De Casteljau's Algorithm Revisited

Jens Gravesen

Abstract. It is demonstrated how all the basic properties of Bézier curves can be derived swiftly and efficiently without any reference to the Bernstein polynomials and essentially with only geometric arguments. This is achieved by viewing one step in de Casteljau's algorithm as an operator (the de Casteljau operator) acting on a sequence of points, producing a sequence with one point less. The properties of Bézier curves are then derived by analysing de Casteljau's operator.

§1. Introduction

We consider one step in de Casteljau's algorithm as an *operator* (the de Casteljau operator), acting on a sequence of points, producing a new sequence with one point less. We produce a Bézier curve by applying de Casteljau's operator a number of times, so it is clear that the properties of a Bézier curve can be found by studying products of de Casteljau's operator with itself. The final ingredient is the observation that de Casteljau's operator is built from the two extremely simple operators, which remove the first and the last points from a sequence, respectively.

A slight modification of the idea has been used by M. Hosaka and M. Kuroda, cf. [3,4], and by J. Hoschek and D. Lasser cf. [5]. Instead of finite sequences they use infinite sequences.

§2. The de Casteljau Algorithm

The de Casteljau algorithm can be described by the scheme

$$
\begin{array}{ccccccccc}
P_0 & \to & P_0^1 & \to & \dots & \to & P_0^{n-1} & \to & P_0^n \\
 & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\
P_1 & \to & P_1^1 & \to & \dots & \to & P_1^{n-1} & & \\
 & \nearrow & & \nearrow & & \nearrow & & & \\
\vdots & & \vdots & & & & & & \\
 & \nearrow & & \nearrow & & & & & \\
P_{n-1} & \to & P_{n-1}^1 & & & & & & \\
 & \nearrow & & & & & & & \\
P_n & & & & & & & &
\end{array} \tag{1}
$$

Mathematical Methods for Curves and Surfaces II
Morten Dæhlen, Tom Lyche, and Larry L. Schumaker (eds.), pp. 221–228.

ISBN 0-8265-1315-8.

Using matrix notation one step in the algorithm (going from one column to the next) can be written as

$$\begin{bmatrix} P_0^{k+1} \\ P_1^{k+1} \\ \vdots \\ P_{n-k-1}^{k+1} \end{bmatrix} = \begin{bmatrix} 1-t & t & 0 & \dots & 0 \\ 0 & 1-t & t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1-t & t \end{bmatrix} \begin{bmatrix} P_0^k \\ P_1^k \\ \vdots \\ P_{n-k}^k \end{bmatrix}$$
$$= \left((1-t) \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} P_0^k \\ P_1^k \\ \vdots \\ P_{n-k}^k \end{bmatrix}.$$

Inspired by this, we introduce two basic operators which act on a sequence of points.

Definition 1. *The* right truncation operator R *and the* left truncation operator L *is defined by*

$$R : (P_0, \dots, P_m) \mapsto (P_0, \dots, P_{m-1}),$$
$$L : (P_0, \dots, P_m) \mapsto (P_1, \dots, P_m).$$

Remark 1. *To be precise we should indicate the length of the sequence on which the operators acts, so the definition reads*

$$R_m : (P_0, \dots, P_m) \mapsto (P_0, \dots, P_{m-1}),$$
$$L_m : (P_0, \dots, P_m) \mapsto (P_1, \dots, P_m).$$

Remark 2. *In this paper all sequences are finite. It is possible to work with infinite sequences,* $(P_0^i, P_1^i, \dots)$*, then the scheme (1) becomes infinite downward and to the right, and our scheme is just the top left triangle of this infinite scheme. The operator* R *is then the identity and the operator* L *is left-shift. We may even use bi-infinite sequences,* $(\dots, P_{-1n}, P_0^i, P_1^i, \dots)$ *in which case the left-shift operator* L *is invertible.*

Even though the definition of R and L is as simple as you could possibly want, they are all we need for the development of the Bézier curve theory. We start by stating some basic properties of R and L.

Lemma 2. *The operators* R *and* L *"commute", i.e.*

$$LR = RL.$$

Proof: This is clear, both LR and RL removes the first and the last point of a sequence. □

Remark. *Strictly speaking the word "commute" is a misnomer, because the* L*'s and the* R*'s on the two sides of the equation is different. The equation should read* $L_{m-1}R_m = R_{m-1}L_m$.

Lemma 3. *If we identify a point P and the sequence (P) consisting of just that point, then,*

$$P_k = L^k R^{m-k}(P_0, \ldots, P_m), \quad \text{for } 0 \leq k \leq m.$$

Proof: The operator $L^k R^{m-k}$ removes the first k points $P_0, \ldots, P_{k-1}$ and the last $m-k$ points $P_{k+1}, \ldots, P_m$, so we are left with the sequence consisting of just one point P_k. □

From the basic operators R and L we can define two new operators.

Definition 4. *The (well known)* forward difference operator Δ *is*

$$\Delta = L - R.$$

Definition 5. *If $t \in \mathbb{R}$, then the* de Casteljau operator $C(t)$ *is*

$$C(t) = (1-t)R + tL = R + t\Delta.$$

By definition

$$C(t) : (P_0, \ldots, P_m) \mapsto ((1-t)P_0 + tP_1, \ldots, (1-t)P_{m-1} + tP_m),$$

and we see that $C(t)$ describes one step in de Casteljau's algorithm. By Lemma 2 we have the following corollary.

Corollary 6. *Let $s, t \in \mathbb{R}$, then the operators R, L, Δ, $C(s)$, $C(t)$ "commute".*

Again the word "commute" should not be taking literally. The operators we have introduced are affine operators:

Lemma 7. *Let $t \in \mathbb{R}$ and let M be any of the operators R, L, Δ, $C(t)$. If $\alpha + \beta = 1$, then*

$$M\big(\alpha(P_0, \ldots, P_n)) + \beta(Q_0, \ldots, Q_n)\big) = \alpha M(P_0, \ldots, P_n)) + \beta M(Q_0, \ldots, Q_n).$$

Proof: As linear combinations of affine operators are affine, we only have to show that the lemma holds for the operators R and L, but this is obvious. □
Finally we have that de Casteljau's operator is affine in it's argument:

Lemma 8. *Let $a, b, t \in \mathbb{R}$ then*

$$C((1-t)a + tb) = (1-t)C(a) + tC(b).$$

Proof: The proof is a straightforward calculation:

$$\begin{aligned} C((1-t)a + tb) &= R + ((1-t)a + tb)(L - R) \\ &= (1-t)R + (1-t)a(L-R) + tR + tb(L-R) \\ &= (1-t)(R + a(L-R)) + t(R + tb(L-R)) \\ &= (1-t)C(a) + tC(b). \quad \square \end{aligned}$$

The de Casteljau's algorithm is encoded in the operator $C(t)$, so we can use it to define a Bézier curve.

Definition 9. *A Bézier curve of degree n with control points $P_0,\dots,P_n$ is defined by*

$$\mathbf{b}(t) = C(t)^n(P_0,\dots,P_n).$$

Remark. *The symbol $C(t)^n$ is a misuse of notation. The definition should read $\mathbf{b}(t) = C_2(t)C_3(t)\dots C_{n+1}(t)(P_0,\dots,P_n)$, but as it is clear from the context what the interpretation of $C(t)^n$ is, we will stick to the simple notation. The same remark holds each time we compose any of our operators and in the following we will use the abusive but simple notation without any further comments.*

§3. Properties of Bézier Curves

As a composition of affine operators gives an affine operator, Lemma 7 immediately give us that the operator $C(t)^n$ is affine. This can be formulated as:

Corollary 10. *Let $(P_0,\dots,P_n)$ be control points for the Bézier curve $\mathbf{b}(t)$ and let $(Q_0,\dots,Q_n)$ be control points for the Bézier curve $\mathbf{c}(t)$. If $\alpha+\beta=1$ then the Bézier curve $\alpha\mathbf{b}(t)+\beta\mathbf{c}(t)$ has control points $(\alpha P_0+\beta Q_0,\dots,\alpha P_n+\beta Q_n)$.*

Next we have the *symmetry* property of Bézier curves:

Theorem 11. *If $\mathbf{b}(t)$ is a Bézier curve with control points $P_0,\dots,P_n$, and $\widetilde{\mathbf{b}}(t)$ is the Bézier curve with control points $P_n,\dots,P_0$, then*

$$\widetilde{\mathbf{b}}(t) = \mathbf{b}(1-t).$$

Proof: Let J denote the operator which reverse the order of a sequence, i.e.

$$J : (P_0,\dots,P_n) \mapsto (P_n,\dots,P_0).$$

We clearly have $RJ = JL$ and $JR = LJ$, and hence

$$\begin{aligned} C(t)J &= ((1-t)R + tL)J = (1-t)RJ + tLJ \\ &= (1-t)JL + tJR = J(tR + (1-t)L) = JC(1-t). \end{aligned}$$

By induction we have

$$C(t)^nJ = JC(1-t)^n,$$

and as J is the identity on a one point sequence, the theorem follows. □

The *derivative* of a Bézier curve is easily found.

Theorem 12. *If $\mathbf{b}(t)$ is a Bézier curve with control points $P_0,\dots,P_n$, then the derivative is given by*

$$\frac{d}{dt}\mathbf{b}(t) = nC(t)^{n-1}\Delta(P_0,\dots,P_n) = n\Delta C(t)^{n-1}(P_0,\dots,P_n).$$

Proof: As $C(t) = R + t\Delta$ we have $\frac{d}{dt}C(t) = \Delta$ and as Δ and $C(t)$ commute,

$$\frac{d}{dt}\left(C(t)^n\right) = \sum_{k=0}^{n-1} C(t)^k \left(\frac{d}{dt}C(t)\right) C(t)^{n-1-k} = \sum_{k=0}^{n-1} C(t)^k \Delta C(t)^{n-1-k}$$
$$= nC(t)^{n-1}\Delta = n\Delta C(t)^{n-1}. \quad \square$$

Remark. *The expression*

$$\frac{d}{dt}\mathbf{b}(t) = nC(t)^{n-1}\Delta(P_0, \ldots, P_n)$$

tells us that the derivative of a Bézier curve is a new Bézier curve where the control points are the forward differences of the original control points multiplied by the degree. The expression

$$\frac{d}{dt}\mathbf{b}(t) = n\Delta C(t)^{n-1}(P_0, \ldots, P_n)$$

shows us that the derivative of a Bézier curve is the forward difference of the two points in the second last step in de Casteljau's algorithm multiplied by the degree.

Corollary 13. *If $\mathbf{b}(t)$ is a Bézier curve with control points $P_0, \ldots, P_n$, then*

$$\frac{d^k}{dt^k}\mathbf{b}(t) = \frac{n!}{(n-k)!}C(t)^{n-k}\Delta^k(P_0, \ldots, P_n)$$
$$= \frac{n!}{(n-k)!}\Delta^k C(t)^{n-k}(P_0, \ldots, P_n).$$

Proof: The corollary follows by induction, using theorem 12. $\square$
As another corollary we get linear precision:

Corollary 14. *If P and Q are two points and the Bézier curve $\mathbf{b}(t)$ has the control points*

$$P_i = \frac{(n-i)}{n}P + \frac{i}{n}Q, \qquad i = 0, \ldots, n,$$

then

$$\mathbf{b}(t) = (1-t)P + tQ.$$

Proof: We only need to show that the derivative of $\mathbf{b}(t)$ is constant $P - Q$, but this is easy. The derivative $\mathbf{b}'(t)$ has control points $n(P_{i+1} - P_i) = P - Q$, and if all the control points of a Bézier curve are one and the same point, then the curve is constant equal to that point. $\square$

The degree elevation formula is the most complicated in the this context, and it is the one instance where it is better to use the Bernstein representation.

Definition 15. *The* degree elevation operator *is defined by*

$$U:(P_0,\ldots,P_n)\mapsto\left(P_0,\frac{nP_1+P_0}{n+1},\ldots,\frac{P_n+nP_{n-1}}{n+1},P_n\right),$$

i.e., the k'th element of $U(P_0,\ldots,P_n)$ *is*

$$\frac{(n+1-k)P_k+kP_{k-1}}{n+1}.$$

Before we prove the degree elevation formula we need a lemma.

Lemma 16. *Acting on sequences of length* $n+1$ *we have,*

$$\Delta U=\frac{n}{n+1}U\Delta.$$

Proof: This is just a calculation. Starting with the k-th element of the sequence $\Delta U(P_0,\ldots,P_n)$, we get

$$\begin{aligned}
&\frac{(n+1-(k+1))P_{k+1}+(k+1)P_k}{n+1}-\frac{(n+1-k)P_k+kP_{k-1}}{n+1}\\
&\quad=\frac{n+1-k-1}{n+1}P_{k+1}-\frac{n+1-k}{n+1}P_k+\frac{k+1}{n+1}P_k-\frac{k}{n+1}P_{k-1}\\
&\quad=\frac{n-k}{n+1}P_{k+1}-\frac{n-k}{n+1}P_k+\frac{k}{n+1}P_k-\frac{k}{n+1}P_{k-1}\\
&\quad=\frac{n}{n+1}\left(\frac{n-k}{n}(P_{k+1}-P_k)+\frac{k}{n}(P_k-P_{k-1})\right).
\end{aligned}$$

and this is exactly the k-th element of the sequence $\frac{n}{n+1}U\Delta(P_0,\ldots,P_n)$. □

We can now formulate the degree elevation formula as

Theorem 17. *Acting on sequences of length* n, *we have*

$$C(t)^nU=C(t)^{n-1}. \tag{2}$$

Proof: This is done by induction on n. If $n=1$, (2) is trivial. So assume $C(t)^nU=C(t)^{n-1}$. We clearly have

$$C(0)^{n+1}U=R^{n+1}U=R^n=C(0)^n,$$

and hence it is sufficient to show that (2) holds after differentiation, i.e. we need to show that

$$(n+1)C(t)^n\Delta U=nC(t)^{n-1}\Delta.$$

By Lemma 16 the left-hand side is $nC(t)^nU\Delta$, but this is the same as the right-hand side by the induction assumption. □

Remark. *Even though both sides of equation (2) can act on sequences of length greater than* n, *it is only valid when acting on sequences of length* n.

An important property of a Bézier curve is given in the *subdivision theorem*, which states that the part of a Bézier curve, which is defined for $t \in [0, c]$, is a Bézier curve with control points $P_0, P_0^1, \ldots, P_0^n$, and the part, which is defined for $t \in [c, 1]$, is a Bézier curve with control points $P_0^n, P_1^{n-1}, \ldots, P_n$, where P_i^k are the intermediate points in de Casteljau's algorithm, evaluated for the parameter value c. We obtain the subdivision theorem by letting $(a, b) = (0, c)$ and $(a, b) = (c, 1)$ in the following theorem.

Theorem 18. *Let* $\mathbf{b}(t)$ *be a Bézier curve with control points* $P_0, \ldots, P_n$*, and let* a, b *be two real numbers, then the restriction of* $\mathbf{b}$ *to the interval* $[a, b]$ *is a Bézier curve with control points* $\widetilde{P}_0, \ldots, \widetilde{P}_n$*, where*

$$\widetilde{P}_k = C(a)^{n-k} C(b)^k (P_0, \ldots, P_n).$$

Proof: If $0 \le k \le n-1$, then the k'th element of the sequence

$$C(t)\big(C(a)^n, C(a)^{n-1}C(b), \ldots, C(b)^n\big) = \big((1-t)R + tL\big)\big(C(a)^n, C(a)^{n-1}C(b), \ldots, C(b)^n\big),$$

is given by

$$\begin{aligned}(1-t)C(a)^{n-k}C(b)^k + tC(a)^{n-k-1}C(b)^{k+1} &= C(a)^{n-k-1}\big((1-t)C(a) + tC(b)\big)C(b)^k \\ &= C(a)^{n-k-1}C\big((1-t)a + tb\big)C(b)^k \\ &= C(a)^{n-k-1}C(b)^k C\big((1-t)a + tb\big).\end{aligned}$$

Thus, we have

$$\begin{aligned}&C(t)\big(C(a)^n, C(a)^{n-1}C(b), \ldots, C(b)^n\big) \\ &\quad = \big(C(a)^{n-1}, C(a)^{n-2}C(b), \ldots, C(b)^{n-1}\big)C\big((1-t)a + tb\big),\end{aligned}$$

and by induction we get

$$C(t)^n\big(C(a)^n, C(a)^{n-1}C(b), \ldots, C(b)^n\big) = C\big((1-t)a + tb\big)^n.$$

Hence $C(t)^n\big(\widetilde{P}_0, \ldots, \widetilde{P}_n\big) = C\big((1-t)a + tb\big)^n (P_0, \ldots, P_n)$ and the proof is complete. □

Finally, we demonstrate how the Bernstein form is a consequence of the binomial formula.

Theorem 19. *Let* $\mathbf{b}(t)$ *be a Bézier curve with control points* $P_0, \ldots, P_n$*. Then*

$$\mathbf{b}(t) = \sum_{k=0}^{n} B_k^n(t) P_k,$$

where

$$B_k^n(t) = \binom{n}{k}(1-t)^{n-k}t^k, \quad k = 0, \ldots, n,$$

are the Bernstein polynomials of degree n.

Proof: We have

$$C(t)^n = \big((1-t)R + tL\big)^n = \sum_{k=0}^{n} \binom{n}{k} \big((1-t)R\big)^{n-k} (tL)^k$$
$$= \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k R^{n-k} L^k = \sum_{k=0}^{n} B_k^n(t) R^{n-k} L^k,$$

so

$$\mathbf{b}(t) = C(t)^n (P_0, \ldots, P_n)$$
$$= \sum_{k=0}^{n} B_k^n(t) R^{n-k} L^k (P_0, \ldots, P_n) = \sum_{k=0}^{n} B_k^n(t) P_k. \quad \square$$

It has recently been popular to view a Bézier curve as a symmetric multi-affine map (the so called *blossom*) evaluated at the diagonal. In our notation this amounts to the following:

Definition 20. *The blossom of a Bézier curve of degree n with control points $P_0, \ldots, P_n$ is defined by*

$$\mathbf{b}(t_1, \ldots, t_n) = C(t_1) \circ \ldots \circ C(t_n)(P_0, \ldots, P_n).$$

Once again we can derive the basic properties of the blossom by utilizing the commutativity of R and L, but we leave this as an exercise for the reader.

Acknowledgments. I am indebted to Lars-Erik Andersson who presented the proof of Theorem 18 (using infinite sequences) at a summer school in computational geometry, held at Linköping University, May 7–11, 1990.

References

1. Andersson, L-E., Private Notes, from a summer school in Computational Geometry, Linköping, 1990.
2. Farin, G., *Curves and Surfaces for Computer Aided Geometric Design,* Academic Press, NY, 1988.
3. Hosaka, M. and M. Kuroda, (in Japanese), J. of Information Processing Society of Japan **17** (1976), 1120–1127.
4. Hosaka, M. and M. Kuroda, A Synthesis Theory of Curves and Surfaces for CAD, Information Processing in Japan **17** (1977), 75–79.
5. Hoschek, J. and D. Lasser, *Fundamentals of Computer Aided Geometric Design,* A. K. Peters, Boston MA, 1993.

Jens Gravesen
Department of Mathematics
Building 303
Technical University of Denmark
DK-2800 Lyngby, DENMARK
J.Gravesen@mat.dtu.dk