Parametrisation in Iso Geometric Analysis, a first report

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1 Introduction

This is a report of our first experiments in the field of iso geometric analysis. The idea was simply to get acquainted with B-splines, finite element analysis, and optimisation. It turned out that already a simple one dimensional example shows that the accuracy of the method is highly dependent of the parametrisation. So we find it worthwhile to report the results.

We consider the simple one dimensional eigenvalue problem of finding the frequencies of longitudinal vibration of a rod, se Figure 1. The governing

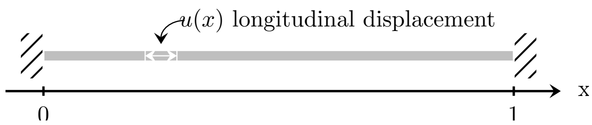

Figure 1: We consider longitudinal vibration of a rod.

equation is Helmholtz's equation with Dirichlet boundary conditions

$$\triangle u = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = -\lambda u, \qquad u(0) = u(1) = 0. \tag{1}$$

The solution is of course well known. The eigenvalues and eigenfunctions are

$$\lambda_n = \omega_n^2 \qquad u_n(x) = \sin(\omega_n x) \qquad \omega_n = n\pi. \tag{2}$$

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If we multiply (1) with a test function v and integrate, then we obtain the weak form of the equations

$$\int_0^1 \lambda u\, v\, \mathrm{d}x = -\int_0^1 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} v\, \mathrm{d}x = -\left[\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} v\right]_0^1 + \int_0^1 \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\, \mathrm{d}x = \int_0^1 \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\, \mathrm{d}x.$$

That is

$$\int_0^1 \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\, \mathrm{d}x = \lambda \int_0^1 u\, v\, \mathrm{d}x. \tag{3}$$

We now write u as a finite linear combination of basis functions ϕ_i, $i = 1, \ldots, N$,

$$u(x) = \sum_{i=1}^{N} u_i \phi_i(x) \tag{4}$$

and let v run through the set of basis functions. We obtain a system of linear equations

$$\sum_{i=1}^{N} u_i \int_0^1 \phi_i'(x)\, \phi_j'(x)\, \mathrm{d}x = \lambda \sum_{i=1}^{N} u_i \int_0^1 \phi_i(x)\, \phi_j(x)\, \mathrm{d}x \quad \text{for all } j = 1, \ldots, N. \tag{5}$$

They can be written in matrix form

$$\mathbf{K}\mathbf{u} = \lambda \mathbf{M}\mathbf{u}, \tag{6}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \tag{7}$$

$\mathbf{K}$ is the stiffness matrix

$$\mathbf{K} = \begin{pmatrix} \langle \phi_1', \phi_1' \rangle & \ldots & \langle \phi_1', \phi_N' \rangle \\ \vdots & & \vdots \\ \langle \phi_N', \phi_1' \rangle & \ldots & \langle \phi_N', \phi_N' \rangle \end{pmatrix}, \tag{8}$$

$\mathbf{M}$ is the mass matrix

$$\mathbf{M} = \begin{pmatrix} \langle \phi_1, \phi_1 \rangle & \ldots & \langle \phi_1, \phi_N \rangle \\ \vdots & & \vdots \\ \langle \phi_N, \phi_1 \rangle & \ldots & \langle \phi_N, \phi_N \rangle \end{pmatrix}, \tag{9}$$

and the inner product between two functions is

$$\langle \phi, \psi \rangle = \int_0^1 \phi(t)\, \psi(t)\, \mathrm{d}t. \tag{10}$$

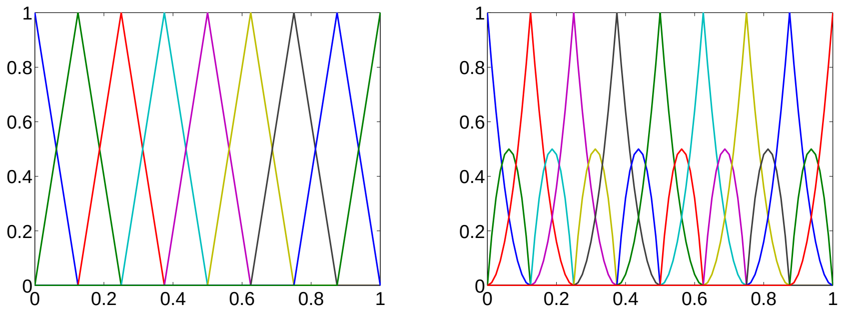

Figure 2: The basis functions in traditional finite element analysis. To the left linear and to the right quadratic.

In traditional finite element analysis the basis functions ϕ_i are chosen as piecewise polynomials that join together in a piecewise C^0 fashion, see Figure 2. The result of using traditional finite element analysis on a uniform grid is shown in Figure 3.

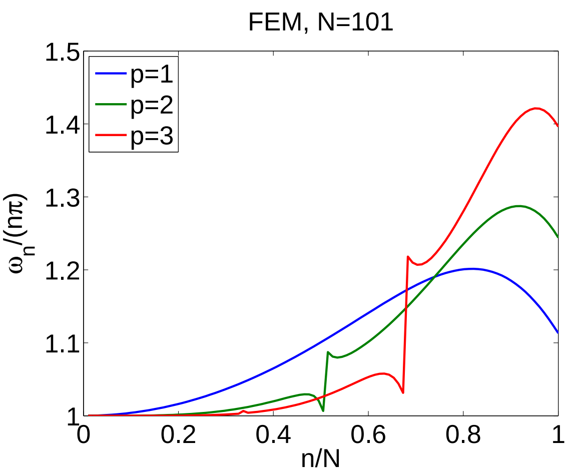

Figure 3: The normalised spectra resulting from finite element analysis with shape functions of degree 1, 2, and 3.

2 Iso Geometric Analysis

In iso geometric analysis the, normally complicated, physical domain Ω is parametrised by a map $F : \widehat{\Omega} \to \Omega$, where the parameter domain $\widehat{\Omega}$ is supposed to be simple and easy to discretize. In the present case we have

$\Omega = \widehat{\Omega} = [0, 1]$. The parametrisation is of the form

$$F(t) = \sum_{i=1}^{N} x_i N_i^p(t), \tag{11}$$

where N_i^p are B-splines of degree p on some knot vector $\mathbf{t}$, see Figure 4. The

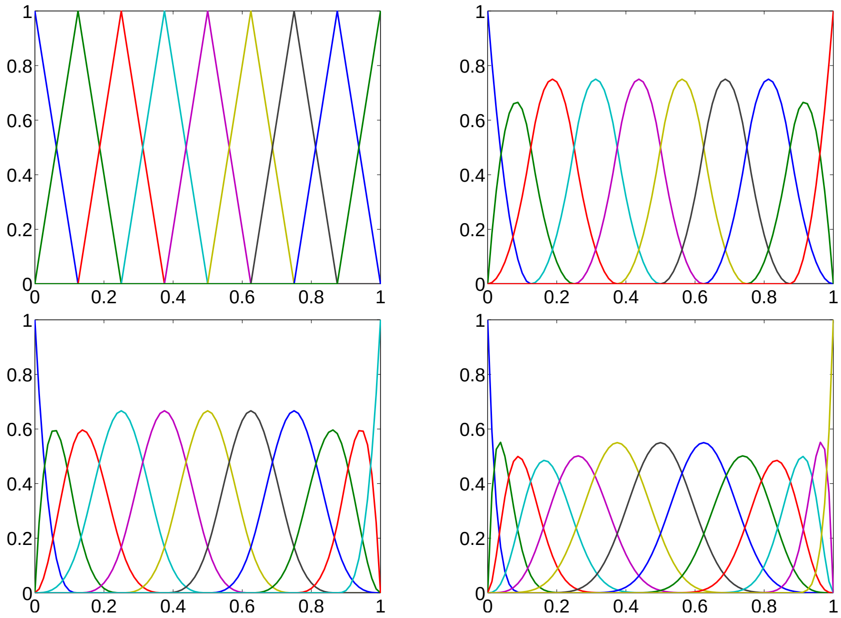

Figure 4: B-splines of degree 1, 2, 3, and 5 on a uniform knot vector.

basis functions are of the form

$$\phi_i = N_i^p \circ F^{-1}, \quad i = 1, \ldots, N. \tag{12}$$

The left hand side of (3) becomes

$$\int_0^1 \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_0^1 \frac{1}{F'(t)}\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t \tag{13}$$

and the right hand side becomes

$$\lambda \int_0^1 u\, v\, \mathrm{d}x = \lambda \int_0^1 u(t)\, v(t)\, F'(t)\, \mathrm{d}t \tag{14}$$

where

$$u(t) = \sum_{i=1}^{N} u_i\, N^p(t). \tag{15}$$

Once more the equation can be written matrix form (6), where the entries of the stiffness and mass matrix now becomes

$$K_{ij} = \int_0^1 \frac{N_i^{p\prime}(t)\, N_j^{p\prime}(t)}{F'(t)}\,\mathrm{d}t, \tag{16}$$

$$M_{ij} = \int_0^1 N_i^p(t)\, N_j^p(t)\, F'(t)\,\mathrm{d}t. \tag{17}$$

We use Gaussian quadrature to evaluate the integrals. Observe that (17) can be evaluated exactly if we use at least $p+1+\lceil p/2 \rceil$ points in the quadrature.

In our example we use a uniform knot vector except for the boundary knots which have full multiplicity. We try four different choice of parametrisation (11) which of course is given by the control points $\mathbf{x} = (x_i)_{i=1\ldots N}$.

1. The control points are chosen such that F is the identity map, i.e., $x = F(t) = t$. The control points are the Greville abscissas, $x_i = (t_{i+1} + \cdots + t_{i+p})/p$.
2. The control points are chosen uniformly, i.e., $x_i = \frac{i-1}{N-1}$.
3. The control points are chosen such that the error of a single eigenvalue λ_k is minimal, i.e. we solve the following optimisation problem

   $$\text{minimise} \left(\frac{\omega_n}{n\pi} - 1 \right)^2. \tag{18}$$

   The eigenvalue $\lambda_n = \omega_n^2$ is chosen as the one with the maximal error in 2.
4. The control points are chosen such that the maximal relative error of the eigenvalues is minimised. We formulate it as the following constrained optimisation problem.

   $$\begin{aligned} &\text{minimise}\, S \\ &\text{such that } \left(\frac{\omega_n}{n\pi} - 1 \right)^2 < S \text{ for all } n = 1, \ldots, N. \end{aligned} \tag{19}$$

All calculations are done in Matlab with the optimisation toolbox. The results can be seen in Figures 5–8.

We have not spend much time on the optimisation and it is very likely that better parametrisations can be obtained by changing the optimisation algorithm. This, however, is not the main point in this report. We simply want to point out that the result of iso geometric analysis can be very sensitive to the parametrisation.

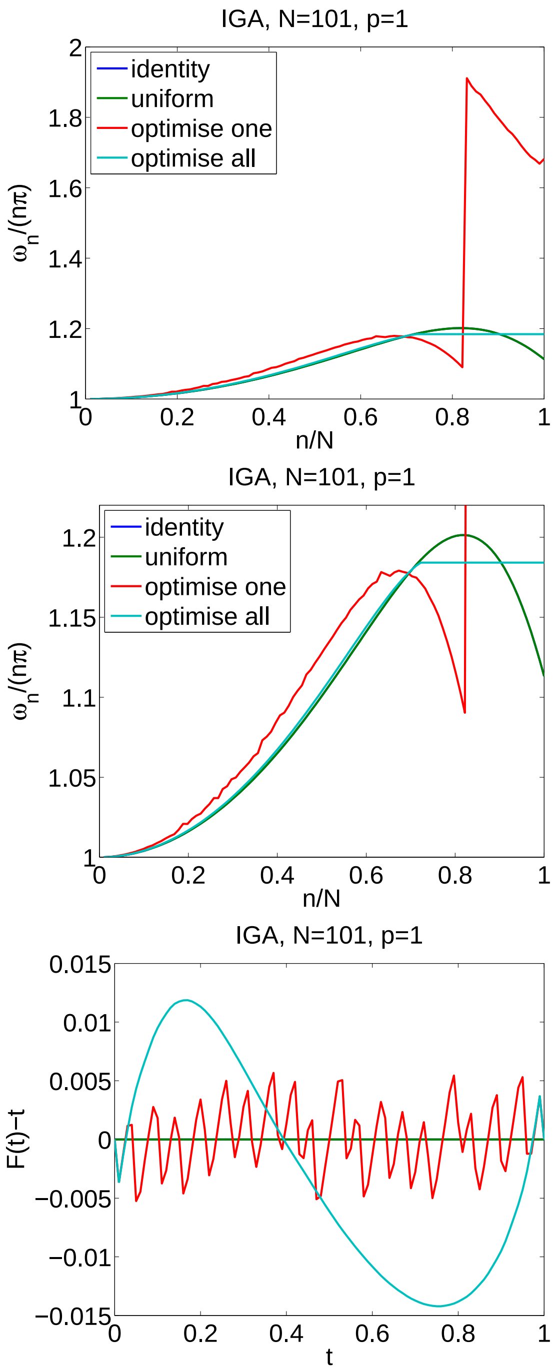

Figure 5: The results of iso geometric analysis using B-splines of degree 1 and four different parametrisations. On the top the normalised spectra, in the middle a zoom near the line $\frac{\omega_n}{n\pi} = 1$, and below the differences between the parametrisations F and the identity.

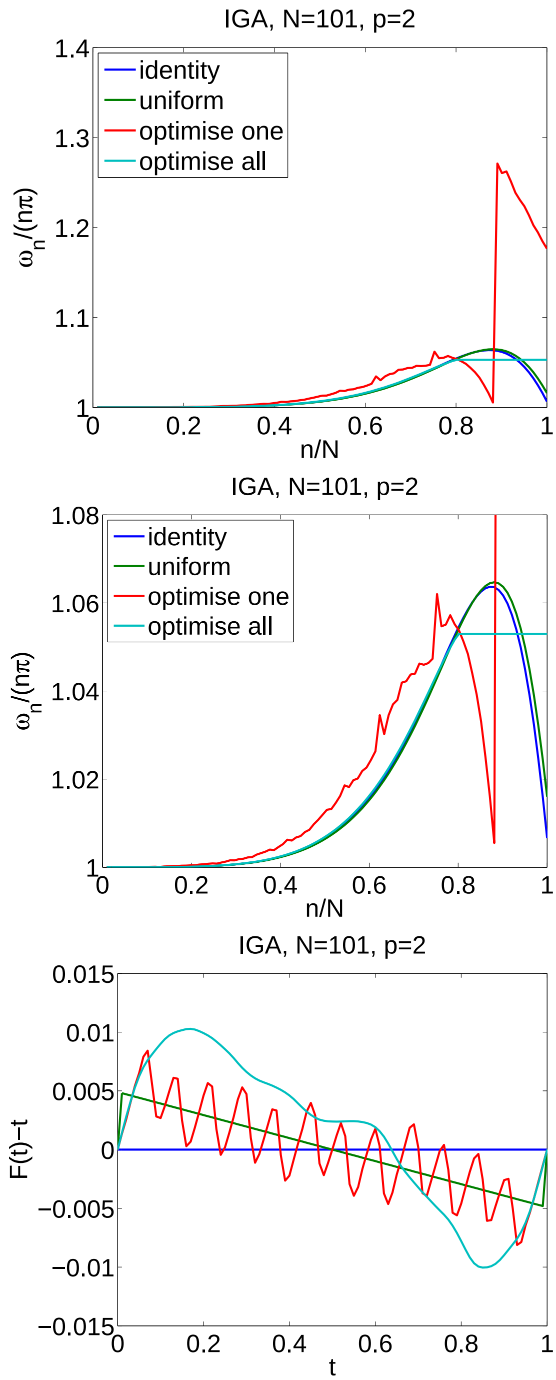

Figure 6: The results of iso geometric analysis using B-splines of degree 2 and four different parametrisations. On the top the normalised spectra, in the middle a zoom near the line $\frac{\omega_n}{n\pi} = 1$, and below the differences between the parametrisations F and the identity.

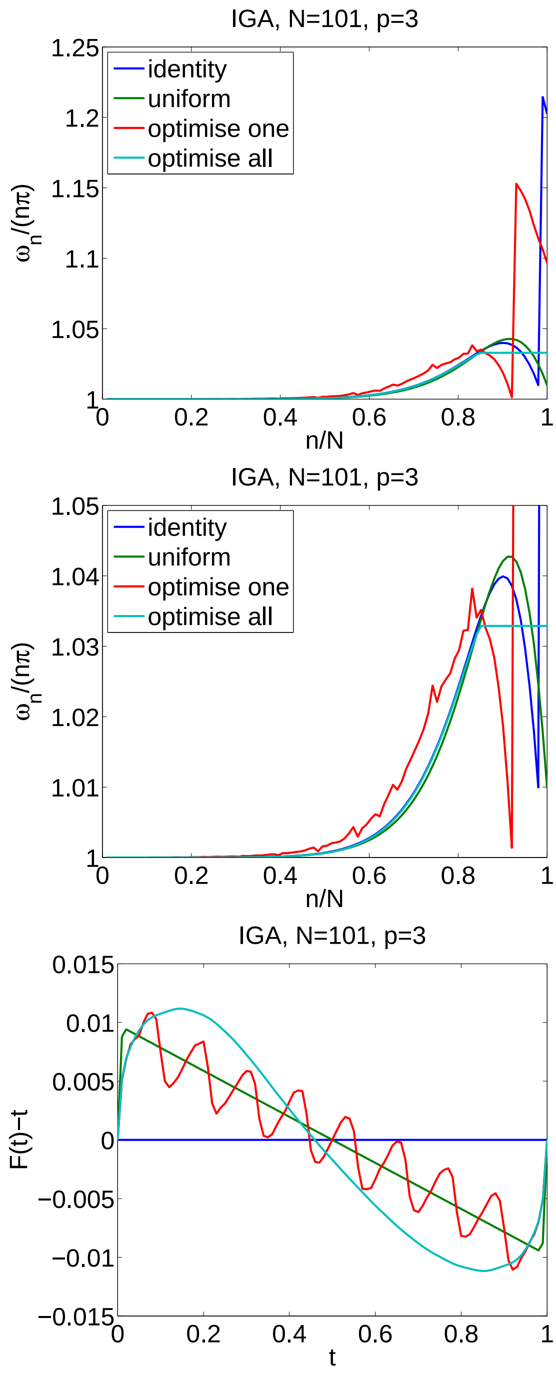

Figure 7: The results of iso geometric analysis using B-splines of degree 3 and four different parametrisations. On the top the normalised spectra, in the middle a zoom near the line $\frac{\omega_n}{n\pi} = 1$, and below the differences between the parametrisations F and the identity.

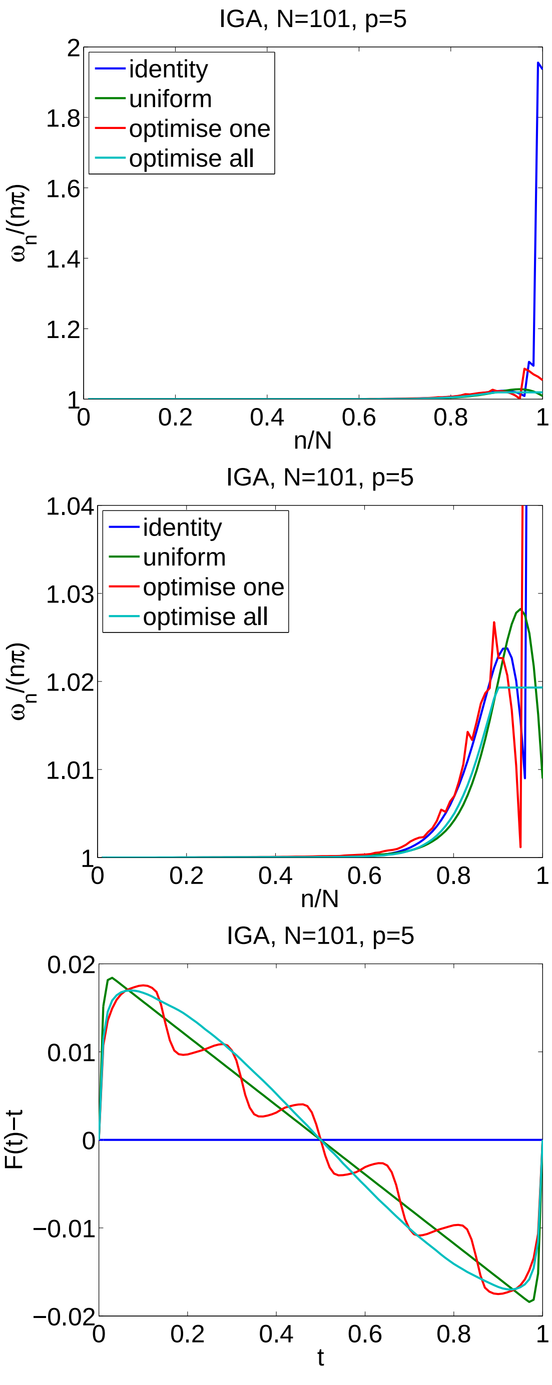

Figure 8: The results of iso geometric analysis using B-splines of degree 5 and four different parametrisations. On the top the normalised spectra, in the middle a zoom near the line $\frac{\omega_n}{n\pi} = 1$, and below the differences between the parametrisations F and the identity.

3 Conclusion

In [1] it was found that using B-splines as basis functions gave considerable better accuracy and robustness than traditional finite elements. We have confirmed these findings, but have also found that in an iso geometric setting the performance is very much dependent on the parametrisation.

References

[1] T.J.R. Hughes, A. Reali, G. Sangalli, Duality and unified analysis of discrete approximations in structural dynamics and wave propagation: Comparison of p-method finite elements with k-method NURBS, *Comput. Methods Appl. Mech. Engrg.* **197**, 4104–4124 (2008).