Third Order Invariants of Surfaces

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Abstract. The classical invariant theory from the 19th century is used to determine a complete system of 3rd order invariants on a surface in three-space. The invariant ring has 18 generators and the ideal of syzygies has 65 generators. The invariants are expressed as polynomials in the components of the first fundamental form, the second fundamental form and the covariant derivative of the latter, or in the case of an implicitly defined surface – $M = f^{-1}(0)$ – as polynomials in the partial derivatives of f up to order three.
As an application some commonly used fairings measures are written in invariant form. It is shown that the ridges and the subparabolic curve of a surface are the zero set of invariant functions and it is finally shown that the Darboux classification of umbilical points can be given in terms of two invariants.

1 Introduction

An *nth order invariant* on a surface M in $\mathbb{R}^3$ is a function $M \to \mathbb{R}$ whose value at a point $P \in M$ depends only on the nth order Taylor expansion of a parameterization of M around P, see Definition 2. E.g. the mean curvature H and the Gauss curvature K are second order invariants, in fact *any* second order invariant can be written as a function of H and K, so they form a *complete system*. In this paper a similar complete minimal system are found for the third order invariants, together with the complete system of relations.

The problem of finding such a system of generators and relations turns out to be a purely algebraic question that was much studied in the 19th century. The literature is immense, so we just refer to the books [1–6], and the survey [7]. Some of the classical algorithms from that time will be used but they will be phrased in the modern language of tensor analysis.

In Sect. 2 we give the precise definition of an invariant and we reduce the problem to a purely algebraic one. The main results are the list of invariants in Table 1 which forms a complete minimal system of generators and Theorem 8 which describe the surprisingly simple structure of the invariant ring.

The proof of Theorem 8 is in three stages. In Sect. 3 we use an algorithm from the 19th century, cf. [1], to determine a minimal set of generators. In Sect. 4 we find a set of relations – called syzygies – among these generators. Finally in Sect. 5 we show that we have found enough syzygies, i.e., they generate the whole ideal of relations. This is done by using Weyls character formula and the residue theorem to calculate the

Hilbert-Molien series which tells the dimension of the space of invariants of a fixed degree.

Section 6 is devoted to implicit surfaces, given by an equation $f(\mathbf{x}) = 0$. It is explained how the invariants can be expressed in terms of the function f.

Once the structure of the invariant ring is established it can be used without knowledge of the proof. So the reader interested in applications only can skip most of the paper and go directly to the examples. The way the theory is used is to perform a calculation using *principal coordinates* in which the first fundamental form, the second fundamental form and its covariant derivative is particular simple. The result is translated into an expression of invariants and then Table 1 can be used to tell what the expression is in an arbitrary parameterization. To demonstrate how this works we present some applications in Sect. 7. First we express some functions used as fairing measures in terms of our invariants. Then we characterize ridges, the subparabolic curve, and the Darboux classification of umbilical points using invariants.

2 Invariants on Surfaces

Let $\mathbf{x}_0$ be a point on a surface $M \subset \mathbb{R}^3$ and denote the unit normal vector and the tangent plane at $\mathbf{x}_0$ by $\mathbf{N}_{\mathbf{x}_0}$ and $T_{\mathbf{x}_0}M$ respectively. Let $\mathbf{r}_1, \mathbf{r}_2$ be a basis for the tangent space $T_{\mathbf{x}_0}M$ and let (x^1, x^2) denote the coordinates on $T_{\mathbf{x}_0}M$ with respect to this basis. Around the point $\mathbf{x}_0$ we can write the surface as a graph of a function on the tangent space. More precisely the map

$$(x^1, x^2) \mapsto \mathbf{x}(x^1, x^2) = \mathbf{x}_0 + x^1\mathbf{r}_1 + x^2\mathbf{r}_2 + h(x^1, x^2)\mathbf{N}_{\mathbf{x}_0} \ , \tag{1}$$

is a local parameterization of M. The inverse map is simply the orthogonal projection $M \to T_{\mathbf{x}_0}M$, and h is the height of the surface over the tangent plane.

Normally the letter g is used for the first fundamental form, but we shall consider three different forms and it seems natural to use the letters a, b, c. So we let $a_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j$ be the components of the first fundamental form. We can Taylor expand the function h to third order:

$$h(x^1, x^2) = \frac{1}{2}b_{ij}x^ix^j + \frac{1}{6}c_{ijk}x^ix^jx^k + \text{higher order terms} \ , \tag{2}$$

where we use *Einsteins summation convention*, so if an index appears once as a subscript and once as a superscript, then it is tacitly understood that we sum over it. We may furthermore assume that b_{ij} and c_{ijk} are symmetric in the indices. We do not need an explicit expression of h in order to determine the coefficients b_{ij} and c_{ijk}. Indeed, we have

Proposition 1. *The coefficients b_{ij} and c_{ijk} are the components of the second fundamental form and the covariant derivative of the second fundamental form respectively, both with respect to the basis $\mathbf{r}_1, \mathbf{r}_2$.*

Proof. In the parameterization (1) we have $h(x^1, x^2) = \big(\mathbf{x}(x^1, x^2) - \mathbf{x}_0\big) \cdot \mathbf{N}_{\mathbf{x}_0}$, so

$$b_{ij} = \left.\frac{\partial^2 h}{\partial x^i \partial x^j}\right|_{(0,0)} = \left.\frac{\partial^2 \mathbf{x}}{\partial x^i \partial x^j}\right|_{(0,0)} \cdot \mathbf{N}_{\mathbf{x}_0} \ ,$$

and this is exactly the components of the second fundamental form at $\mathbf{x}_0$. To first order we have that $\frac{\partial \mathbf{x}}{\partial x^1} = \mathbf{r}_i + b_{ij}x^j \mathbf{N}_{\mathbf{x}_0}$ and as $\mathbf{N}_{\mathbf{x}_0} \perp \mathbf{r}_i$ the components of the first fundamental are constant to first order: $\frac{\partial \mathbf{x}}{\partial x^i} \cdot \frac{\partial \mathbf{x}}{\partial x^j} \approx a_{ij}$. Likewise $\left|\frac{\partial \mathbf{x}}{\partial x^1} \times \frac{\partial \mathbf{x}}{\partial x^2}\right|^2 \approx |\mathbf{r}_1 \times \mathbf{r}_2|^2$. Finally

$$\begin{aligned}
&\frac{\partial^2 \mathbf{x}}{\partial x^i \partial x^j} \cdot \left(\frac{\partial \mathbf{x}}{\partial x^1} \times \frac{\partial \mathbf{x}}{\partial x^2}\right) \\
&\approx (b_{ij} + c_{ijk}x^k)\mathbf{N}_{\mathbf{x}_0} \cdot \big((\mathbf{r}_1 + (b_{11}x^1 + b_{12}x^2)\mathbf{N}_{\mathbf{x}_0}) \times (\mathbf{r}_2 + (b_{12}x^1 + b_{22}x^2)\mathbf{N}_{\mathbf{x}_0})\big) \\
&\qquad = (b_{ij} + c_{ijk}x^k)\mathbf{N}_{\mathbf{x}_0} \cdot (\mathbf{r}_1 \times \mathbf{r}_2) = |\mathbf{r}_1 \times \mathbf{r}_2| \, (b_{ij} + c_{ijk}x^k) \ .
\end{aligned}$$

So the components of the second fundamental form are to first order $b_{ij} + c_{ijk}x^k$. Moreover, the ordinary derivative has components c_{ijk} and as the Christoffel symbols vanishes at $\mathbf{x}_0$ the covariant derivative at $\mathbf{x}_0$ agrees with the ordinary derivative and has components c_{ijk} too. □

We have two interpretations of the quantities a_{ij}, b_{ij}, and c_{ijk}, as the coefficients of a homogeneous polynomial in two variables (called a *binary form*) and as the components of a k-form on M. We will also use a third interpretation, namely as the components of an element of the space $S^k\mathbb{R}^2$ of *symmetric k-tensors* on $\mathbb{R}^2$.

We have said that a third order invariant is a function that depends only on the third order behaviour of the surface. We can now make this precise:

Definition 2. *Let $\mathbf{r} : U \to M \subset \mathbb{R}^3$ be a parameterization of a surface. Let the components of the first fundamental form be $\mathbf{a} = a_{11}, a_{12}, a_{22}$, let the components of the second fundamental form be $\mathbf{b} = b_{11}, b_{12}, b_{22}$, and let the components of the covariant derivative of the second fundamental form be $\mathbf{c} = c_{111}, c_{112}, c_{122}, c_{222}$. A third order invariant is a function $f : M \to \mathbb{R}$ that can be written on the form $f(\mathbf{r}(u, v)) = F(\mathbf{a}(u, v), \mathbf{b}(u, v), \mathbf{c}(u, v))$, where $F : S^2\mathbb{R}^2 \times S^2\mathbb{R}^2 \times S^3\mathbb{R}^2 \to \mathbb{R}$.*

The function f is a function on the surface and is thus independent of the parameterization. On the other hand, if we change the parameterization of the surface we change the basis in the tangent plane and this in turn changes the components $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of the three forms on the tangent plane. So F can not be arbitrary; it has to be invariant under the change of basis, i.e., under the action of $GL_2(\mathbb{R})$ on $S^2\mathbb{R}^2 \times S^2\mathbb{R}^2 \times S^3\mathbb{R}^2$.

We want to determine a finite set $F_1, \ldots, F_n$ of such invariant functions such that an arbitrary invariant function F can be written

$$F(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \widehat{F}(F_1(\mathbf{a}, \mathbf{b}, \mathbf{c}), \ldots, F_n(\mathbf{a}, \mathbf{b}, \mathbf{c})).$$

We will in fact find a set of invariant *polynomials* such that any invariant polynomial can be written in the form above. Then the same is true for arbitrary invariant functions too, because such a set of polynomials separates orbits, see [3, Theorem 8.17]. The advantage is that the polynomial problem is a purely algebraic problem.

3 The Generators

We first consider a slightly different problem. We will consider forms or symmetric tensors over the complex numbers so we are given three binary forms $a_{ij}x^ix^j$, $b_{ij}x^ix^j$, and $c_{ijk}x^ix^jx^k$ where $(x^1, x^2) = \mathbf{x} \in \mathbb{C}^2$, and we ask for polynomials in the variables $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}$ that are invariant under the action of $SL_2(\mathbb{C})$. More precisely we want to determine the structure of the *invariant ring* $\mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}]^{SL_2(\mathbb{C})}$. A polynomial is the sum of components, homogeneous in each set of variables $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}$, and it is invariant if and only if each of its homogeneous components is invariant.

In the classical literature a *joint covariant* of *multi-degree* (d_1, d_2, d_3) and *order* k is a homogeneous invariant polynomial which has degree d_1 in $\mathbf{a}$, degree d_2 in $\mathbf{b}$, degree d_3 in $\mathbf{c}$, and degree k in $\mathbf{x}$. In particular, the forms themselves are covariants, and a *joint invariant* is a joint covariant of order 0.

An SL_2-invariant will in general not be invariant under the action of GL_2. Indeed, a diagonal matrix $\left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right)$ acts on x^i by multiplication with t^{-1}, on $a_{i,j}, b_{i,j}$ by multiplication with t^2, and on $c_{i,j,k}$ by multiplication with t^3, so an $A \in GL_2$ acts on a joint covariant by multiplication with $\det A^\rho$, where $2\rho = 2d_1 + 2d_2 + 3d_3 - k$, we say that it is a *relative GL_2-covariant* of *weight* (or *index*) ρ.

The description of the invariant ring $\mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}]^{SL_2}$ is a classical problem that was studied intensely in the nineteenth century, and the two basic problems are the following

- Find a set of basic covariants $C_1, \ldots, C_p$, called a *complete system*, such that any covariant can be written as a polynomial in the basic covariants. I.e., such that the map $\phi : \mathbb{C}[X_1, \ldots, X_p] \to \mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}]$, given by $\phi(P) = P(C_1, \ldots, C_p)$ maps onto $\mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}]^{SL_2}$.
- Find all syzygies among the basic covariants, i.e., find the kernel $\mathcal{S}$ of ϕ. That is, all polynomials with $P(C_1, \ldots, C_p) = 0$.

In 1890 Hilbert showed that there always exists a finite system of generators and relations, see [8]. I.e., there exists generators $C_1, \ldots, C_p \in \mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}]$ and syzygies $S_1, \ldots, S_q \in \mathbb{C}[X_1, \ldots, X_p]$ such that the map $X_i \mapsto C_i$ gives an isomorphism

$$\mathbb{C}[X_1, \ldots, X_p] \,\big/\, (S_1, \ldots, S_q) \cong \mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}]^{SL_2} \;,$$

where $(S_1, \ldots, S_q)$ denotes the ideal generated by $S_1, \ldots, S_q$.

Before Hilbert the emphasis was on the explicit construction of covariants, often using *transvectants*. They can be defined symbolically, see [1, Chapter III] or [2, (20.18)] or they can be defined using differential operators, but we will use contraction of tensors to define them. We can assume that the components $f_{i_1,\ldots,i_n}$ of a polynomial $f = f_{i_1,\ldots,i_n}x^{i_1}\ldots x^{i_n}$ are symmetric in the indices so we may consider them as components of a symmetric tensor $f_{i_1,\ldots,i_n}x^{i_1} \otimes \cdots \otimes x^{i_n} \in S^n(\mathbb{C}^2)$. The *rth transvectant* of two symmetric tensors f and g is denoted $(f, g)^{(r)}$ and is defined by having components

$$(f, g)^{(r)}_{i_{r+1},\ldots,i_n,j_{r+1},\ldots,j_m} = S\big(\varepsilon^{i_1j_1}\ldots\varepsilon^{i_rj_r} f_{i_1,\ldots,i_n} g_{j_1,\ldots,j_m}\big) \;, \tag{3}$$

where ε^{ij} is the completely anti symmetric symbol $\varepsilon^{11} = \varepsilon^{22} = 0$ and $\varepsilon^{12} = -\varepsilon^{21} = 1$. The symbol S stands for symmetrization of the free indices. Observe that the symmetry of f and g implies, that up to a sign, this is the only non zero contraction. As a contraction of a tensor is a new tensor we see that the transvectants of two covariants is a new covariant. Obviously

$$\begin{aligned}\deg(f,g)^{(r)} &= \deg f + \deg g \ ,\\ \mathrm{order}(f,g)^{(r)} &= \mathrm{order}\, f + \mathrm{order}\, g - 2r \ ,\\ \mathrm{weight}(f,g)^{(r)} &= \mathrm{weight}\, f + \mathrm{weight}\, g + r \ .\end{aligned}$$

It should be mentioned that all this is part of the representation theory of $SL_2\mathbb{C}$ (or $SU(2)$), see [9]. The irreducible representations are the spaces $S^n\mathbb{C}^2$ of symmetric tensors, and a tensor product of two of these has the following decomposition

$$S^n\mathbb{C}^2 \otimes S^m\mathbb{C}^2 = \bigoplus_{r=0}^{\lfloor \frac{n+m}{2} \rfloor} S^{n+m-2r}\mathbb{C}^2 \ .$$

The rth transvectant is exactly the projection from $S^n\mathbb{C}^2 \otimes S^m\mathbb{C}^2$ to $S^{n+m-2r}\mathbb{C}^2$.

The following theorem tells us how to get a complete system for a single binary form, see [1, § 86] or [2, Theorem 24.3].

Theorem 3. *Any covariant of a binary form f of degree d in its coefficients can be written as a linear combination of transvectants of the form itself and covariants of degree $d-1$.*

As the only covariant of degree 1 is the form itself this shows that we can find a complete system of covariants of single form consisting of transvectants. E.g. a complete system of covariants for a single quadratic binary form a consists of the form itself and its discriminant:

$$a = a_{ij}x^i x^j \ , \qquad (a,a)^{(2)} = 2(a^{11}a^{22} - a^{12}a^{12}) \ , \tag{4}$$

see [1, 2, 4, 5].

A complete system of single cubic form c consists of the form itself, its Hessian, the discriminant of the Hessian and the Jacobian of the form with its Hessian:

$$c \ , \qquad H = (c,c)^{(2)} \ , \qquad D = (H,H)^{(2)} \ , \qquad T = (H,c)^{(1)} \ , \tag{5}$$

see [1, 2, 4, 5]. The joint covariants of a collection of forms can be created from complete subsystems, see [1, § 103].

Theorem 4. *If S_1 and S_2 are two finite and complete systems of forms, then there exists a finite and complete system consisting of transvectants of products of elements of S_1 and products of forms of S_2.*

The problem with this theorem is that we don't know how many products we need to form, before we take transvectants. But if one of the systems above is the complete system (4) of a single binary quadratic form then more is true, see [1, § 141].

Theorem 5. *If S_1 is the system* (4) *for a quadratic form a, and S_2 is an arbitrary system, then the irreducible transvectants belong to one of the three classes, $(C,a^r)^{(2r-1)}$, $(C,a^r)^{(2r)}$, and $(C_1C_2,a^r)^{(2r)}$, where C_1 and C_2 have odd order and the order of the product C_1C_2 is $2r$.*

By using this result twice we can find a complete system for two quadratic and one cubic binary form, i.e., a complete system for the third order invariants of a surface. The result is a rather large system and not all elements are needed. To get rid of the superfluous elements we use the following important result, see [6, Chapter VIII, §7], or [4, Chapter 4.3].

Theorem 6. *If $C, C_1, \dots, C_n$ are covariants and $C = p_1 C_1 + \cdots + p_n C_n$ for some polynomials p_i, then we can assume that p_i are covariants too. In other words, if a covariant is contained in the ideal generated by some covariants then it is contained in the algebra generated by the same covariants:*

$$C \in (C_1, \dots, C_n) \Rightarrow C \in \mathbb{C}[C_1, \dots, C_n] \ .$$

So using the results above we first find a large set of generators. Then we sort them such that the partial ordering induced by the order and the multi degree is respected, i.e., $k \le k' \wedge d_1 \le d_1' \wedge d_2 \le d_2' \wedge d_3 \le d_3' \implies C \le C'$. Finally we take each covariant in turn and if it is contained in the ideal generated by the previous ones then we throw it away otherwise we keep it. All this was done using the algebra program "Singular" [10]. The result is a minimal system of generators consisting of 18 invariants, 13 linear covariants, 6 quadratic covariants, and 4 cubic covariants. The 18 invariants are

$$\begin{gathered}(a,a)^{(2)}, \quad (a,b)^{(2)}, \quad (b,b)^{(2)}, \quad \big((c,c)^{(2)},a\big)^{(2)}, \quad \big((c,c)^{(2)},b\big)^{(2)}, \\ \Big(\big((c,c)^{(2)},b\big)^{(1)},a\Big)^{(2)}, \quad \big(c^2,a^3\big)^{(6)}, \quad \big(c(c,b)^{(2)},a^2\big)^{(4)}, \quad \big((c,b)^{(2)\,2},a\big)^{(2)}, \\ \big(c^2,b^3\big)^{(6)}, \quad \big(c(c,b)^{(1)},a^3\big)^{(6)}, \quad \Big(c\big(c,b^2\big)^{(3)},a^2\Big)^{(4)}, \quad \Big((c,b)^{(2)}\big(c,b^2\big)^{(3)},a\Big)^{(2)}, \\ \big((c,c)^{(2)},(c,c)^{(2)}\big)^{(2)}, \quad \Big(c\big((c,c)^{(2)},c\big)^{(1)},a^3\Big)^{(6)}, \quad \Big(c\Big(\big((c,c)^{(2)},c\big)^{(1)},b\Big)^{(2)},a^2\Big)^{(4)}, \\ \Big((c,b)^{(2)}\Big(\big((c,c)^{(2)},c\big)^{(1)},b\Big)^{(2)},a\Big)^{(2)}, \quad \Big(c\big((c,c)^{(2)},c\big)^{(1)},b^3\Big)^{(6)} \ .\end{gathered}$$

The symmetrization in (3) means that we in general will get a sum of different contractions. But each contraction in such a sum is an invariant and at least one of them is irreducible. So we can obtain a complete system where each generator is a single contraction. In Table 1 we have listed one possible choice of generators along with their multi-degree and weight. There are at first sight up to 2^{18} terms, but the symmetries reduces this number to 54, see [11] where the sums have been expanded.

4 The Syzygies

We now turn to the problem of finding all syzygies, i.e., all relations between the basic invariants in Table 1. In this section we present a set of syzygies and in the next section we prove that this set generates the ideal of syzygies.

Proposition 7. *There are 39 syzygies of the form*

$$J_i J_j = Q_{ij}^0 + \sum_{k=1}^{2} Q_{ij}^k J_k \ , \qquad 1 \le i \le j \le 2 \text{ or } 3 \le i \le j \le 10 \ , \tag{6}$$

Table 1. The basic invariants, their multi-degree, and their weight. See [11] for expanded expressions.

$I_0 = \frac{1}{2}\varepsilon^{i_1j_1}\varepsilon^{i_2j_2}a_{i_1i_2}a_{j_1j_2}$	$(2,0,0)$	2
$I_1 = \varepsilon^{i_1j_1}\varepsilon^{i_2j_2}a_{i_1i_2}b_{j_1j_2}$	$(1,1,0)$	2
$I_2 = \frac{1}{2}\varepsilon^{i_1j_1}\varepsilon^{i_2j_2}b_{i_1i_2}b_{j_1j_2}$	$(0,2,0)$	2
$I_3 = \frac{1}{2}\varepsilon^{i_1j_3}\varepsilon^{i_2k_3}\varepsilon^{j_1k_1}\varepsilon^{j_2k_2}a_{i_1i_2}c_{j_1j_2j_3}c_{k_1k_2k_3}$	$(1,0,2)$	4
$I_4 = \frac{1}{2}\varepsilon^{i_1j_3}\varepsilon^{i_2k_3}\varepsilon^{j_1k_1}\varepsilon^{j_2k_2}b_{i_1i_2}c_{j_1j_2j_3}c_{k_1k_2k_3}$	$(0,1,2)$	4
$I_5 = \varepsilon^{i_1l_1}\varepsilon^{i_2m_1}\varepsilon^{j_1l_2}\varepsilon^{j_2m_2}\varepsilon^{k_1l_3}\varepsilon^{k_2m_3}a_{i_1i_2}a_{j_1j_2}a_{k_1k_2}c_{l_1l_2l_3}c_{m_1m_2m_3}$	$(3,0,2)$	6
$I_6 = \varepsilon^{i_1l_1}\varepsilon^{i_2m_1}\varepsilon^{j_1l_2}\varepsilon^{j_2m_2}\varepsilon^{k_1l_3}\varepsilon^{k_2m_3}b_{i_1i_2}b_{j_1j_2}b_{k_1k_2}c_{l_1l_2l_3}c_{m_1m_2m_3}$	$(0,3,2)$	6
$I_7 = \frac{1}{2}\varepsilon^{i_1j_1}\varepsilon^{i_2j_2}\varepsilon^{k_1l_1}\varepsilon^{k_2l_2}\varepsilon^{i_3k_3}\varepsilon^{j_3l_3}c_{i_1i_2i_3}c_{j_1j_2j_3}c_{k_1k_2k_3}c_{l_1l_2l_3}$	$(0,0,4)$	6
$J_1 = \varepsilon^{i_1l_1}\varepsilon^{i_2m_1}\varepsilon^{j_1l_2}\varepsilon^{j_2m_2}\varepsilon^{k_1l_3}\varepsilon^{k_2m_3}a_{i_1i_2}a_{j_1j_2}b_{k_1k_2}c_{l_1l_2l_3}c_{m_1m_2m_3}$	$(2,1,2)$	6
$J_2 = \varepsilon^{i_1l_1}\varepsilon^{i_2m_1}\varepsilon^{j_1l_2}\varepsilon^{j_2m_2}\varepsilon^{k_1l_3}\varepsilon^{k_2m_3}a_{i_1i_2}b_{j_1j_2}b_{k_1k_2}c_{l_1l_2l_3}c_{m_1m_2m_3}$	$(1,2,2)$	6
$J_3 = \varepsilon^{i_1j_1}\varepsilon^{i_2k_3}\varepsilon^{j_2l_3}\varepsilon^{k_1l_1}\varepsilon^{k_2l_2}a_{i_1i_2}b_{j_1j_2}c_{k_1k_2k_3}c_{l_1l_2l_3}$	$(1,1,2)$	5
$J_4 = \varepsilon^{i_1l_1}\varepsilon^{i_2m_1}\varepsilon^{j_1m_2}\varepsilon^{j_2n_1}\varepsilon^{k_1m_3}\varepsilon^{k_2n_2}\varepsilon^{l_2n_3}a_{i_1i_2}a_{j_1j_2}a_{k_1k_2}b_{l_1l_2}c_{m_1m_2m_3}c_{n_1n_2n_3}$	$(3,1,2)$	7
$J_5 = \varepsilon^{i_1k_1}\varepsilon^{i_2m_1}\varepsilon^{j_1m_2}\varepsilon^{j_2n_1}\varepsilon^{k_2m_3}\varepsilon^{l_1n_2}\varepsilon^{l_2n_3}a_{i_1i_2}a_{j_1j_2}b_{k_1k_2}b_{l_1l_2}c_{m_1m_2m_3}c_{n_1n_2n_3}$	$(2,2,2)$	7
$J_6 = \varepsilon^{i_1k_1}\varepsilon^{i_2m_1}\varepsilon^{j_1m_2}\varepsilon^{j_2n_1}\varepsilon^{k_2m_3}\varepsilon^{l_1n_2}\varepsilon^{l_2n_3}a_{i_1i_2}b_{j_1j_2}b_{k_1k_2}b_{l_1l_2}c_{m_1m_2m_3}c_{n_1n_2n_3}$	$(1,3,2)$	7
$J_7 = \varepsilon^{i_1l_1}\varepsilon^{i_2m_1}\varepsilon^{j_1l_2}\varepsilon^{j_2m_2}\varepsilon^{k_1l_3}\varepsilon^{k_2n_1}\varepsilon^{m_3p_1}\varepsilon^{n_2p_2}\varepsilon^{n_3p_3}a_{i_1i_2}a_{j_1j_2}a_{k_1k_2}c_{l_1l_2l_3}c_{m_1m_2m_3}c_{n_1n_2n_3}c_{p_1p_2p_3}$	$(3,0,4)$	9
$J_8 = \varepsilon^{i_1l_1}\varepsilon^{i_2m_1}\varepsilon^{j_1l_2}\varepsilon^{j_2m_2}\varepsilon^{k_1l_3}\varepsilon^{k_2n_1}\varepsilon^{m_3p_1}\varepsilon^{n_2p_2}\varepsilon^{n_3p_3}a_{i_1i_2}a_{j_1j_2}b_{k_1k_2}c_{l_1l_2l_3}c_{m_1m_2m_3}c_{n_1n_2n_3}c_{p_1p_2p_3}$	$(2,1,4)$	9
$J_9 = \varepsilon^{i_1l_1}\varepsilon^{i_2m_1}\varepsilon^{j_1l_2}\varepsilon^{j_2m_2}\varepsilon^{k_1l_3}\varepsilon^{k_2n_1}\varepsilon^{m_3p_1}\varepsilon^{n_2p_2}\varepsilon^{n_3p_3}a_{i_1i_2}b_{j_1j_2}b_{k_1k_2}c_{l_1l_2l_3}c_{m_1m_2m_3}c_{n_1n_2n_3}c_{p_1p_2p_3}$	$(1,2,4)$	9
$J_{10} = \varepsilon^{i_1l_1}\varepsilon^{i_2m_1}\varepsilon^{j_1l_2}\varepsilon^{j_2m_2}\varepsilon^{k_1l_3}\varepsilon^{k_2n_1}\varepsilon^{m_3p_1}\varepsilon^{n_2p_2}\varepsilon^{n_3p_3}b_{i_1i_2}b_{j_1j_2}b_{k_1k_2}c_{l_1l_2l_3}c_{m_1m_2m_3}c_{n_1n_2n_3}c_{p_1p_2p_3}$	$(0,3,4)$	9

and 16 of the form

$$J_i J_j = \sum_{k=3}^{10} Q_{ij}^k J_k \ , \qquad 1 \leq i \leq 2 \textit{ and } 3 \leq j \leq 10 \ , \tag{7}$$

where Q_{ij}^k *is a polynomial in* $I_0, \dots, I_7$ *of the form*

$$Q_{ij}^k = \sum_{\sum \deg I_{i_p} = \deg J_i + \deg J_j - \deg J_k} d_{i_1 \dots i_k} I_{i_1} \dots I_{i_k} \ ,$$

($\deg J_0 = \mathbf{0}$)*. We furthermore have two syzygies of the form*

$$Q_i^1 J_1 + Q_i^2 J_2 = Q_i^0 \ , \qquad i = 1, 2 \ , \tag{8}$$

and eight of the form

$$Q_i^3 J_3 + \dots + Q_i^{10} J_{10} = 0 \ , \qquad i = 3, \dots, 10 \ , \tag{9}$$

where Q_i^k *are polynomials of the form*

$$Q_i^k = \sum_{\sum \deg I_{i_p} = (\gamma_1^i, \gamma_2^i, \gamma_3^i) - \deg J_k} d_{i_1 \dots i_k} I_{i_1} \dots I_{i_k} \ ,$$

($\deg J_0 = \mathbf{0}$)*, and the degrees* $(\gamma_1^i, \gamma_2^i, \gamma_3^i)$ *are*

$$\begin{array}{ccccc} (3,2,6) \ , & (2,3,6) \ , & (4,1,4) \ , & (3,2,4) \ , & (2,3,4) \ , \\ (1,4,4) \ , & (3,1,6) \ , & (2,2,6) \ , & (1,3,6) \ , & (3,3,6) \ . \end{array} \tag{10}$$

Proof. We only sketch the proof. Equations (6) and (7) are finite dimensional inhomogeneous linear equations in the coefficients of the polynomials Q_{ij}^k. Using Maple or a similar system it is not hard to solve these equations, see [11].

When the degrees (10) are known then the existence of the polynomials Q_i^k in (8) and (9) reduces to a finite dimensional linear algebra problem, but we have to be careful. If we take syzygies of degree $(1,3,6)$, $(3,1,6)$, $(2,3,4)$, and $(3,2,4)$ and multiply with I_0, I_2, I_3, and I_4 respectively we obtain four syzygies of degree $(3,3,6)$. So when we solve (9) to find the polynomials Q_{10}^k, the space of solutions has dimension greater than one. We need to pick a solution that is not a $\mathbb{C}[I_0, \dots, I_7]$ linear combinations of syzygies of lower degree, but this is not hard to do using a computer algebra system, see [11]. In fact, this is how the 65 syzygies were found in the first place. Starting with low degree, Maple was used to determine syzygies of a fixed degree that can't be expressed as a $\mathbb{C}[I_0, \dots, I_7]$ linear combinations of the syzygies previously found. This process was iterated until no new syzygies emerged for some degrees. We might at this point believe we have all syzygies, but it is not proved – that will be done in the next section.

To simplify the calculations we can pick a good basis for the tangent plane, so we may assume that $a_{ij} = \delta_{ij}$ and b_{ij} is diagonal and obtain the expressions in Table 2, c.f. Sect. 7. □

Note that if we eliminate $\mathbf{a}, \mathbf{b}, \mathbf{c}$ from the ideal $(X_0 - I_0, \ldots, X_7 - I_7, Y_1 - J_1, \ldots, Y_{10} - J_{10})$ in the ring $\mathbb{C}[\mathbf{X}, \mathbf{Y}, \mathbf{a}, \mathbf{b}, \mathbf{c}]$ then we obtain a system of generators for the ideal of syzygies. It is in principle possible to use Gröbner basis methods to do this, but the present problem is apparently too large to be solved this way. At least my implementation in Singular ran for months and never terminated.

Now consider the ring $\mathbb{C}[\mathbf{X}, \mathbf{Y}] = \mathbb{C}[X_0, \ldots, X_7, Y_1, \ldots, Y_{10}]$. We introduce a triple grading by letting

$$\deg(X_i) = \deg(I_i) = (\alpha_1^i, \alpha_2^i, \alpha_3^i), \quad \deg(Y_i) = \deg(J_i) = (\beta_1^i, \beta_2^i, \beta_3^i). \tag{11}$$

The values of $(\alpha_1^i, \alpha_2^i, \alpha_3^i)$ and$(\beta_1^i, \beta_2^i, \beta_3^i)$ can be found in Table 1. We can in an obvious manner consider Q_{ij}^k and Q_i^k as polynomials in $\mathbf{X}$, i.e., as elements in $\mathbb{C}[\mathbf{X}]$ and we now put

$$\begin{aligned} S_{ij} &= Y_i Y_j - \Big(Q_{ij}^0(\mathbf{X}) + \sum_{k=1}^{10} Q_{ij}^k(\mathbf{X}) Y_k \Big) \ , && 1 \le i \le j \le 10 \ , \\ S_i &= Q_i^1(\mathbf{X}) Y_1 + Q_i^2(\mathbf{X}) Y_2 - Q_i^0(\mathbf{X}) \ , && i = 1, 2 \ , \\ S_i &= Q_i^3(\mathbf{X}) Y_3 + \cdots + Q_i^{10}(\mathbf{X}) Y_{10} \ , && i = 3, \ldots, 10 \ , \end{aligned}$$

then $\deg S_i = (\gamma_1^i, \gamma_2^i, \gamma_3^i)$ is given by (10). Now let $\mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ denote the polynomial ring in the variable a_{ij}, b_{ij}, c_{ijk}, and consider the map $\phi : \mathbb{C}[\mathbf{X}, \mathbf{Y}] \to \mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ given by $\phi(X_i) = I_i$ and $\phi(Y_i) = J_i$. It preserves the grading, and maps onto the invariant ring. The polynomials S_{ij} and S_i are in the kernel of ϕ, so if $\mathcal{S}$ is the ideal generated by S_{ij} and S_i then we have a surjective map

$$\mathbb{C}[\mathbf{X}, \mathbf{Y}]/\mathcal{S} \to \mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}]^{SL_2(\mathbb{C})} \ , \tag{12}$$

and we want to show it is an isomorphism. If the degree is fixed then we have a linear map between finite dimensional vector spaces, so we need only to show that the two spaces have the same dimension. This is done in the next section.

5 The Structure of the Invariant Ring

Consider the subspace of invariants of multi-degree $\mathbf{d} = (d_1, d_2, d_3)$ and denote the dimension by $D_{\mathbf{d}}$. The *Hilbert-Molien series* is $H(\mathbf{z}) = \sum D_{\mathbf{d}} \mathbf{z}^{\mathbf{d}}$, where $\mathbf{z}^{\mathbf{d}} = z_1^{d_1} z_2^{d_2} z_3^{d_3}$. From the point of view of Lie group theory we have a representation of $SL_2(\mathbb{C})$ on $S^{\mathbf{d}}\big(S^2(\mathbb{C}^2) \times S^2(\mathbb{C}^2) \times S^3(\mathbb{C}^2)\big)$ and the space of invariants of multi-degree $\mathbf{d}$ is exactly the subspace where $SL_2(\mathbb{C})$ acts trivially. We can split $\mathcal{S}^{\mathbf{d}}$ in a direct sum of irreducible representations and the number of times the trivial representation occur is $D_{\mathbf{d}}$. This number can be computed by Weyls character formula, see [9]. Let $g_n : SL_2(\mathbb{C}) \to GL\big(S^n(\mathbb{C}^2)\big)$ be the n'th symmetric representation of $SL_2(\mathbb{C})$, let T be a maximal torus in $SL_2(\mathbb{C})$ and let $\mathrm{d}\mathbf{t}$ be a Haar measure on T, then with $(n_1, n_2, n_3) = (2, 2, 3)$, we have

$$H(z_1, z_2, z_3) = \int_{\mathbf{t} \in T} \frac{\prod_{1 \le i < j \le 2} \big(1 - \frac{t_i}{t_j}\big)}{\prod_{k=1}^3 \det(1 - z_k g_{n_k}(\mathbf{t}))} \, \mathrm{d}\mathbf{t}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-e^{-2i\theta}}{\prod_{k=1}^{3}\bigl(\prod_{l=0}^{n_k}(1-z_k e^{(2l-n_k)i\theta})\bigr)}\,\mathrm{d}\theta$$
$$= \frac{1}{2\pi i}\int_{S^1} \frac{1-\zeta^{-2}}{\prod_{k=1}^{3}\bigl(\prod_{l=0}^{n_k}(1-z_k \zeta^{2l-n_k})\bigr)}\zeta^{-1}\,\mathrm{d}\zeta$$
$$= \frac{1}{2\pi i}\int_{S^1} \frac{\zeta^5(\zeta^2-1)}{Q(\zeta)}\,\mathrm{d}\zeta,$$

where

$$Q(\zeta) = (\zeta^2-z_1)(1-z_1)(1-z_1\zeta^2)(\zeta^2-z_2)(1-z_2)(1-z_2\zeta^2)$$
$$(\zeta^3-z_3)(\zeta-z_3)(1-z_3\zeta)(1-z_3\zeta^3).$$

We will use the residue theorem to evaluate the last integral, and if we assume that $|z_1|, |z_2|, |z_3| < 1$ and that $\xi_1^2 = z_1$, $\xi_2^2 = z_2$, and $\eta^3 = z_3$ then the poles inside the unit circle are $\pm\xi_1, \pm\xi_2, \eta, e^{\pm 2\pi i/3}\eta, z_3$. We use Maple to calculate the residues for the eight poles and sum the results. The details can be found in [11] and the final result is

$$H(z_1,z_2,z_3) = \frac{1+\sum_{j=1}^{10} z_1^{\beta_1^i} z_2^{\beta_2^i} z_3^{\beta_3^i} - \sum_{k=1}^{10} z_1^{\gamma_1^k} z_2^{\gamma_2^k} z_3^{\gamma_3^k} - z_1^4 z_2^4 z_3^8}{\prod_{i=0}^{7}\left(1-z_1^{\alpha_1^i} z_2^{\alpha_2^i} z_3^{\alpha_3^i}\right)} \quad , \tag{13}$$

where the exponents α_i^j, β_i^j, and γ_i^j are given in (11) and (10).

We now consider the corresponding series – called the *Hilbert series* – for the ring at the left hand side of (12), but first we find a simple description of the ring. We define the ideals $\mathcal{S}_0 = (\dots, S_{ij}, \dots)$ and $\mathcal{S}_1 = (S_1, \dots, S_{10})$ and the rings $R_0 = \mathbb{C}[\mathbf{X}] = \mathbb{C}[X_0, \dots, X_7]$ and $R_1 = \mathbb{C}[\mathbf{X},\mathbf{Y}]/\mathcal{S}_0 = R_0[\mathbf{Y}]/\mathcal{S}_0$. Then $\mathcal{S} = (\mathcal{S}_0 \cup \mathcal{S}_1)$ and $\mathbb{C}[\mathbf{X},\mathbf{Y}]/\mathcal{S} = \bigl(\mathbb{C}[\mathbf{X}][\mathbf{Y}]/\mathcal{S}_0\bigr)/\mathcal{S}_1 = R_1/\mathcal{S}_1$. The syzygies S_{ij} tells us that in the ring R_1 any element can be uniquely written as $p = p_0 + \sum_{i=1}^{10} p_i Y_i$ where $p_i \in R_0$. Put an other way, as an R_0 module we have $R_1 = R_0 \oplus R_0 Y_1 \oplus \cdots \oplus R_0 Y_{10}$. We now proceed to look at $\mathcal{S}_1$ as an R_0 module. As an R_1 module it is generated by $S_1, \dots, S_{10}$ so as an R_0 module it is generated by $S_1, \dots, S_{10}$ and all products $S_i Y_j$. We put $S_0 = Y_1 S_2$, and using a computer algebra system we find that $Y_i S_j$ is contained in $\mathrm{span}_{R_0}\{S_0, \dots, S_{10}\}$ for all i, j, see [11]. Observe that $(\gamma_1^0, \gamma_2^0, \gamma_3^0) = \deg S_0 = \deg Y_1 + \deg S_2 = (4,4,8)$ which is the last exponent in the Hilbert-Molien series (13). So $\mathcal{S}_1$ is generated by $S_0, \dots, S_{10}$ as an R_0 module. Furthermore, we can write

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} = \mathbf{A}_1 \begin{bmatrix} 1 \\ Y_1 \\ Y_2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} S_3 \\ \vdots \\ S_{10} \end{bmatrix} = \mathbf{A}_2 \begin{bmatrix} Y_3 \\ \vdots \\ Y_{10} \end{bmatrix} \quad ,$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are matrices with elements in R_0. We find that $\det \mathbf{A}_2 = 2 \det \mathbf{A}_1 \neq 0$ in R_0, see [11]. So $\mathcal{S}_1$ is a free R_0 module: $\mathcal{S}_1 = R_0 S_0 \oplus \cdots \oplus R_0 S_{10}$. The Hilbert series for the polynomial ring R_0 is

$$H_0(z_1,z_2,z_3) = \left(\prod_{i=0}^{7}\bigl(1-z_1^{\alpha_1^i} z_2^{\alpha_2^i} z_3^{\alpha_3^i}\bigr)\right)^{-1} \quad .$$

The ring R_1 is a free R_0 module and has the Hilbert series

$$H_1(z_1,z_2,z_3) = H_0(z_1,z_2,z_3)\left(1+\sum_{i=1}^{10} z_1^{\beta_1^i} z_2^{\beta_2^i} z_3^{\beta_3^i}\right) \quad .$$

The ideal $\mathcal{S}_1$ is a free R_0 module too and has the Hilbert series

$$H_2(z_1,z_2,z_3) = H_0(z_1,z_2,z_3)\left(\sum_{i=0}^{10} z_1^{\gamma_1^i} z_2^{\gamma_2^i} z_3^{\gamma_3^i}\right) \quad .$$

So all in all we have that the Hilbert series for $R_1/\mathcal{S}_1$ is

$$\begin{aligned} H(z_1,z_2,z_3) &= H_1(z_1,z_2,z_3) - H_2(z_1,z_2,z_3) \\ &= \left(\prod_{i=0}^{7}\left(1 - z_1^{\alpha_1^i} z_2^{\alpha_2^i} z_3^{\alpha_3^i}\right)\right)^{-1} \left(1+\sum_{i=1}^{10} z_1^{\beta_1^i} z_2^{\beta_2^i} z_3^{\beta_3^i} - \sum_{i=0}^{10} z_1^{\gamma_1^i} z_2^{\gamma_2^i} z_3^{\gamma_3^i}\right) \quad . \end{aligned}$$

This is exactly the same as the Hilbert-Molien series (13) for the ring of SL_2 invariants. As a consequence we have that the kernel of ϕ is $\mathcal{S}$. This is our main result and we formulate it as the following theorem.

Theorem 8. *The map ϕ given by $X_i \mapsto I_i$ and $Y_i \mapsto J_i$ induces an isomorphism*

$$\mathbb{C}[\mathbf{X},\mathbf{Y}] \;/\; \mathcal{S} \cong \mathbb{C}[\mathbf{a},\mathbf{b},\mathbf{c}]^{SL_2} \quad .$$

I.e., any SL_2-invariant can be written as a polynomial in $I_0,\ldots,I_7$, $J_1,\ldots,J_{10}$ and any syzygy can be written as a $\mathbb{C}[\mathbf{X},\mathbf{Y}]$-linear combination of S_i and S_{ij}, $i,j = 1,\ldots 10$. The map also induces an isomorphism

$$\left(\mathbb{C}[\mathbf{X}] \oplus \mathbb{C}[\mathbf{X}]Y_1 \oplus \ldots \mathbb{C}[\mathbf{X}]Y_{10}\right) \;/\; \mathcal{S}_1 \cong \mathbb{C}[\mathbf{a},\mathbf{b},\mathbf{c}]^{SL_2} \quad .$$

I.e., any SL_2-invariant I can be written as

$$I = p_0(I_0,\ldots,I_7) + p_1(I_0,\ldots,I_7)J_1 + \cdots + p_{10}(I_0,\ldots,I_7)J_{10} \quad ,$$

and any syzygy among these is a $\mathbb{C}[\mathbf{X}]$-linear combination of $S_0,\ldots,S_{10}$.

We really want to study invariants of binary forms over the reals, but this makes no difference. A polynomial is invariant under the action of $SL_2(\mathbb{R})$ if and only if it vanishes under the induced action of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. The one-to-one correspondence between homogeneous polynomials and symmetric tensors means that if $\mathcal{I}_{\mathbf{d}}$ denotes the space of homogeneous $SL_2(\mathbb{R})$-invariant polynomials of degree $\mathbf{d}$ (and order 0) then we have

$$\mathcal{I}_{\mathbf{d}} \subseteq S^{\mathbf{d}}\big(S^{2,2,3}(\mathbb{R}^2)\big) = S^{\mathbf{d}}\big(S^2(\mathbb{R}^2)\times S^2(\mathbb{R}^2)\times S^3(\mathbb{R}^2)\big) \subseteq \mathbb{R}[\mathbf{a},\mathbf{b},\mathbf{c}] \quad ,$$

where $S^{\mathbf{d}}$ denotes the symmetric product. The invariant ring is $\mathbb{R}[\mathbf{a},\mathbf{b},\mathbf{c}]^{SL_2} = \bigoplus_{\mathbf{d}} \mathcal{I}_{\mathbf{d}}$. If we complexify we get

$$\mathcal{I}_{\mathbf{d}} \otimes_{\mathbb{R}} \mathbb{C} \subseteq S^{\mathbf{d}}\big(S^{2,2,3}(\mathbb{R}^2)\big) \otimes_{\mathbb{R}} \mathbb{C} = S^{\mathbf{d}}\big(S^{2,2,3}(\mathbb{C}^2)\big) \subseteq \mathbb{C}[\mathbf{a}] \quad ,$$

and as $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ we see that the complex polynomials of multi-degree $\mathbf{d}$ that vanishes under the action of $\mathfrak{sl}_2(\mathbb{C})$ is $\mathcal{I}_{\mathbf{d}} \otimes_{\mathbb{R}} \mathbb{C}$. In other words there is a one to one correspondence between $SL_2(\mathbb{R})$-invariants and $SL_2(\mathbb{C})$-invariants.

Furthermore, we are in fact interested in GL_2-invariants, i.e., invariants with weight 0. None of the polynomial invariants has weight 0, but as $I_0 > 0$ we can obtain absolute GL_2-invariants by dividing with a suitable power of I_0. If we let ρ_i and δ_i be the weight of I_i and J_i respectively, and put

$$\widehat{I}_i = \frac{I_i}{I_0^{\rho_i/2}} \qquad \text{and} \qquad \widehat{J}_i = \frac{I_i}{I_0^{\delta_i/2}} \ , \tag{14}$$

then $\widehat{I}_1, \dots, \widehat{I}_7$, and $\widehat{J}_1, \widehat{J}_2$ are rational $GL_2(\mathbb{R})$-invariants, but $\widehat{J}_3, \dots, \widehat{J}_{10}$ are only invariant under the action of $GL_2^+(\mathbb{R})$. They changes sign if the linear transformation has a negative determinant, i.e., if the orientation is reversed. This gives the following corollary.

Corollary 9. *In Theorem 8 the field* $\mathbb{C}$ *can be replaced by* $\mathbb{R}$. *Furthermore, any rational* $GL_2^+(\mathbb{R})$*-invariant can be written as*

$$\frac{p_0 + p_1\widehat{J}_1 + \cdots + p_{10}\widehat{J}_{10}}{q_0 + q_1\widehat{J}_1 + \cdots + q_{10}\widehat{J}_{10}} \ ,$$

and any rational $GL_2(\mathbb{R})$*-invariant can be written as*

$$\frac{p_0 + p_1\widehat{J}_1 + p_2\widehat{J}_2}{q_0 + q_1\widehat{J}_1 + q_2\widehat{J}_2} + \frac{p_3\widehat{J}_3 + \cdots + p_{10}\widehat{J}_{10}}{q_3\widehat{J}_3 + \cdots + q_{10}\widehat{J}_{10}} \ ,$$

where p_i *and* q_i *are polynomials in* $\widehat{I}_1, \dots, \widehat{I}_7$.

6 Implicit Surfaces

We now consider an implicitly defined surface $M = h^{-1}(0)$, where $h : \mathbb{R}^3 \to \mathbb{R}$. We first assume that $|\nabla h| = 1$ in some neighbourhood of M, so $h(x)$ is the *signed distance* from M to x for x in that neighbourhood. If $\mathrm{I\!I}$ and $\nabla\mathrm{I\!I}$ denotes the second fundamental form and its covariant derivative respectively, then $\mathrm{d}^2 h = -\mathrm{I\!I} \circ \pi$ and $\mathrm{d}^3 h = -\nabla\mathrm{I\!I} \circ \pi$ where $\pi : \mathbb{R}^3 \to T_pM$ is the orthogonal projection onto the tangent space. If we assume that ∇h is the third basis vector for $\mathbb{R}^3$, then $a_{ij} = \delta_{ij}$, $b_{ij} = h_{ij}$, and $c_{ijk} = h_{ijk}$ for $i, j, k = 1, 2$. Furthermore

$$\varepsilon^{ij} = \varepsilon^{ijk} h_k \quad \text{where} \quad \varepsilon^{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

So to express the invariants in Table 1 in terms of the signed distance function we simply make the above substitutions. At first we have to sum from 1 to 2 only, but as

$\varepsilon^{3jk}h_k = \varepsilon^{i3k}h_k = 0$ we may sum from 1 to 3. The expression is now invariant so it holds for an arbitrary direction of ∇h.

We now consider an arbitrary C^3 function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $M = f^{-1}(0)$ and $\lambda = |\nabla f| \neq 0$ in a neighbourhood of M. By differentiating the equation $\mathrm{d}h(\nabla h) = |\nabla h|^2 = 1$ we see that ∇h is a null vector for the higher order derivatives of h, and using this fact we obtain

$$h_i = \frac{1}{\lambda} f_i \ , \tag{16}$$

$$h_{ij} = \frac{1}{\lambda} f_{ij} - \frac{1}{\lambda^3}\left(f_{ik}f^k f_j + f_{jk}f^k f_i\right) + \frac{1}{\lambda^5}\left(f_{kl}f^k f^l\right) f_i f_j \ , \tag{17}$$

$$h_{ijk} = \frac{1}{\lambda} f_{ijk} - \frac{1}{\lambda^3}\left(f_{il}f^l f_{jk} + f_{kl}f^l f_{ij} + f_{jl}f^l f_{ki}\right) + \text{terms with } f_i, f_j, f_k \ , \tag{18}$$

where $f^l = \delta^{kl} f_k = f_l$. As $\varepsilon^{ijk} f_i f_k = \varepsilon^{ijk} f_j f_k = 0$ we can discard any term in (17) and (18) that contains f_i, f_j, or f_k as a factor. Summing up we have the following result:

Theorem 10. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be C^3 functions such that $\lambda = |\nabla f| \neq 0$ in a neighbourhood of the implicitly defined surface $M = f^{-1}(0)$. The invariants in Table 1 can be found by making the substitutions*

$$\varepsilon^{ij} \mapsto \frac{1}{\lambda}\varepsilon^{ijk} f_k \ , \qquad a_{ij} \mapsto \delta_{ij} \ ,$$
$$b_{ij} \mapsto \frac{1}{\lambda} f_{ij} \ , \qquad c_{ijk} \mapsto \frac{1}{\lambda} f_{ijk} - \frac{1}{\lambda^3}\left(f_{il}f^l f_{jk} + f_{kl}f^l f_{ij} + f_{jl}f^l f_{ki}\right) \ .$$

7 Applications

Coordinates on a surface where – at some point – the first fundamental form to first order is δ_{ij} and the second fundamental form to first order is diagonal are called *principal coordinates* at that point. E.g. if we in (1) have that the basis $\mathbf{r}_1, \mathbf{r}_2$ for the tangent plane is orthonormal and in the principal directions, then we have principal coordinates at $\mathbf{x}_0$. Now $a_{11} = a_{22} = 1$, $a_{12} = b_{12} = 0$ so $I_0 = 1$, $\widehat{I_i} = I_i$, $\widehat{J_i} = J_i$. If we put $b_{11} = L$, $b_{22} = N$, $c_{111} = P$, $c_{112} = Q$, $c_{122} = S$, and $c_{222} = T$ then we get the expressions in Table 2.

The equations $(*)$ is a system of linear equations in $P^2, PS, S^2, Q^2, QT, T^2$

$$\begin{bmatrix} 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & -N & -L & 0 & N & L \\ 1 & 3 & 3 & 1 & 0 & 0 \\ N^3 & 3LN^2 & 3L^2N & L^3 & 0 & 0 \\ N & L+2N & 2L+N & L & 0 & 0 \\ N^2 & 2LN+N^2 & L^2+2LN & L^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} P^2 \\ Q^2 \\ S^2 \\ T^2 \\ PS \\ QT \end{bmatrix} = \begin{bmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \\ J_1 \\ J_2 \end{bmatrix} \ , \tag{19}$$

and the equations $(**)$ is a system of linear equations in PQ, PT, QS, ST

$$(L-N)\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ N & L+N & L & 0 \\ N^2 & 2LN & L^2 & 0 \end{bmatrix} \begin{bmatrix} PQ \\ QS \\ ST \\ PT \end{bmatrix} = \begin{bmatrix} J_3 \\ J_4 \\ J_5 \\ J_6 \end{bmatrix} \ . \tag{20}$$

Table 2. The basic invariants in principal coordinates

$$
\begin{aligned}
& & I_1 &= L + N \ , \\
& & I_2 &= LN \ , \\
&(*) & I_3 &= PS - S^2 - Q^2 + QT \ , \\
&(*) & I_4 &= NPS - LS^2 - NQ^2 + LQT \ , \\
&(*) & I_5 &= P^2 + 3Q^2 + 3S^2 + T^2 \ , \\
&(*) & I_6 &= N^3P^2 + 3LN^2Q^2 + 3L^2NS^2 + L^3T^2 \ , \\
& & I_7 &= 3Q^2S^2 - P^2T^2 - 4PS^3 - 4Q^3T + 6PQST \ , \\
&(*) & J_1 &= NP^2 + (L+2N)Q^2 + (2L+N)S^2 + LT^2 \ , \\
&(*) & J_2 &= N^2P^2 + \left(2LN + N^2\right)Q^2 + \left(L^2 + 2LN\right)S^2 + L^2T^2 \ , \\
&(**) & J_3 &= (L-N)(PT - QS) \ , \\
&(**) & J_4 &= (L-N)(PQ + 2QS + ST) \ , \\
&(**) & J_5 &= (L-N)(NPQ + (L+N)QS + LST), \\
&(**) & J_6 &= (L-N)\left(N^2PQ + 2LNQS + L^2ST\right) \ , \\
& & J_7 &= 2\left(PQ^3 - S^3T\right) + \left(P^2 + 3Q^2 - 3S^2 - T^2\right)PT + 3\left((Q+T)^2 - (P+S)^2\right)QS \ , \\
& & J_8 &= 2\left(NPQ^3 - LS^3T\right) + \left(NP^2 + (L+2N)Q^2 - (2L+N)S^2 - LT^2\right)PT \\
& & &\quad + \left((L+2N)Q^2 + 2(2L+N)QT + 3LT^2 - NP^2 - 2(L+2N)PS - (2L+N)S^2\right)QS \ , \\
& & J_9 &= 2\left(N^2Q^2 - N(2L+N)S^2\right)PQ + \left(N\left(NP^2 + (2L+N)Q^2\right) - L\left((2N+L)S^2 + LT^2\right)\right)PT \\
& & &\quad - \left(3N^2P^2 - N(2L+N)Q^2 + L(L+2N)S^2 - 3L^2T^2\right)QS + \left(L(2N+L)Q^2 - L^2S^2\right)ST \ , \\
& & J_{10} &= 2N^2\left(NQ^2 - 3LS^2\right)PQ + \left(N^3P^2 + 3LN^2Q^2 - 3L^2NS^2 - L^3T^2\right)PT \\
& & &\quad - 3\left(N^3P^2 - LN^2Q^2 + L^2NS^2 - L^3T^2\right)QS + 2L^2\left(3NQ^2 - LS^2\right)ST \ .
\end{aligned}
$$

The determinant of the matrix is in both cases $(L-N)^7$, so if $L \neq N$, i.e, at a non-umbilical point, we can solve the equations, and the solutions are

$$\begin{bmatrix} PQ \\ QS \\ ST \\ PT \end{bmatrix} = \frac{1}{(L-N)^3} \begin{bmatrix} 0 & L^2 & -2L & 1 \\ 0 & -LN & L+N & -1 \\ 0 & N^2 & -2N & 1 \\ (L-N)^2 & -LN & L+N & -1 \end{bmatrix} \begin{bmatrix} J_3 \\ J_4 \\ J_5 \\ J_6 \end{bmatrix} \quad . \tag{21}$$

and

$$\begin{aligned} P^2 &= \frac{L^3 I_5 - I_6 - 3L^2 J_1 + 3LJ_2}{(L-N)^3} \quad , \\ Q^2 &= \frac{-L^2 N I_5 + I_6 + (L+2N)LJ_1 - (2L+N)J_2}{(L-N)^3} \quad , \\ S^2 &= \frac{LN^2 I_5 - I_6 - (2L+N)NJ_1 + (L+2N)J_2}{(L-N)^3} \quad , \\ T^2 &= \frac{-N^3 I_5 + I_6 + 3N^2 J_1 - 3NJ_2}{(L-N)^3} \quad , \\ PS &= Q^2 + \frac{LI_3 - I_4}{L-N} \quad , \\ QT &= S^2 - \frac{NI_3 - I_4}{L-N} \quad . \end{aligned} \tag{22}$$

Expressions similar to (22) were also found in [12], but there a different set of invariants was used namely:

$$\begin{aligned} \Lambda_1 &= \frac{I_5}{I_0^3} \quad , & \Lambda_2 &= \frac{2I_0 I_3 + I_5}{I_0^3} \quad , \\ \Lambda_3 &= \frac{I_1 I_5 - I_0 J_1}{I_0^4} \quad , & \Lambda_4 &= \frac{2I_0 I_1 I_3 + I_1 I_5 - 2I_0^2 I_4 - I_0 J_1}{I_0^4} \quad , \\ \Lambda_5 &= \frac{I_1^2 I_5 - 2I_0 I_1 J_1 + I_0^2 J_2}{I_0^5} \quad , & \Lambda_6 &= \frac{I_1^3 I_5 - I_0^3 I_6 - 3I_1^2 J_1 + 3I_0^2 I_1 J_2}{I_0^6} \quad . \end{aligned}$$

In principal coordinates the *principal curvatures* are simply $\kappa_1 = L$ and $\kappa_2 = N$, and the *principal directions* $\mathbf{e}_1$ and $\mathbf{e}_2$ are the coordinates directions. Furthermore, at a non umbilical point – where $\kappa_1 \neq \kappa_2$ – the directional derivatives of the principal curvatures are given by $\partial_{\mathbf{e}_1}\kappa_1 = P$, $\partial_{\mathbf{e}_2}\kappa_1 = Q$, $\partial_{\mathbf{e}_1}\kappa_2 = S$, and $\partial_{\mathbf{e}_2}\kappa_2 = T$, see [12].

We will now give a couple of examples to demonstrate how this can be used.

7.1 Fairing

Over the years there have been many suggestions of functions which should estimate the 'fairness' of a surface, see [13] for an extensive treatment. As a simple example we can take $|\nabla H|^2$, where $H = \frac{1}{2} I_1 / I_0$ is the mean curvature, just as $K = I_2 / I_0$ is the Gauss curvature. In principal coordinates we have

$$H(x^1, x^2) = \frac{1}{2}\left((L + x^1 P + x^2 Q) + (N + x^1 S + x^2 T) + \text{higher order terms}\right) \quad ,$$

so at $(0,0)$ we have $\nabla H = \frac{1}{2}(P+S, Q+T)$ and hence

$$|\nabla H|^2 = \frac{P^2+2PS+S^2+Q^2+2QT+T^2}{4}$$

If we now substitute (22) into the expression then we get

$$\begin{aligned}
|\nabla H|^2 &= \frac{P^2+3Q^2+3S^2+T^2}{4} + \frac{LI_3 - I_4}{2(L-N)} - \frac{NI_3 - I_4}{2(L-N)} \\
&= \frac{L^3 I_5 - I_6 - 3L^2 J_1 + 3LJ_2}{4(L-N)^3} \\
&\quad + 3\frac{-L^2 N I_5 + I_6 + (L+2N)LJ_1 - (2L+N)J_2}{4(L-N)^3} \\
&\quad + 3\frac{LN^2 I_5 - I_6 - (2L+N)NJ_1 + (L+2N)J_2}{4(L-N)^3} \\
&\quad + \frac{-N^3 I_5 + I_6 + 3N^2 J_1 - 3NJ_2}{4(L-N)^3} + \frac{I_3}{2} \\
&= \frac{(L^3 - 3L^2 N + 3LN^2 - N^3)I_5}{4(L-N)^3} + \frac{(-1+3-3+1)I_6}{4(L-N)^3} \\
&\quad + \frac{(-3L^2 + 3(L^2+2LN) - 3(2LN+N^2) + 3N^2)J_1}{4(L-N)^3} \\
&\quad + \frac{3L - (2L+N) + (L+2N) - 3N)J_2}{4(L-N)^3} + \frac{I_3}{2} \\
&= \frac{I_5}{4} + \frac{I_3}{2} = \frac{\widehat{I}_5}{4} + \frac{\widehat{I}_3}{2} = \frac{2I_0 I_3 + I_5}{4I_0^3} \quad .
\end{aligned}$$

As $I_5 = P^2 + 3Q^2 + 3S^2 + T^2$ in principal coordinates we could have done the calculation faster. In any case, we have performed the calculation using special coordinates, but as both sides of the equality are invariant the equality holds in any parameterization. In a similar manner – see [12] – it can be shown that

$$\begin{aligned}
|\nabla K|^2 &= \frac{2I_0 I_2 I_3 + I_0 J_2}{I_0^4} \quad , \\
\big(\nabla(|\kappa_1| + |\kappa_2|)\big)^2 &= \begin{cases} \frac{2I_0 I_3 + I_5}{I_0^3} & \text{if } I_2 > 0 \quad , \\ \frac{2I_0(4I_0 I_2 - I_1^2)I_3 + I_1^2 I_5 - 4I_0 I_1 J_1 + 4I_0^2 J_2}{I_0^3(I_1^2 - 4I_0 I_2)} & \text{if } I_2 < 0 \quad , \end{cases} \\
\big(\nabla\big(\kappa_1^2 + \kappa_2^2\big)\big)^2 &= 4\frac{2I_0^2 I_2 I_3 + I_1^2 I_5 - 2I_0 I_1 J_1 + I_0^2 J_2}{I_0^5} \quad , \\
\big(\partial_{\mathbf{e}_1}\kappa_1\big)^2 + \big(\partial_{\mathbf{e}_2}\kappa_2\big)^2 &= \frac{(I_1^2 - I_0 I_2)I_5 - 3I_0 I_1 J_1 + 3I_0^2 J_2}{I_0^3(I_1^2 - 4I_2)} \quad , \\
\frac{1}{\pi}\int_0^\pi \left(\frac{\mathrm{d}\kappa_n}{\mathrm{d}s}\right)^2 \mathrm{d}\phi &= \frac{6I_0 I_3 + 5I_5}{16 I_0^3} \quad .
\end{aligned}$$

7.2 Ridges and the Subparabolic Curve

A surface has two *focal surfaces* or *evolutes* given as the locus of the two principal centre of curvature. The focal surfaces will in general have cuspidal edges called *ribs* lying over curves, called *ridges*, in the original surface, see [14]. The parabolic curve - where the Gaussian curvature is zero – in the focal surfaces lies over a curve, called the *subparabolic curve*, in the original surface, see [14].

If we in the tangent plane use rectangular coordinates (x, y) such that the axes are in the principal directions, then we can write (2) as

$$z = \frac{1}{2}\left(Lx^2 + Ny^2\right) + \frac{1}{6}\left(Px^3 + 3Qx^2y + 3Sxy^2 + Ty^3\right) + \text{higher order terms} \ .$$

The unit normal is to first order $\mathbf{N} \approx (-Lx, -Ny, 1)$ and if $L \neq N$ then the principal curvatures are to first order $\kappa_1 \approx L + Px + Qy$ and $\kappa_2 \approx N + Sx + Ty$. The two sheets of the focal surface are to first order given by

$$\left(0, \frac{L-N}{L}y, \frac{1}{L} - \frac{P}{L^2}x - \frac{Q}{L^2}y\right) \quad \text{and} \quad \left(\frac{N-L}{N}x, 0\frac{1}{N} - \frac{S}{N^2}x - \frac{T}{N^2}y\right) \ .$$

The cross products of the partial derivatives are

$$\left(\frac{L-N}{L^3}P, 0, 0\right) \quad \text{and} \quad \left(0, \frac{N-L}{N^3}T, 0\right)$$

respectively. At a non umbilical point (21) shows that

$$\begin{aligned}(L-N)^3PT &= (L-N)^2J_3 - LNJ_4 + (L+N)J_5 - J_6 \\ &= \left((L+N)^2 - 4LN\right)J_3 - LNJ_4 + (L+N)J_5 - J_6 \\ &= \left(\widehat{I}_1{}^2 - 4\widehat{I}_2\right)\widehat{J}_3 - \widehat{I}_2\widehat{J}_4 + \widehat{I}_1\widehat{J}_5 - \widehat{J}_6 \ .\end{aligned}$$

We have assumed that $L \neq N$, but if $L = N$ then $J_3 = J_4 = J_5 = J_6 = 0$ so the equation holds in this case too. This give us the required invariant description of the ridges:

Theorem 11. *The ridges of a surface is the zero set of the invariant function*

$$\frac{\left(I_1^2 - 4I_0I_2\right)J_3 - I_2J_4 + I_1J_5 - I_0J_6}{I_0^{9/2}} \ .$$

We saw above that the ridges at non umbilical points are given by $\partial_{\mathbf{e}_1}\kappa_1 = 0$ or $\partial_{\mathbf{e}_2}\kappa_2 = 0$. Similar the subparabolic lines are given by $\partial_{\mathbf{e}_2}\kappa_1 = 0$ or $\partial_{\mathbf{e}_1}\kappa_2 = 0$, see [14]. In principal coordinates we get the equation $QS = 0$, and at a non umbilical point (21) shows that

$$(L-N)^3QS = -LNJ_4 + (L+N)J_5 - J_6 = -\widehat{I}_2\widehat{J}_4 + \widehat{I}_1\widehat{J}_5 - \widehat{J}_6 \ .$$

Just as before this give us the invariant description of the subparabolic curve:

Theorem 12. *The subparabolic curve of a surface is the zero set of the invariant function*

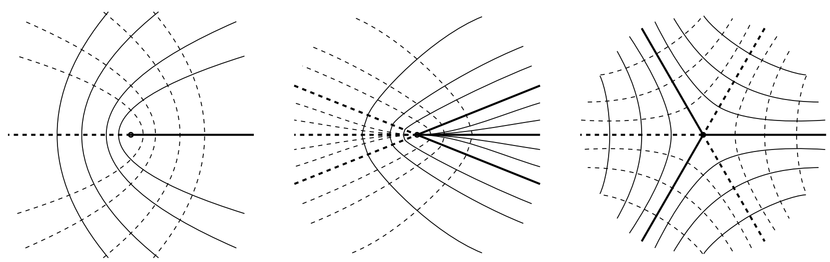
$$\frac{-I_2J_4 + I_1J_5 - I_0J_6}{I_0^{9/2}} \ .$$

7.3 Darboux's Classification of Umbilical Points

At an umbilical point the first and second fundamental form are proportional and then 13 of the 18 basic invariants can be expressed as linear combination of the others. We are left with only 5 invariants I_0, I_3, I_5, I_7, and J_7, and a single syzygy $J_7^2 = Q(I_0, I_3, I_5, I_7)$. We have essentially the joint invariants of one quadratic and one cubic binary form.

The Darboux's classification depends on the pattern of the lines of curvatures around the umbilical point, which in turn depends on whether there are one or three real root lines of the cubic form $c = \nabla\mathrm{I\!I}$, and in the latter case whether the three root lines are contained in a right angle or not, see [14]. A root line of a cubic form is a direction where it vanishes, and there is at least one root line which we can assume it is the x-axis. I.e., we may assume that $P = 0$. Then $I_7 = 3Q^2S^2 - 4Q^3T$ and $c = (3Qx^2 + 3Sxy + Ty^2)y$. The quadratic factor has the discriminant $\frac{3}{4}\big(4QT - 3S^2\big) = -\frac{3}{4}I_7/Q^2$ so we have

$$I_7 < 0 \iff c \text{ has 3 distinct real root lines,}$$
$$I_7 > 0 \iff c \text{ has exactly 1 real root line.}$$



lemon $I_7 > 0$ | (le)monstar $I_7 < 0, 3I_0I_3 + I_5 > 0$ | star $I_7 < 0, 3I_0I_3 + I_5 < 0$

Fig. 1. Curvature lines around an isolated umbilical point

In the case $I_7 < 0$ we can assume that the three real root directions are $\mathbf{v}_1 = (\alpha_1, \beta_1)$, $\mathbf{v}_2 = (\alpha_2, \beta_2)$, and $\mathbf{v}_3 = (\alpha_3, \beta_3) = (1, 0)$. The cubic form is then

$$c = \prod_{i=1}^{3}(\beta_i x - \alpha_i y) = \big(-\beta_1\beta_2 x^2 + (\alpha_1\beta_2 + \beta_1\alpha_2)xy - \alpha_1\alpha_2 y^2\big)y \ .$$

So $3Q = -\beta_1\beta_2$ and $T = -\alpha_1\alpha_2$. Hence

$$(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{v}_3)(\mathbf{v}_2 \cdot \mathbf{v}_3) = (\alpha_1\alpha_2 + \beta_1\beta_2)\alpha_1\alpha_2 = T^2 + 3QT = \frac{3I_0I_3 + I_5}{I_0^3} \ .$$

We can now see that if $I_7 < 0$, then we have

$$3I_0I_3 + I_5 > 0 \iff \text{The root lines of } c \text{ are contained in a right angle,}$$
$$3I_0I_3 + I_5 < 0 \iff \text{The root lines of } c \text{ aren't contained in a right angle.}$$

This gives the classification in Fig. 1, where we have sketched the three generic patterns possible for the lines of curvature around an isolated umbilical point.

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