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CLASSROOM NOTES IN APPLIED MATHEMATICS

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This section contains brief notes which are essentially self-contained applications of mathematics that can be used in the classroom. New applications are preferred, but exemplary applications not well known or readily available are accepted.

Both "modern" and "classical" applications are welcome, especially modern applications to current real world problems.

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CATASTROPHE THEORY AND CAUSTICS*

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Abstract. In this paper it is shown by elementary methods that in codimension two and under the assumption that light rays are straight lines, a caustic is the catastrophe set for a timefunction. The general case is also discussed.

1980 AMS Mathematics subject classification 58C28.

Key words. caustic, catastrophe theory, envelope

1. Introduction. Since its appearance, there has been an intense debate about catastrophe theory, not about the mathematical contents of the theory, but about some of the applications of the theory in biology, medicine, sociology, etc. In this paper catastrophe theory is applied to the theory of caustics. This is considered to be one of the more sound applications of catastrophe theory, and it has not been questioned. It has long been known empirically that a caustic has only a finite number of possible shapes. Catastrophe theory has confirmed this result and has even shown that the known lists of stable caustics is complete.

In his book *Stabilité structurelle et morphogénèse,* René Thom explains [7, p. 63] how the elementary catastrophes occur as singularities of propagating wave fronts. Later Klaus Jähnich [8], among others, examined this case more closely and gave a complete proof of Thom's conjecture.

In this paper we present an elementary proof of the special case, where the problem can be considered as two-dimensional and the light rays as straight lines. In addition the general theorem will be made plausible.

The paper is an extension of notes prepared for a seminar on catastrophe theory arranged by the Association of Mathematics Teachers in Denmark, summer, 1979. I wish to thank my teacher Professor Vagn Lundsgaard Hansen who planned the course and encouraged the present work, and my fellow instructors Martin Philip Bendsøe and Henrik Pedersen for valuable discussions.

2. The classification theorem. We start by stating Thom's theorem. A more extensive introduction to catastrophe theory can be found in Callahan [2], [3], and proofs in Zeeman and Trotman [8].

Let $f: \mathbb{R}^n \times \mathbb{R}^r \rightsquigarrow \mathbb{R}$ be a smooth function, i.e. of class C^∞. We let $\mathbf{x} = (x_1, \cdots, x_n)$ denote an element belonging to $\mathbb{R}^n$ and $\mathbf{y} = (y_1, \cdots, y_r)$ denote an element belonging to

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$\mathbb{R}^r$. Define $M_f \subseteq \mathbb{R}^n \times \mathbb{R}^r$ by

$$M_f = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^r \;\middle|\; \frac{\partial f}{\partial x_i}(\mathbf{x}, \mathbf{y}) = 0, \text{ all } i = 1, \cdots, n \right\}.$$

Generically M_f is an r-manifold (r-dimensional surface in $\mathbb{R}^{n+r}$) because it is the null-space of n equations. Let

$$\chi_f : M_f \longrightarrow \mathbb{R}^n \times \mathbb{R}^r \longrightarrow \mathbb{R}^r$$

be the map induced by the projection $\mathbb{R}^n \times \mathbb{R}^r \longrightarrow \mathbb{R}^r$. The map χ_f is called the *catastrophe map* of f. Let F denote the space of smooth functions on $\mathbb{R}^{n+r}$, with the Whitney C^∞-topology (Two functions are close if the values of the functions as well as the values of the derivatives are close, see Zeeman and Trotman [8, p. 316] or Callahan [2, p. 222].

THEOREM 1 (Thom). *If $r \leqq 5$, there exists an open dense set $F_* \subseteq F$ called the set of generic functions. If f is generic, then:*

(1) *M_f is an r-manifold.*

(2) *Any singularity of χ_f is equivalent (see below) to one of a finite number of types called elementary catastrophes.*

(3) *χ_f is locally stable with respect to small pertubations of f.*

Two maps $\chi : M \curvearrowright \mathbb{R}^r$ and $\chi' : M' \curvearrowright \mathbb{R}^r$ are equivalent if there exists diffeomorphisms $h : M \curvearrowright M'$ and $k : \mathbb{R}^r \curvearrowright \mathbb{R}^r$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\;h\;} & M' \\ \chi \big\downarrow & & \big\downarrow \chi' \\ \mathbb{R}^r & \xrightarrow{\;k\;} & \mathbb{R}^r \end{array}$$

A diffeomorphism can be considered as a curvilinear change of coordinates. The set of curvilinear singular points $(\mathbf{x}, \mathbf{y}) \in M_f$ of χ_f is denoted Δ_f and is given by

$$\Delta_f = \left\{ (\mathbf{x}, \mathbf{y}) \in M_f \;\middle|\; \det \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}, \mathbf{y}) \right] = 0 \right\}.$$

The image $\chi_f(\Delta_f)$ is called the *catastrophe set* and is denoted D_f. If χ_f and $\chi_{f'}$ are equivalent and h, k are the associated diffeomorphisms then $M_{f'} = h(M_f)$, $\Delta_{f'} = h(\Delta_f)$ and $D_{f'} = k(D_f)$, so locally D_f and $D_{f'}$ have the same shape.

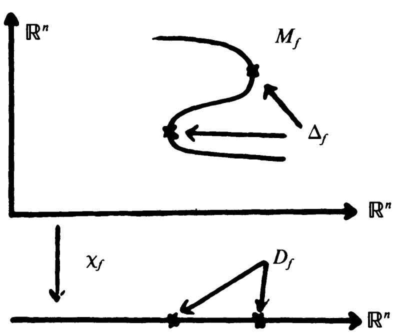

FIG. 1

3. The envelope. We next investigate the possible shape of a caustic formed by a given bundle of light. In order to solve the problem by elementary methods we make two assumptions. Assume the existence of a direction in which the light bundle is translation invariant, and that the speed of light is constant. Then the light rays are straight lines. Thus we consider a family of lines in $\mathbb{R}^2$.

Let a, b and c be real smooth functions defined on an open interval $I \subseteq \mathbb{R}$, such that

$$\det\begin{bmatrix} a(x) & b(x) \\ a'(x) & b'(x) \end{bmatrix} \neq 0 \quad \text{for all } x \in I.$$

For every $x \in I$ let $L(x)$ denote the straight line in $\mathbb{R}^2$ given by the equation

$$a(x)u + b(x)v = c(x) \quad (\text{where } (u, v) \text{ are the coordinates in } \mathbb{R}^2).$$

Assume that $x \in I$ and $x + \Delta x \in I$, with $\Delta x \neq 0$. We will find a condition that ensures that $L(x)$ and $L(x + \Delta x)$ intersect. By the mean value theorem there exist real numbers ξ, ζ, η between x and $x + \Delta x$ such that

$$\begin{aligned} a(x + \Delta x) &= a(x) + a'(\xi)\Delta x, \\ b(x + \Delta x) &= b(x) + b'(\zeta)\Delta x, \\ c(x + \Delta x) &= c(x) + c'(\eta)\Delta x. \end{aligned}$$

If (u, v) is a point on both $L(x)$ and $L(x + \Delta x)$ we then have

$$\begin{aligned} &a(x)u + b(x)v = c(x), \\ &(a(x) + a'(\xi)\Delta x)u + (b(x) + b'(\zeta)\Delta x)v = c(x) + c'(\eta)\Delta x. \end{aligned}$$

Since $\Delta x \neq 0$ these equations are equivalent to the equations

$$\begin{aligned} a(x)u + b(x)v &= c(x), \\ a'(\xi)u + b'(\zeta)v &= c'(\eta). \end{aligned}$$

It is well known that the latter pair of equations has a solution if

$$\det\begin{bmatrix} a(x) & b(x) \\ a'(\xi) & b'(\zeta) \end{bmatrix} \neq 0.$$

This is fulfilled when Δx is sufficiently small, because $\det\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right) \neq 0$ and a, b, a' and b' are continuous. The point of intersection is given by

$$(u, v) = \left(\frac{\det\begin{bmatrix} c(x) & b(x) \\ c'(\eta) & b'(\zeta) \end{bmatrix}}{\det\begin{bmatrix} a(x) & b(x) \\ a'(\xi) & b'(\zeta) \end{bmatrix}}, \frac{\det\begin{bmatrix} a(x) & c(x) \\ a'(\xi) & c'(\eta) \end{bmatrix}}{\det\begin{bmatrix} a(x) & b(x) \\ a'(\xi) & b'(\zeta) \end{bmatrix}} \right).$$

If we let $\Delta x \to 0$, then ξ, ζ and η converge to x, and by continuity, the point of intersection between $L(x)$ and $L(x + \Delta x)$ converges to

$$(u(x), v(x)) = \left(\frac{\det \begin{bmatrix} c(x) & b(x) \\ c'(x) & b'(x) \end{bmatrix}}{\det \begin{bmatrix} a(x) & b(x) \\ a'(x) & b'(x) \end{bmatrix}}, \frac{\det \begin{bmatrix} a(x) & c(x) \\ a'(x) & c'(x) \end{bmatrix}}{\det \begin{bmatrix} a(x) & b(x) \\ a'(x) & b'(x) \end{bmatrix}} \right).$$

We see that $(u(x), v(x))$ is the unique solution to the equations

$$a(x)u + b(x)v = c(x),$$

$$a'(x)u + b'(x)v = c'(x).$$

The curve $I \curvearrowright \mathbb{R}^2{:}x \longrightarrow (u(x), v(x))$ is called the envelope of the family $(L(x))_{x \in I}$ of lines. We have shown that for sufficiently small Δx, $L(x + \Delta x)$ will intersect $L(x)$ near the envelope. If the lines, as in our case, represent light rays, this implies that the light is concentrated on the envelope, i.e. the envelope is the caustic for the light rays.

4. Caustics as catastrophe sets, the special case. Besides describing light as rays, we can describe light as waves. If the waves are known we get the rays as the normals to the wavefronts. Thus the caustic of a wavefront is the envelope for the normals.

Let V be a wavefront in $\mathbb{R}^2$, that is, locally it is nothing but a C^∞-curve $(\tilde{u}, \tilde{v})$: $I \curvearrowright \mathbb{R}^2{:}x \longrightarrow (\tilde{u}(x), \tilde{v}(x))$. As we are only interested in local properties, we assume that all of V is given by $(\tilde{u}, \tilde{v})$. We can now define the timefunction T associated to V. T measures the time it takes a light ray to travel from a point belonging to V to a point belonging to $\mathbb{R}^2$. We have assumed that the speed of light is constant so we can use distance as a measure for time. We define

$$T{:}I \times \mathbb{R}^2 \curvearrowright \mathbb{R},$$

$$T{:}(x, u, v) \longrightarrow \sqrt{(\tilde{u}(x) - u)^2 + (\tilde{v}(x) - v)^2}.$$

Clearly T is smooth on $I \times (\mathbb{R}^2 \backslash V)$. If we let C denote the caustic of the wavefront V we have

THEOREM 2.

$$C \backslash V = \left\{ (u, v) \in \mathbb{R}^2 \backslash V \;\middle|\; \exists\, x \in I{:} \frac{\partial T}{\partial x}(x, u, v) = 0 \textit{ and } \frac{\partial^2 T}{\partial x^2}(x, u, v) = 0 \right\} = D_T \backslash V.$$

Proof. The last equality is simply the definition of D_T; see §2. The first equality is seen in the following way:

$$\frac{\partial T}{\partial x} = \frac{1}{T}[(\tilde{u} - u)\tilde{u}' + (\tilde{v} - v)\tilde{v}']$$

and

$$\frac{\partial^2 T}{\partial x^2} = -\frac{1}{T^2}\frac{\partial T}{\partial x}[(\tilde{u} - u)\tilde{u}' + (\tilde{v} - v)\tilde{v}'] + \frac{1}{T}[\tilde{u}'^2 + (\tilde{u} - u)\tilde{u}'' + \tilde{v}'^2 + (\tilde{v} - v)\tilde{v}'']$$

$$= \frac{1}{T}\left[-\left(\frac{\partial T}{\partial x}\right)^2 + \tilde{u}'^2 + (\tilde{u} - u)\tilde{u}'' + \tilde{v}'^2 + (\tilde{v} - v)\tilde{v}'' \right].$$

For $(u, v) \in \mathbb{R}^2 \backslash V$ we have that

$$\frac{\partial T}{\partial x}(x, u, v) = 0 \quad \text{and} \quad \frac{\partial^2 T}{\partial x^2}(x, u, v) = 0$$

is equivalent to

$$(\tilde{u}(x) - u)\tilde{u}'(x) + (\tilde{v}(x) - v)\tilde{v}'(x) = 0,$$

$$\tilde{u}'(x)^2 + (\tilde{u}(x) - u)\tilde{u}''(x) + \tilde{v}'(x)^2 + (\tilde{v}(x) - v)\tilde{v}''(x) = 0.$$

By rearranging we get

$$\tilde{u}'(x)u + \tilde{v}'(x)v = \tilde{u}(x)\tilde{u}'(x) + \tilde{v}(x)\tilde{v}'(x),$$

$$\tilde{u}''(x)u + \tilde{v}''(x)v = \tilde{u}'(x)^2 + \tilde{u}(x)\tilde{u}''(x) + \tilde{v}'(x)^2 + \tilde{v}(x)\tilde{v}''(x),$$

or equivalently

$$\tilde{u}'(x)u + \tilde{v}'(x)v = (\tilde{u}\tilde{u}' + \tilde{v}\tilde{v}')(x),$$

$$\tilde{u}''(x)u + \tilde{v}''(x)v = (\tilde{u}\tilde{u}' + \tilde{v}\tilde{v}')'(x).$$

The last equations give the envelope for the normals to V, because the normals to the curve $(\tilde{u}, \tilde{v}): x \longrightarrow (\tilde{u}(x), \tilde{v}(x))$ are given by the equation

$$(\tilde{u}(x) - u)\tilde{u}'(x) + (\tilde{v}(x) - v)\tilde{v}'(x) = 0$$

or equivalently

$$\tilde{u}'(x)u + \tilde{v}'(x)v = (\tilde{u}\tilde{u}' + \tilde{v}\tilde{v}')(x).$$

As the caustic of V is the envelope for the normals to V, the theorem follows. □

5. The general case. Let V be a wavefront in $\mathbb{R}^3$. Locally it is a C^∞-map $\tilde{\mathbf{y}}: I^2 \curvearrowright \mathbb{R}^3$: $\mathbf{x} \longrightarrow \tilde{\mathbf{y}}(\mathbf{x})$. ($\mathbf{x} = (x_1, x_2)$ denotes a point belonging to $I^2 = I \times I \subseteq \mathbb{R}^2$ and $\mathbf{y} = (y_1, y_2, y_3)$ denotes a point belonging to $\mathbb{R}^3$.) As in the preceding paragraphs we are only interested in local properties, so we assume that all of V is parametrized by $\tilde{\mathbf{y}}$.

The only assumption made is the existence of a (smooth) timefunction T associated to V, i.e. for a point $\tilde{\mathbf{y}} \in V$ and a point $\mathbf{y} \in \mathbb{R}$ we can determine the time it takes light to travel from $\tilde{\mathbf{y}}$ to $\mathbf{y}$. T can be regarded as a smooth function: $I^2 \times \mathbb{R}^3 \curvearrowright \mathbb{R}$.

Let $\mathbf{y} \in \mathbb{R}^3$. We wish to determine the points $\tilde{\mathbf{y}} \in V$ emitting light passing through $\mathbf{y}$.

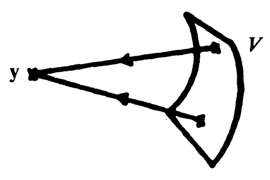

FIG. 2

According to Fermat's principle a light ray travels in such a way that the time taken is the least possible. The points $\tilde{\mathbf{y}}(\mathbf{x})$ emitting light hitting $\mathbf{y}$ are thus given by the equations

$$\frac{\partial T}{\partial x_i}(\mathbf{x}, \mathbf{y}) = 0, \qquad i = 1, 2.$$

By definition, this means that $(\mathbf{x}, \mathbf{y})$ belongs to M_T. We conclude, $\mathbf{y} \in \mathbb{R}^3$ is hit by a light ray from $\tilde{\mathbf{y}}(\mathbf{x})$ if and only if $(\mathbf{x}, \mathbf{y}) \in \chi_T^{-1}(\mathbf{y})$. It is clear that the light is concentrated on the critical values of χ_T, so the caustic is the catastrophe set D_T.

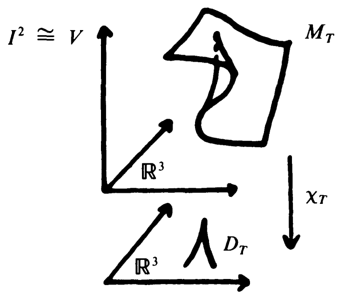

FIG. 3

6. Fermat's principle and the timefunction. We will give a brief discussion of the timefunction. In order to do this we state *Fermat's principle* or *the principle of least time*, (see [4, Chap. 26]). The most common variant is: "Out of all possible paths that it might take to get from one point to another, light takes the path which requires the shortest time." To give the precise statement we must look at the space of all paths between the two given points. Then "light takes a path which is a critical point for the timefunction." Similarly we have that light emitted by a wavefront takes a path to a given point which is a critical point for the timefunction, this time defined on the space of all paths between the wavefront and the given point.

From Fermat's principle we get the well-known facts that in a medium with constant speed of light the light rays are straight lines and that light rays are orthogonal to the wavefronts. From Fermat's principle we can also deduce the laws of reflection and refraction.

Consider a wavefront V. In the preceding paragraphs we said that V has a timefunction if for a point $\mathbf{x}$ belonging to V and a point $\mathbf{y}$ belonging to $\mathbb{R}^3$ it is possible to determine the time it takes light to travel from $\mathbf{x}$ to $\mathbf{y}$. So in order to define a timefunction for V there must only exist one path from $\mathbf{x}$ to $\mathbf{y}$ which is a critical point for the timefunction. Notice that this unique path or light ray is not necessarily a light ray emitted by the wavefront V.

We will consider two examples that indicate that catastrophe theory also applies to some systems without a timefunction.

Consider an arrangement with a mirror; see Fig. 4. Clearly it is impossible to define a timefunction for this system, because there are two possible light rays from $\tilde{\mathbf{y}}(\mathbf{x})$ to $\mathbf{y}$ so

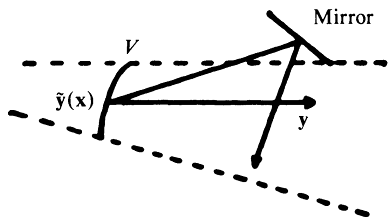

FIG. 4

$T(\mathbf{x}, \mathbf{y})$ would be doublevalued. But if we only consider light rays or paths which do not hit the mirror, it is possible to define a timefunction. The light rays emitted by the wavefront V plus the nearby ones do not hit the mirror, so we can use Fermat's principle and the discussion in the preceding paragraph also applies to this case.

Consider an arrangement with a lens; see Fig. 5. Again it is impossible to define a timefunction, because $T(\mathbf{x}, \mathbf{y})$ would be multivalued. In the first example we disregarded all but one light ray between every pair of points $(\mathbf{x}, \mathbf{y})$ belonging to $V \times \mathbb{R}^3$. If we do this in this example, we would ignore light rays arbitrarily close to the remaining light ray, and

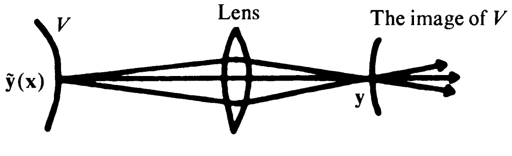

FIG. 5

thus make the use of Fermat's principle impossible. Instead, we observe that we can define a timefunction if we only look at points $\mathbf{y}$ outside the image of V. We conclude that the part of the caustic outside the image of V is the catastrophe set for a timefunction. By choosing another wavefront V' with image disjoint from the image of V, we see that all of the caustic locally is the catastrophe set for a timefunction.

The two examples above do not contradict Fermat's principle. They simply indicate that it is impossible to define a timefunction depending only on the initial points, which is required in our application of catastrophe theory. It is of course possible to define the timefunction on the set of paths and thus make use of Fermat's principle.

7. Conclusion. We have shown that if a given bundle of light has a timefunction, then the caustic is the catastrophe set of this timefunction. Theorem 1 now gives that a stable caustic, locally, only can have a finite number of shapes (stable with respect to small pertubations of the timefunction, i.e., small pertubations of both the wavefront and the media). We have to be careful, because to a given point on the caustic we shall look locally not only on the caustic around the point, but also on the wavefront around the points, from which the light rays come. It is possible that caustics from different locations on the wavefront appear on the same spot, so we can get a caustic, consisting of several elementary catastrophe sets. If we use that stable intersections between surfaces in $\mathbb{R}^3$ only occur as intersections between two or three 2-dimensional surfaces or between a 1-dimensional and a 2-dimensional surface, we get that around a given point the shape of a stable caustic must have one of the basic forms shown in Figs. 6-13; see Callahan [3] or Poston and Stewart [6, Chap. 9].

We have shown that a stable caustic locally at most can have one of the eight shapes mentioned above. In [5] K. Jänich shows that all eight shapes can be realized as caustics. Some of them appear in photos in M.V. Berry [1], which also contains a discussion of the unstable caustics.

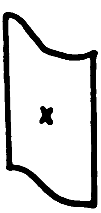

FIG. 6. *The fold.*

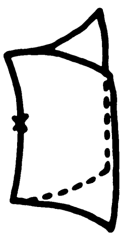

FIG. 7. *The cusp.*

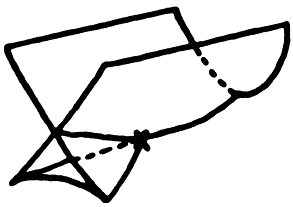

FIG. 8. *The swallow tail.*

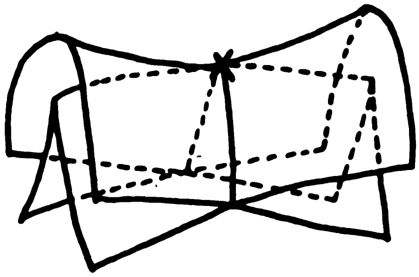

FIG. 9. *The hyperbolic umbilic.*

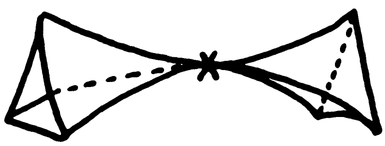

FIG. 10. *The elliptic umbilic.*

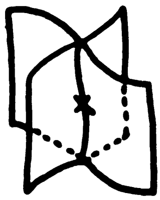

FIG. 11. *Intersection between two folds.*

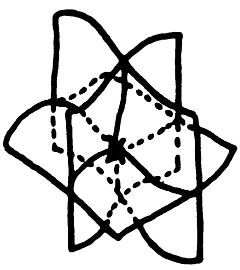

FIG. 12. *Intersection between three folds.*

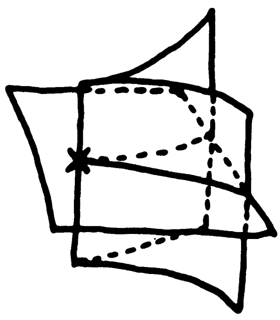

FIG. 13. *Intersection between a fold and a cusp.*

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