \overline{\mathbf{w}} \rangle}{|\mathbf{v}|^2} + O(h^4)\right) \mathrm{dt}$$
$$= h|\mathbf{v}| \left(1 + \cdots + h^3 \frac{\langle \mathbf{v}, \overline{\mathbf{w}} \rangle}{|\mathbf{v}|^2} \int_0^1 B_1^{2\prime}(t)\,\mathrm{dt}\right) + O(h^5),$$

and as $\int_0^1 B_1^{2\prime}(t)\,\mathrm{dt} = B_1^2(1) - B_1^2(0) = 0$, we have the same result as for the polynomial curves in [2]. The energy is treated the same way:

$$\begin{aligned}
|\mathbf{p}'(t)|^2 &= h^2|\mathbf{v}|^2 + \cdots + O(h^4),\\
|\mathbf{p}''(t)|^2 &= 16h^4|\mathbf{w}|^2 + O(h^6),\\
\langle \mathbf{p}'(t), \mathbf{p}''(t) \rangle &= 4h^3 \langle \mathbf{v}, \mathbf{w} \rangle + \cdots + O(h^5),\\
|\mathbf{p}'(t)|^{-5} &= h|\mathbf{v}| \left(1 + \cdots + O(h^3)\right).
\end{aligned}$$

Hence

$$|\mathbf{p}'(t)|^2 |\mathbf{p}''(t)|^2 - \langle \mathbf{p}'(t), \mathbf{p}''(t) \rangle^2 = 16h^6 \left(|\mathbf{v}|^2|\mathbf{w}|^2 - \langle \mathbf{v}, \mathbf{w} \rangle^2\right) + \cdots + O(h^8),$$

and

$$\mathcal{E} = \frac{1}{2} \int \kappa^2\,\mathrm{ds} = \frac{1}{2} \int_0^1 \frac{|\mathbf{p}'(t)|^2 |\mathbf{p}''(t)|^2 - \langle \mathbf{p}'(t), \mathbf{p}''(t) \rangle^2}{|\mathbf{p}'(t)|^5}\,\mathrm{dt}$$
$$= 8 \frac{|\mathbf{v}|^2|\mathbf{w}|^2 - \langle \mathbf{v}, \mathbf{w} \rangle^2}{|\mathbf{v}|^5} h + \cdots + O(h^3).$$

I.e., once more we have the same result as for the polynomial curves in [2]. □

As the total arc-length (or energy) of the curve γ is the sum of the arc-length (or energy) of the 2^k segments obtained after k-times repeated subdivision, we easily obtain:

Theorem 3.3. *Let $\gamma(t)$, $t \in [t, 0]$ be a rational Bézier curve of degree n, with positive weights and with $\gamma'(t) \neq 0$, all $t \in [t, 0]$. Let*

$$\gamma_j^k(t) = r((j - 1 + t)/2^k), \qquad j = 1 \ldots, 2^k,$$

be the 2^k segments obtained after k-times repeated subdivision. Let $P_{j;0}^k \ldots, P_{j;n}^k$ be the control points for the segment γ_j^k and put

$$\begin{aligned}
\mathbf{v}_j^k &= P_{j;n}^k - P_{j;0}^k,\\
\mathbf{w}_l^k &= \frac{3}{n+1} \left(\left(\sum_{i=1}^{n-1} P_{j;i}^k \right) - (n-1) \left(P_{j;0}^k + \frac{1}{2} \mathbf{v}_j^k \right) \right),
\end{aligned}$$

Then we have the following approximations for the arc-length and energy of the curve γ:

$$\mathcal{L}(\gamma) = \sum_{j=1}^{2^k} \frac{2L_c(\gamma_j^k) + (n-1)L_p(\gamma_j^k)}{n+1} + O\big(2^{-4k}\big) \tag{15}$$

$$\mathcal{L}(\gamma) = \sum_{j=1}^{2^k} |\mathbf{v}_j^k| \left(1 + \frac{2}{3}\frac{|\mathbf{v}_j^k|^2|\mathbf{w}_j^k|^2 - \langle \mathbf{v}_j^k, \mathbf{w}_j^k\rangle^2}{|\mathbf{v}_j^k|^4}\right) + O\big(2^{-4k}\big), \tag{16}$$

$$\mathcal{E}(\gamma) = \sum_{j=1}^{2^k} 12\frac{n-1}{n+1}\frac{L_p(\gamma_j^k) - L_c(\gamma_j^k)}{L_c^2(\gamma_j^k)} + O\big(2^{-2k}\big) \tag{17}$$

$$\mathcal{E}(\gamma) = \sum_{j=1}^{2^k} 12\frac{n-1}{n+1}8\frac{|\mathbf{v}_j^k|^2|\mathbf{w}_j^k|^2 - \langle \mathbf{v}_j^k, \mathbf{w}_j^k\rangle^2}{|\mathbf{v}_j^k|^5} + O\big(2^{-2k}\big), \tag{18}$$

Proof. The prof of (15) is a straight forward calculation using Theorem 3.1:

$$\begin{aligned}
\mathcal{L}(\gamma) &= \sum_{j=1}^{2^k} \mathcal{L}(\gamma_j^k) \\
&= \sum_{j=1}^{2^k} \left(|\mathbf{v}_j^k| \left(1 + \frac{2}{3}\frac{|\mathbf{v}_j^k|^2|\mathbf{w}_j^k|^2 - \langle \mathbf{v}_j^k, \mathbf{w}_j^k\rangle^2}{|\mathbf{v}_j^k|^4}\right) + O\big(2^{-5k}\big)\right) \\
&= \sum_{j=1}^{2^k} \left(\frac{2L_c(\gamma_j^k) + (n-1)L_p(\gamma_j^k)}{n+1} + O\big(2^{-5k}\big)\right) \\
&= \sum_{j=1}^{2^k} \frac{2L_c(\gamma_j^k) + (n-1)L_p(\gamma_j^k)}{n+1} + O\big(2^{-4k}\big)
\end{aligned}$$

and similar for (16)–(18). □

The ingredients L_c and L_p in (15) and (17) require the calculations of $n+1$ square-roots, while the ingredients in (16) and (18) only need one square-root, (to calculate $|\mathbf{v}|$). On the other hand, the vector $\mathbf{w}$ has no immediate geometric interpretation, in contrast to the chord-length L_c and the polygon-length L_p.

4 Algorithms for the arc-length

The convergence in (15) and (16) is so fast that it is obvious to use these results as the basis for an algorithm. Suppose we want to calculate the arc-length of a curve γ with control points $P_0, \ldots, P_n$. As an approximation to the arc-length we can then use either

$$L = \frac{2L_c(\gamma) + (n-1)L_p(\gamma)}{n+1} \quad \text{or} \quad \widetilde{L} = |\mathbf{v}| \left(1 + \frac{|\mathbf{v}|^2|\mathbf{w}|^2 - \langle \mathbf{v}, \mathbf{w} \rangle^2}{|\mathbf{v}|^4}\right),$$

where $\mathbf{v} = P_n - P_0$ and $\mathbf{w} = \frac{3}{n+1} \sum_{i=1}^{n-1} \left(P_n - P_0 - \frac{i}{n}\mathbf{v}\right)$. If we estimate the error of the approximation to be sufficiently small, then we just use this approximation, else we subdivide the curve, calculates an approximation for each piece and add the two values to get an approximation to the total arc-length of the curve. In Figure 2 and Figure 3 we present pseudo code for

```
length(b): real
    b: record of BezierCurve; Bezier curve: degree, control points, weights, etc.
begin
   Lp ← poly_length(b);        The length of the control polygon.
   Lc ← chord_length(b);       The length of the chord.
   n ← degree(b);              The degree of b.
   if good approximation
   then
       return (2*Lc+(n-1)*Lp)/(n+1);
   else begin
       b1, b2 ← subdivide(b);  The two halfs of b.
       return length(b1) + length(b2);
       end;
end length
```

Figure 2: Pseudo code for the calculation of arc-length, using (15).

this adaptive and recursive method. We have not considered the question of how to determine whether we have a “good approximation” or not, but the discussion in [2] of this issue apply just as well to the present rational case.

5 Conclusion

We have extended earlier work on the arc-length and energy of polynomial Bézier curves to the case of rational Bézier curves. There is no difference in the final results, so the earlier algorithm for polynomials curves works equally well in the case of rational curves. We have furthermore presented an alternative algorithm which uses fewer square-roots, but for which the geometric interpretation is less obvious.

```
length(b): real
    b: record of BezierCurve;        Bezier curve: degree, control points, weights, etc.
begin
    v ← first_order_term(b);         The vector v.
    w ← second_order_term(b,v);      The vector w.
    if good approximation
    then
        v2 ← scalar_product(v,v);    The square length |v|^2.
        w2 ← scalar_product(w,w);    The square length |w|^2
        vw ← scalar_product(v,w);    The inner product ⟨v, w⟩
        return sqrt(v2)(1+2/3*(v2*w2-vw*vw)/(v2*v2));
    else begin
        b1, b2 ← subdivide(b);       The two halfs of b.
        return length(b1) + length(b2);
        end;
end length
```

Figure 3: Pseudo code for the calculation of arc-length, using (16).

References

[1] Gerald Farin. *Curves and Surfaces for Computer Aided Geometric Design. A Practical Guide*. Academic Press, London, 1988.

[2] Jens Gravesen. Adaptive subdivision and the length and energy of Bézier curves. *Computational Geometry*, 8:13–31, 1997.

[3] B. Guenter and R. Parent. Computing the arc length of parametric curves. *IEEE Computer Graphics and Applications*, 10(3):72–78, May 1990.