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Constructing invariant fairness measures for surfaces

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The paper proposes a rational method to derive fairness measures for surfaces. It works in cases where isophotes, reflection lines, planar intersection curves, or other curves are used to judge the fairness of the surface. The surface fairness measure is derived by demanding that all the given curves should be fair with respect to an appropriate curve fairness measure. The method is applied to the field of ship hull design where the curves are plane intersections. The method is extended to the case where one considers, not the fairness of one curve, but the fairness of a one parameter family of curves. Six basic third order invariants by which the fairing measures can be expressed are defined. Furthermore, the geometry of a plane intersection curve is studied, and the variation of the total, the normal, and the geodesic curvature and the geodesic torsion is determined.

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1. Introduction

An appealing approach to fairing is the variational, where a surface $\mathcal{M}$ is faired by minimizing a functional in form of an integral $\int_{\mathcal{M}} \Psi \, d\Omega$, where Ψ is a function which measures the local fairness of the surface. A crucial point is the definition of the fairness measure Ψ, and over the years there have been many suggestions. For a recent survey on modelling based on this principle, see [10]. Brunet et al. [1] who are particular interested in ship hull design have a long discussion of existing fairing algorithms. Nowacki et al. [16] have investigated and compared several fairing measures. The variety of suggestions reflects the fact that there is no universal accepted mathematical definition of fairness. One definition we think everybody can agree on is the following:

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A surface is fair if the designer is satisfied.

It implies that that no *single* universal definition of fairness exists. It might be that even though two naval architects can agree on what a fair surface is, they may both disagree with a car designer, that is:

The criteria for fairness of a surface are application-dependent.

E.g., if a designer uses isophotes to judge the fairness of a surface, then we should derive a fairness measure that captures the fairness of all isophotes. The last point we want to make is:

The fairness of a surface depends only on the geometry of the surface.

This is also expressed by saying that the fairness measure is *invariant*. I.e., the value of the fairness measure is the same no matter how the surface is parametrized. Even if practical and computational considerations make it necessary to replace the invariant measure with a more simple function, it is advantageously to know the 'correct' function, and then choose the simple function as an approximation, cf. [9].

The case of curve fairing is in a better shape. Historically thin, elastic wooden beams were used to design curves. The shape of these *physical splines* is determined by minimizing the elastic bending energy. This energy is proportional to $\int \kappa^2 \, \mathrm{d}s$, where κ is the curvature of the curve. Euler [5] solved this problem of variation, and the resulting curves are known as Euler elastica. In [12] Mehlum introduced nonlinear splines based on this idea. Later Moreton and Séquin [14] replaced the curvature κ with the curvature variation $\mathrm{d}\kappa/\mathrm{d}s$. Brunnett and Wendt [2] introduced a tension parameter σ and minimized $\int (\kappa^2 + \sigma) \, \mathrm{d}s$. The familiar cubic spline, which minimize $\int |\mathbf{x}''(t)|^2 \, \mathrm{d}t$, can be considered as an approximation to the Euler elastica. Splines in tension, introduced by Schweikert [20], can be considered as an approximation of elastic splines in tension. All prevailing fairing schemes for curves tries, in one way or the other, to obtain a nice curvature plot. There seems to be consensus that fairness measures for curves based on the curvature or curvature variation give good results. This is perhaps not so surprising, after all any second order invariant function on a plane curve is a function of the curvature, $\psi = \psi(\kappa)$, and any third order invariant function is a function of the curvature and curvature variation, $\psi = \psi(\kappa, \mathrm{d}\kappa/\mathrm{d}s)$.

If we consider space curves we get the *torsion* τ as a new third order invariant, so now any third order invariant function can be written as $\psi = \psi(\kappa, \mathrm{d}\kappa/\mathrm{d}s, \tau)$. Typically some combination of κ^2, $(\kappa')^2$, and τ^2 is used, see [6].

One of the reasons for the many different proposed fairness measures for surfaces is that the variety of invariant functions for surfaces is much larger than for curves. A second order invariant function on a surface is a symmetric function of the principal curvatures $\Psi = \Psi(\kappa_1, \kappa_2)$, or more manageable a function of the mean curvature H and the Gaussian curvature K, $\Psi = \Psi(H, K)$.

If we want a third order method, it is not even clear what the set of possible invariant functions looks like. At non-umbilical points, a third order invariant function is a suitable symmetric function of the principal curvatures κ_i, and their derivatives in the principal directions $\partial_{\mathbf{e}_j}\kappa_i$. But the principal directions are only defined up to a sign, so the same is true for $\partial_{\mathbf{e}_j}\kappa_i$. Thus we should only consider functions of the six quadratic terms $(\partial_{\mathbf{e}_k}\kappa_i)(\partial_{\mathbf{e}_k}\kappa_j)$. If we have an orientation then the last four quadratic terms $(\partial_{\mathbf{e}_1}\kappa_i)(\partial_{\mathbf{e}_2}\kappa_j)$ are well defined too. We will in this paper only use functions of the first six products. More seriously, at an umbilical point, where $\kappa_1 = \kappa_2$, any direction is principal, and the whole approach breaks down. So we need to find third order equivalents of H and K. In this paper we find 6 third order invariant functions $\Lambda_1, \dots, \Lambda_6$ which are well defined at each point of the surface, and at a non-umbilical point the products $(\partial_{\mathbf{e}_k}\kappa_i)(\partial_{\mathbf{e}_k}\kappa_j)$ can be expressed in terms of $\kappa_1, \kappa_2, \Lambda_1, \dots, \Lambda_6$, see theorem 7.

Furthermore, like H and K, the functions $\Lambda_1, \dots, \Lambda_6$ are rational functions in the components of the first and second fundamental forms and the covariant derivative of the latter, so, in contrast to $\partial_{\mathbf{e}_j}\kappa_i$, they are easy to find.

A word of warning, the eight functions H, K, $\Lambda_1, \dots, \Lambda_6$ are not a complete system of invariants in the sense of classical invariant theory, see [7]. It can be shown that such a complete system consists of 18 basic invariant polynomials and the ideal of relations (the syzygies) has 65 generators. If invariance under orientation reversing is requested then there are 10 basic invariants and 6 basic syzygies. The proof of these facts would take us too far a field and will appear elsewhere. In this paper the abovementioned six invariants will suffice.

As soon as we have a complete system of invariants, we can rewrite invariant expressions like $(\partial_{\mathbf{e}_1}\kappa_1)^2 + (\partial_{\mathbf{e}_2}\kappa_2)^2$ in terms of the complete system. This gives an analytical expressions for the invariant that can be used in any parametrization of the surface. It also enables us to perform calculations on the surface using a particular parametrization, cf. definition 3, and later express the results in an arbitrary parametrization, see the proof of (18) for an example of this technique.

A designer often judges the fairness of a surface by inspection of isophotes, reflection lines, etc. In these cases we will say that a surface is fair if all of the curves are fair. So now assume we have a collection $\mathcal{C}$ of curves on the surface M and a good curve fairness measure ψ. We then use the functional

$$\int_{c\in\mathcal{C}} \int_c \psi \,\mathrm{d}s \,\mathrm{d}I = \int_{p\in\mathcal{M}} \int_{\{c\in\mathcal{C}|p\in c\}} \psi(c, p) \,\mathrm{d}I_p \,\mathrm{d}\Omega = \int_{\mathcal{M}} \Psi \,\mathrm{d}\Omega, \tag{1}$$

as a measure for the fairness of the surface. The first equality is obtained by changing the order of integration; but it should not be taken too literately – that require a precise definition of both $\mathrm{d}I$ and $\mathrm{d}I_p$, and this can only be done case by case, see below. The importance of the equation is that we can read off our surface fairness measure:

$$\Psi(p) = \int_{\{c\in\mathcal{C}|p\in c\}} \psi(c, p) \,\mathrm{d}I_p. \tag{2}$$

Here we do need a precise definition of $\mathrm{d}I_p$. An isophote, say, through a point is determined by the direction to the light source so in order to get all possible isophotes we need to consider all possible directions, i.e., we have sphere worth of isophotes and in this case a natural choice of $\mathrm{d}I_p$ would be the usual area measure on S^2. The case of planar intersections is similar. Now that we know $\mathrm{d}I_p$ we could take (1) as a defining equation for $\mathrm{d}I$.

If the value of the function (2) is small at all points, then all the curves we are considering are fair with respect to the chosen curve fairness measure, or at least very few of them are unfair, and this is exactly the goal.

As an example of this general principle we will in this paper derive surface fairness measures suitable for the design of ship hulls. If one considers the hull of modern commercial vessel, then 90% of the hull is quite simple. The midship section is more or less a simple cylinder; but the bulbous bow and the stern with the propeller bossing has a very complex geometry and the transition from the simple cylindrical shape to the complex double curved shapes at the bow and stern gives the naval architect many problems. Traditionally the fairness of a ship hull is judged by looking at plane intersections. So we let the class of curves be all plane intersections and as the curve fairness measures we use either the square of the curvature or the square of the curvature variation. Using the square of the curvature leads to a second order measure and the square of the curvature variation leads to a third order measure, see definition 13. They turn out to be very similar to the ones defined by Mehlum and Tarrou in [13]. In fact, the results of [13] can be considered as an other application of our main principle. The class of curves just has to be changed to the set of geodesics. In the same spirit, the fairness functional considered by Moreton and Séquin in [14] is obtained by letting the class of curves consist of all lines of curvature.

A somewhat different approach is the one by Rando and Roulier. In [18] they secure invariance by considering the area of some derived surfaces. In the end they too obtain third order functions on the surface. During the analysis they claim that it is possible to "refine" a parametrization by lines of curvature such that the parameter becomes arc-length on the lines of curvature. This is obviously wrong, it would imply that the surface is isometric to the plane and thus is developable. The result of the analysis is still correct though. The "impossible parametrization" is only needed to second order at a given point, and this can be achieved.

Often a naval architect not only looks at a single plane intersection, but intersects the ship hull with a family of parallel planes, see figure 2. More precisely, the intersection curves are projected onto one of the parallel planes, and the designer judges the fairness of the hull by looking at this family of plane curves. Such a family of curves can be generated by the flow of a vector field $\mathbf{v}$ orthogonal to the intersection curves. If $|\mathbf{v}|$ is constant, then the curves are parallel and this we will consider fair. Consequently, we use $(\mathrm{d}|\mathbf{v}|/\mathrm{d}s)^2$ and $(\mathrm{d}^2|\mathbf{v}|/\mathrm{d}s^2)^2$ as fairness measures on the intersection curves, see definition 14.

At this point one might object that a naval architect normally looks at axis-parallel intersections so it is a mistake to integrate over all possible intersections. But from time to time other intersections are considered and if asked a naval architect wants all plane intersections to be fair.

The scheme proposed by Brunet et al. [1] is somewhat in the same spirit as ours. Their main idea is to consider a finite number of planar intersections and apply curve fairing to those curves taking the relative position of neighbouring curves into account. In the next step they change the surface, using a least square fit, such that planar intersections in the surface approximate the previous faired curves. One might say that they directly use the left hand side of (1) as the definition of surface fairness, except that they don't work directly on the surface; but via the intersection curves.

In section 2 we define the 6 third order invariants, and in section 3 we study the geometry of a plane intersection. These two sections are rather technical and may be skipped at a first reading. We assume knowledge of tensor notation and the classical differential geometry of curves and surfaces, and we will just write enough to fix the notation. Otherwise we refer the reader to the vast literature on the subject, e.g., [3,8,11,21]. In the book [17] by Porteous there is thorough discussion of various third order structures like ridges and the classification of umbilical points. The machinery from differential geometry makes it relatively simple to determine the abovementioned fairness measures, which is done in section 4. A preliminary test of the fairness measures is presented in section 5.

2. 3rd order invariants

Consider a surface $\mathcal{M}$ with a regular parametrization $\mathbf{x}(u, v)$. It will be useful also to use the notation $u^1 = u$ and $u^2 = v$. The metric on the surface is a symmetric covariant 2-tensor with coefficients equal to the components of the first fundamental form:

$$g_{11} = E = \mathbf{x}_u \cdot \mathbf{x}_u, \qquad g_{12} = g_{21} = F = \mathbf{x}_u \cdot \mathbf{x}_v, \qquad g_{22} = G = \mathbf{x}_v \cdot \mathbf{x}_v, \tag{3}$$

i.e., $g_{ij} = \mathbf{x}_{u^i} \cdot \mathbf{x}_{u^j}$. We will use the Einstein summation convention, i.e., if an index appears once as a subscript and once as superscript, then it is tacitly understood that we sum over it. So if $\mathbf{v}$ and $\mathbf{w}$ are tangent vectors with coefficients v^i, respectively w^j, then we can write $\mathbf{v} = v^i \mathbf{x}_{u^i} = \sum_{i=1}^{2} v^i \mathbf{x}_{u^i}$ and $\mathbf{v} \cdot \mathbf{w} = g_{ij} v^i w^j = \sum_{i,j=1}^{2} g_{ij} v^i w^j$. The reciprocal of the metric tensor is a contravariant 2-tensor with coefficients g^{ij}, which is defined by $g^{i\alpha} g_{\alpha j} = \delta^i_j$, where δ^i_j is the Kronecker delta symbol, which is 1 if $i = j$ and zero otherwise. It is clear that as matrices we have $[g^{ij}] = [g_{ij}]^{-1}$, and hence

$$g^{11} = \frac{G}{EG - F^2}, \qquad g^{12} = g^{21} = \frac{-F}{EG - F^2}, \qquad g^{22} = \frac{E}{EG - F^2}.$$

Let B denote the symmetric covariant 2-tensor with coefficients equal to the components of the second fundamental form:

$$\begin{aligned} b_{11} &= L = \mathbf{x}_{uu} \cdot \mathbf{N} = -\mathbf{x}_u \cdot \mathbf{N}_u, \\ b_{12} &= b_{21} = M = \mathbf{x}_{uv} \cdot \mathbf{N} = -\mathbf{x}_u \cdot \mathbf{N}_v = -\mathbf{x}_v \cdot \mathbf{N}_u, \\ b_{22} &= N = \mathbf{x}_{vv} \cdot \mathbf{N} = -\mathbf{x}_v \cdot \mathbf{N}_v, \end{aligned}$$

i.e., $b_{ij} = \mathbf{x}_{u^i u^j} \cdot \mathbf{N} = -\mathbf{x}_{u^i} \cdot \mathbf{N}_{u^j}$. The first and second fundamental form can now be written as $\mathrm{I}(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v} = g_{ij} v^i v^j$ and $\mathrm{II}(\mathbf{v}) = B(\mathbf{v}, \mathbf{v}) = b_{ij} v^i v^j$. The normal

$\mathbf{N} = (\mathbf{x}_u \times \mathbf{x}_v)/\sqrt{EG - F^2}$ can be considered as a map $M \to S^2$, the so-called Gauss map. The tangent planes T_pM and $T_{\mathbf{N}(p)}S^2$ are parallel planes in $\mathbb{R}^3$, so the differential of the Gauss map is a linear map of T_pM to itself. The bilinear form B can now be written as $B(\mathbf{v}, \mathbf{w}) = -\mathrm{d}\mathbf{N}(\mathbf{v}) \cdot \mathbf{w}$. The mean and the Gaussian curvature are in this notation given by $H = \frac{1}{2}\mathrm{tr}\,(-\mathrm{d}\mathbf{N})$ and $K = \det(-\mathrm{d}\mathbf{N})$. The linear map $-\mathrm{d}\mathbf{N}$ is a 2-tensor of type (1,1) and as such it has coefficients given by $b^i_j = g^{i\alpha}b_{\alpha j}$. We say that b^i_j is obtained from b_{ij} by *raising* an index. We can go one step further and raise the other index too:

$$b^{ij} = g^{j\beta}b^i_\beta = g^{i\alpha}g^{j\beta}b_{\alpha\beta}.$$

The first and second fundamental forms completely determine the surface, so any invariant must be a function of these two quantities alone. E.g., the two basic second order invariants H and K are rational expressions in the components of the first and second fundamental form. In order to get higher order invariants we need to differentiate, and as we want invariant expressions we have to use covariant differentiation. For this we need the Christoffel symbols of the first and second kind, which are defined by

$$\Gamma_{ijk} = \mathbf{x}_{u^iu^j} \cdot \mathbf{x}_{u^k}, \qquad \Gamma^k_{ij} = g^{k\alpha}\Gamma_{ij\alpha}.$$

The covariant derivative of the first fundamental form vanishes, so any third order invariant has to be constructed algebraically from the covariant differential of the second fundamental form, together with the first and second fundamental form.

Lemma 1. The covariant differential ∇B is a symmetric covariant 3-tensor with coefficients $b_{ij,k}$ given by:

$$b_{11,1} = P = L_u + 2\mathbf{x}_{uu} \cdot \mathbf{N}_u = \mathbf{x}_{uuu} \cdot \mathbf{N} + 3\mathbf{x}_{uu} \cdot \mathbf{N}_u, \tag{4a}$$

$$b_{11,2} = Q = L_v + 2\mathbf{x}_{uv} \cdot \mathbf{N}_u = \mathbf{x}_{uuv} \cdot \mathbf{N} + \mathbf{x}_{uu} \cdot \mathbf{N}_v + 2\mathbf{x}_{uv} \cdot \mathbf{N}_u \tag{4b}$$

$$= b_{12,1} = b_{21,1} = M_u + \mathbf{x}_{uu} \cdot \mathbf{N}_v + \mathbf{x}_{uv} \cdot \mathbf{N}_u, \tag{4c}$$

$$b_{22,1} = S = N_u + 2\mathbf{x}_{uv} \cdot \mathbf{N}_v = \mathbf{x}_{uvv} \cdot \mathbf{N} + \mathbf{x}_{vv} \cdot \mathbf{N}_u + 2\mathbf{x}_{uv} \cdot \mathbf{N}_v \tag{4d}$$

$$= b_{12,2} = b_{21,2} = M_v + \mathbf{x}_{vv} \cdot \mathbf{N}_u + \mathbf{x}_{uv} \cdot \mathbf{N}_v, \tag{4e}$$

$$b_{22,2} = T = N_v + 2\mathbf{x}_{vv} \cdot \mathbf{N}_v = \mathbf{x}_{vvv} \cdot \mathbf{N} + 3\mathbf{x}_{vv} \cdot \mathbf{N}_v. \tag{4f}$$

Proof. This is essentially the Codazzi equations and holds in large generality, see [4, chapter 6, remark 3.5]. In our simple case it is a straightforward calculation:

$$\begin{aligned} b_{ij,k} &= \partial_{u^k}b_{ij} - \Gamma^\alpha_{ik}b_{\alpha j} - \Gamma^\alpha_{jk}b_{i\alpha} = \partial_{u^k}b_{ij} - g^{\alpha\beta}\Gamma_{ik\beta}b_{\alpha j} - g^{\alpha\beta}\Gamma_{jk\beta}b_{i\alpha} \\ &= \partial_{u^k}(\mathbf{x}_{u^iu^j} \cdot \mathbf{N}) - \mathbf{x}_{u^iu^k} \cdot \mathbf{x}_{u^\beta}b^\beta_j - \mathbf{x}_{u^ju^k} \cdot \mathbf{x}_{u^\beta}b^\beta_i \\ &= \mathbf{x}_{u^iu^ju^k} \cdot \mathbf{N} + \mathbf{x}_{u^iu^j} \cdot \mathbf{N}_{u^k} + \mathbf{x}_{u^iu^k} \cdot \mathbf{N}_{u^j} + \mathbf{x}_{u^ju^k} \cdot \mathbf{N}_{u^i}. \end{aligned}$$

The last expression is clearly symmetric in i, j, k and (4) follows. □

Remark 2. The symbols P, Q, S, T were introduced in [13].

We get invariant functions by contracting the tensor product $(\nabla B)\otimes(\nabla B)$. As ∇B is symmetric there are only two different contractions:

$$\Lambda_1 = g^{i\alpha} g^{j\beta} g^{k\gamma} b_{ij,k} b_{\alpha\beta,\gamma}, \tag{5}$$

$$\Lambda_2 = g^{ij} g^{\alpha\beta} g^{k\gamma} b_{ij,k} b_{\alpha\beta,\gamma}. \tag{6}$$

These two invariants are actually the only third order invariants which we later will use, but in order to have a complete set of invariants we need more functions. We will get those by contracting the tensor products $B \otimes (\nabla B) \otimes (\nabla B)$, $B \otimes B \otimes (\nabla B) \otimes (\nabla B)$, and $B \otimes B \otimes B \otimes (\nabla B) \otimes (\nabla B)$. That corresponds to replacing a number of the g^{ij}'s by b^{ij}'s in (5) and (6):

$$\Lambda_3 = g^{i\alpha} g^{j\beta} b^{k\gamma} b_{ij,k} b_{\alpha\beta,\gamma} = g^{i\alpha} g^{j\beta} g^{kl} g^{\delta\gamma} b_{l\delta} b_{ij,k} b_{\alpha\beta,\gamma}, \tag{7}$$

$$\Lambda_4 = g^{ij} g^{\alpha\beta} b^{k\gamma} b_{ij,k} b_{\alpha\beta,\gamma} = g^{ij} g^{\alpha\beta} g^{kl} g^{\delta\gamma} b_{l\delta} b_{ij,k} b_{\alpha\beta,\gamma}, \tag{8}$$

$$\Lambda_5 = b^{i\alpha} b^{j\beta} g^{k\gamma} b_{ij,k} b_{\alpha\beta,\gamma} = g^{il} g^{\alpha\delta} g^{jm} g^{\beta\zeta} g^{k\gamma} b_{l\delta} b_{m\zeta} b_{ij,k} b_{\alpha\beta,\gamma}, \tag{9}$$

$$\Lambda_6 = b^{i\alpha} b^{j\beta} b^{k\gamma} b_{ij,k} b_{\alpha\beta,\gamma} = g^{il} g^{\alpha\delta} g^{jm} g^{\beta\zeta} g^{kn} g^{\gamma\eta} b_{l\delta} b_{m\zeta} b_{n\eta} b_{ij,k} b_{\alpha\beta,\gamma}. \tag{10}$$

The above Λ_i's can be calculated directly in any parametrization and the result is independent of the parametrization.

2.1. Principal coordinates

The expressions for curvature and curvature variation of a curve is very simple in the arc-length parametrization. If we try to find a similar parametrization of a surface, then we discover that it is impossible, unless the surface is developable. It can be done pointwise to second order and we now define special coordinates on a surface which makes the expressions for the invariants as simple as possible. The exponential map $\exp_P : T_P\mathcal{M} \to \mathcal{M}$ is defined by demanding that $\gamma(t) = \exp_p(t\mathbf{v})$ is a geodesic with $\gamma(0) = p$ and $\gamma'(0) = \mathbf{v}$, cf. [3, p. 284].

Definition 3. Let $\mathbf{x}(u, v) = \exp_p(u\mathbf{e}_1 + v\mathbf{e}_2)$, where $\mathbf{e}_1, \mathbf{e}_2$ is an orthonormal basis in the principal directions. Then (u, v) are called *principal coordinates* on $\mathcal{M}$ at p.

At an umbilical point any direction is principal, so here the only condition on $\mathbf{e}_1$ and $\mathbf{e}_2$ is the orthonormality. The convenience of these coordinates is due to the following results.

Lemma 4. If (u, v) are principal coordinates on a surface $\mathcal{M}$, then the first and second fundamental form can be expanded as

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{3}K(p) \begin{bmatrix} v^2 & -uv \\ -uv & u^2 \end{bmatrix} + \text{ higher order terms}, \tag{11}$$

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} + u \begin{bmatrix} P & Q \\ Q & S \end{bmatrix} + v \begin{bmatrix} Q & S \\ S & T \end{bmatrix} + \text{ higher order terms}, \tag{12}$$

where $K(p)$ is the Gaussian curvature at p, and κ_1, κ_2, P, Q, S, and T are evaluated at p.

Proof. If we put $(u, v) = (r\cos\theta, r\sin\theta)$, then we obtain geodesic polar coordinates on the surface, see [8, pp. 184–187]. Equation (11) now follows from the corresponding expansion of the first fundamental form in geodesic polar coordinates. We now know that the derivatives of the components of the first fundamental form vanishes at p. Then the Christoffel symbols vanish too, and the second order derivatives $\mathbf{x}_{u^i u^j}$ are orthogonal to the tangent plane. From the equations (4) we now have $P = L_u$, $Q = L_v = M_u$, $S = N_u = M_v$, and $T = N_v$, and as $\mathbf{e}_1$ and $\mathbf{e}_2$ are in the principal directions (12) follows. □

Lemma 5. In principal coordinates the third order invariants are, at the point of origo, given by

$$\Lambda_1 = P^2 + 3Q^2 + 3S^2 + T^2, \tag{13a}$$
$$\Lambda_2 = (P+S)^2 + (Q+T)^2, \tag{13b}$$
$$\Lambda_3 = \kappa_1\bigl(P^2 + 2Q^2 + S^2\bigr) + \kappa_2\bigl(Q^2 + 2S^2 + T^2\bigr), \tag{13c}$$
$$\Lambda_4 = \kappa_1(P+S)^2 + \kappa_2(Q+T)^2, \tag{13d}$$
$$\Lambda_5 = \kappa_1^2\bigl(P^2 + Q^2\bigr) + 2\kappa_1\kappa_2\bigl(Q^2 + S^2\bigr) + \kappa_2^2\bigl(S^2 + T^2\bigr), \tag{13e}$$
$$\Lambda_6 = \kappa_1^3 P^2 + 3\kappa_1^2\kappa_2 Q^2 + 3\kappa_1\kappa_2^2 S^2 + \kappa_2^3 T^2. \tag{13f}$$

Proof. We only have to notice that, at the point p, we have that $g^{ii} = 1$, $b^{ii} = \kappa_i$, and $g^{ij} = b^{ij} = 0$ if $i \neq j$, hence using (4) we obtain

$$\Lambda_1 = g^{i\alpha} g^{j\beta} g^{k\gamma} b_{ij,k} b_{\alpha\beta,\gamma} = \sum_{i,j,k=1}^{2} (b_{ij,k})^2 = P^2 + 3Q^2 + 3S^2 + T^2,$$

and similar for the other five invariants. □

Lemma 6. If we use principal coordinates at a non-umbilical point p and $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors in the principal directions, then

$$P = \partial_{\mathbf{e}_1}\kappa_1, \qquad Q = \partial_{\mathbf{e}_2}\kappa_1, \qquad S = \partial_{\mathbf{e}_1}\kappa_2, \qquad T = \partial_{\mathbf{e}_2}\kappa_2.$$

Proof. If we differentiate the equations

$$LN - M^2 = K\bigl(EG - F^2\bigr) \quad \text{and} \quad GL - 2FM + EN = 2H\bigl(EG - F^2\bigr),$$

with respect to u^i, and use (11) and (12), then we obtain the equations

$$\kappa_2 L_{u^i} + \kappa_1 N_{u^i} = K_{u^i} \quad \text{and} \quad L_{u^i} + N_{u^i} = 2H_{u^i}. \tag{14}$$

If the point p is a non-umbilical point, i.e., $\kappa_1 \neq \kappa_2$, then κ_1 and κ_2 are smooth functions in a neighbourhood of p. Furthermore, $K_{u^i} = \kappa_2 \kappa_{1u^i} + \kappa_1 \kappa_{2u^i}$, $2H_{u^i} = \kappa_{1u^i} + \kappa_{2u^i}$, and we see that the solution to (14) is $L_{u^i} = \kappa_{1u^i}$ and $N_{u^i} = \kappa_{2u^i}$. □

At an umbilical point were $\kappa_1 = \kappa_2$ only Λ_1 and Λ_2 are distinct, so we cannot have a complete system of invariants. But at a non-umbilical point the invariants H, K, and $\Lambda_1, \dots, \Lambda_6$ forms a complete system in the following sense:

Theorem 7. Let p be a non-umbilical point on a surface, then the invariants H, K, and $\Lambda_1, \dots, \Lambda_6$ uniquely determine the principal curvatures and the products $(\partial_{\mathbf{e}_k}\kappa_i)(\partial_{\mathbf{e}_k}\kappa_j)$ of their derivatives.

Proof. First we note that the mean curvature H and the Gaussian curvature K determine the principal curvatures κ_1 and κ_2. By lemma 6 equations (13a), (13c), (13e), and (13f) are a set of linear equations in $(\partial_{\mathbf{e}_j}\kappa_i)^2$. It is easily to be seen that the determinant of the system is $(\kappa_1 - \kappa_2)^6$ and that the solution is:

$$(\partial_{\mathbf{e}_1}\kappa_1)^2 = P^2 = \frac{-\kappa_2^3\Lambda_1 + 3\kappa_2^2\Lambda_3 - 3\kappa_2\Lambda_5 + \Lambda_6}{(\kappa_1 - \kappa_2)^3}, \tag{15a}$$

$$(\partial_{\mathbf{e}_2}\kappa_1)^2 = Q^2 = \frac{\kappa_1\kappa_2^2\Lambda_1 - (2\kappa_1\kappa_2 + \kappa_2^2)\Lambda_3 + (\kappa_1 + 2\kappa_2)\Lambda_5 - \Lambda_6}{(\kappa_1 - \kappa_2)^3}, \tag{15b}$$

$$(\partial_{\mathbf{e}_1}\kappa_2)^2 = S^2 = \frac{-\kappa_1^2\kappa_2\Lambda_1 + (\kappa_1^2 + 2\kappa_1\kappa_2)\Lambda_3 - (2\kappa_1 + \kappa_2)\Lambda_5 + \Lambda_6}{(\kappa_1 - \kappa_2)^3}, \tag{15c}$$

$$(\partial_{\mathbf{e}_2}\kappa_2)^2 = T^2 = \frac{\kappa_1^3\Lambda_1 - 3\kappa_1^2\Lambda_3 + 3\kappa_1\Lambda_5 - \Lambda_6}{(\kappa_1 - \kappa_2)^3}. \tag{15d}$$

From (13b) and (13d) we have

$$(\partial_{\mathbf{e}_1}\kappa_1)^2 + 2(\partial_{\mathbf{e}_1}\kappa_1)(\partial_{\mathbf{e}_1}\kappa_2) + (\partial_{\mathbf{e}_1}\kappa_2)^2 = (P + S)^2 = \frac{\Lambda_4 - \kappa_2\Lambda_2}{\kappa_1 - \kappa_2}, \tag{16a}$$

$$(\partial_{\mathbf{e}_2}\kappa_1)^2 + 2(\partial_{\mathbf{e}_2}\kappa_1)(\partial_{\mathbf{e}_2}\kappa_2) + (\partial_{\mathbf{e}_2}\kappa_2)^2 = (Q + T)^2 = \frac{\Lambda_4 - \kappa_1\Lambda_2}{\kappa_2 - \kappa_1}. \tag{16b}$$

As $(\partial_{\mathbf{e}_j}\kappa_i)^2$ is given by (15a)–(15d) we immediately see that $(\partial_{\mathbf{e}_1}\kappa_1)(\partial_{\mathbf{e}_1}\kappa_2)$ and $(\partial_{\mathbf{e}_2}\kappa_1)(\partial_{\mathbf{e}_2}\kappa_2)$ are determined by $\Lambda_1, \dots, \Lambda_6$. □

Remark 8. The expressions above become singular at umbilical points where $\kappa_1 = \kappa_2$. It is not because the invariants $\Lambda_1, \dots, \Lambda_6$ become singular, but because they become linearly dependent.

Remark 9. At a non-umbilical point any third order invariant function on a surface can be written

$$\Psi = \Psi(H, K, \Lambda_1, \dots, \Lambda_6),$$

but the six third order invariants $\Lambda_1, \ldots, \Lambda_6$ are not independent, by (16a) and (16b) they satisfy two quadratic equations.

To illustrate how any invariant can be expressed in terms of the Λ_i's we have taken the third order invariant fairness measures investigated in [16], and have rewritten them as follows:

$$(\mathrm{grad}H)^2 = \frac{1}{4}\Lambda_2, \tag{17}$$

$$(\mathrm{grad}K)^2 = \big(4H^2 - K\big)\Lambda_1 + K\Lambda_2 - 4H\Lambda_3 + \Lambda_5, \tag{18}$$

$$\big(\mathrm{grad}\big(|\kappa_1| + |\kappa_2|\big)\big)^2 = \Lambda_2 + \frac{|K| - K}{|K|}\left(\Lambda_1 - \Lambda_2 + \frac{1}{2}\frac{K\Lambda_1 - 2H\Lambda_3 + \Lambda_5}{H^2 - K}\right), \tag{19}$$

$$\big(\mathrm{grad}\big(\kappa_1^2 + \kappa_2^2\big)\big)^2 = 4\big(\Lambda_5 - K(\Lambda_1 - \Lambda_2)\big), \tag{20}$$

$$(\partial_{\mathbf{e}_1}\kappa_1)^2 + (\partial_{\mathbf{e}_2}\kappa_2)^2 = \Lambda_1 + \frac{3}{4}\frac{K\Lambda_1 - 2H\Lambda_3 + \Lambda_5}{H^2 - K}, \tag{21}$$

$$\frac{1}{\pi}\int_0^{\pi}\left(\frac{\mathrm{d}\kappa_{\mathrm{n}}}{\mathrm{d}s}\right)^2 \mathrm{d}\phi = \frac{2\Lambda_1 + 3\Lambda_2}{16}. \tag{22}$$

Proof of (18). If we use principal coordinates then (11) and (12) imply that we have the following expansion of the Gaussian curvature

$$K = \frac{LN - M^2}{EG - F^2} = (\kappa_1 + uP + vQ)(\kappa_2 + uS + vT) - (uQ + vS)^2 + \cdots.$$

For $(u, v) = (0, 0)$ we get

$$(\mathrm{grad}K)^2 = (\kappa_2 P + \kappa_1 S)^2 + (\kappa_2 Q + \kappa_1 T)^2.$$

If we expand this expression and substitute (15a)–(16b) we obtain

$$(\mathrm{grad}K)^2 = (\kappa_1 - \kappa_2)^2\Lambda_1 + \kappa_1\kappa_2\Lambda_2 - 2(\kappa_1 + \kappa_2)\Lambda_3 + \Lambda_5,$$

and this is exactly (18). □

The other cases are similar.

3. The geometry of a plane intersection

We are given a surface $\mathcal{M}$ which we intersect with a plane α. We parametrize the intersection curve by arc length $s \mapsto \mathbf{r}(s) = \mathbf{x}(u(s), v(s))$. We denote the tangent vector of the curve by $\mathbf{t}$, and the binormal of the curve by $\mathbf{b}$, it is of course also a normal to the plane α. We put $\mathbf{n} = \mathbf{b} \times \mathbf{t}$ which gives us the *Frenet frame* $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. We put $\mathbf{T} = \mathbf{t}$ and $\mathbf{U} = \mathbf{N} \times \mathbf{T}$, where $\mathbf{N}$ is the normal to the surface, this gives us the *Darboux frame* $\{\mathbf{T}, \mathbf{U}, \mathbf{N}\}$. Finally we denote the angle between $\mathbf{N}$ and $\mathbf{b}$ by θ, and the angle between $\mathbf{x}_u$

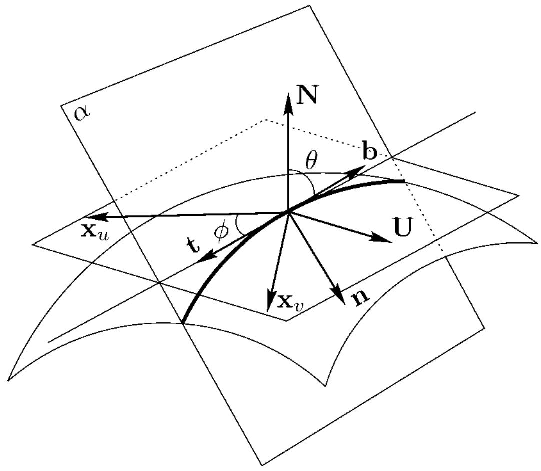

Figure 1. Intersecting the surface with a plane.

and **t** by ϕ, see figure 1. The transition from the Darboux frame to the Frenet frame is given by

$$\begin{bmatrix}\mathbf{t}\\\mathbf{n}\\\mathbf{b}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}\mathbf{T}\\\mathbf{U}\\\mathbf{N}\end{bmatrix}. \tag{23}$$

The derivative of the Frenet and the Darboux frame is given by the Frenet–Serret formula, and its generalization

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix}\mathbf{t}\\\mathbf{n}\\\mathbf{b}\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\-\kappa&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\\mathbf{n}\\\mathbf{b}\end{bmatrix},\qquad \frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix}\mathbf{T}\\\mathbf{U}\\\mathbf{N}\end{bmatrix}=\begin{bmatrix}0&\kappa_\mathrm{g}&\kappa_\mathrm{n}\\-\kappa_\mathrm{g}&0&-\tau_\mathrm{g}\\-\kappa_\mathrm{n}&\tau_\mathrm{g}&0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\\mathbf{U}\\\mathbf{N}\end{bmatrix}, \tag{24}$$

where κ is the curvature, κ_n is the normal curvature, κ_g is the geodesic curvature and τ_g is the geodesic torsion, see [8, section 57]. The quantities κ, κ_n, κ_g, τ_g, and θ are invariants of the curve and the surface and we have the following relations:

$$\kappa_\mathrm{n}=\kappa\sin\theta,\qquad \kappa_\mathrm{g}=\kappa\cos\theta,\qquad \frac{\mathrm{d}\theta}{\mathrm{d}s}=\tau_\mathrm{g}. \tag{25}$$

The last equality holds because we consider a plane curve which has vanishing torsion, see [8, section 56]. The normal curvature κ_n and the geodesic torsion τ_g depend only on the direction in the tangent plane, and it is well known that the normal curvature is given as the quotient between the second and first fundamental form:

$$\kappa_\mathrm{n}=\frac{L\,\mathrm{d}u^2+2M\,\mathrm{d}u\,\mathrm{d}v+N\,\mathrm{d}v^2}{E\,\mathrm{d}u^2+2F\,\mathrm{d}u\,\mathrm{d}v+G\,\mathrm{d}v^2}, \tag{26}$$

the geodesic torsion is a less known quantity, but there is a similar formula:

$$\tau_g = -\frac{(EM - FL)\,du^2 + (EN - GL)\,du\,dv + (FN - GM)\,dv^2}{\sqrt{EG - F^2}(E\,du^2 + 2F\,du\,dv + G\,dv^2)}, \tag{27}$$

see [8, pp. 106, 160]. The angle ϕ obviously depends on the parametrization, and its derivative can in general be expressed in terms of the geodesic curvature of the curve and the two parameter curves. To simplify the calculation we will assume that we have principal coordinates at a point p on the surface.

Lemma 10. Let $\mathbf{r}(s) = \mathbf{x}(u(s), v(s))$ be a curve on the surface parametrized by arc length. Using principal coordinates, then at the point of origo:

$$(u', v') = (\cos\phi, \sin\phi), \tag{28}$$

$$(u'', v'') = (-\kappa_g \sin\phi, \kappa_g \cos\phi), \tag{29}$$

$$\frac{d\phi}{ds} = \kappa_g. \tag{30}$$

Proof. As $\mathbf{t} = \mathbf{r}' = u'\mathbf{x}_u + v'\mathbf{x}_v$ is a unit vector, and $\mathbf{x}_u, \mathbf{x}_v$ is an orthonormal basis at p, we immediately have (28). We have

$$\kappa\mathbf{n} = \mathbf{r}'' = (u^i)''\mathbf{x}_{u^i} + (u^i)'(u^j)'\mathbf{x}_{u^i u^j},$$

and as $\mathbf{x}_{u^i} \perp \mathbf{x}_{u^j u^k}$, and $\mathbf{x}_{u^i} \cdot \mathbf{x}_{u^j} = \delta_{ij}$ at p we find

$$(u^i)'' = \kappa\mathbf{n} \cdot \mathbf{x}_{u^i} = (\kappa_g\mathbf{U} + \kappa_n\mathbf{N}) \cdot \mathbf{x}_{u^i} = \kappa_g\mathbf{U} \cdot \mathbf{x}_{u^i}.$$

As $\mathbf{U} = -\sin\phi\,\mathbf{x}_{u^1} + \cos\phi\,\mathbf{x}_{u^2}$, and $\mathbf{x}_{u^i} \cdot \mathbf{x}_{u^j} = \delta_{ij}$ at p we get (29). Finally, we have

$$\mathbf{r}' \cdot \mathbf{x}_u = |\mathbf{x}_u| \cos\phi = \sqrt{E}\cos\phi, \quad \text{and} \quad \mathbf{r}' \cdot \mathbf{x}_v = |\mathbf{x}_v| \sin\phi = \sqrt{G}\sin\phi.$$

As $E_u = E_v = G_u = G_v = 0$ and $E = G = 1$ at p, we see that

$$\mathbf{r}'' \cdot \mathbf{x}_u + \mathbf{r}' \cdot (u'\mathbf{x}_{uu} + v'\mathbf{x}_{uv}) = -\phi' \sin\phi,$$

$$\mathbf{r}'' \cdot \mathbf{x}_v + \mathbf{r}' \cdot (u'\mathbf{x}_{uv} + v'\mathbf{x}_{vv}) = \phi' \cos\phi.$$

As $\mathbf{r}' \perp \mathbf{x}_{u^i u^j}$ at p, we have on the one hand that

$$\mathbf{r}'' \cdot \mathbf{x}_u = -\phi' \sin\phi \quad \text{and} \quad \mathbf{r}'' \cdot \mathbf{x}_v = \phi' \cos\phi,$$

and on the other hand we have from above that

$$\mathbf{r}'' \cdot \mathbf{x}_u = u'' = -\kappa_g \sin\phi \quad \text{and} \quad \mathbf{r}'' \cdot \mathbf{x}_v = v'' = \kappa_g \cos\phi,$$

all in all we obtain (30). □

We are now in a position to find the curvature of the intersection curve and the derivative with respect to arc length of the curvature:

Lemma 11. Suppose that we have, at a point p on a surface, principal coordinates $\mathbf{x}(u, v) = \exp_p(u\mathbf{e}_1 + v\mathbf{e}_2)$, and intersect the surface with a plane α through p. Let θ denote the angle between the tangent plane and α, and let ϕ denote the angle between $\mathbf{e}_1$ and the line of intersection, see figure 1. At the point p we have the following expressions:

$$\kappa_\mathrm{n} = \kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi, \tag{31}$$

$$\kappa = \frac{1}{\sin\theta}\big(\kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi\big), \tag{32}$$

$$\kappa_\mathrm{g} = \cot\theta\big(\kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi\big), \tag{33}$$

$$\tau_\mathrm{g} = (\kappa_1 - \kappa_2)\cos\phi\sin\phi, \tag{34}$$

$$\begin{aligned}\frac{\mathrm{d}\kappa_\mathrm{n}}{\mathrm{d}s} &= P\cos^3\phi + 3Q\cos^2\phi\sin\phi + 3S\cos\phi\sin^2\phi + T\sin^3\phi \\ &\quad + 2\cot\theta\big(\kappa_1\cos^2\phi + \kappa_2\sin^2\phi\big)(\kappa_2 - \kappa_1)\cos\phi\sin\phi,\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}s} &= \frac{1}{\sin\theta}\big(P\cos^3\phi + 3Q\cos^2\phi\sin\phi + 3S\cos\phi\sin^2\phi + T\sin^3\phi \\ &\quad + 3\cot\theta\big(\kappa_1\cos^2\phi + \kappa_2\sin^2\phi\big)(\kappa_2 - \kappa_1)\cos\phi\sin\phi\big),\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\mathrm{d}\kappa_\mathrm{g}}{\mathrm{d}s} &= \cot\theta\big(P\cos^3\phi + 3Q\cos^2\phi\sin\phi + 3S\cos\phi\sin^2\phi + T\sin^3\phi\big) \\ &\quad + \frac{1 + 2\cos^2\theta}{\sin^2\theta}\big(\kappa_1\cos^2\phi + \kappa_2\sin^2\phi\big)(\kappa_2 - \kappa_1)\cos\phi\sin\phi,\end{aligned} \tag{37}$$

$$\frac{\mathrm{d}\tau_\mathrm{g}}{\mathrm{d}s} = -Q\cos^3\phi + (P - 2S)\cos^2\phi\sin\phi - (T - 2Q)\cos\phi\sin^2\phi + S\sin^3\phi. \tag{38}$$

Proof. Equation (31) follows from (26), and it is simply the well-known Euler's theorem which can be found in any standard text on differential geometry. As $\kappa_\mathrm{n} = \kappa\sin\theta$ and $\kappa_\mathrm{g} = \kappa\cos\theta$, (32) and (33) readily follow. Equation (34) follows from (27), it is less well known, but it is still a classical result, see [8, p. 161]. As (11) holds at p differentiation of (26) gives

$$\begin{aligned}\frac{\mathrm{d}\kappa_\mathrm{n}}{\mathrm{d}s} &= L_u\big(u'\big)^3 + L_v\big(u'\big)^2v' + 2M_u\big(u'\big)^2v' + 2M_vu'\big(v'\big)^2 + N_uu'\big(v'\big)^2 + N_v\big(v'\big)^3 \\ &\quad + 2Lu'u'' + 2M\big(u''v' + u'v''\big) + 2Nv'v''.\end{aligned}$$

Using (12) yields

$$\frac{\mathrm{d}\kappa_\mathrm{n}}{\mathrm{d}s} = P\big(u'\big)^3 + 3Q\big(u'\big)^2v' + 3Su'\big(v'\big)^2 + T\big(v'\big)^3 + 2\kappa_1u'u'' + 2\kappa_2v'v'',$$

and inserting (28), (29), and (33) gives us (35). We have $\kappa = \kappa_\mathrm{n}\sin^{-1}\theta$ and hence

$$\frac{\mathrm{d}\kappa}{\mathrm{d}s} = \frac{1}{\sin\theta}\frac{\mathrm{d}\kappa_\mathrm{n}}{\mathrm{d}s} - \kappa_\mathrm{n}\frac{\cos\theta}{\sin^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{\sin\theta}\left(\frac{\mathrm{d}\kappa_\mathrm{n}}{\mathrm{d}s} - \kappa_\mathrm{g}\,\tau_\mathrm{g}\right).$$

Using (33)–(35) we obtain (36). Likewise $\kappa_g = \kappa_n \cot\theta$, so

$$\frac{d\kappa_g}{ds} = \cot\theta \frac{d\kappa_n}{ds} - \kappa_n \frac{1}{\sin^2\theta}\frac{d\theta}{ds} = \cot\theta \frac{d\kappa_n}{ds} - \frac{1}{\sin^2\theta}\kappa_n\tau_g,$$

and using (31), (34), and (35) we obtain (37). Finally, once more using (11) a differentiation of (27) gives

$$\begin{aligned}\frac{d\tau_g}{ds} &= -\big((M_u u' + M_v v')(u')^2 + (N_u u' + N_v v' - L_u u' - L_v v')u'v' \\ &\quad - (M_u u' + M_v v')(v')^2\big), \\ &= -Q(u')^3 + (P - 2S)(u')^2 v' - (T - 2Q)u'(v')^2 + S(v')^3,\end{aligned}$$

and we obtain (38). □

4. Fairness measures for surfaces

In this section we will derive several surface fairness measures based on the previous analysis of a plane intersection.

4.1. Intersection with one plane

Here we will use the curve fairness measures $\psi = \kappa^2$ and $\psi = (d\kappa/ds)^2$, to derive the surface fairness measure Ψ. In order to determine $\Psi(p)$ we need to integrate $\psi(p)$ over all intersections with a plane through p. These planes are parametrized by the angles θ and ϕ, see figure 1. I.e., we would like to put

$$\Psi(p) = \int_0^{\pi/2}\int_0^{2\pi} \psi(\theta,\phi)\sin\theta \, d\phi \, d\theta. \tag{39}$$

Using (32) and (36) it is easy to perform the ϕ-integration:

Lemma 12. Let s and κ denote the arc length and the curvature, respectively of the intersection curve between a surface and a plane which forms the angle θ with the tangent plane at a point p. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa^2 \, d\phi = \frac{3H^2 - K}{2\sin^2\theta}, \tag{40}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{d\kappa}{ds}\right)^2 d\phi = \frac{2\Lambda_1 + 3\Lambda_2}{16\sin^2\theta} + \frac{9\cos^2\theta(H^2 - K)(5H^2 - K)}{8\sin^4\theta}. \tag{41}$$

Observe that for $\theta = \pi/2$ we have the result of Mehlum and Tarrou [13].

Proof. Using principal coordinates at the point p and (31), we get

$$\frac{1}{2\pi}\int_0^{2\pi}\kappa_n^2\,\mathrm{d}\phi = \frac{1}{2\pi}\int_0^{2\pi}\left(\kappa_1^2\cos^4\phi + 2\kappa_1\kappa_2\cos^2\phi\sin^2\phi + \kappa_2^2\sin^4\phi\right)\mathrm{d}\phi$$
$$= \frac{3}{8}\kappa_1^2 + \frac{1}{4}\kappa_1\kappa_2 + \frac{3}{8}\kappa_2^2 = \frac{3}{8}(\kappa_1+\kappa_2)^2 - \frac{1}{2}\kappa_1\kappa_2 = \frac{1}{2}\left(3H^2 - K\right),$$

and by (32) we have (40). Likewise, using (35) and lemma 5, we get

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\mathrm{d}\kappa_n}{\mathrm{d}s}\right)^2\mathrm{d}\phi = \frac{1}{16}\left(5P^2 + 6PS + 9S^2 + 5T^2 + 6QT + 9Q^2\right)$$
$$+ \frac{9\cot^2\theta}{128}(\kappa_2-\kappa_1)^2\left(5\kappa_1^2 + 6\kappa_1\kappa_2 + 5\kappa_2^2\right)$$
$$= \frac{1}{16}(2\Lambda_1 + 3\Lambda_2) + \frac{9\cot^2\theta}{8}\left(H^2 - K\right)\left(5H^2 - K\right),$$

which by (36) establishes (41). □

We still need to integrate over θ, and then the denominators $\sin^2\theta$ and $\sin^4\theta$ pose a problem. Geometrically the intersection plane becomes tangential when θ is zero, and in general the intersection curve becomes singular. Thus it does not make sense to ask about the fairness of the intersection curve if the intersecting plane is close to tangential. Hence we introduce a cutoff angle δ and consider only $\theta \in [\delta, \pi/2]$. This leads to the fairing measures

$$\Psi_1(p) = \left[\int_\delta^{\pi/2}\frac{\sin\theta}{2\sin^2\theta}\,\mathrm{d}\theta\right]\left(3H^2 - K\right),$$
$$\Psi_2(p) = \left[\int_\delta^{\pi/2}\frac{\sin\theta}{16\sin^2\theta}\,\mathrm{d}\theta\right](2\Lambda_1 + 3\Lambda_2)$$
$$+ \left[\int_\delta^{\pi/2}\frac{9\cos^2\theta\sin\theta}{8\sin^4\theta}\,\mathrm{d}\theta\right]\left(H^2 - K\right)\left(5H^2 - K\right).$$

The main point is that the integrals in the brackets above are independent of both the surface and the point on the surface. Furthermore, a fairness measure is only interesting when we compare its value at different points or on different surfaces. So multiplying a fairness measure by some constant does not change anything essential. Hence we arrive at the following definitions:

Definition 13. The second order invariant fairness measure based on κ^2 is given by

$$\Psi_1 = 3H^2 - K. \tag{42}$$

The third order invariant fairness measure based on $(\mathrm{d}\kappa/\mathrm{d}s)^2$ is given by

$$\Psi_2 = \alpha\Psi_2^a + \beta\Psi_2^b, \tag{43}$$

where $\alpha, \beta \in \mathbb{R}$, and

$$\Psi_2^a = 2\Lambda_1 + 3\Lambda_2, \tag{44}$$
$$\Psi_2^b = \left(H^2 - K\right)\left(5H^2 - K\right). \tag{45}$$

In (43) a large value of α corresponds to putting emphasis on intersections with planes almost normal to the surface, while a large value of β puts emphasis on intersections with planes almost tangential to the surface. One could also introduce a weight function in θ to cancel the singularity, but it leads to the same result as the cutoff angle.

4.2. Intersection with a family of parallel planes

Besides considering intersection curves by themselves a naval architect often considers plots like the one in figure 2. It is made by intersecting the ship hull with parallel planes at regular intervals and then plotting the resulting curves. The curves can also be considered as isolines for the function which measures distance along the normal to the parallel planes.

We want to quantify how the intersection curves vary from plane to plane. The first order term of the distance function is the gradient, so if we consider the gradient along an intersection curve or isoline, then we have the infinitesimal variation of the curve. In fact, the gradient is a vector field on the surface and the flow of this vector field maps the intersection curves to each other. The gradient $\mathbf{V}$ is orthogonal to the isolines, so we can write it as $\mathbf{V} = V\mathbf{U}$. Recall that if $\mathbf{T}$ is the unit tangent vector of the isoline, and $\mathbf{N}$ is the unit normal to the surface, then $\mathbf{U} = \mathbf{N} \times \mathbf{T}$, see figure 1. Let $\mathbf{b}$ be the common unit normal vector to the parallel intersection planes, and consequently the binormal vector of the intersection curves or isolines. If we measure the distance in the direction of $\mathbf{b}$, we clearly have that the projection of the gradient on the normal line of the intersection planes is $\mathbf{b}$, i.e., $\mathbf{V} \cdot \mathbf{b} = 1$. As $\mathbf{U} \cdot \mathbf{b} = \sin\theta$, see figure 1, we have $V = \sin^{-1}\theta$ and $\mathbf{V} = \sin^{-1}\theta\, \mathbf{U}$.

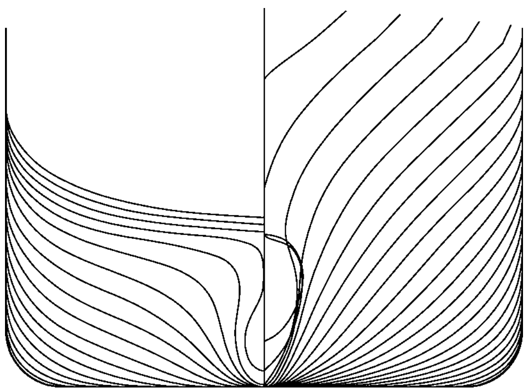

Figure 2. Station lines for a large modern container vessel. To the left the aft part and to the right the fore part.

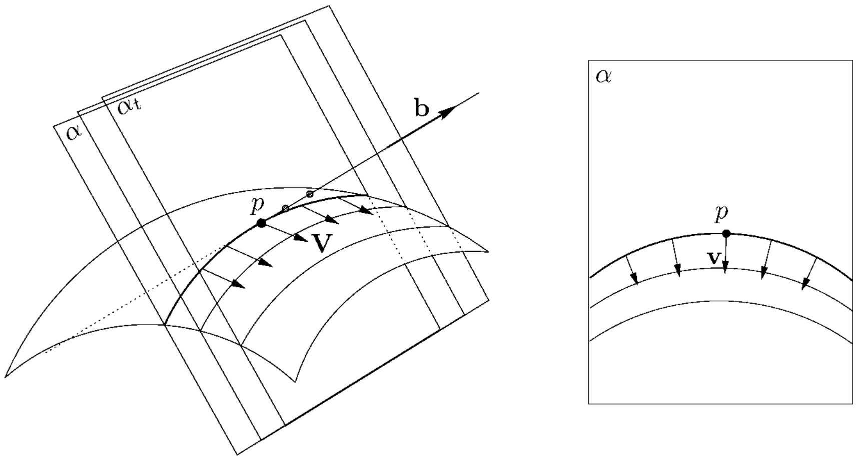

Figure 3. Intersecting a surface with a family of parallel planes. To the right the projection on one of the planes, this would be the view of the ship designer.

We now project everything onto one of the intersecting planes, see figure 3. We obtain hereby a family of plane curves and a vector field **v** orthogonal to these curves. Due to the linearity of the projection, the flow of **v** generates the projections of the intersection curves. As **v** is orthogonal to **t**, it is a multiple of **n**, and as $\mathbf{U} \cdot \mathbf{n} = \cos\theta$, we finally obtain

$$|\mathbf{v}| = \cot\theta. \tag{46}$$

If $|\mathbf{v}|$ is constant, then the curves are parallel, which we consider fair. So we will use $\mathrm{d}|\mathbf{v}|/\mathrm{d}s$ and $\mathrm{d}^2|\mathbf{v}|/\mathrm{d}s^2$ as measures of fairness. We immediately obtain

$$\frac{\mathrm{d}|\mathbf{v}|}{\mathrm{d}s} = -\frac{1}{\sin^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{-\tau_\mathrm{g}}{\sin^2\theta} = \frac{1}{\sin^2\theta}(\kappa_2 - \kappa_1)\cos\phi\sin\phi, \tag{47}$$

and

$$\frac{\mathrm{d}^2|\mathbf{v}|}{\mathrm{d}s^2} = \frac{1}{\sin^2\theta}\left(2\cot\theta\,\tau_\mathrm{g}\frac{\mathrm{d}\theta}{\mathrm{d}s} - \frac{\mathrm{d}\tau_\mathrm{g}}{\mathrm{d}s}\right) = \frac{1}{\sin^2\theta}\left(2\cot\theta\,\tau_\mathrm{g}^2 - \frac{\mathrm{d}\tau_\mathrm{g}}{\mathrm{d}s}\right). \tag{48}$$

Just as in the case of the curve fairing measures, the expressions

$$\psi = \left(\frac{\mathrm{d}|\mathbf{v}|}{\mathrm{d}s}\right)^2 \quad \text{and} \quad \psi = \left(\frac{\mathrm{d}^2|\mathbf{v}|}{\mathrm{d}s^2}\right)^2$$

are functions along the intersection curves. Hence we can proceed with the integration exactly as before. As in that case the ϕ-integration is straightforward, and we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\mathrm{d}|\mathbf{v}|}{\mathrm{d}s}\right)^2\mathrm{d}\phi = \frac{1}{8\sin^4\theta}(\kappa_2 - \kappa_1)^2 = \frac{1}{2\sin^4\theta}\big(H^2 - K\big), \tag{49}$$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\mathrm{d}^2|\mathbf{v}|}{\mathrm{d}s^2}\right)^2\mathrm{d}\phi &= \frac{1}{\sin^4\theta}\left(\frac{3\cot^2\theta}{8}(\kappa_2-\kappa_1)^4\right.\\ &\quad\left.+\frac{1}{16}\left(P^2-2PS+5S^2+5Q^2-2QT+T^2\right)\right)\\ &= \frac{1}{\sin^4\theta}\left(\frac{3}{2}\cot^2\theta\left(H^2-K\right)^2+\frac{1}{16}(2\Lambda_1-\Lambda_2)\right). \end{aligned} \tag{50}$$

The remarks before definition 13 regarding the θ-integration apply here too, so we arrive at the following definitions:

Definition 14. The second order invariant fairness measure based on $(\mathrm{d}|\mathbf{v}|/\mathrm{d}s)^2$ is given by

$$\Psi_3 = H^2 - K. \tag{51}$$

The third order invariant fairness measure based on $(\mathrm{d}^2|\mathbf{v}|/\mathrm{d}s^2)^2$ is given by

$$\Psi_4 = \alpha\Psi_4^a + \beta\Psi_4^b, \tag{52}$$

where $\alpha, \beta \in \mathbb{R}$, and

$$\Psi_4^a = 2\Lambda_1 - \Lambda_2, \tag{53}$$

$$\Psi_4^b = \left(H^2 - K\right)^2. \tag{54}$$

In (52) a large value of α corresponds to putting emphasis on intersections with planes almost normal to the surface, while a large value of β puts emphasis on intersections with planes almost tangential to the surface.

5. Testing the fairness measures

At many shipyards the hull is defined by a network of curves, station lines, water lines, buttocks, and some auxiliary feature lines on the hull. This network of curves is faired by hand, which is a very time consuming process and requires an experienced naval architect. A set of B-spline patches are then fitted to (some of) the curves. The fit is not particular smooth, only continuity is secured; but in practice the joins are close to G^1. The resulting surface forms the basis for the subsequent production.

We have taken such a collection of B-spline patches forming the hull of a large container vessel, and have colour-coded them according to the fairness measures. It was our experience that the second order measures were of little help, areas with high values of the second order measures were not necessarily unfair, but both of the third order measures did well. Even though the hull was supposedly fair the third order measures discovered ‘bad spots’ on the surface, and subsequent intersections with planes revealed that the hull indeed was unfair in these areas.

One such example is depicted in figure 4, it is a piece of the hull (6 m $\times$ 8 m), near the bulbous bow, consisting of four B-spline patches, each with 20×20 control

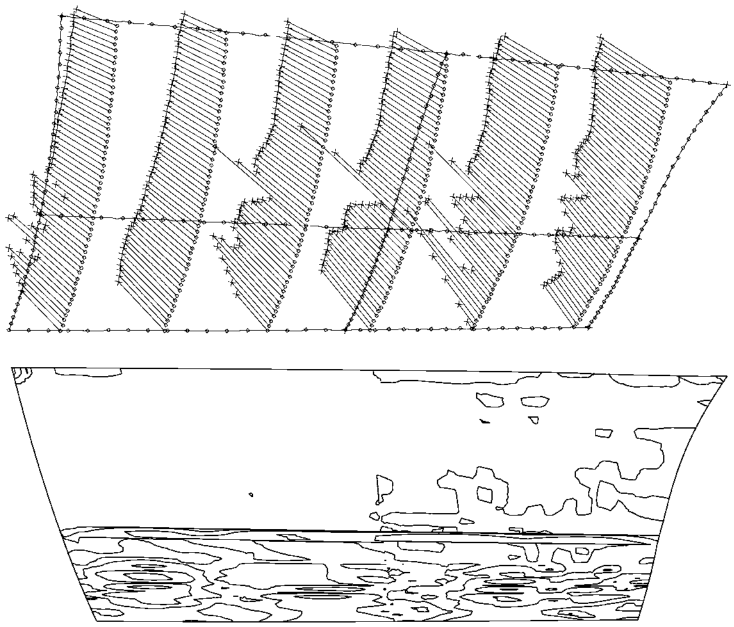

Figure 4. Before optimization. Below isolines for the fairness measure (at the values 4, 1, $1/4$, $1/16$, and $1/64$) and above six plane intersection curves with their curvature vectors. The point of view is different in the two images. There are large discontinuities along the boundary between the patches, visible in both pictures, this is caused by the patches not being joined G^2.

points. In this example we have used the third order measure (43) based on $\mathrm{d}^2\kappa/\mathrm{d}s^2$ with $\alpha = 1$ and $\beta = 57$, corresponding to a cutoff angle of 10°. There is nothing special about this value, and we will leave to the designer to determine how close he want to go to a tangential intersection. Instead of the colour coding we have drawn isolines for the values 4, 1, $1/4$, $1/16$, and $1/64$. The maximum value (above 4) is attained in the middle of lower right patch, and as can be seen in figure 4, a plane intersection through this area has a bad curvature plot.

Encouraged by the third order measures ability to pinpoint the unfair parts of the hull we have tried to use the same measure for fairing the hull. We have minimized the integral of the fairness measure over the surface. The optimization is simply a steepest descent method and the integral is determined by 4 point Gaussian quadrature on each knot interval. Allowing the control points of the B-spline patches to move up to 10 mm, (compare with the dimension of the patch) we succeeded in reducing the value of the fairness measure with a factor of 64, and the curvature of the plane intersections had accordingly a much nicer behaviour, see figure 5.

We are at this stage only interested in the potential of the method, and it is premature to have any precise timing of the calculations. E.g., the optimization methods is chosen for its ease of implementation and not its speed of convergence. We can say that the time for the colour coding of the four patches is measured in seconds on a SGI-onyx, and the optimization took approximately 1 hour on the same machine.

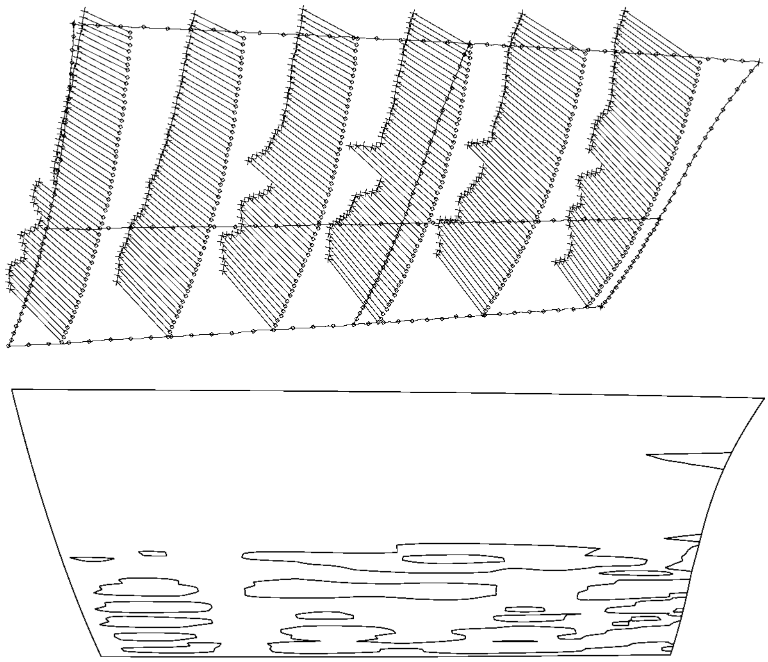

Figure 5. After optimization, only the isolines for the values $1/16$ and $1/64$ shows up. We have not addressed the lack of G^2-continuity between the patches so the discontinuities in the curvature plot persists.

These first tests are very promising and the goal is to be able to fair larger parts of the hull. However, in the present surface representation the hull consist of more than 200 B-spline patches, that gives more than 80,000 control points. So before we start the fairing we want to simplify the surface description. We will also replace the hand fairing of the network of curves with an automated fairing. Depending on the size of the surface representation it might be, that we need some approximation to the fairing measure, e.g., the data dependent approximation of Greiner [9].

6. Conclusion

We have described a general principle by which we in a rational way, from a class of curves and a curve fairness measure, can construct invariant fairness measures for surfaces.

For the design of ship hulls the principle has been applied to planar intersection curves, and the curve fairness measures given by the square of the curvature and the square of the curvature variation as curve fairness measures. This leads to a second and third order invariant fairness measures for surfaces.

Still with ship design in mind, we have defined a second and a third order fairness measure for a family of plane curves and used these measures to derive new second and third order invariants fairness measures for surfaces.

We have defined six third order invariants by which ours and previous fairing measures can be expressed.

We have determined the variation in curvature, normal curvature, geodesic curvature, and geodesic torsion of an arbitrary plane intersection.

Tests of the fairness measures shows that they indeed can detect the unfair parts of a ship hull and that an optimization procedure can fair the hull.

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References

[1] P. Brunet, A. Vinacua, M. Vivo, N. Pla and A. Rodriguez, Surface fairing for ship hull design application, Math. Engrg. Industry 7 (1998) 179–193.

[2] G. Brunnett and J. Wendt, Elastic splines with tension control, in: *Mathematical Methods for Curves and Surfaces*, eds. M. Dæhlen, T. Lyche, and L.L. Schumaker (Vanderbilt Univ. Press, Nashville, 1998).

[3] M.P. do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice-Hall, Englewood Cliffs, NJ, 1976).

[4] M.P. do Carmo, *Riemannian Geometry* (Birkhäuser, Boston, 1992).

[5] L. Euler, De curvis elasticis, in: *Additamentum I zum Methodus Inveniendi Lineas Curvas Maxima Minimivi Proprietate Gaudentes* (Lausanne, 1744). Translated to German by H. Linsenart in *Abhandlungen über das Gleichgewicht und die Schwingungen der ebenen elastischen Kurven* (Ostwalds Klassiker der exakten Wissenschaften, No. 175, Leipzig, 1910).

[6] A. Ginnis and S. Wahl, Benchmark results in the area of curve fairing, in: [15], pp. 29–54.

[7] J.H. Grace and A. Young, *The Algebra of Invariants* (Cambridge Univ. Press, Cambridge, 1903).

[8] W.C. Graustein, *Differential Geometry* (Dover, New York, 1966).

[9] G. Greiner, Variational design and fairing of spline surfaces, Computer Graphics Forum 13 (1994) 143–154.

[10] G. Greiner, Modelling of curves and surfaces based on optimization techniques, in: [15], pp. 11–27.

[11] M. Lipschutz, *Differential Geometry* (McGraw-Hill, New York, 1969).

[12] E. Mehlum, Nonlinear splines, in: *Computer Aided Geometric Design*, eds. R.E. Barnhill and R.F. Riesenfeld (Academic Press, New York, 1974) pp. 173–208.

[13] E. Mehlum and C. Tarrou, Invariant smoothness measures for surfaces, Adv. Comput. Math. 8 (1998) 49–63.

[14] H.P. Moreton and C.H. Séquin, Functional optimization for fair surface design, ACM Computer Graphics 26 (1992) 167–176.

[15] H. Nowacki and P.D. Kaklis, eds., *Creating Fair and Shape-Preserving Curves and Surfaces* (B.G. Teubner, Stuttgart/Leipzig, 1998).

[16] H. Nowacki, G. Westgaard and J. Heimann, Creation of fair surfaces based on higher order fairness measures with interpolation constraints, in: [15], pp. 141–161.

[17] I.R. Porteous, *Geometric Differentiation, for the Intelligence of Curves and Surfaces* (Cambridge Univ. Press, Cambridge, 1994).

[18] J.A. Roulier and T. Rando, Designing faired parametric surfaces, Computer-Aided Design 23 (1991) 492–497.
[19] R.F. Sarraga, Recent methods for surface optimization, Computer Aided Geom. Design 15 (1998) 417–436.
[20] G.G. Schweikert, An interpolating curve using a spline in tension, J. Math. Phys. 45 (1966) 312–317.
[21] D.J. Struik, *Lectures on Classical Differential Geometry* (Addison-Wesley, London, 1961).