The Intrinsic Equation of Planar Curves and G^2 Hermite Interpolation

Jens Gravesen

Abstract. This paper presents the intrinsic equation of a planar curve, as a practical tool for curve design. It is demonstrated that the tangent direction, the arc length, and the curvature are easy to obtain. It is furthermore shown that a parametrization of the curve by tangent direction is easily obtained, as are the evolute, the involutes, and the offsets. It is finally shown how Hermite data can be fitted in a local and convexity preserving way.

§1. Introduction

In this paper a planar curve with a non-vanishing curvature is represented, not by an explicit parametrization or by an implicit equation in the plane, but by an *intrinsic equation* of the form $\frac{\mathrm{d}s}{\mathrm{d}\varphi} = \varrho(\varphi)$, where s is *arc length*, φ is *tangent direction*, and ϱ is the *radius of curvature*. In the presence of a point with vanishing curvature, the intrinsic equation takes the form $\frac{\mathrm{d}s}{\mathrm{d}u} = nu^{n-1}\varrho(u)$, where $\varphi = \varphi_0 + u^n$.

A parametrization (by tangent direction) is obtained by an integration, that in many cases can be done analytically, and the resulting map $\mathbf{x}$ into $\mathbb{R}^2$ depends linearly on the function ϱ (or h). That implies, among other things, that the interpolation problem becomes linear. The arc length is also obtained by a simple integration and many geometric constructions such as: *the evolute*, *involutes*, *offset curves* and *Minkowski sums* are very simple in this representation. In fact, the starting point of this work was a problem posed by the Danfoss at the 32nd European Study Group with Industry in 1998. The intrinsic equation was used for the geometric modelling of the so-called *scroll compressor* and it enabled us to express all the important geometric characteristics of the scroll compressor in closed analytical form, see [4].

In the present paper we will use the intrinsic equation to give a local convexity preserving solution to the G^2 Hermite interpolation problem in

Geometric Modeling and Computing: Seattle 2003
M. Lucian and M. Neamtu (eds.), pp. 295–310.

ISBN 0-0-9728482-3-1

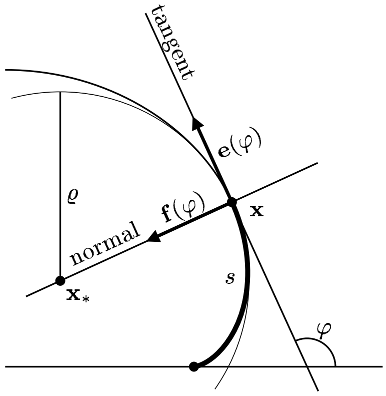

Fig. 1. Some basic geometric concepts: the arc length s, the tangent direction φ, and the radius of curvature $\varrho = \mathrm{d}s/\mathrm{d}\varphi$.

the plane. That is, we will interpolate points, tangents and curvatures in such way that each segment of the interpolant depends only on the data at the end points and preserves convexity of the data.

The papers [9] and [1] used the intrinsic equation on the form $\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \kappa(s)$, where κ is the curvature. The drawback here is that a parametrization (by arc length) can be found only by numeric integration and the resulting map $\mathbf{x}$ into $\mathbb{R}^2$ depends in a non-linear fashion on κ.

Parametrization by tangent direction was used by Meek and Walton in the papers [7] and [8] to match G^1 and G^2 Hermite data by spirals.

Some of the advantages of the intrinsic equation can also be obtained by the Pythagorean-hodograph curves introduced by R. Farouki in [3]. They give foe example polynomial arc length and rational offset curves.

It should be noted that if $\varrho(\varphi)$ is piecewise constant then we obtain the well known arc splines, see [2] and [6], and if $\varrho(\varphi)$ is piecewise linear or piecewise quadratic we have the involute curves of Kuroda and Mukai, see [5].

§2. The Intrinsic Equation of a Planar Convex Curve

Figure 1 illustrates the definition of the *tangent direction* φ as the angle between the tangent and a fixed line, say the x-axis, and the *radius of curvature* ϱ as the radius in the *circle of curvature*, *i.e.*, the circle with the highest order of contact with the curve. It is well known that $\varrho = 1/\kappa$ where κ is the *curvature* of the curve, and that the latter is defined as the rate of change of the tangent vector, *i.e.*, $\kappa = \frac{\mathrm{d}\varphi}{\mathrm{d}s}$. If the curve is *convex*, *i.e.*, if the curve has non-vanishing curvature, say $\kappa > 0$, then φ can be used as a parameter on the curve and the radius of curvature is given by

$\varrho = \frac{\mathrm{d}s}{\mathrm{d}\varphi}$.

The *intrinsic equation* of curve is an equation linking the arc length and the tangent direction, and the mathematical significance of it is that it gives a unique definition of a curve, in contrast to a parametrization, which is far from unique. In this paper we will use the intrinsic equation in the form of the differential equation

$$\frac{\mathrm{d}s}{\mathrm{d}\varphi} = \varrho(\varphi).$$

We immediately have that

$$s = \int \varrho \, \mathrm{d}\varphi,$$

and if ϱ is (piecewise) polynomial in φ then so is s. We introduce the two unit vectors $\mathbf{e}(\varphi) = (\cos\varphi, \sin\varphi)$ and $\mathbf{f}(\varphi) = (-\sin\varphi, \cos\varphi)$. The condition that φ is the tangent direction can now be formulated as the equation

$$\mathbf{t} = \mathbf{e}(\varphi) = (\cos\varphi, \sin\varphi).$$

The *normal vector* is then $\mathbf{n} = \mathbf{f}(\varphi)$. If $\varphi \mapsto \mathbf{x}(\varphi)$ is a parametrization of a curve by tangent direction, then as $\frac{\mathrm{d}s}{\mathrm{d}\varphi} = \varrho(\varphi)$, we have

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\varphi} = \frac{\mathrm{d}s}{\mathrm{d}\varphi}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \varrho(\varphi)\mathbf{e}(\varphi),$$

and we can recapture the parametrization by a simple integration

$$\mathbf{x} = \int \varrho(\varphi)\mathbf{e}(\varphi) \, \mathrm{d}\varphi.$$

Observe that if ϱ is piecewise polynomial in φ then we get a closed expression for the parametrization. Furthermore, if we have a fixed parameter value φ_0, then the corresponding point $\mathbf{x}(\varphi_0)$ on the curve depends *linearly* on the function ϱ.

To illustrate the convenience of this representation we formulate some classical constructions in this framework. Recall that the *evolute* of curve is the locus of the centre of curvature, *i.e.,* the centre $\mathbf{x}_*$ of the circle of curvature, see Figure 1. The centre of curvature is $\mathbf{x}_* = \mathbf{x} + \varrho\mathbf{f}(\varphi)$, and by differentiation we obtain

$$\frac{\mathrm{d}\mathbf{x}_*}{\mathrm{d}\varphi} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\varphi} + \frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\mathbf{f}(\varphi) + \varrho\frac{\mathrm{d}\mathbf{f}(\varphi)}{\mathrm{d}\varphi} = \varrho\mathbf{e}(\varphi) + \frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\mathbf{f}(\varphi) - \varrho\mathbf{e}(\varphi) = \frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\mathbf{f}(\varphi).$$

From this we see that the tangent direction of the evolute is $\varphi_* = \varphi \pm \frac{\pi}{2}$, where the sign is the same as that of $\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}$, and that the radius of curvature is $\varrho_* = \left|\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right|$. That is,

$$\varrho_{\mathbf{x}_*}(\varphi_*) = \pm\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\left(\varphi_* \mp \frac{\pi}{2}\right).$$

An *involute* of the curve is given by $\mathbf{x}^* = \mathbf{x} - (s+c)\mathbf{e}(\varphi)$ where $c \in \mathbb{R}$ is an arbitrary constant. By differentiation we have

$$\frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}\varphi} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\varphi} - \frac{\mathrm{d}s}{\mathrm{d}\varphi}\mathbf{e}(\varphi) - (s+c)\frac{\mathrm{d}\mathbf{e}(\varphi)}{\mathrm{d}\varphi} = -(s+c)\mathbf{f}(\varphi).$$

We now see that if $s + c > 0$, then the tangent-direction of the involute is $\varphi^* = \varphi - \frac{\pi}{2}$, and that the radius of curvature is $\varrho^* = s + c$. That is,

$$\varrho^*(\varphi^*) = c + \int_{\frac{\pi}{2}}^{\varphi^*+\frac{\pi}{2}} \varrho(\tau)\,\mathrm{d}\tau.$$

An *offset curve* is given by $\mathbf{x}_c = \mathbf{x} - c\mathbf{f}(\varphi)$, and then

$$\frac{\mathrm{d}\mathbf{x}_c}{\mathrm{d}\varphi} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\varphi} - c\frac{\mathrm{d}\mathbf{f}(\varphi)}{\mathrm{d}\varphi} = \varrho\mathbf{e}(\varphi) + c\mathbf{e}(\varphi) = (\varrho + c)\mathbf{e}(\varphi).$$

That is, the offset curves are obtained by adding a constant to $\varrho(\varphi)$:

$$\varrho_c(\varphi) = \varrho(\varphi) + c.$$

This is a special case of the *Minkowski* sum of two curves. When tangent direction is the parameter, the Minkowski sum of two curves $\mathbf{x}_1$ and $\mathbf{x}_2$ is simply the ordinary sum: $(\mathbf{x}_1, \mathbf{x}_2) \mapsto \mathbf{x}_1 + \mathbf{x}_2$. If ϱ_1 and ϱ_2 are the radius of curvature of $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively then the Minkowski sum has radius of curvature

$$\varrho_{\mathbf{x}_1+\mathbf{x}_2} = \varrho_1 + \varrho_2.$$

The curve is *closed* if the function ϱ is periodic with period 2π and

$$\int_0^{2\pi} \varrho(\varphi)\mathbf{e}(\varphi)\,\mathrm{d}\varphi = 0.$$

From a design or engineering point of view it is not so much the function $\varrho(\varphi)$ which interest us; but rather the *curvature profile* $\kappa(s)$, c.f., [9]. As mentioned above we have direct access to both $s(\varphi)$ and $\kappa = 1/\varrho(\varphi)$ so it is easy to plot the graph of $\kappa(s)$. In the same spirit, the integrals $\int \kappa^2\,\mathrm{d}s$ and $\int \left(\frac{\mathrm{d}\kappa}{\mathrm{d}s}\right)^2 \mathrm{d}s$ are often used as measures for the *fairness* of a planar curve, and these integrals are easy to calculate in the present framework. Indeed, we have

$$\int \kappa^2\,\mathrm{d}s = \int \frac{1}{\varrho^2}\frac{\mathrm{d}s}{\mathrm{d}\varphi}\,\mathrm{d}\varphi = \int \frac{1}{\varrho}\,\mathrm{d}\varphi,$$

and as

$$\frac{\mathrm{d}\kappa}{\mathrm{d}s} = -\frac{1}{\varrho^2}\frac{\mathrm{d}\varrho}{\mathrm{d}s} = -\frac{1}{\varrho^2}\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = -\frac{1}{\varrho^3}\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi},$$

we have

$$\int \left(\frac{\mathrm{d}\kappa}{\mathrm{d}s}\right)^2 \mathrm{d}s = \int \frac{1}{\varrho^6}\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2 \frac{\mathrm{d}s}{\mathrm{d}\varphi}\,\mathrm{d}\varphi = \int \frac{1}{\varrho^5}\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2 \mathrm{d}\varphi.$$

Once more, if $\varrho(\varphi)$ is (piecewise) polynomial in φ the integrals can be determined in closed analytical form.

We now look at a specific example we will need for the interpolation.

Lemma 1. *Consider the function*

$$\varrho(\varphi) = \frac{(\varphi_1 - \varphi)\varrho_0 + (\varphi - \varphi_0)\varrho_1}{\varphi_1 - \varphi_0}, \qquad \varphi \in [\varphi_0, \varphi_1].$$

If we put $\Delta\varphi = \varphi_1 - \varphi_0$,

$$\begin{aligned} a(\Delta\varphi) &= \frac{1}{\Delta\varphi} - \frac{\cos\Delta\varphi}{\sin\Delta\varphi}, \\ b(\Delta\varphi) &= \frac{1}{\sin\Delta\varphi} - \frac{1}{\Delta\varphi}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{v} &= a(\Delta\varphi)\mathbf{e}(\varphi_0) + b(\Delta\varphi)\mathbf{e}(\varphi_1) \approx \frac{\Delta\varphi}{2}\mathbf{e}\left(\varphi_0 + \tfrac{1}{3}\Delta\varphi\right), \\ \mathbf{w} &= b(\Delta\varphi)\mathbf{e}(\varphi_0) + a(\Delta\varphi)\mathbf{e}(\varphi_1) \approx \frac{\Delta\varphi}{2}\mathbf{e}\left(\varphi_1 - \tfrac{1}{3}\Delta\varphi\right), \end{aligned}$$

then

$$\mathbf{x}(\varphi_1) - \mathbf{x}(\varphi_0) = \int_{\varphi_0}^{\varphi_1} \varrho(\varphi)\mathbf{e}(\varphi)\,\mathrm{d}\varphi = \varrho_0\mathbf{v} + \varrho_1\mathbf{w}.$$

Furthermore,

$$\begin{aligned} \frac{\Delta\varphi}{2} - \frac{\Delta\varphi^3}{72} &\le |\mathbf{v}|, |\mathbf{w}| \le \frac{\Delta\varphi}{2}, \\ \varphi_0 + \frac{\Delta\varphi}{3} - \frac{\Delta\varphi^3}{810} &\le \arg\mathbf{v} \le \varphi_0 + \frac{\Delta\varphi}{3}, \\ \varphi_1 - \frac{\Delta\varphi}{3} &\le \arg\mathbf{w} \le \varphi_1 - \frac{\Delta\varphi}{3} + \frac{\Delta\varphi^3}{810}. \end{aligned}$$

Furthermore the length, energy, and curvature variation is given by

$$\int_{s_0}^{s_1} \mathrm{d}s = \int_{\varphi_0}^{\varphi_1} \varrho(\varphi)\,\mathrm{d}\varphi = \frac{\varrho_0+\varrho_1}{2}(\varphi_1-\varphi_0),$$

$$\int_{s_0}^{s_1} \kappa^2\,\mathrm{d}s = \frac{\varphi_1-\varphi_0}{\varrho_1-\varrho_0}\log\left(\frac{\varrho_1}{\varrho_0}\right),$$

$$\int_{s_0}^{s_1} \left(\frac{\mathrm{d}\kappa}{\mathrm{d}s}\right)^2 \mathrm{d}s = \frac{(\varrho_0+\varrho_1)(\varrho_0^2+\varrho_1^2)}{4\varrho_0^4\varrho_1^4}\,\frac{(\varrho_1-\varrho_0)^2}{\varphi_1-\varphi_0}.$$

Proof: The integrals are found by a straightforward calculation, where we use that $\dfrac{\mathrm{d}}{\mathrm{d}s} = \dfrac{1}{\varrho}\dfrac{\mathrm{d}}{\mathrm{d}\varphi}$ and that $\mathrm{d}s = \varrho\,\mathrm{d}\varphi$. The inequalities stem from simple Taylor expansions. □

2.1. Examples

The circle has constant radius of curvature and as integration of ρ leads to an involute we see that if $\varrho(\varphi)$ is a polynomial then the curve is an iterated involute of a circle.

The *logarithmic spiral* is given by $\mathbf{x} = ce^{d\theta}\mathbf{e}(\theta)$, where c, d are real positive numbers. We have $\mathbf{x}' = ce^{d\theta}(d\mathbf{e}(\theta) + \mathbf{f}(\theta))$ so $\varphi = \theta + \operatorname{arccot} d$ and $\frac{\mathrm{d}s}{\mathrm{d}\varphi} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = |\mathbf{x}'| = ce^{d\theta}\sqrt{1+d^2} = c\sqrt{1+d^2}e^{-d\operatorname{arccot} d}e^{d\varphi}$. Thus the intrinsic equation of a *logarithmic spiral* is $\frac{\mathrm{d}s}{\mathrm{d}\varphi} = \alpha\, e^{\beta\varphi}$, where $\alpha = c\sqrt{1+d^2}e^{-d\operatorname{arccot} d}$ and $\beta = d$.

In [8] the *Clothoid* is expressed as a function of tangent direction and the derivative is given as $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\varphi} = \frac{1}{\sqrt{2\pi\varphi}}\mathbf{e}(t)$. From this it is seen that the Clothoid has the intrinsic equation $\frac{\mathrm{d}s}{\mathrm{d}\varphi} = \frac{1}{\sqrt{2\pi\varphi}}$.

The *Tschirnhausen cubic* is treated too and the curvature as a function of tangent direction is given as $\kappa = \frac{2}{(1+\tan^2\varphi/2)^2}$. Hence the intrinsic equation is $\frac{\mathrm{d}s}{\mathrm{d}\varphi} = \frac{(1+\tan^2\varphi/2)^2}{2}$.

§3. Inflection Points

Suppose that the curvature of a curve has a simple zero for $s = 0$, *i.e.*, that $\kappa(s) = as + \cdots$. Then, up to a rotation, we have $\varphi = \frac{1}{2}as^2 + \cdots$. It is obviously impossible to use φ as parameter, so instead we use a parameter u such that $\varphi = u^2$. As $u^2 = \frac{1}{2}as^2 + \cdots$, $s = \frac{\sqrt{2}}{\sqrt{a}}u + \cdots$, the curvature is $\kappa = \sqrt{2a}\,u + \cdots$, and the radius of curvature is of the form $\varrho = \frac{h(u)}{2u}$. We have

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u} = \frac{\mathrm{d}\varphi}{\mathrm{d}u}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\varphi} = 2u\varrho\,\mathbf{e}(\varphi) = h(u)\,\mathbf{e}(u^2).$$

All this generalize to higher order zeros of the curvature.

Theorem 1. *Consider a point on a smooth curve where the curvature vanishes. Let s be the arc-length measured from that point and assume that the curvature is given by $\kappa = as^{n-1}$ + higher order terms. Then there exists a local parameter u such that the tangent direction is given by $\varphi = \varphi_0 + u^n$, where φ_0 is the tangent direction at the given point of the curve. Furthermore, if $h(u) = \frac{\mathrm{d}s}{\mathrm{d}u}$, then*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u} = h(u)\mathbf{e}(\varphi_0 + u^n) \quad \text{and} \quad \kappa(u) = \frac{nu^{n-1}}{h(u)}.$$

Proof: In the following $\varepsilon(s)$ denotes a continuous function with $\varepsilon(0) = 0$. As $\kappa = as^{n-1}(1 + \varepsilon(s))$ the tangent direction is $\varphi = \varphi_0 + \frac{a}{n}s^n(1 + \varepsilon(s))$. If we define the new parameter by $u = s\sqrt[n]{(\varphi - \varphi_0)/s^n} = \sqrt[n]{\frac{a}{n}}\, s(1 + \varepsilon(s))$ we see that we have an allowable change of parameter. If $h(u) = \frac{\mathrm{d}s}{\mathrm{d}u}$, then $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u} = \frac{\mathrm{d}s}{\mathrm{d}u}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = h(u)\mathbf{e}$. Finally $\kappa(u) = \frac{\mathrm{d}\varphi}{\mathrm{d}s} = nu^{n-1}\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{nu^{n-1}}{h(u)}$. □

As before we get the parametrization by integration

$$\mathbf{x}(u) = \mathbf{x}_0 + \int_0^u h(t)\mathbf{e}(\varphi_0 + t^n)\,\mathrm{d}t.$$

If $h(u) = u^{kn+n-1}$, then we can use the substitution $t = u^n$ to find the integral but for a general polynomial it is generally impossible to find the integral in terms of elementary functions. Instead we can restrict ourself to a certain class of polynomials. Partial integration gives

$$\begin{aligned}
&\int u^{kn+\ell}\mathbf{e}(\varphi_0 + u^n)\,\mathrm{d}u \\
&\quad = \frac{u^{kn+\ell+1}}{kn+\ell+1}\mathbf{e}(\varphi_0 + u^n) - \int \frac{nu^{(k+1)n+\ell}}{kn+\ell+1}\mathbf{f}(\varphi_0 + u^n)\,\mathrm{d}u \\
&= \frac{u^{kn+\ell+1}}{kn+\ell+1}\mathbf{e}(\varphi_0 + u^n) - \frac{nu^{(k+1)n+\ell+1}}{(kn+\ell+1)((k+1)n+\ell+1)}\mathbf{f}(\varphi_0 + u^n) \\
&\qquad - \int \frac{n^2u^{(k+2)n+\ell}}{(kn+\ell+1)((k+1)n+\ell+1)}\mathbf{e}(\varphi_0 + u^n)\,\mathrm{d}u.
\end{aligned}$$

So,

$$\begin{aligned}
&\int \left(u^{kn+\ell} + \frac{n^2u^{(k+2)n+\ell}}{(kn+\ell+1)((k+1)n+\ell+1)}\right)\mathbf{e}(\varphi_0 + u^n)\,\mathrm{d}u \\
&= \frac{u^{kn+\ell+1}}{kn+\ell+1}\mathbf{e}(\varphi_0 + u^n) - \frac{nu^{(k+1)n+\ell+1}}{(kn+\ell+1)((k+1)n+\ell+1)}\mathbf{f}(\varphi_0 + u^n).
\end{aligned}$$

Hence we can let $h(u)$ be a linear combination of the polynomials

$$u^{kn+n-1} \quad \text{and} \quad u^{kn+\ell} + \frac{n^2u^{(k+2)n+\ell}}{(kn+\ell+1)((k+1)n+\ell+1)}, \qquad 0 \le \ell \le n-2,$$

where $k \in \mathbb{N}_0$. In particular, if we have a simple inflexion point where $n = 2$, then we can use linear combinations of the polynomials

$$u^{2k+1} \qquad \text{and} \qquad u^{2k} + \frac{4u^{2k+4}}{(2k+1)(2k+3)}, \qquad k \in \mathbb{N}_0 .$$

We now consider the simplest case with $h(u) = 1 + \frac{4}{3}u^4$ and $u \in [0, u_1]$. If the tangent direction at $u = 0$ is φ_0 and $\Delta\varphi = \varphi - \varphi_0$, then the tangent direction and radius of curvature are $\varphi = \varphi_0 + u^2$ and $\varrho = \frac{h(u)}{2u} = \frac{1}{2u} + \frac{2}{3}u^3 = \frac{1}{\sqrt{\Delta\varphi}}\left(\frac{1}{2} + \frac{2}{3}\Delta\varphi^2\right)$ respectively. As $\mathbf{f}(\varphi) = \frac{-1}{\sin\Delta\varphi}\mathbf{e}(\varphi_0) + \frac{\cos\Delta\varphi}{\sin\Delta\varphi}\mathbf{e}(\varphi))$ we have

$$\begin{aligned}
\mathbf{x} - \mathbf{x}_0 &= \int_0^u \left(1 + \frac{4}{3}t^4\right) \mathbf{e}(\varphi_0 + t^2)\,\mathrm{d}t = u\mathbf{e}(\varphi_0 + u^2) - \frac{2}{3}u^3\mathbf{f}(\varphi_0 + u^2) \\
&= \sqrt{\Delta\varphi}\left(\mathbf{e}(\varphi) - \frac{2}{3}\Delta\varphi\mathbf{f}(\varphi)\right) \\
&= \sqrt{\Delta\varphi}\left(\frac{2\Delta\varphi}{3\sin\Delta\varphi}\mathbf{e}(\varphi_0) + \frac{3\sin\Delta\varphi - 2\Delta\varphi\cos\Delta\varphi}{3\sin\Delta\varphi}\mathbf{e}(\varphi)\right).
\end{aligned}$$

As for Lemma 1, straightforward calculations yield the following result.

Lemma 2. *Let $\Delta\varphi = \varphi_1 - \varphi_0$ and consider the function*

$$h(u) = \varrho_1 \frac{2\sqrt{\Delta\varphi}}{3 + 4\Delta\varphi^2}\left(3 + 4u^4\right), \quad u \in \left[0, \sqrt{\Delta\varphi}\right]. \tag{1}$$

The radius of curvature at $u = \sqrt{\Delta\varphi}$ is ϱ_1 and

$$\mathbf{x}\left(\sqrt{\Delta\varphi}\right) - \mathbf{x}(0) = \int_0^{\sqrt{\Delta\varphi}} h(u)\mathbf{e}(\varphi_0 + u^2)\,\mathrm{d}u = \varrho_1\mathbf{u},$$

where

$$\begin{aligned}
\mathbf{u} &= \frac{6\Delta\varphi}{3 + 4\Delta\varphi^2}\left(\frac{2\Delta\varphi}{3\sin\Delta\varphi}\mathbf{e}(\varphi_0) + \left(1 - \frac{2\Delta\varphi}{3\sin\Delta\varphi}\cos\Delta\varphi\right)\mathbf{e}(\varphi)\right) \\
&\approx 2\Delta\varphi\,\mathbf{e}\left(\varphi_0 + \frac{1}{3}\Delta\varphi\right).
\end{aligned}$$

Furthermore, the length, energy, and curvature variation are given by

$$\begin{aligned}
\int_{s_0}^{s_1} \mathrm{d}s &= \int_0^{\sqrt{\Delta\varphi}} h(u)\,\mathrm{d}u = \varrho_1 \frac{2\Delta\varphi}{3 + 4\Delta\varphi^2}\left(3 + \frac{4}{5}\Delta\varphi^2\right), \\
\int_{s_0}^{s_1} \kappa^2\,\mathrm{d}s &= \int_0^{\sqrt{\Delta\varphi}} \frac{4u^2}{h(u)}\mathrm{d}u = \frac{K_0(\Delta\varphi)}{\varrho_1}, \\
\int_{s_0}^{s_1} \left(\frac{\mathrm{d}\kappa}{\mathrm{d}s}\right)^2 \mathrm{d}s &= \int_0^{\sqrt{\Delta\varphi}} 4\frac{(h(u) - uh'(u))^2}{h(u)^5}\mathrm{d}u = \frac{K_1(\Delta\varphi)}{\varrho_1^3},
\end{aligned}$$

where

$$K_0(t) = \frac{3+4t^2}{\sqrt{t}} \int_0^{\sqrt{t}} \frac{2u^2}{3+4u^4} \mathrm{d}u \approx \frac{3}{2}t + \frac{32}{63}t^3,$$
$$K_1(t) = \frac{9}{2}\left(\frac{3+4t^2}{\sqrt{t}}\right)^3 \int_0^{\sqrt{t}} \frac{(4u^4-1)^2}{(4u^4+3)^5} \mathrm{d}u \approx \frac{1}{2}t^{-1} + \frac{8}{15}t + \frac{32}{15}t^3.$$

§4. Interpolation of Points, Tangents, and Curvatures

We present a *local* solution to the G^2 Hermite interpolation problem so we only need to consider two points A, B, with two tangent directions φ_A, φ_B, and two curvatures κ_A, κ_B. We let $\mathbf{r} = \overrightarrow{AB} = |\mathbf{r}|\mathbf{e}(\theta)$ and search for a curve from A to B that fits these data.

4.1. The Convex Case

We now assume that $\kappa_A, \kappa_B > 0$ and that $\varphi_A < \theta < \varphi_B$. We will find a piecewise linear function $\varrho : [\varphi_A, \varphi_B] \to \mathbb{R}_+$, with $\varrho(\varphi_A) = 1/\kappa_A$, $\varrho(\varphi_B) = 1/\kappa_B$, and $\int_{\varphi_A}^{\varphi_B} \varrho\mathbf{e}\,\mathrm{d}\varphi = \mathbf{r}$. We put $\varrho_0 = 1/\kappa_A$, $\varrho_4 = 1/\kappa_B$, $\varphi_0 = \varphi_A$, and $\varphi_4 = \varphi_B$, introduce the three breakpoints $\varphi_1, \varphi_2, \varphi_3 \in [\varphi_0, \varphi_4]$, and denote the radius of curvature at these three points by ϱ_1, ϱ_2, and ϱ_3 respectively. As in Lemma 1 we put

$$\begin{aligned} \mathbf{v}_i &= a(\Delta\varphi_i)\mathbf{e}(\varphi_{i-1}) + b(\Delta\varphi_i)\mathbf{e}(\varphi_i) \\ \mathbf{w}_i &= b(\Delta\varphi_i)\mathbf{e}(\varphi_{i-1}) + a(\Delta\varphi_i)\mathbf{e}(\varphi_i) \end{aligned} \qquad i = 0, \ldots, 4,$$

where $\Delta\varphi_i = \varphi_i - \varphi_{i-1}$. The integration yields

$$\varrho_0\mathbf{v}_1 + \varrho_1(\mathbf{w}_1 + \mathbf{v}_2) + \varrho_2(\mathbf{w}_2 + \mathbf{v}_3) + \varrho_3(\mathbf{w}_3 + \mathbf{v}_4) + \varrho_4\mathbf{w}_4 = \mathbf{r},$$

or equivalently

$$\varrho_1(\mathbf{w}_1 + \mathbf{v}_2) + \varrho_2(\mathbf{w}_2 + \mathbf{v}_3) + \varrho_3(\mathbf{w}_3 + \mathbf{v}_4) = \mathbf{r} - \varrho_0\mathbf{v}_1 - \varrho_4\mathbf{w}_4. \qquad (2)$$

This is a linear equation in $(\varrho_1, \varrho_2, \varrho_3)$ which is easily solved. The only problem is that we want positive solutions, and in order to secure this we choose φ_1, φ_2, and φ_3 carefully. We put $\varphi_2 = \theta$, and then we have

Lemma 3. *If* $\mathbf{r} = x\mathbf{e}(\varphi_0) + y\mathbf{e}(\varphi_4)$, *then*

$$\varphi_2 - \varphi_0 \geq \frac{y}{x+y}\sin(\varphi_4 - \varphi_0),$$
$$\varphi_4 - \varphi_2 \geq \frac{x}{x+y}\sin(\varphi_4 - \varphi_0).$$

Proof: As

$$\mathbf{e}(\varphi_2) = \mathbf{e}(\theta) = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\mathbf{e}(\varphi_0) + y\mathbf{e}(\varphi_4)}{|x\mathbf{e}(\varphi_0) + y\mathbf{e}(\varphi_4)|},$$

we have

$$\begin{bmatrix} \cos\varphi_0 & \cos\varphi_4 \\ \sin\varphi_0 & \sin\varphi_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sin(\varphi_4 - \varphi_0)} \begin{bmatrix} \sin\varphi_4 & -\cos\varphi_4 \\ -\sin\varphi_0 & \cos\varphi_0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{3}$$

Thus

$$\cot(\varphi_2 - \varphi_0) = \frac{x + y\cos(\varphi_4 - \varphi_0)}{y\sin(\varphi_4 - \varphi_0)}.$$

The inequality

$$\operatorname{arccot}\left(k\frac{1-t}{t}\right) \geq \frac{t}{k}, \quad \text{for } 0 \leq t \leq \frac{2k^2}{1+k^2},$$

is easily established by differentiating the two sides. As

$$\frac{x + y\cos(\varphi_4 - \varphi_0)}{y\sin(\varphi_4 - \varphi_0)} = \frac{1 - \cos(\varphi_4 - \varphi_0)}{\sin(\varphi_4 - \varphi_0)} \frac{1 - \frac{y}{x+y}(1 - \cos(\varphi_4 - \varphi_0))}{\frac{y}{x+y}(1 - \cos(\varphi_4 - \varphi_0))}$$

and

$$\frac{2\left(\frac{1-\cos(\varphi_4-\varphi_0)}{\sin(\varphi_4-\varphi_0)}\right)^2}{1 + \left(\frac{1-\cos(\varphi_4-\varphi_0)}{\sin(\varphi_4-\varphi_0)}\right)^2} = 1 - \cos(\varphi_4 - \varphi_0),$$

we obtain the first inequality. The second inequality is similar. □

Lemma 4. *If $\varphi_2 - \varphi_0, \varphi_4 - \varphi_2 < \pi/2$ and $\Delta\varphi_1$ and $\Delta\varphi_4$ are sufficiently small then there are positive solutions to* (2).

Proof: If $\Delta\varphi_1 = \Delta\varphi_4 = 0$ then $\mathbf{v}_1 = \mathbf{w}_1 = \mathbf{v}_4 = \mathbf{w}_4 = \mathbf{0}$, so (2) becomes

$$\varrho_1\mathbf{v}_2 + \varrho_2(\mathbf{w}_2 + \mathbf{v}_3) + \varrho_3\mathbf{w}_3 = \mathbf{r},$$

and we have the situation in Figure 2. As $0 < \varphi_2 - \varphi_0, \varphi_4 - \varphi_2 < \pi/2$ we can write $\mathbf{r} = \varrho_1\mathbf{v}_2 + \varrho_3\mathbf{w}_3$ with $\varrho_1, \varrho_3 > 0$ and if ϱ_2 is sufficiently small then ϱ_1 and ϱ_3 are still positive. Thus for $\Delta\varphi_1, \Delta\varphi_4$ in a neighbourhood of 0 there are positive solutions. □

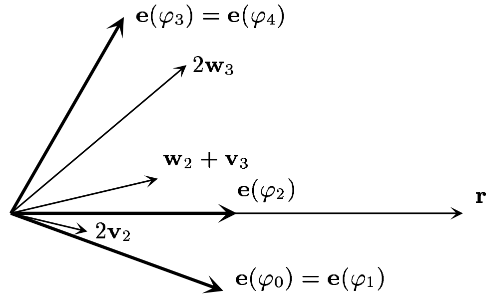

Fig. 2. The limit situation $\varphi_1 = \varphi_0$ and $\varphi_3 = \varphi_4$.

In order to choose φ_1 and φ_3 we will require that $\varrho_0\mathbf{v}_1 + \varrho_4\mathbf{w}_4 \approx \alpha\mathbf{r}$ for some $\alpha \in (0,1)$. We have

$$\begin{aligned}
\mathbf{e}(\varphi_1) &= \mathbf{e}(\varphi_0 + \Delta\varphi_1) = \cos\Delta\varphi_1\mathbf{e}(\varphi_0) + \sin\Delta\varphi_1\mathbf{f}(\varphi_0) \\
&= \left(\cos\Delta\varphi_1 - \frac{\sin\Delta\varphi_1\cos(\varphi_4-\varphi_0)}{\sin(\varphi_3-\varphi_0)}\right)\mathbf{e}(\varphi_0) + \frac{\sin\Delta\varphi_1}{\sin(\varphi_4-\varphi_0)}\mathbf{e}(\varphi_4) \\
\mathbf{e}(\varphi_3) &= \mathbf{e}(\varphi_4 - \Delta\varphi_4) = \cos\Delta\varphi_4\mathbf{e}(\varphi_4) - \sin\Delta\varphi_4\mathbf{f}(\varphi_4) \\
&= \frac{\sin\Delta\varphi_4}{\sin(\varphi_4-\varphi_0)}\mathbf{e}(\varphi_0) + \left(\cos\Delta\varphi_4 - \frac{\sin\Delta\varphi_4\cos(\varphi_4-\varphi_0)}{\sin(\varphi_4-\varphi_0)}\right)\mathbf{e}(\varphi_4)
\end{aligned}$$

To first order in $\Delta\varphi_1$ and $\Delta\varphi_4$ we have

$$\begin{aligned}
\mathbf{v}_1 &= a(\Delta\varphi_1)\mathbf{e}(\varphi_0) + b(\Delta\varphi_1)\mathbf{e}(\varphi_1) \approx \frac{1}{2}\Delta\varphi_1\mathbf{e}(\varphi_0) \\
\mathbf{w}_4 &= a(\Delta\varphi_4)\mathbf{e}(\varphi_3) + b(\Delta\varphi_4)\mathbf{e}(\varphi_4) \approx \frac{1}{2}\Delta\varphi_4\mathbf{e}(\varphi_4)
\end{aligned}$$

Using this approximation, the equation $\varrho_0\mathbf{v}_1 + \varrho_4\mathbf{w}_4 = \alpha(x\mathbf{e}(\varphi_0) + y\mathbf{e}(\varphi_4))$, where x and y are given by (3), yields

$$\Delta\varphi_1 = \frac{2\alpha x}{\varrho_0} \quad \text{and} \quad \Delta\varphi_4 = \frac{2\alpha y}{\varrho_4}$$

We need $\Delta\varphi_1 < \varphi_2 - \varphi_0$ and $\Delta\varphi_4 < \varphi_4 - \varphi_2$ so we make the following definitions

$$\begin{aligned}
\widetilde{x} &= \frac{x\sin(\varphi_4-\varphi_0)}{\varrho_0} = \frac{r_1\sin\varphi_4 - r_2\cos\varphi_4}{\varrho_0}, \\
\widetilde{y} &= \frac{y\sin(\varphi_4-\varphi_0)}{\varrho_4} = \frac{-r_1\sin\varphi_0 + r_2\cos\varphi_0}{\varrho_4},
\end{aligned}$$

and

$$\widetilde{\alpha} = \min\left\{1, \frac{\varphi_2-\varphi_0}{2\widetilde{x}}, \frac{\varphi_4-\varphi_2}{2\widetilde{y}}\right\}.$$

Finally, we put

$$\Delta\varphi_1 = \widetilde{\alpha}\widetilde{x} \quad \text{and} \quad \Delta\varphi_4 = \widetilde{\alpha}\widetilde{y}.$$

The three cases correspond to $\alpha = \frac{\sin(\varphi_4-\varphi_0)}{2}$, $\alpha = \frac{(\varphi_2-\varphi_0)\varrho_0}{4x}$, and $\alpha = \frac{(\varphi_4-\varphi_2)\varrho_4}{4y}$, respectively. In all three cases Lemma 3 shows that $\Delta\varphi_1 \leq \frac{\varphi_2-\varphi_0}{2}$ and $\Delta\varphi_4 \leq \frac{\varphi_4-\varphi_2}{2}$. We can now start with these values of $\Delta\varphi_1$ and $\Delta\varphi_4$ and solve the linear equation (2). If there are positive solutions we stop, otherwise we keep halving $\Delta\varphi_1$ and $\Delta\varphi_4$ until we get positive solutions.

At this point we have a one-parameter family of solutions and an open parameter interval where the solution is positive, we call this interval the *feasible interval.* There are many ways to fix the parameter in the feasible interval. Here we give four suggestions:

1. Take the midpoint of the feasible interval.
2. Minimize $\sum_{i=1}^{3} \varrho_i^2$ over the feasible interval.
3. Minimize $\sum_{i=1}^{4}(\varrho_i - \varrho_{i-1})^2$ over the feasible interval.
4. Minimize the curvature variation $\sum_{i=1}^{4} \frac{(\varrho_{i-1}+\varrho_i)(\varrho_{i-1}^2+\varrho_i^2)(\varrho_i-\varrho_{i-1})^2}{4\varrho_{i-1}^4\varrho_i^4(\varphi_i-\varphi_{i-1})}$ over the feasible interval.

4.2. The Zero Curvature Case

We now assume that $\kappa_A = 0$, but that we otherwise have the same situation as in the previous section. We also use the same notation and choose three breakpoints and the radius of curvature at these points. In the first interval we use Theorem 1 with the function h given by (1). The equation (2) is then replaced by

$$\varrho_1(\mathbf{u}_1 + \mathbf{v}_2) + \varrho_2(\mathbf{w}_2 + \mathbf{v}_3) + \varrho_3(\mathbf{w}_3 + \mathbf{v}_4) = \mathbf{r} - \varrho_4\mathbf{w}_4, \tag{4}$$

where

$$\mathbf{u}_1 = \frac{2\Delta\varphi_1}{3+4\Delta\varphi_1^2}\left(\frac{2\Delta\varphi_1}{3\sin\Delta\varphi_1}\mathbf{e}(\varphi_0) + \left(1 - \frac{2\Delta\varphi_1\cos\Delta\varphi_1}{3\sin\Delta\varphi_1}\right)\mathbf{e}(\varphi_1)\right).$$

If we choose $\varphi_2 = \theta$ then the same proof as for Lemma 4 gives

Lemma 5. *If $\varphi_2 - \varphi_0, \varphi_4 - \varphi_2 < \pi/2$ and $\Delta\varphi_4$ is sufficiently small then there are positive solutions to (4).*

There is only one auxiliary term on the right hand side of equation (4), namely $\varrho_4\mathbf{w}_4$, which only depends on $\Delta\varphi_4$. So we just set

$$\Delta\varphi_1 = \frac{\varphi_2 - \varphi_0}{2} \quad \text{and} \quad \Delta\varphi_4 = \frac{\varphi_4 - \varphi_0}{2}.$$

As before, we start with these values of $\Delta\varphi_1$ and $\Delta\varphi_4$ and solve the linear equation (4). If there are positive solutions we stop, otherwise we keep halving $\Delta\varphi_4$ until we have. We once more have a one-parameter family of solutions and we fix the solution in a way similar to above. In case 3 the objective function is replaced by $\sum_{i=2}^{4}(\varrho_i - \varrho_{i-1})^2$ and in case 4 the objective function is replaced by

$$\frac{K_1(\Delta\varphi_1)}{\varrho_1^3} + \sum_{i=2}^{4} \frac{(\varrho_{i-1}+\varrho_i)(\varrho_{i-1}^2+\varrho_i^2)(\varrho_i-\varrho_{i-1})^2}{4\varrho_{i-1}^4\varrho_i^4(\varphi_i-\varphi_{i-1})},$$

where K_1 is defined in Lemma 2. If we also have $\kappa_B = 0$ then (4) becomes

$$\varrho_1(\mathbf{u}_1+\mathbf{v}_2)+\varrho_2(\mathbf{w}_2+\mathbf{v}_3)+\varrho_3(\mathbf{w}_3+\mathbf{u}_4)=\mathbf{r},$$

where

$$\mathbf{u}_4 = \frac{2\Delta\varphi_4}{3+4\Delta\varphi_4^2}\left(\frac{2\Delta\varphi_4}{3\sin\Delta\varphi_4}\mathbf{e}(\varphi_4) + \left(1 - \frac{2\Delta\varphi_4\cos\Delta\varphi_4}{3\sin\Delta\varphi_4}\right)\mathbf{e}(\varphi_3)\right).$$

If $\varphi_2 - \varphi_0, \varphi_4 - \varphi_2 < \pi/2$ and we choose $\varphi_2 = \theta$ then we have positive solutions. If we let $\varphi_3 = \varphi_B$ and only choose two break points then we get the equation

$$\varrho_1(\mathbf{u}_1+\mathbf{v}_2)+\varrho_2(\mathbf{w}_2+\mathbf{u}_3)=\mathbf{r},$$

where $\mathbf{u}_3$ is defined as $\mathbf{u}_4$. If φ_1 and φ_2 are chosen symmetrically around θ then $\mathbf{v}_2$ and $\mathbf{w}_2$ is on the same side of $\mathbf{r}$ as $\mathbf{u}_1$ and $\mathbf{u}_3$ respectively, and we obtain a unique solution which is positive if $\theta - \varphi_0, \varphi_3 - \theta < \pi/2$.

4.3. The Non-Convex Case

We now assume that $\kappa_A < 0 < \kappa_B$ and that $\theta < \varphi_A, \varphi_B$. We put $\varphi_3^- = \varphi_A$, $\varphi_3^+ = \varphi_B$, $\varrho_3^- = -1/\kappa_A$, and $\varrho_3^+ = 1/\kappa_B$. There has to be an inflexion point somewhere on the curve from A to B, we denote this point by Q, the tangent direction by φ_0 and put $\mathbf{r}^- = \overrightarrow{QA}$ and $\mathbf{r}^+ = \overrightarrow{QB}$. The curve from Q to B is called $\mathbf{x}^+$ and the curve from Q to A is called $\mathbf{x}^-$. We choose break points $\varphi_0 < \varphi_1^- < \varphi_2^- < \varphi_3^-$ and $\varphi_0 < \varphi_1^+ < \varphi_2^+ < \varphi_3^+$ and denote the corresponding radius of curvatures by $\varrho_1^\pm, \varrho_2^\pm$. The tangent direction of $\mathbf{x}^-$ is opposite of the curve at hand so when considering this piece we have to add π to the tangent direction. That correspond to changing sign of $\mathbf{e}(\varphi)$. As in the previous section, we use Theorem 1 with functions $h^\pm$ given by (1) on the intervals $\left[0, \sqrt{\varphi_1^\pm - \varphi_0}\right]$. We then obtain

$$\mathbf{r}^\pm = \pm\left(\varrho_1^\pm(\mathbf{u}_1^\pm+\mathbf{v}_2^\pm)+\varrho_2^\pm(\mathbf{w}_2^\pm+\mathbf{v}_3^\pm)+\varrho_3^\pm\mathbf{w}_3^\pm\right).$$

As $\mathbf{r} = \mathbf{r}^+ - \mathbf{r}^-$, we get

$$\begin{aligned}\varrho_1^+(\mathbf{u}_1^+ + \mathbf{v}_2^+) + \varrho_1^-(\mathbf{u}_1^- + \mathbf{v}_2^-) + \varrho_2^+(\mathbf{w}_2^+ + \mathbf{v}_3^+) + \varrho_2^-(\mathbf{w}_2^- + \mathbf{v}_3^-)\\ = \mathbf{r} - \varrho_3^+\mathbf{w}_3^+ - \varrho_3^-\mathbf{w}_3^-, \quad (5)\end{aligned}$$

If we choose $\varphi_1^\pm = \theta$ and $\varphi_0 < \theta$ then the same proof as for Lemma 4 and 5 give

Lemma 6. *If $\varphi_A - \theta, \varphi_B - \theta < \pi/2$ and $\Delta\varphi_3^-$ and $\Delta\varphi_3^+$ are sufficiently small then there are positive solutions to* (5).

As in the zero curvature case we start by letting

$$\Delta\varphi_3^\pm = \frac{\varphi_3^\pm - \varphi_1^\pm}{2},$$

and solving (5). If there are positive solutions we stop, otherwise we keep halving $\Delta\varphi_3^\pm$. We obtain a two-parameter family of solutions where the feasible set is given by four linear inequalities in the parameter plane. We fix the solution the same way as before, except that we in case 1 replace 'midpoint' by 'center of mass', in case 3 use the objective function $\sum_{i=2}^3(\varrho_i^- - \varrho_{i-1}^-)^2 + \sum_{i=2}^3(\varrho_i^+ - \varrho_{i-1}^+)^2$, and in case 4 use the objective function

$$\frac{K_1(\Delta\varphi_1^-)}{{\varrho_1^-}^3} + \sum_{i=2}^3 \frac{(\varrho_{i-1}^- + \varrho_i^-)({\varrho_{i-1}^-}^2 + {\varrho_i^-}^2)(\varrho_i^- - \varrho_{i-1}^-)^2}{4{\varrho_{i-1}^-}^4{\varrho_i^-}^4(\varphi_i^- - \varphi_{i-1}^-)} + \frac{K_1(\Delta\varphi_1^+)}{{\varrho_1^+}^3} + \sum_{i=2}^3 \frac{(\varrho_{i-1}^+ + \varrho_i^+)({\varrho_{i-1}^+}^2 + {\varrho_i^+}^2)(\varrho_i^+ - \varrho_{i-1}^+)^2}{4{\varrho_{i-1}^+}^4{\varrho_i^+}^4(\varphi_i^+ - \varphi_{i-1}^+)},$$

where K_1 is defined in Lemma 2. The cases where one or both of the curvatures κ_A and κ_B vanish can be handled in the same way as in the previous section.

§5. Examples

In Figure 3 we have plotted the interpolant to a set of data where all three convexity situations occur. In the non-convex case we have used $\varphi_0 = 2\theta - (\varphi_A + \varphi_B)/2$. The points are marked by small circles and the tangent and curvature can be inferred from the porcupine plot. The difference between the four cases is the way the free parameters are fixed.

In cases 2 and 3 the objective function is quadratic which leads to a linear problem and they generally give a more fair result than case 1, but there are points where ϱ is close to zero. This means the curvature is huge, leading to 'spikes' in the porcupine plot. This situation occurs when the free parameter is close to an end point of the feasible interval, or in the non-convex case when the pair of free parameters is close to the boundary of the feasible set. By shrinking the feasible set in the optimization, in the convex case we could for example use the middle third of the original feasible interval, the minimum of the resulting ϱ would be larger and thus reduce the size of the spikes.

The best result is obtained in case 4, but it is also the most expensive. Observe that circles are reproduced in cases 3 and 4.

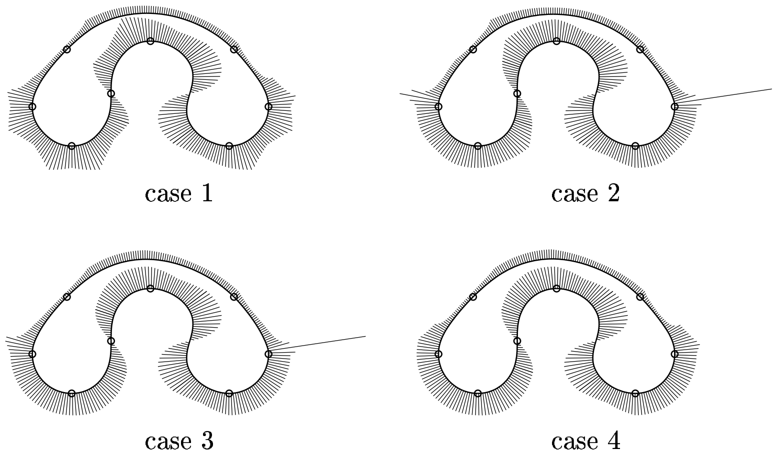

Fig. 3. Porcupine plots of curves interpolating G^2 Hermite data. The two spikes in case 2 and 3 have been truncated.

References

1. Adams, J. A., The intrinsic method for curve definition, Computer-Aided Design **7** (1975), 243–249.
2. Bolton, K. M., Biarc curves, Computer-Aided Design **7** (1975), 89–92.
3. Farouki, R. T., and T. Sakkalis, Pythagorean hodographs, IBM J. Res. Develop. **34** (1990), 736–752.
4. Gravesen, J., and C. Henriksen, The geometry of the scroll compressor, SIAM Review **43** (2001), 113–126.
5. Kuroda, M., and S. Mukai, Interpolating involute curves, in *Curve and Surface Fitting: Saint-Malo 1999*, Albert Cohen, Christophe Rabut, and Larry L. Schumaker (eds.), Vanderbilt University Press, Nashville, 2000, 273–280.
6. Meek, D. S., and D. J. Walton, Approximating smooth planar curves by arc splines, J. Comput. Appl. Math. **59** (1995), 221–231.
7. Meek, D. S., and D. J. Walton, Planar G^1 Hermite interpolation with spirals, Comput. Aided Geom. Design **15** (1998), 787–801.
8. Meek, D. S., and D. J. Walton, Planar spirals that match G^2 Hermite data, Comput. Aided Geom. Design **15** (1998), 103–126.
9. Nutbourne, A. W., P. M. McLellan, and R. M. Kensit, Curvature profiles for plane curves, Computer-Aided Design **4** (1972), 176–184.

Jens Gravesen
Department of Mathematics
Technical University of Denmark
Matematiktorvet Building 303
DK-2800 Kgs. Lyngby
Denmark
J.Gravesen@mat.dtu.dk
http://www.mat.dtu.dk