The geometry of the Moineau pump

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Abstract

The Moineau pump was invented in 1931 by the French engineer René Moineau and exhibits an intriguing geometry. The original design is based on hypo- and epicycloids and all except one design has either cusps or less severe inflexion points with infinite curvature. By using the support function to represent planar curves it is possible to make an explicit analysis of a general design and we can show that points of infinite curvature are unavoidable.

Key words: Moineau pump, envelope, support function.

1 Introduction

The Moineau pump is an invention from 1931 by the French engineer René Moineau, see [11]. The Moineau pump has two parts rotating relative to each other in an eccentric motion. The shapes in an axial cross section are in the original design based on epi- and hypo-cycloids. In 2006 the large Danish pump manufacturer Grundfos wanted an investigation of other possible designs and brought the problem to the 57th European Study Group with Industry held at the Technical University of Denmark. All but one of the classical designs possesses points with infinite curvature and it is of particular interest too see if that can be avoided.

In a previous analysis of the scroll compressor [3] planar curves was represented by specifying the radius of curvature as a function of tangent direction. In the present work we specify the support as a function of normal direction. In both cases it is trivial to find a point with a given tangent or normal and this makes

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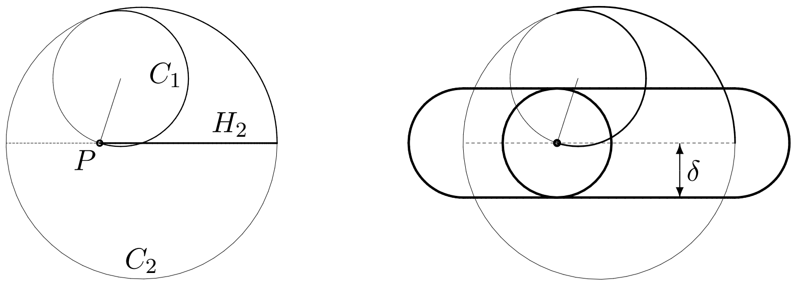


Fig. 1. The circle C_1 rolls inside the circle C_2 and the point P rolls back and forth on the diameter H_2. To the right P and H_2 are offset with δ.

it possible to get closed analytical expressions for the envelope of a moving curve.

The support function is a classical tool in convex geometry [2] and was first suggested for use in geometric design by M. Sabin in 1974 [12]. Lately there has been renewed interest in this representation [1,5,7,10,13,14]. Using the support function it is possible to analyse a general design and show that infinite curvature can not be avoided.

The paper is organised as follows. In Section 2 we introduce the original designs and explain how the pump works. In Section 3 we introduce the representation by the support function and give some properties of the representation. In Section 4 we analyse a general design, show that points of infinite curvature can not be avoided and give a few examples of new designs.

2 The original design

If a circle C_1 with radius 1 rolls inside a circle C_n of radius n then a fixed point P on C_1 traces a hypo-cycloid H_n with n cusps, see Fig. 3. In the case where $n = 2$ the hypo-cycloid is simply a diameter of C_2 traced twice, once in each direction, see Fig. 1. The rolling of C_1 inside C_2 also defines a motion in the plane and under this motion the point P moves along H_2. If we offset both P and H_2 with the same amount δ we obtain the rotor, a circle, moving back and forth in the stator, two parallel lines with semicircles attached at the ends. The picture is a horizontal section in the pump and the areas to each side of the rotor are sections in two pump chambers.

To construct the pump we now lift the right hand picture in Fig. 1 to horizontal planes $z =$ constant, see Fig. 2 to the left. We then roll each copy of the circle C_1, to which the rotor is attached, a distance proportional to the height z, see the second picture in Fig. 2. Finally we rotate each horizontal plane, with both the rotor and the stator, such that all copies of the circle C_1 are back to the original position, see the third picture in Fig. 2. The stator is now formed by

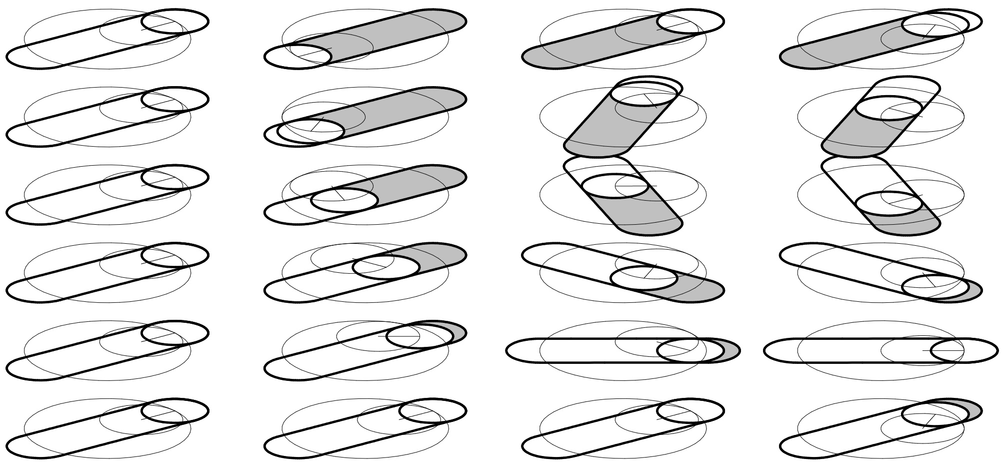


Fig. 2. The construction of the pump is illustrated in the first three pictures and the pumping in the last two pictures.

the thick “rounded rectangles”, the rotor is formed by the thick small circles, and the space in between forms two series of pump chambers. In Fig. 2 the horizontal sections in one of the pump chambers are coloured in gray but only a portion corresponding to half the height of a pump chamber is shown, (any other portion can be found by symmetry). As all copies of C_1 now are above each other they form a cylinder as do the copies of C_2. If the C_1 cylinder now rolls inside the C_2 cylinder the thick small circles moves back and forth inside the “rounded rectangles”. As we see from the difference between the third and the last picture this has same effect as a vertical translation followed by a rotation, i.e., the pump chambers are moved up (or down) by a screw motion. For more pictures and animations, see [8,9].

2.1 The $n+1$:n hypo-cycloid construction

Consider Fig. 3. We have three circles C_1, C_n and C_{n+1} with radii 1, n, and $n+1$ respectively. Rolling C_1 inside C_n and C_{n+1} produces the two hypo-cycloids H_n and H_{n+1} respectively. Now consider the motion generated by letting C_n roll inside C_{n+1} and let H_n follow the motion. It is easily seen that H_{n+1} is (part of) the envelope for this moving curve. Furthermore, a straightforward calculation shows that the cusps of H_n stays on H_{n+1} during the motion, see [6]. We can now perform the same procedure as above. Lift the construction to horizontal planes $z =$ constant, roll each copy of C_n a distance proportional to z, and rotate each horizontal plane such that the copy of C_n returns to its original position. This way $n+1$ series of pump chambers are formed and when the cylinder formed by the copies of C_n rolls inside the cylinder formed by the copies of C_{n+1} the pump chambers move up or down by a screw motion, see [8,9].

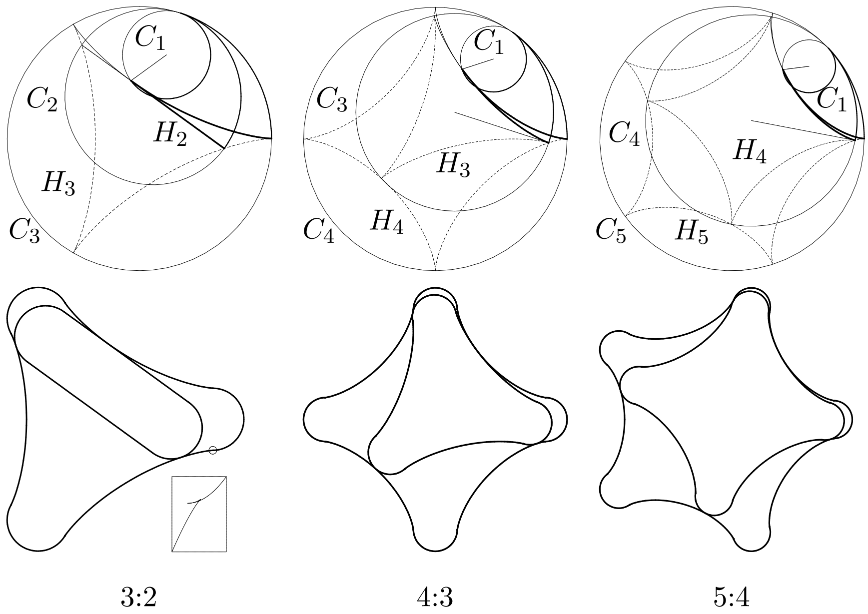
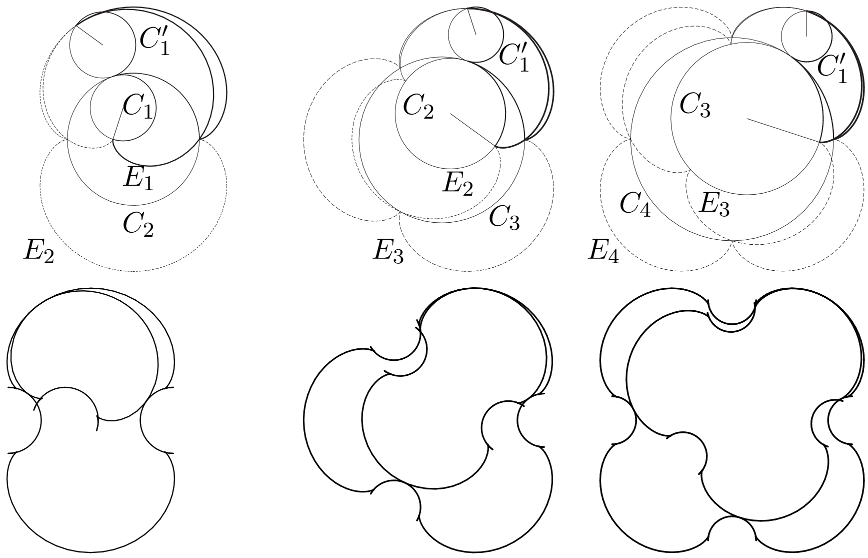


Fig. 3. Above the $n+1{:}n$ hypo-cycloid construction for $n = 2, 3, 4$. Below offsets of the same constructions. In the 3:2 construction we have zoomed in on a point where a semicircle connects to the offset H_3. The zoom uses different scales in the x and y direction.

In the patent [11] the construction is offset, probably to avoid the cusps. If an offset of a curve passes through a centre of curvature, i.e., intersects the evolute, then a cusp is formed. So to avoid cusps a curve can at most be offset out to the minimum radius of curvature. The hypo-cycloids have zero radius of curvature at the cusps so even though the offset is visually fine the offsets do have cups, see Fig. 3.

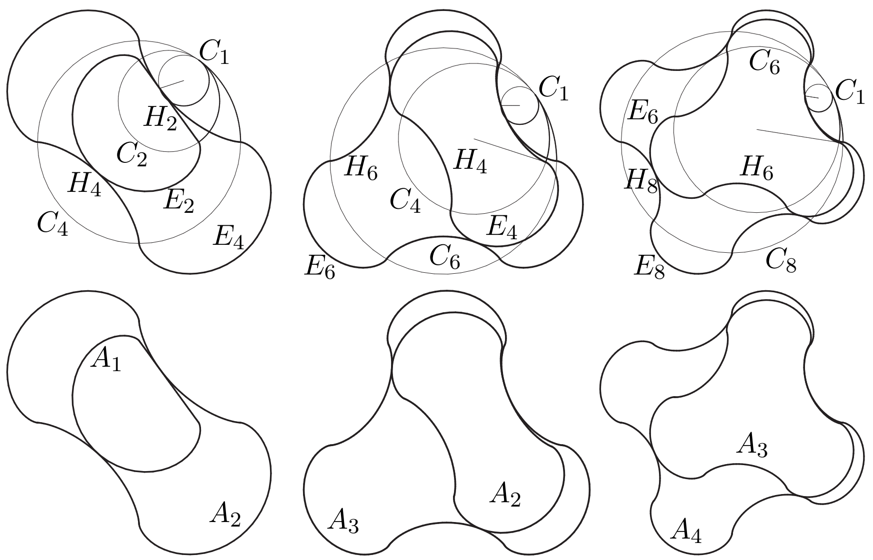
2.2 The $n+1{:}n$ epi-cycloid construction

Consider Fig. 4. We have three circles C'_1, C_n and C_{n+1} with radii 1, n, and $n+1$ respectively. Rolling C'_1 outside C_n and C_{n+1} produces the two epi-cycloids E_n and E_{n+1} respectively. Now consider the motion generated by letting C_n roll inside C_{n+1} and let E_n follow the motion. As in the previous cases E_{n+1} is (part of) the envelope for this moving curve and the cusps of E_{n+1} stays on E_n during the motion, see [6]. A pump is constructed by the same procedure as in the hypo-cycloid case and $n+1$ series of pump chambers are formed. Once again the cusps of a epi-cycloid has infinite curvature, so an offset has cusps too, as is clearly seen in Fig. 4.



2:1 3:2 4:3

Fig. 4. Above the $n+1{:}n$ epi-cycloid construction for $n = 1, 2, 3$, below offsets.



2:1 3:2 4:3

Fig. 5. The construction consisting of alternating arcs of hypo- and epi-cycloids.

2.3 The $n+1{:}n$ hypo-epi-cycloid construction

Consider Fig. 5. We have three circles C_1, C_{2n}, C_{2n+2} with radii 1, $2n$, and $2n+2$ respectively. If we take every second arc of the hypo-cycloid H_{2n} and every second arc of the epi-cycloid E_{2n} we obtain a curve A_n consisting of alternating arcs of hypo- and epi-cycloids. Each of these arcs are obtained by rolling C_1 on the inside or the outside of C_{2n}. We now let A_n follow the motion generated by letting C_{2n} roll inside C_{2n+2}. It can be shown that the outer part of the envelope is A_{n+1}, that the tangents agree at each point in $A_{2n} \cap A_{n+1}$, and that there in general are $n+1$ points of contact except at specific instances where two points of contacts come together, see [6]. By the same procedure as in the case hypo-cycloid or epi-cycloids, $n+1$ series of pump chambers are formed and when the cylinder formed by the copies of C_{2n} rolls inside the cylinder formed by the copies of C_{2n+2} the pump chambers move up or down by a screw motion, see [8,9]. This construction is without cusps but there are n inflexion points on A_n and here the curvature is infinite.

3 The support function

We first introduce a rotating frame $(\mathbf{e}, \mathbf{f})$ and the rotation matrix $\mathbf{R}$,

$$\mathbf{e}(\phi) = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}, \qquad \mathbf{f}(\phi) = \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}, \qquad \mathbf{R}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \tag{1}$$

Now let $h : \mathbb{R} \to \mathbb{R}$ be a real C^2 function and consider the parametrisation

$$\mathbf{x}(\phi) = h(\phi)\,\mathbf{e}(\phi) + h'(\phi)\,\mathbf{f}(\phi). \tag{2}$$

Differentiation yields $\mathbf{x}'(\phi) = (h(\phi) + h''(\phi))\,\mathbf{f}(\phi)$ so $\mathbf{x}$ is regular if and only if

$$\frac{\mathrm{d}s}{\mathrm{d}\phi} = h(\phi) + h''(\phi) \neq 0. \tag{3}$$

In that case the tangent is $\mathbf{t} = \pm\mathbf{f}(\phi)$ and the normal is $\mathbf{n} = \mathbf{e}(\phi)$ [1]. Hence

$$h = \mathbf{x}\cdot\mathbf{f} = \mathbf{x}\cdot\mathbf{n} = \text{distance from origo to the tangent line.} \tag{4}$$

That is, h is the *support function* for the curve parametrised by $\mathbf{x}$. We have

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}s^2} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\phi} = (h + h'')^{-1}\,\mathbf{f}' = -(h + h'')^{-1}\,\mathbf{e}. \tag{5}$$

We see that if h is a C^k function with $k \geq 2$ and satisfying (3) then the parametrisation by arc length is C^k even though the parametrisation (2) is only C^{k-1}. We also see that the signed radius of curvature is given by

$$\rho = \frac{\mathrm{d}s}{\mathrm{d}\phi} = h(\phi) + h''(\phi). \tag{6}$$

Any planar curve with non vanishing curvature, i.e., without inflexion points, can be given as (2). If h is the support function of curve $\mathbf{x}$ then the rotated

[1] If $\rho > 0$ then $(\mathbf{t}, \mathbf{n}) = (\mathbf{f}, \mathbf{e})$ is a negatively oriented basis

curve $\mathbf{R}(t)\,\mathbf{x}$ has the support function $\phi \mapsto h(\phi - t)$, the translated curve $\mathbf{x}+\mathbf{a}$ has the support function $\phi \mapsto h(\phi) + \mathbf{a}\cdot\mathbf{e}(\phi)$, and the scaled curve $c\mathbf{x}$ has the support function ch, see [1,5,7,10,12–14].

We can not use the normal direction as a parameter across an inflexion point, but the modification in [4] allows us that.

Theorem 1 *Consider an ordinary inflexion point (x_0, y_0) with curvature $\kappa = \kappa_1 s + \frac{1}{2}\kappa_2 s^2 + O(s^3)$, where s is arc length and $\kappa_1 > 0$. Denote the normal direction by ϕ and let ϕ_0 be the normal direction at the inflexion point. We can introduce a parameter u such that $u^2 = \phi - \phi_0$ and the support function can be written*

$$h = x_0 + y_0 u^2 + \frac{2}{3}\sqrt{\frac{2}{\kappa_1}}\,u^3 + O(u^4). \tag{7}$$

Furthermore,

$$\frac{\mathrm{d}h}{\mathrm{d}\phi} = y_0 + \sqrt{\frac{2}{\kappa_1}}\,u + O(u^2) \quad \textit{and} \quad \frac{\mathrm{d}^2 h}{\mathrm{d}\phi^2} = \sqrt{\frac{1}{2\kappa_1}}\,\frac{1}{u} + O(1). \tag{8}$$

Proof. By a rotation we may assume that $\phi_0 = 0$. Then we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \kappa = \kappa_1 s + \frac{\kappa_2}{2}s^2 + O(s^3) \quad \text{and} \quad u^2 = \phi = \frac{\kappa_1}{2}s^2 + \frac{\kappa_2}{6}s^3 + O(s^3).$$

Thus

$$s = \sqrt{\frac{2}{\kappa_1}}u - \frac{\kappa_2}{3\kappa_1^2}u^2 + O(u^3), \tag{9}$$

and

$$\cos(\phi) = 1 - \frac{1}{8}\kappa_1^2 s^4 - \frac{1}{12}\kappa_1\kappa_2 s^5 + O(s^6), \quad \sin(\phi) = \frac{1}{2}\kappa_1 s^2 + \frac{1}{6}\kappa_2 s^3 + O(s^4).$$

The curve can now be parametrised as

$$\mathbf{x} = \int \mathbf{t}\,\mathrm{d}s = \int \mathbf{f}(\phi)\,\mathrm{d}s = \begin{bmatrix} x_0 - \frac{1}{6}\kappa_1 s^3 - \frac{1}{24}\kappa_2 s^5 + O(s^5) \\ y_0 + s - \frac{1}{40}\kappa_1^2 s^5 + O(s^6) \end{bmatrix}.$$

Finally, the support function can be written

$$h = \mathbf{x}\cdot\mathbf{n} = \mathbf{x}\cdot\mathbf{e}(\phi) = x_0 + \frac{y_0\kappa_1}{2}s^2 + \left(\frac{2\kappa_1 + y_0\kappa_2}{6}\right)s^3 + O(s^4).$$

Substituting (9) into this expression shows (7). Using that $\frac{\mathrm{d}}{\mathrm{d}u} = 2u\frac{\mathrm{d}}{\mathrm{d}\phi}$, equations (8) is a now a straightforward calculation. □

4 General designs

We consider designs as in Fig. 5, i.e., designs consisting of arcs with alternating signs of curvature. We start with one positively curved arc of the rotor. This arc is then moved around by the motion generated by a circle rolling inside another circle and we find the stator as the envelope of this moving curve. Then we reverse the viewpoint and consider the rotor as fixed while the stator is moving. Now the full rotor can be found as an envelope of this new moving curve.

The motion generated by rolling a circle of radius b inside a circle of radius a is given by

$$\mathbf{x} \mapsto \mathbf{R}(t)\,\mathbf{x} + c\,\mathbf{e}(\alpha t), \tag{10}$$

where $c = a - b$ and $\alpha = b/(b-a)$. In particular, if we have a circle of radius n rolling inside a circle of radius $n+1$ then $c = 1$ and $\alpha = -n$. Observe that $\alpha < 0$ if $a > b$. Now consider the motion of the curve (2),

$$\mathbf{X}(\phi, t) = \mathbf{R}(t)\,\mathbf{x}(\phi) + c\,\mathbf{e}(\alpha t) = h(\phi)\,\mathbf{e}(\phi + t) + h'(\phi)\,\mathbf{f}(\phi + t) + c\,\mathbf{e}(\alpha t). \tag{11}$$

We have a point on the envelope if and only if the velocity and the tangent are parallel. The partial derivatives of $\mathbf{X}$ are

$$\frac{\partial \mathbf{X}}{\partial \phi} = (h(\phi) + h''(\phi))\,\mathbf{f}(\phi + t), \tag{12}$$

$$\frac{\partial \mathbf{X}}{\partial t} = h(\phi)\,\mathbf{f}(\phi + t) - h'(\phi)\,\mathbf{e}(\phi + t) + \alpha c\,\mathbf{f}(\alpha t). \tag{13}$$

So we have a point on the envelope if and only if $\partial\mathbf{X}/\partial t$ is orthogonal to $\mathbf{e}(\phi + t)$, i.e., if and only if

$$0 = \frac{\partial \mathbf{X}}{\partial t} \cdot \mathbf{e}(\phi + t) = -h'(\phi) + c\alpha \sin(\phi + t - \alpha t) \tag{14}$$

or

$$\sin(\phi + (1-\alpha)t) = \frac{h'(\phi)}{c\alpha}. \tag{15}$$

We can solve this equation with respect to t and obtain the complete solution

$$t_{\sigma,k} = \frac{2k\pi + \frac{1-\sigma}{2}\pi + \sigma \arcsin\left(\frac{h'(\phi)}{c\alpha}\right) - \phi}{1-\alpha}, \quad \sigma = \pm 1, \quad k \in \mathbb{Z}. \tag{16}$$

The value of k only matters in a few instances, so in the following we will just write t_σ. Observe that

$$\cos(\phi + (1-\alpha)t_\sigma) = \cos\left(\frac{1-\sigma}{2}\pi + \sigma \arcsin\left(\frac{h'(\phi)}{c\alpha}\right)\right) = \sigma\sqrt{1 - \left(\frac{h'(\phi)}{c\alpha}\right)^2}. \tag{17}$$

The normal of the envelope $\mathbf{X}(\phi, t_\sigma)$ is the normal of $\mathbf{x}(\phi)$ rotated through the angle t_σ, i.e., it is $\mathbf{e}(\phi + t_\sigma)$. So the normal direction of the envelope is

$$\phi_\sigma = \phi + t_\sigma = \frac{1}{1-\alpha}\left(2k\pi + \frac{1-\sigma}{2}\pi + \sigma \arcsin\left(\frac{h'(\phi)}{c\alpha}\right) - \alpha\phi\right), \tag{18}$$

and the derivative with respect to ϕ is

$$\frac{\mathrm{d}\phi_\sigma}{\mathrm{d}\phi} = \frac{\sigma h''(\phi) - c\alpha^2\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}}{(1-\alpha)c\alpha\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}}. \tag{19}$$

The support function of the envelope is $h_\sigma = \mathbf{X}(\phi, t_\sigma) \cdot \mathbf{e}(\phi_\sigma)$ which becomes

$$h_\sigma = h(\phi) + c\cos(\phi_\sigma - \alpha t_\sigma) = h(\phi) + \sigma c\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}. \tag{20}$$

The first derivative with respect to normal direction is

$$\begin{aligned}\frac{\mathrm{d}h_\sigma}{\mathrm{d}\phi_\sigma} &= h'(\phi_\sigma - t_\sigma)\left(1 - \frac{\mathrm{d}t_\sigma}{\mathrm{d}\phi_\sigma}\right) - c\sin(\phi_\sigma - \alpha t_\sigma)\left(1 - \alpha\frac{\mathrm{d}t_\sigma}{\mathrm{d}\phi_\sigma}\right) \\ &= h'(\phi)\left(1 - \frac{\mathrm{d}t_\sigma}{\mathrm{d}\phi_\sigma}\right) - c\frac{h'(\phi)}{c\alpha}\left(1 - \alpha\frac{\mathrm{d}t_\sigma}{\mathrm{d}\phi_\sigma}\right) = \frac{\alpha-1}{\alpha}h'(\phi),\end{aligned} \tag{21}$$

and the second derivative is

$$\frac{\mathrm{d}^2h_\sigma}{\mathrm{d}\phi_\sigma^2} = \frac{\alpha-1}{\alpha}h''(\phi)\left(\frac{\mathrm{d}\phi_\sigma}{\mathrm{d}\phi}\right)^{-1} = \frac{c(1-\alpha)^2h''(\phi)\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}}{-\sigma h''(\phi) + c\alpha^2\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}}. \tag{22}$$

If we interchange what is moving and what is fixed and reverse the time then we just have to interchange a and b. Then c is replaced with $-c$ and α with $1-\alpha$. Making the appropriate modifications to (18) we find the normal directions

$$\phi_{\sigma,\mu} = \phi + \frac{1}{\alpha}\left(\left(2\ell - 2k + \frac{\sigma-\mu}{2}\right)\pi + (\mu - \sigma)\arcsin\left(\frac{h'(\phi)}{c\alpha}\right)\right). \tag{23}$$

The support functions are

$$h_{\sigma,\mu} = h(\phi) + (\sigma-\mu)c\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}, \tag{24}$$

and the first and second derivative are

$$\frac{\mathrm{d}h_{\sigma,\mu}}{\mathrm{d}\phi_{\sigma,\mu}} = h'(\phi), \tag{25}$$

$$\frac{\mathrm{d}^2 h_{\sigma,\mu}}{\mathrm{d}\phi_{\sigma,\mu}^2} = \frac{h''(\phi)c\alpha^2\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}}{(\mu-\sigma)h''(\phi)+c\alpha^2\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}}. \tag{26}$$

We collect the information in the following theorem

Theorem 2 *Let a positively curved arc of the rotor have the support function $\phi \mapsto h(\phi)$ and let the motion be generated by a circle of radius n rolling inside a circle of radius $n+1$, where $n \in \mathbb{N}$. Let*

$$\begin{aligned} \phi_1(\phi) - \frac{1}{n+1}\left(n\phi - \arcsin(h'(\phi)/n)\right), \\ \phi_{-1}(\phi) = \frac{1}{n+1}\left(\pi + n\phi + \arcsin(h'(\phi)/n)\right), \\ \phi_{-1,1}(\phi) = \phi + \frac{1}{n}\left(\pi - 2\arcsin(h'(\phi)/n)\right), \end{aligned}$$

and

$$\begin{aligned} h_1(\phi) &= h(\phi) + \frac{1}{n}\sqrt{n^2-(h'(\phi))^2}, \\ h_{-1}(\phi) &= h(\phi) - \frac{1}{n}\sqrt{n^2-(h'(\phi))^2}, \\ h_{-1,1}(\phi) &= h(\phi) - \frac{2}{n}\sqrt{n^2-(h'(\phi))^2}. \end{aligned}$$

Then the positively and negatively curved arcs of the rotor have the support functions

$$\phi \mapsto h(\phi + 2k\pi/n), \quad \phi \mapsto h_{-1,1}\left(\phi_{-1,1}^{-1}(\phi + 2k\pi/n)\right), \quad k = 0, \dots, n-1,$$

respectively, and the positively and negatively curved arcs of the stator have the support functions

$$\phi \mapsto h_1\left(\phi_1^{-1}(\phi + 2k\pi/n)\right), \quad \phi \mapsto h_{-1}\left(\phi_{-1}^{-1}(\phi + 2k\pi/n)\right), \quad k = 0, \dots, n,$$

respectively. We can parametrise the arcs as

$$\begin{aligned} \mathbf{x}(\phi) &= h(\phi)\,\mathbf{e}\left(\phi + \tfrac{2k\pi}{n}\right) + h'(\phi)\,\mathbf{f}\left(\phi + \tfrac{2k\pi}{n}\right), \\ \mathbf{x}_{-1,1}(\phi) &= h_{-1,1}(\phi)\,\mathbf{e}\left(\phi_{-1,1}(\phi) + \tfrac{2k\pi}{n}\right) + h'(\phi)\,\mathbf{f}\left(\phi_{-1,1}(\phi) + \tfrac{2k\pi}{n}\right), \\ \mathbf{x}_1(\phi) &= h_1(\phi)\,\mathbf{e}\left(\phi_1(\phi) + \tfrac{2k\pi}{n+1}\right) + \frac{n+1}{n}h'(\phi)\,\mathbf{f}\left(\phi_1(\phi) + \tfrac{2k\pi}{n+1}\right), \\ \mathbf{x}_{-1}(\phi) &= h_{-1}(\phi)\,\mathbf{e}\left(\phi_{-1}(\phi) + \tfrac{2k\pi}{n+1}\right) + \frac{n+1}{n}h'(\phi)\,\mathbf{f}\left(\phi_{-1}(\phi) + \tfrac{2k\pi}{n+1}\right). \end{aligned}$$

A straightforward calculation shows that

$$\rho_\sigma = \rho - \frac{\left(h'' - \sigma c\alpha\sqrt{1-\left(\frac{h'}{c\alpha}\right)^2}\right)^2}{h'' - \sigma c\alpha^2\sqrt{1-\left(\frac{h'}{c\alpha}\right)^2}}, \tag{27}$$

$$\rho_{\sigma,-\sigma} = \rho - 2\frac{(h'')^2 + c^2\alpha^2 - (h')^2 - 2\sigma c h''\sqrt{1-\left(\frac{h'}{c\alpha}\right)^2}}{2h'' - \sigma c\alpha^2\sqrt{1-\left(\frac{h'}{c\alpha}\right)^2}}. \tag{28}$$

We will now analyse the situation at the endpoints of the arcs of the horizontal sections of the pump.

Theorem 3 *Consider a Moineau pump design where the horizontal sections of both the stator and rotor consists of smooth arcs with alternating strictly positive and strictly negative curvature in the interior. If the support function of the positively curved arc of the rotor either has an expansion*

$$h = h_0 + h_1(\phi - \phi_0) + \frac{h_m}{m!}(\phi - \phi_0)^m + O((\phi - \phi_0)^{m+1}), \tag{29}$$

where $h_m \neq 0$, or an expansion

$$h = h_0 + h_1 u^2 + \frac{2h_3}{3}u^3 + \frac{h_4}{2}u^4 + O(u^5), \tag{30}$$

where $u^2 = \phi - \phi_0$ and $h_3 \neq 0$. Then there are points with infinite curvature in the design.

Proof. As the arcs of the stator, with support functions h_1 and h_{-1} respectively, have to connect, the normal directions ϕ_1 and ϕ_{-1} have to agree at the endpoints. Hence $(h')^2 = c^2\alpha^2$ at the endpoints. By a rotation we may assume that $\phi = 0$ at the endpoint we consider.

First we consider the case (29). If $m = 2$ then it is easily seen that

$$\rho_\sigma = \rho - \frac{h_2^2 + O(\phi^{1/2})}{h_2 + O(\phi^{1/2})} \to \rho - h_2 = h_0, \quad \text{for } \phi \to 0. \tag{31}$$

If $m \geq 4$ then we have

$$\rho_\sigma = \rho - \frac{c^2\alpha^2\frac{-2h_m}{h_1(m-1)!}\phi^{m-1} + O\left(\phi^{m+1}\right)}{-\sigma c\alpha^2\sqrt{\frac{-2h_m}{h_1(m-1)!}}\phi^{\frac{m-1}{2}} + O\left(\phi^{\frac{m+1}{2}}\right)} \to \rho, \quad \text{for } \phi \to 0. \tag{32}$$

If $m = 3$ and $h_3 \neq -\alpha^2 h_1$ then we have

$$\rho_\sigma = \rho - \frac{\left(h_3 - \sigma c\alpha\sqrt{-h_3/h_1}\right)^2 \phi^2 + O(\phi^3)}{\left(h_3 - \sigma c\alpha^2\sqrt{-h_3/h_1}\right)\phi + O(\phi^2)} \to \rho, \quad \text{for } \phi \to 0. \tag{33}$$

Finally, if $m = 3$ and $h_3 = -\alpha^2 h_1$ then we obtain

$$\rho_{-1,1} = \rho + \frac{(\alpha^2 - 1)\alpha^4 c^2 \phi^2 + O(\phi^3)}{\alpha^2 h_1 \phi + O(\phi^2)} \to \rho, \quad \text{for } \phi \to 0. \tag{34}$$

We now consider the case (30). As $\frac{\mathrm{d}}{\mathrm{d}u} = 2u\frac{\mathrm{d}}{\mathrm{d}\phi}$ we have

$$\frac{\mathrm{d}h}{\mathrm{d}\phi} = h_1 + h_3 u + h_4 u^2 + O(u^3) \quad \text{and} \quad \frac{\mathrm{d}^2 h}{\mathrm{d}\phi^2} = h_3\frac{1}{u} + h_4 + O(u).$$

Substituting this into (27) yields

$$\rho_\sigma = h_0 + O(u) \to h_0, \quad \text{for } u \to 0. \tag{35}$$

As $\rho_1 \geq 0$ and $\rho_{-1} \leq 0$, we can conclude that $\rho_1 = \rho_{-1} = 0$ at the endpoint of the arcs of the stator in the cases (31), (32), (33), and (35). Similar $\rho \geq 0$ and $\rho_{-1,1} \leq 0$ so in the case (34) we have $\rho_{-1,1} = \rho = 0$ at the endpoint of arcs of the rotor. In all cases the design has points of infinite curvature. □

4.1 Examples

We will consider a deformation of the 3:2 hypo-epi-cycloid design. We let $c = 1$ and $\alpha = -2$. An epi-cycloid arc of the rotor has the support function

$$h(\phi) = 3\cos\left(\frac{2\phi}{3}\right), \quad h'(\phi) = -2\sin\left(\frac{2\phi}{3}\right), \quad h''(\phi) = -\frac{4}{3}\cos\left(\frac{2\phi}{3}\right),$$

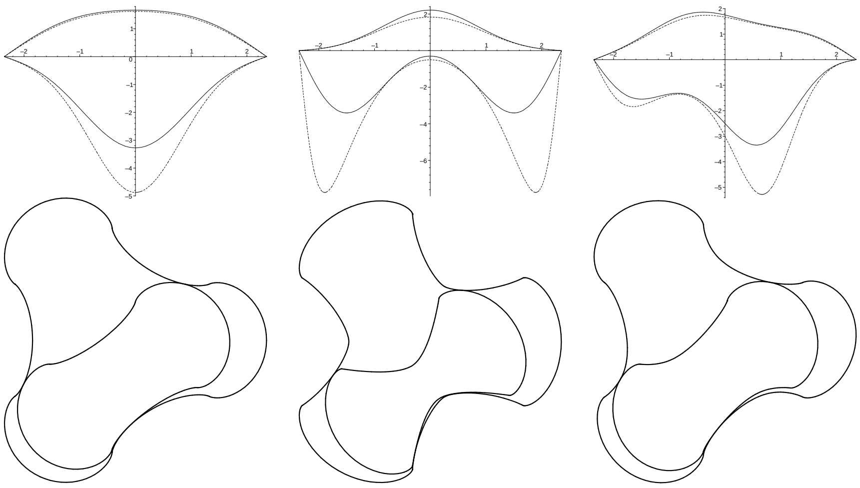
with $\phi \in \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$. If we preserve the values of h, h', and h'' at the endpoints then we consider deformations $h + \eta$, where

$$\eta : \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right] \to \mathbb{R}, \quad \text{with} \quad \eta\left(\pm\frac{4\pi}{3}\right) = \eta'\left(\pm\frac{4\pi}{3}\right) = \eta''\left(\pm\frac{4\pi}{3}\right) = 0.$$

One possibility is

$$\eta(\phi) = \cos^3\left(\frac{2\phi}{3}\right)\left(a_0 + \sum_{\ell=1}^{N}\left(a_\ell \cos^\ell\left(\frac{2\phi}{3}\right) + b_\ell \sin^\ell\left(\frac{2\phi}{3}\right)\right)\right).$$

See Figure 6 for a few examples of such deformations.



$(a_0, b_1) = \left(\frac{1}{8}, 0\right)$ $(a_0, b_1) = \left(-\frac{1}{2}, 0\right)$ $(a_0, b_1) = \left(0, \frac{1}{6}\right)$

Fig. 6. Below new designs of a Moineau pump. Above the radius of curvature of the arcs of the stator and the arcs of the rotor are shown with solid and dotted lines respectively.

5 Conclusion

The support function representation of planar curves are used to define and analyse designs of Moineau pumps. In particular, it is proved that if the rotor and stator consists of alternating arcs with positive and negative curvature then it is impossible to avoid points with infinite curvature in the design.

It is an open problem whether the inclusion of straight lines in the design allows for other curvature bounded designs than the 2:1 hypo-cycloid construction in Figure 1.

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