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ISOGEOMETRIC ANALYSIS AND SHAPE OPTIMISATION

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Summary. We look at some succesfull examples of shape optimisation using isogeometric analysis. We also addresses some problems which we encountered.

1 Introduction

In isogeometric analysis the physical domain $\Omega \subseteq \mathbb{R}^2$ is parametrised by a map $\mathbf{x} : [0,1]^2 \to \Omega$. The map $\mathbf{x}$, as well as all physical fields, are given in terms of B-splines or NURBS,

$$\mathbf{x}(u,v) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{c}_{i,j} M_i(u)\, N_j(v), \tag{1}$$

where $\mathbf{c}_{i,j}$ are the control points. When u or v becomes 0 or 1 we obtain the four boundary curves $\mathbf{x}_1, \dots, \mathbf{x}_4$. The shape of Ω is determined by the boundary so shape optimisation is done by adjusting the four boundary curves or rather the boundary control points $\mathbf{c}_{0,j}, \mathbf{c}_{m,j}, \mathbf{c}_{i,0}, \mathbf{c}_{i,n}$. How the inner control points are determined is addressed in Section 4.

2 Optimisation of the frequencies of a drum

In the first example we consider the design of a drum. That is, given N required frequencies $\widehat{\lambda}_i$, $i = 1, \dots, N$, we want to design a vibrating membrane such that the lower eigenfrequences are exactly as required. Mathematically we specify the lower eigenvalues of the Laplace operator. Not even the full spectrum of the Laplace operator determines the domain so we minimise the length of the perimeter and treat the specified eigenvalues as constraints, see [8] and references therein. If $\lambda_1 \leq \lambda_2 \leq \dots$ are the eigenvalues of the Laplace operator $\triangle$, then we consider the following optimisation problem,

$$\text{minimise} \sum_{i=1}^{4} \int_0^1 \left| \frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} \right| \mathrm{d}t, \quad \text{such that} \quad \begin{array}{ll} \lambda_i = \widehat{\lambda}_i, & i = 1, \dots, N, \\ \triangle f_i = \lambda_i f_i, & i = 1, \dots, N. \end{array} \tag{2}$$

In the specific example shown in Figure 1 we want the first four frequencies or eigenvalues to be in the harmonic proportion 2 : 3 : 3 : 4. The problem with the double eigenvalue is solved by replacing (2) for the case $i = 2, 3$ with $\lambda_2 + \lambda_3 = \widehat{\lambda_2} + \widehat{\lambda_2}$ and $\lambda_2 \lambda_3 = \widehat{\lambda_2 \lambda_2}$.

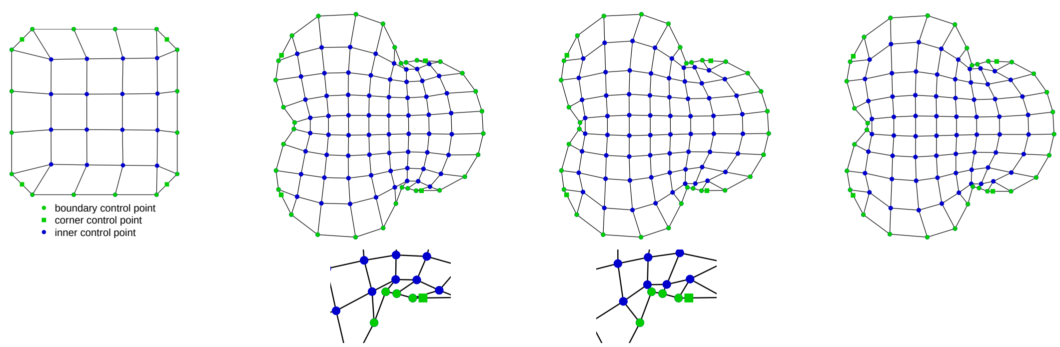

Figure 1: Minimising the perimeter of a harmonic drum. After 50 iterations the parametrisation became nearly singular and if we continued it started to fold over in the indicated area. After improving the parametrisation by using the Winslow functional the optimisation converged after another 16 iterations.

One problem we encountered during the optimisation was that the map $\mathbf{x}$ became singular, i.e., it was no longer a parametrisation. So there is the need to have an reliable method to determine the inner control points, see Section 4.

3 Optimisation of a pipe bend

In the second example we look at a 2D Stokes flow problem where a pipe bend has to be designed such that the internal energy loss is minimised under constraints on the area of the pipe bend, see [9]. If (u_1, u_2) is the velocity of the fluid the we have the following problem,

$$\text{minimise} \int_\Omega (\|\nabla u_1\| + \|\nabla u_2\|)\,\mathrm{d}x\,\mathrm{d}y \quad \text{such that} \quad \text{area}(\Omega) \leq A \tag{3}$$

The optimised design, see Figure 2, is in agreement with the result obtained by topology op-

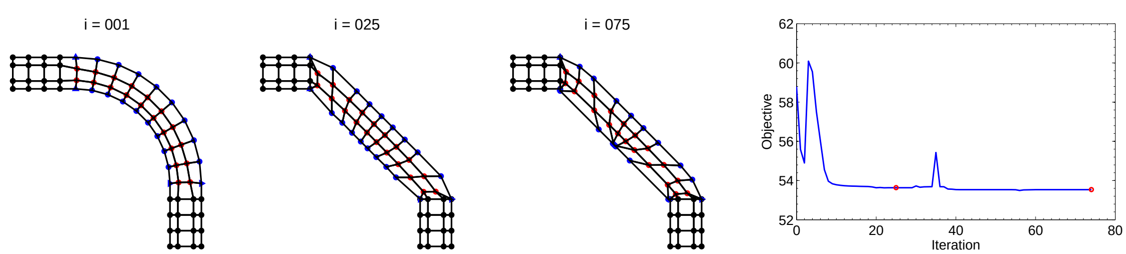

Figure 2: Optimisation of a pipe bend. After 25 iterations the geometry is determined. During the remaining iterations the optimisation only changes the parametrisation, making it worse.

timisation, c.f., [3]. We see that the design is obtained after 25 iterations. But, the optimiser continues its work, not changing the design but clustering the control points and thereby creating a poorer parametrisation. This introduces numerical errors that makes the objective function smaller. It is possible because the design contains a straight line and the control points can move freely on this line without changing the geometry. It is a well known problem in shape optimisation and has previously been dealt with by filtering techniques, extra contributions to the objective function, or extra constraints such as a minimum distance between control points, see [1]. The latter will unfortunately also prevents the sharp corners at the inlet and outlet that are part of the design. Another solution is to detect and remove superfluous control points, in the present case eight on the inside and six on the outside of the bend.

4 Parametrisation

We now consider the parametrisation problem. That is, given a parametrisation $\mathbf{y} : \partial[0,1]^2 \to \partial\Omega$ of the boundary of a domain Ω extend it to a parametrisation $\mathbf{x} : [0,1]^2 \to \Omega$ of the whole domain. Or, in terms of the control points given boundary control points determine the inner control points, see Figure 3. The simplest way of obtaining a map $\mathbf{x}$ is by considering the control

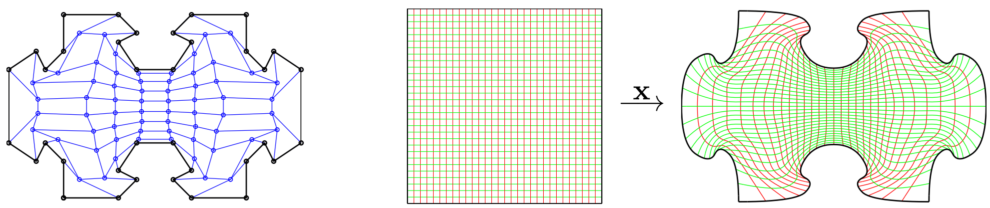

Figure 3: The parametrisation problem: Given the (black) boundary control points, determine the (blue) inner control points such that the map $\mathbf{x}$ is a parametrisation.

net as a set of springs with the same spring constant. Then every inner control points is the average of its four neighbours. This is a linear system of equations which are easily solved. By adjusting the spring constants one can make a given reference configuration in balance and then use the equilibrium equations to get the inner control points after a change of the boundary control points. One way of doing this is by using the mean value coordinates of Floater, see [4]. Another way to use a reference configuration is to demand that the configuration of an inner control point and its four neighbours should be a scaled and rotated version of the one in the reference net. This leads to an overdetermined set of equations which has to be solved in the least square sense.

The map $\mathbf{x}$ is a parametrisation if and only if the determinant of the Jacobian is non vanishing. The determinant of the Jacobian is piecewise polynomial, so we can write it in terms of B-splines

$$\det J = \sum_{i,j=1}^{m,n} \sum_{k,\ell=1}^{m,n} \det(\mathbf{c}_{i,j}, \mathbf{c}_{k,\ell})\, M_i{}'(u)\, N_j(v)\, M_k(u)\, N_\ell{}'(v) = \sum_{i,j=1}^{\widetilde{m},\widetilde{n}} d_{i,j}\, \widetilde{M}_i(u)\, \widetilde{N}_j(v), \tag{4}$$

A *sufficient* condition for the positivity of $\det J$ is the positivity of all the coefficients $d_{i,j}$. They depend quadratically on the control points $\mathbf{c}_{i,j}$. The solution to the following problem

$$\underset{\text{inner control points}}{\text{maximise}} \quad S, \quad \text{such that} \quad d_{i,j} \geq S. \tag{5}$$

gives a valid parametrisation if the control net is sufficiently refined. Even though the parametrisation is valid it need not be very good. One way to improve it is to make it as conformal as possible and this can be done by minimising the *Winslow* functional:

$$\underset{\text{inner control points}}{\text{minimise}} \int_0^1 \int_0^1 \frac{\|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2}{\det(\mathbf{x}_u, \mathbf{x}_v)} \, \mathrm{d}u \, \mathrm{d}v, \tag{6}$$

see [5] for details. To make sure that we have a valid parametrisation, and a positive denominator, we add the constraints, $d_{i,j} \geq \delta S_0$, where $\delta \in [0,1]$ and $d_{i,j}$ and S_0 are given by (4) and (5) respectively. If we let $\mathbf{r} = \mathbf{x}^{-1}$ be the inverse map and change variables from (u, v) to (x, y) in (6) then we obtain the following linearly constrained quadratic optimisation problem

$$\underset{\mathbf{r}}{\text{minimise}} \int_{\Omega} \left(\|\mathbf{r}_x\|^2 + \|\mathbf{r}_y\|^2\right) \, \mathrm{d}x \, \mathrm{d}y, \quad \text{such that} \quad \mathbf{r}|_{\partial\Omega} = \mathbf{y}^{-1}. \tag{7}$$

It has a unique minimum realised by a pair of harmonic functions. By the Kneser-Rado-Choquet Theorem, [2, 6, 10], this is a diffeomorphism. So the original problem (6) has a unique minimum too.

If we square the numerator in (6) then we obtain the modified Liao functional which is well known from grid generation, [7], but in our experience the Winslow functional behaves better for our purpose.

It is quite expensive to solve the problems (5) and (6) so we do not do this in each optimisation cycle. If the parametrisation becomes close to singular then we do it and obtain hereby a good reference parametrisation, or control net, $\mathbf{x}_0$. We then propose to linearise the problem (6) and solve the linear equation

$$H_{\mathbf{x}_0}(\mathcal{W})\mathbf{x} = H_{\mathbf{x}_0}(\mathcal{W})\mathbf{x}_0 - \nabla_{\mathbf{x}_0}\mathcal{W}, \tag{8}$$

where $\mathcal{W}$ denotes the Winslow functional and $\nabla_{\mathbf{x}_0}\mathcal{W}$ and $H_{\mathbf{x}_0}(\mathcal{W})$ are the gradient and Hessian evaluated at $\mathbf{x}_0$, respectively.

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