Numerical bifurcation of Hamiltonian relative periodic orbits

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Abstract

Relative periodic orbits (RPOs) are ubiquitous in symmetric Hamiltonian systems and occur for example in celestial mechanics, molecular dynamics and rigid body motion. RPOs are solutions which are periodic orbits of the symmetry-reduced system. In this paper we analyze certain symmetry-breaking bifurcations of Hamiltonian relative periodic orbits and show how they can be detected and computed numerically. These are turning points of RPOs, relative period-doubling and relative period-halving bifurcations along branches of RPOs. In a comoving frame the latter correspond to symmetry-breaking/symmetry-increasing pitchfork bifurcations or to period doubling/period-halving bifurcations. We apply our methods to the family of rotating choreographies which bifurcate from the famous Figure Eight solution of the three body problem as angular momentum is varied. We find that the Eights rotating around the $e_2$-axis bifurcate to the family of rotating Eights that connect to the Lagrange relative equilibrium. Moreover, we find several relative period-doubling bifurcations and a turning point of the planar rotating choreography which bifurcates from the Figure Eight solution when the third component of the angular momentum vector is varied.

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1 Introduction

Relative periodic orbits (RPOs) are ubiquitous in symmetric Hamiltonian systems. They are periodic orbits of the symmetry reduced system. In the original phase space they represent a periodic vibrational dynamics superimposed with a drift along the symmetry group, e.g. superimposed with a rotation or translation. In recent years a lot of progress has been made in the bifurcation theory of Hamiltonian RPOs, see [14, 16, 22, 20]. But a general theory of generic bifurcations of RPOs so far only exists for dissipative systems [10, 21]. The additional structure of symmetric Hamiltonian systems changes the generic behaviour compared to general systems dramatically. As a result of this, a general bifurcation theory of Hamiltonian RPOs and the parallel development of numerical methods for the detection and computation of those bifurcations are an open problem. Recent progress in the continuation of normal periodic orbits of symmetric Hamiltonian systems has been made by Muñoz-Almaraz et al [4, 15]. Chenciner et al [3] continue rotating choreographies of the three-body problems which bifurcate from the famous Figure Eight solution [2].

In [20] we developed a persistence theory for nondegenerate RPOs with generic drift momentum pair (see Section 2 below for definitions of these terms). In [25] we have extended the numerical methods for the continuation of periodic orbits of general symmetric systems from [23] to Hamiltonian systems based on the theoretical persistence results from [20].

In this paper we prove a persistence result for transversal Hamiltonian RPOs of compact group actions with generic drift-momentum pair. These include turning points of RPOs in energy or momentum. Moreover, we prove a theorem on relative period doubling of RPOs with regular drift momentum pair. We then present underdetermined nonlinear systems of equations that are satisfied by Hamiltonian relative equilibria and RPOs respectively, have regular derivative at their solutions and are therefore amenable to standard numerical path-following methods. Moreover, we develop algorithms for the computation of turning points, relative period doubling and relative period halving bifurcations of Hamiltonian RPOs. We continue in a conserved quantity of the system, which is either the energy or a component of the momentum map. The list of bifurcations of Hamiltonian RPOs which we study is by no means exhaustive. In particular, in this paper we assume that the spatial symmetry of the RPO is trivial (by reducing the dynamics on the fixed point space of the spatial symmetry). Hence we do not deal with bifurcations breaking the spatial symmetry at all. Although we present new theoretical results, the emphasis in this paper is the “translation” of theoretical persistence and bifurcation results into efficient algorithms for the numerical path-following and the numerical computation of bifurcations.
The paper is structured as follows: The topic of Section 2 is the continuation of transversal relative equilibria and RPOs. First, in Section 2.1, we review the notion of transversal relative equilibria and their persistence and present a numerical method for their computation. In Section 2.2 we introduce so-called transversal RPOs. These generalize the concept of transversal relative equilibria introduced in [17]. The nondegeneracy condition which we required in earlier works [24, 20, 25] is a more restrictive condition. We then generalize our earlier numerical continuation methods [25] for nondegenerate RPOs to transversal RPOs of compact group actions in Sections 2.3, 2.4. In Section 2.5 we numerically analyze turning points of relative equilibria and RPOs which are continued in some component of the momentum or in energy (in the case of RPOs). These occur at transversal, but degenerate relative equilibria rps. RPOs. We present numerical methods for their detection and computation.  

In Section 3 we present a theorem on relative period doubling bifurcations of nondegenerate RPOs with generic drift momentum pair. In a comoving frame these correspond to period-doubling or symmetry breaking pitchfork bifurcations (as analyzed in [5] for non-Hamiltonian systems with discrete symmetries). We then present numerical methods for the detection and computation of relative period doubling and relative period halving bifurcations of Hamiltonian relative periodic orbits.

In Section 4 we apply our results to rotating choreographies of the three-body problem which bifurcate from the famous Figure Eight solution of Chenciner and Montgomery [2]. It is well-known that one of the non-planar rotating Figure Eight families to the Lagrange relative equilibrium (for example, see the discussion in [3]). We find that the other non-planar family of rotating Figure Eights bifurcates in a relative period halving bifurcation to the family of rotating choreographies that connect to the Lagrange relative equilibrium. Moreover, we find several relative period-doubling bifurcations and a turning point of the planar rotating choreography which bifurcates from the Figure Eight solution when the third component of the angular momentum vector is varied.

## 2 Persistence and numerical continuation of transversal RPOs

In this section we present methods for the continuation of transversal RPOs extending results of [20, 23]. We start with the simpler case of continuation of transversal relative equilibria in Section 2.1. Then we prove a persistence result for transversal RPOs (Section 2.2. In Sections 2.3 and 2.4 we develop a method for the path-following of transversal RPOs, and in Section 2.5 we show how to detect and compute turning points of relative equilibria and RPOs.

### 2.1 Continuation of transversal Hamiltonian relative equilibria

We consider a Hamiltonian system

\[ \dot{x} = f_H(x) = J \nabla H(x) \]  

with Hamiltonian (energy) \( H(x) \) on a finite-dimensional symplectic vector space \( \mathcal{X} = \mathbb{R}^{2d} \) with symplectic structure matrix \( J \) (i.e., \( J \) is skew-symmetric and invertible). Let

\[ \Omega(v, w) = (J^{-1}v, w) \]

be the symplectic form generated by \( J \). Let \( \Phi^t(x_0) \) denote the flow of (2.1) by , i.e., \( x(t) = \Phi^t(x_0) \) is a solution of (2.1) with initial value \( x(0) = x_0 \). Then the energy \( H(x) \) is a conserved quantity of (2.1): \( H(\Phi^t(x_0)) = H(x_0) \) for all \( x_0, t \). We assume that a finite-dimensional compact Lie
group $\Gamma$ acts on $X$ symplectically symplectically (i.e., $\Omega$ is $\Gamma$-invariant) and that the Hamiltonian $H$ is $\Gamma$-invariant, which implies that (2.1) is $\Gamma$-equivariant, i.e., $f_H(\gamma x) = \gamma f_H(x)$, $x \in X$, $\gamma \in \Gamma$. So whenever $x(t)$ is a solution of (2.1) then so is $\gamma x(t)$. We call the elements of $\Gamma$ the symmetries of (2.1). Let $g = T_d\Gamma$ denote the Lie algebra of $\Gamma$. By Noether’s theorem locally there is a conserved quantity $J_\xi$ of (2.1) for each continuous symmetry $\xi \in g$ of the system such that $J_\xi$ is the Hamiltonian for the symplectic flow $x \rightarrow \exp(t\xi)x$, see, e.g., [1, 12]. The map $J_\xi(x) = J(x)(\xi)$ is linear in $\xi$, so that $J$ is a map from a neighbourhood of each $x \in X$ to $g^*$, called a momentum map. Let $\text{Ad}_\gamma$, $\gamma \in \Gamma$, denote the adjoint action of $\Gamma$ on $g$: $\text{Ad}_\gamma \xi = \gamma \xi \gamma^{-1}$, $\xi \in g$, $\gamma \in \Gamma$. Then the coadjoint action of $\Gamma$ on $g^*$ is given by

\[ \gamma \mu = (\text{Ad}^*_\gamma)^{-1} \mu, \quad \gamma \in \Gamma. \quad (2.2) \]

We assume throughout the paper that $J$ is defined on the whole of $X$ and is $\Gamma$-equivariant with respect to the $\Gamma$-action on $X$ and the coadjoint action on $g^*$. Moreover, we choose an $\text{Ad}$-invariant inner product on $g$ such that the adjoint action on $g$ is by orthogonal matrices and the adjoint and coadjoint actions can be identified.

As usual (c.f. [7]), for an action of a group $\Gamma$ on a space $X$ we define the isotropy group of $x \in X$ as $\Gamma_x = \{ \gamma \in \Gamma \mid \gamma x = x \}$. For any subgroup $K$ or element $\gamma \in \Gamma$ of $\Gamma$ we define the fixed point space of $K$ r.s.p. $\gamma$ as $\text{Fix}_X(K) = \{ x \in X \mid \gamma x = x \ \forall \gamma \in K \}$ and $\text{Fix}_X(\gamma) = \{ x \in X \mid \gamma x = x \}$ respectively. We denote by $N(K)$ the normalizer of the subgroup $K$ of $\Gamma$. For any group $\Gamma$ define $\Gamma^{id}$ to be the identity component of $\Gamma$.

**Example 2.1** In the case of rotational symmetries $\Gamma = \text{SO}(3)$ the space of momenta is $g^* = \text{so}(3)^* \equiv \mathbb{R}^3$ and $J : X \rightarrow \mathbb{R}^3$ is the angular momentum, see Section 4 below for an example from celestial mechanics. In this $g = \text{so}(3) \simeq \mathbb{R}^3$ and the adjoint and coadjoint actions are just the usual multiplication by matrices in $\text{SO}(3)$. Here the identification $\text{so}(3) \simeq \mathbb{R}^3$ is given by the map

\[ \xi = (\xi_1, \xi_2, \xi_3) \rightarrow \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}. \quad (2.3) \]

The Lie bracket becomes $[\xi, \eta] = \xi \times \eta$, where $\xi, \eta \in \mathbb{R}^3 \simeq \text{so}(3)$.

A point $x \in X$ lies on a relative equilibrium $\Gamma x$ if there is some $\xi \in g$ such that $\xi x = f_H(x)$, i.e., the relative equilibrium through $\xi x$ is an equilibrium of the Hamiltonian system (2.1) in a frame moving with velocity $\xi$. We call $\xi$ the drift velocity of the relative equilibrium at $x$.

Let $x$ lie on a relative equilibrium with drift velocity $\xi$ and momentum $\mu = J(\xi)$ at $x$. There is a simple relation between the drift velocity and momentum of a relative equilibrium:

\[ \text{ad}^*_\xi \mu = 0, \quad (2.4) \]

which is implied by momentum conservation: $\dot{\mu} = J(\dot{x}) = J(\Phi_t(\xi)) = J(\exp(t\xi)x) = \text{Ad}^*_{\exp(-t\xi)} \mu$, c.f. [17, 24]. This relation is crucial for the problem of persistence to nearby momentum values, as we will see below.

**2.1.1 Persistence of transversal relative equilibria**

Before we come to the numerical continuation of transversal RPOs we first present a persistence result for transversal relative equilibria with regular velocity-momentum pair and show how to continue them numerically.
Definition 2.2 [17, 24]

(i) We call pairs \((\xi, \mu) \in \mathfrak{g} \times \mathfrak{g}^*\) satisfying (2.4) velocity-momentum pairs and denote the space of velocity-momentum pairs by
\[
(\mathfrak{g} \oplus \mathfrak{g}^*)^\circ := \{ (\xi, \mu) \in \mathfrak{g} \oplus \mathfrak{g}^*, \; \text{ad}_\xi^\ast \mu = 0 \}.
\]

(ii) We define an action of \(\Gamma\) on the space of velocity-momentum pairs as follows:
\[
\gamma(\xi, \mu) = (\text{Ad}_\gamma \xi, (\text{Ad}_\gamma^\ast \mu)^{-1}), \quad \gamma \in \Gamma, \quad (\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^\circ.
\]
For later purposes we define the isotropy subgroup \(\Gamma_{(\xi,\mu)}\) of \((\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^\circ\) with respect to this action as
\[
\Gamma_{(\xi,\mu)} = \{ \gamma \in \Gamma, \; (\gamma(\xi), \gamma^0) = (\xi, \mu) \},
\]
denote its Lie algebra by \(\mathfrak{g}_{(\xi,\mu)}\) and let \(\dim \mathfrak{g}_{(\xi,\mu)} = \dim \mathfrak{g}_{(\xi,\mu)}\). Moreover, we define the isotropy subgroup of \(\xi \in \mathfrak{g}\) as \(\Gamma_\xi = \Gamma_{(0,0)} = \{ \gamma \in \Gamma, \; \text{Ad}_\gamma \xi = \xi \}\) and the momentum isotropy subgroup of \(\mu \in \mathfrak{g}^*\) by \(\Gamma_\mu = \Gamma_{(0,\mu)} = \{ \gamma \in \Gamma, (\text{Ad}_\gamma^\ast)^{-1} \mu = \mu \}\), denote their Lie algebras by \(\mathfrak{g}_\xi\) and \(\mathfrak{g}_\mu\) respectively, and define \(\dim \mathfrak{g}_\xi = \dim \mathfrak{g}_\xi\), \(\dim \mathfrak{g}_\mu = \dim \mathfrak{g}_\mu\).

(iii) We call a velocity-momentum pair \((\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^\circ\) regular if \(\dim \mathfrak{g}_{(\xi,\mu)}\) is locally constant in the space of velocity-momentum pairs (2.5).

(iv) We call \(\xi \in \mathfrak{g}\) regular if \(\dim \mathfrak{g}_\xi\) is locally constant in \(\mathfrak{g}\).

(v) We call \(\mu \in \mathfrak{g}^*\) regular if \(\dim \mathfrak{g}_\mu\) is locally constant in \(\mathfrak{g}^*\).

Remark 2.3 As shown in [17, 24], \((\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^\circ\) is regular if and only if \(\mathfrak{g}_{(\xi,\mu)}\) is the Lie algebra of a maximal torus. In particular, for a regular velocity-momentum pair \((\xi, \mu)\) the isotropy subalgebra \(\mathfrak{g}_{(\xi,\mu)}\) is abelian. Regular velocity-momentum pairs \((\xi, \mu)\) are generic in the space of velocity-momentum pairs \((\mathfrak{g} \oplus \mathfrak{g}^*)^\circ\). Regular \(\mu \in \mathfrak{g}^*\) are generic in \(\mathfrak{g}^*\) and regular \(\xi \in \mathfrak{g}\) are generic in \(\mathfrak{g}\). The velocity of a regular velocity momentum pair is generically regular. In this case \(\mathfrak{g}_{(\xi,\mu)} = \mathfrak{g}_\xi\) holds. Similarly, the momentum \(\mu\) of a regular velocity momentum pair \((\xi, \mu)\) is generically regular. In this case \(\mathfrak{g}_{(\xi,\mu)} = \mathfrak{g}_\mu\) holds. These statements are needed later on.

Next, we describe the form that (2.1) takes, in symmetry-adapted coordinates. This is needed for the derivation of the persistence result and the numerical continuation method for relative equilibria. Denote by \(N\) a normal space transverse to \(\Gamma \tilde{x}\) at \(\tilde{x}\), i.e., \(\mathcal{X} = \Gamma_{\tilde{x}} \mathcal{X} \oplus N\). Then \(N\) is a model for the space of group orbits \(\mathcal{X}/\Gamma\) near \(\tilde{x}\). Moreover there are a choice of normal space \(N\) and coordinates \(x \simeq (\gamma, \nu), \; \gamma \in \Gamma, \; \nu \in N, \) near \(\Gamma \tilde{x}\) such that \(\tilde{x} \simeq (\text{id}, 0)\), and the dynamics in these coordinates takes the form [18]:
\[
\dot{\gamma} &= \gamma f_\Gamma (\nu, w) = \gamma D_\nu h (\nu, w), \\
\dot{\nu} &= f_{\mathcal{X}_\Gamma}(\nu, w) = \text{ad}_{\nu}^\ast h (\nu, w), \\
\dot{w} &= f_{\mathcal{X}_N}(\nu, w) = J_{\mathcal{X}_N} w = \mathfrak{J}_{\mathcal{X}_N} w.
\]
(2.6)

Here \(\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1\) and \(\nu \in N\) is decomposed as \(\nu = (\nu, w), \; \nu \in \mathcal{X}_0, \; w \in \mathcal{X}_1\). The space \(\mathcal{X}_1 = \ker DJ(\tilde{x}) \cap \mathcal{X}\) is symplectic with symplectic structure matrix \(J_{\mathcal{X}_1}\) and is called symplectic normal space. Let \(n_\mu\) be a \(\Gamma_{\mu}\)-invariant complement to \(\mathfrak{g}_\mu\) in \(\mathfrak{g}^*\). Then the annihilator \(\text{ann}_{\mathfrak{g}^*}(n_\mu)\) of \(n_\mu\) in \(\mathfrak{g}\) is a \(\Gamma_{\mu}\)-invariant section transverse to the momentum group orbit \(\Gamma \tilde{\mu}\) at \(\tilde{\mu}\) in \(\mathfrak{g}^*\) and \(\mathcal{X}_0 \simeq \mathfrak{g}_\mu^* \simeq \text{ann}(n_\mu)\). Moreover \(h(\nu, w)\) is the Hamiltonian in the coordinates \(x \simeq (\gamma, \nu, w)\) The original relative equilibrium corresponds to the equilibrium \((\nu, w) = 0\) of the \((\nu, w)-\text{subsystem}\).
of (2.6), i.e., \( D_w h(0) = 0 \). Moreover \( f_{r}(0,0) = \dot{\xi} \) where \( \dot{\xi} \) is the drift velocity of the relative equilibrium at \( \bar{x} \). The momentum map in these coordinates takes the form

\[
j(\gamma, \nu, w) = \gamma (\bar{\mu} + \nu).
\]

(2.7)

With respect to the decomposition \( \mathcal{X} = \mathcal{T}_x \Gamma \bar{x} \oplus N_0 \oplus N_1 \), the linearization \( A = D(f_H(\bar{x}) - \dot{\xi}) \) at the relative equilibrium in a frame moving with its drift velocity \( \dot{\xi} \) is given by

\[
A = \begin{pmatrix}
-\text{ad}_\xi & D^2_w h(0) & D^2_{w} h(0) \\
0 & 0 & 0 \\
J_{N_1} D^2_{J,w} h(0) & J_{N_1} D^2_{w} h(0) & 0
\end{pmatrix},
\]

(2.8)

see \[18\]. Decompose \( \nu \in \mathfrak{g}_\mu^* \) as \( \nu = \chi + \zeta \) where \( \chi \in \ker \text{ad}_\xi \mid \mathfrak{g}_\mu^* \simeq \mathfrak{g}_\mu^*(\xi, \bar{\mu}) \) and \( \zeta \in \text{ann}_{\mathfrak{g}_\mu^*}(\mathfrak{g}_\mu^*(\xi, \bar{\mu})) \).

Here we use that \( \ker \text{ad}_\xi \mid \mathfrak{g}_\mu^* = \text{ann}_{\mathfrak{g}_\mu^*}(\text{ad}_\xi \mathfrak{g}_\mu) \), that \( \mathfrak{g}_\mu^*(\xi, \bar{\mu}) = \ker \text{ad}_\xi \mid \mathfrak{g}_\mu^* \) and that

\[
\ker \text{ad}_\xi \mid \mathfrak{g}_\mu^* \oplus \text{ad}_\xi \mathfrak{g}_\mu = \mathfrak{g}_\mu.
\]

(2.9)

The latter is true because \( \Gamma \) is compact and \( \text{ad}_\xi \) is semisimple.

We see from (2.8) that Jordan blocks of \( A = D(f_H(\bar{x}) - \dot{\xi}) \) to the eigenvalue 0 of \( A \) corresponding to the kernel \( \mathfrak{g}_\mu^*(\xi, \bar{\mu}) \) of \( A_0 := \text{ad}_\xi \mid \mathfrak{g}_\mu^* \) and the kernel of \( A_1 = D_w f_{N_1}(0) \) occur in the matrix \( A_{10} := D_v f_{N_1}(0) \) and may result in \( [A_{10}, A_1] \) having full rank even if \( A_1 \) does not have full rank. This motivates the following definition:

**Definition 2.4** A relative equilibrium \( \Gamma \bar{x} \) with regular velocity-momentum pair is called transversal if the matrix

\[
[D_{\chi} f_{N_1}(\nu, w), D_w f_{N_1}(\nu, w)]|_{(\nu, w) = 0}
\]

has full rank.

As in \[24\] we call a relative equilibrium nondegenerate if \( D_w f_{N_1}(0) \) is invertible. Our definition of a transversal relative equilibrium is equivalent to the definition: By \[17, \text{Theorem 4} \] a relative equilibrium of a compact group action with regular velocity-momentum pair is transversal if in our notation the matrix \( D_w f_{N_1}(0) \) is either invertible or has a semisimple eigenvalue 0 and if \( D_{\chi} f_{N_1}(0), \chi \in \mathfrak{g}_\mu^*(\xi, \bar{\mu}), \nu = (\chi, \zeta) \in \mathfrak{g}_\mu^* \), maps \( \mathfrak{g}_\mu^*(\xi, \bar{\mu}) \) onto the kernel of \( D_w f_{N_1}(0) \). Equation (5) of \[17\] gives, in our notation,

\[
A_1 A_{10} = -A_{10} A_1.
\]

This implies that the image of \( D_{\chi} f_{N_1}(0) \) lies in the kernel of \( D_w f_{N_1}(0) \). Therefore the definition of transversality of \[17\] is equivalent to the condition (2.10) of Definition 2.4.

**Remark 2.5** Let \( S_\mu \) be a nonlinear slice transverse to \( \Gamma \bar{x} \) in \( J^{-1}(\bar{\mu}) \) where \( \bar{\mu} = J(\bar{x}) \). Then \( T_x S_\mu \simeq N_1 \) and \( S_\mu \) is called the Marsden-Weinstein reduced phase space. Then the relative equilibrium \( \Gamma \bar{x} \) is a critical point of \( H|_{S_\mu} \), i.e., \( DH|_{S_\mu}(\bar{x}) = 0 \). If \( \bar{\mu} \) is regular then \( S_\mu \) has constant dimension for \( \mu \approx \bar{\mu} \) and we can choose \( S_\mu \) to depend smoothly on \( \mu \). In this case the transversality condition (2.10) is equivalent to the condition that \( D_{\chi, \bar{x}} DH|_{S_{\bar{\mu}}, \bar{x}}(\bar{x}) \) has full rank.

As shown in \[17\], near a transversal relative equilibrium \( \Gamma \bar{x} \) of a compact group action with regular velocity momentum pair \( (\xi, \bar{\mu}) \) as defined in Definition 2.4, there is an \( r \)-dimensional manifold of relative equilibria, with \( r = \dim \mathfrak{g}_\mu^*(\xi, \bar{\mu}) \):

**Theorem 2.6** Let \( \bar{x} \) lie on a transversal relative equilibrium relative equilibrium \( \Gamma \bar{x} \) of (2.1) with regular velocity-momentum pair \( (\xi, \bar{\mu}) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c \). Let \( r = r(\xi, \bar{\mu}) \). Then there is an \( r \)-dimensional family \( \Gamma \chi(s), s \in \mathbb{R}^r \), of relative equilibria near \( \Gamma \bar{x} \) such that \( x(0) = \bar{x}, \) with momentum \( \bar{\mu} + \chi(s) \), \( \chi(s) \in \mathfrak{g}_\mu^*(\xi, \bar{\mu}) \) where \( \chi(0) = 0 \), and with drift velocity \( \xi(s) \in \mathfrak{g}_\mu^*(\xi, \bar{\mu}) \) close to \( \xi(0) = \xi \). If the relative equilibrium \( \Gamma \bar{x} \) is nondegenerate then we can choose \( s = \chi \in \mathfrak{g}_\mu^*(\xi, \bar{\mu}) \).

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Proof. For sake of completeness and to see the connection with the numerical continuation method of Section 2.1.2 we include a proof which is only valid for compact groups and different from the proof given in [17].

We see from (2.8) that \( \tilde{x} \) is a transversal relative equilibrium if we only reduce by the group \( \tilde{\Gamma} = \Gamma_\xi \). Denote by \( \tilde{\mathcal{N}} = \tilde{\mathcal{N}}_0 \oplus \tilde{\mathcal{N}}_1 \) a normal space transverse to the group orbit such that in the coordinates \( x \cong (\tilde{\gamma}, \tilde{\nu}, \tilde{w}) \), \( \tilde{\gamma} \in \tilde{\Gamma} \) the vectorfield (2.1) takes the form (2.6). Since \( g_{\xi(\tilde{\nu},\tilde{\mu})} \) is abelian by Remark 2.3 we have \( \tilde{f}_{\tilde{\mathcal{N}}_0}(\tilde{\nu}, \tilde{w}) \equiv 0 \) where \( \tilde{\nu} \in g^*_{\xi(\tilde{\nu},\tilde{\mu})} \), and \([\tilde{D}_{\tilde{w}} \tilde{f}_{\tilde{\mathcal{N}}_1}(0), \tilde{D}_{\tilde{\nu}} \tilde{f}_{\tilde{\mathcal{N}}_1}(0)]\) has full rank. So there is a manifold of equilibria \((\tilde{\nu}(s), \tilde{w}(s))\) of \( \tilde{f}_{\tilde{\mathcal{N}}_i}, s \in \mathbb{R}^r \), which gives a manifold of relative equilibria \( x(s) \) of (2.1).}

2.1.2 Numerical computation of transversal relative equilibria

Let \( \Gamma \tilde{x} \) be a transversal relative equilibrium with regular velocity-momentum pair \( (\tilde{\xi}, \tilde{\mu}) \) at \( \tilde{x} \). Let \( e^1_\xi, \ldots, e^q_\xi \) denote a basis of \( g_{\xi(\tilde{\nu},\tilde{\mu})} \). Identify \( (\xi_1, \ldots, \xi_r) \in \mathbb{R}^r \) with \( \xi = \sum_{i=1}^q \xi_i e^i_\xi \). Let \( J_i(x) := J_{e^i_\xi}(x) \). Because of Remark 2.3 typically the drift velocity \( \xi \) of the relative equilibrium is regular, and we have \( g_{\xi} = g_{\xi(\tilde{\nu},\tilde{\mu})} \).

The manifold of relative equilibria from Theorem 2.6 can then be computed by solving the underdetermined system

\[
F(x, \xi, \lambda_\mu) = f_H(x) - \sum_{i=1}^r \lambda_\mu, i \nabla J_i(x) - \sum_{i=1}^r \xi_i e^i_\xi x = 0, \tag{2.11}
\]

where \( F : X \times \mathbb{R}^{2r} \to X \):

**Theorem 2.7** Let \( \tilde{x} \) lie on a transversal relative equilibrium \( \Gamma \tilde{x} \) with regular velocity \( \tilde{\xi} \in g \) and momentum \( \tilde{\mu} \) and let \( r = \dim g_{\xi(\tilde{\nu},\tilde{\mu})} \). Then the derivative \( DF(y) \) of (2.11) has full rank at any solution \( y = (x, \xi, \lambda_\mu) \) of \( F = 0 \) close to \((\tilde{x}, \tilde{\xi}, 0)\), and any such solution satisfies \( \lambda_\mu = 0 \) and, hence, is a relative equilibrium of (2.1).

**Proof.** From (2.8) we see that, to take account of the symmetry-induced kernel vectors of \( Df_H(\tilde{x}) - \tilde{\xi} \), it suffices to reduce by \( \tilde{\Gamma} := Y_{\xi} \), as in the proof of Theorem 2.6. From

\[
DF(\tilde{x}, \tilde{\xi}, 0) = (Df_H(\tilde{x}) - \tilde{\xi}), e^1_\xi \tilde{x}, \ldots, e^q_\xi \tilde{x}, \nabla J_1(\tilde{x}), \ldots, \nabla J_r(\tilde{x}) \),
\]

we see that \( DF(\tilde{x}, \tilde{\xi}, 0) \) has full rank if and only if \([D_{\tilde{w}} f_{\mathcal{N}_1}(0), D_{\tilde{\nu}} f_{\mathcal{N}_1}(0)]\) has image \( \tilde{\mathcal{N}}_1 \) (as defined in the proof of Theorem 2.6. This condition is satisfied if and only if the relative equilibrium is transversal.

The solution manifold of (2.11) has dimension \( 2r \). By Theorem 2.6 the points \((\gamma_{x(s)}, \xi(s), 0)\), where \( \gamma \in \Gamma^m_{\xi(\tilde{\nu},\tilde{\mu})} \), are solutions of \( F(x, \xi, 0) = 0 \) with \( F \) from (2.11), and this set is also \( 2r \)-dimensional. Therefore we have \( \lambda_\mu = 0 \) in any solution of (2.11) near \((\tilde{x}, \tilde{\xi}, 0)\).}

Under the assumptions of Theorem 2.7, the underdetermined system (2.11) is amenable to standard numerical methods; for example, the Gauss-Newton method applied to (2.11) converges for initial values \( y = (x, \xi, \lambda_\mu) \) close to \((\tilde{x}, \tilde{\xi}, 0)\).

**Remark 2.8** In Theorem 2.6 we only assume that the relative equilibrium \( \Gamma \tilde{x} \) has a regular velocity-momentum pair \( (\xi, \mu) \) and not necessarily a regular velocity \( \xi \), so in general \( g_\xi \neq g_{\xi(\tilde{\nu},\tilde{\mu})} \). In this case let \( e^1_\xi, \ldots, e^q_\xi \) denote a basis of \( g_\xi \) such that \( \text{span}(e^{i+1}_\xi, \ldots, e^q_\xi) = n_\mu \cap g_\xi \) and \( \text{span}(e^1_\xi, \ldots, e^q_\xi) = g_{\xi(\tilde{\nu},\tilde{\mu})} \). As before, identify \( \xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q \) with \( \sum_{i=1}^q \xi_i e^i_\xi \).
Because of Remark 2.3 we typically have \( q = r \). In the case that \( q > r \) at \( \bar{x} \) then the derivative \( D \tilde{F}(\bar{x}, \xi, 0) \) of (2.11) does not have full rank, and convergence of the Gauss-Newton method applied to (2.11) is expected to be slow, also at relative equilibria nearby. Instead of (2.11) it is then advantageous to solve

\[
F(x, \xi, \lambda_\mu) = \begin{pmatrix}
J_r(x) - \sum_{i=1}^{r} \lambda_{\mu_i} \nabla J_i(x) - \sum_{i=1}^{q} \xi_i e^i_{\xi}
\end{pmatrix} = 0,
\]

where \( F : \mathcal{X} \times \mathbb{R}^q \times \mathbb{R}^r \to \mathcal{X} \times \mathbb{R}^{q-r} \). Note that \( \tilde{\mu}(e^i_{\xi}) = 0 \) for \( j = r + 1, \ldots, q \) since \( \tilde{\mu} \in g^* \). Because of (2.8) the derivative of (2.12)

\[
D F(\bar{x}, \xi, 0) = \begin{pmatrix}
D(f_H(\bar{x}) - \tilde{\xi}) & e^1_{\xi}\bar{x} & \cdots & e^r_{\xi}\bar{x} & \nabla J_1(\bar{x}) & \cdots & \nabla J_r(\bar{x}) \\
\text{DiJ}_{r+1}(\bar{x}) & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{DiJ}_q(\bar{x}) & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

has full rank, and so the solution manifold of (2.12) is 2\( r \)-dimensional. Moreover, all solutions close to the original relative equilibrium satisfy \( \xi_i = 0 \), \( i = r + 1, \ldots, q \). To see this note that, as in the proof of Theorem 2.7, the points \((\gamma x(s), \xi(s), 0), s \in \mathbb{R}^r, \) form a 2\( r \)-dimensional manifold of solutions of (2.12) as their momentum values \( \mu(s) = J(x(s)) \) satisfy \( \mu(s) - \tilde{\mu} \in g^* \subseteq \text{ann}(n_\mu) \) and hence \( \mu_j(s) = 0 \), \( j = r + 1, \ldots, q \).

Remark 2.9 Muñoz-Almaraz et al (see equation (6.9) of [15]) call a relative equilibrium \( \Gamma \bar{x} \) of (2.1) with drift velocity \( \bar{\xi} \) normal if in our notation

\[
\mathcal{N}(D(f_H(\bar{x}) - \tilde{\xi})) + \bar{z} \bar{x} = \ker \text{DiJ}(\bar{x})
\]

for some subgroup \( Z \) which lies in the centre of \( \Gamma^\text{id} \) and satisfies \( \bar{\xi} \in \mathcal{Z} \). Here \( \mathcal{Z} \) is the Lie algebra of \( Z \). In particular they require \( \bar{\xi} \) to commute with all of \( g \). Under this assumption they prove in [15, Theorem 16] that the \( \bar{x} \) persists to an \( m \)-dimensional manifold of points \( x \) on relative equilibria which lie in a section \( \mathcal{N} \) transverse to \( \Gamma \bar{x} \) at \( \bar{x} \). Here \( m = \text{dim} \ Z \). They show that this manifold of relative equilibria can be computed numerically by solving a system of the form (2.11) with (in our notation) \( \bar{\xi} \in \mathcal{Z}, \lambda_\mu \in g^* \cong \mathbb{R}^q \), and with \( g = \text{dim} \Gamma \) phase conditions. Note that by projecting (2.13) to \( \mathcal{N}_1 \) we see that normal relative equilibria are nondegenerate.

2.1.3 Numerical path-following of relative equilibria

Let, as before, \( \bar{x} \) lie on a transversal relative equilibrium \( \Gamma \bar{x} \) with regular velocity drift pair \((\bar{\xi}, \tilde{\mu})\). Let \( \{e^i_{\xi}, i = 1, \ldots, g\} \) be a basis of \( g \) such that \( \text{span}\{e^i_{\xi}, i = q + 1, \ldots, g\} = \text{ad}_{\bar{\xi}} g \), and \( \text{span}\{e^i_{\xi}, i = 1, \ldots, q\} = g^* \), \( \text{span}\{e^i_{\xi}, i = 1, \ldots, r\} = g^*_{(\bar{\xi}, \tilde{\mu})} \), as in (2.9). For \( \mu \in g^* \) let \( \mu_j = \mu(\xi_j), j = 1, \ldots, g \). We fix \( g - 1 \) momentum components, without loss of generality, the components \( \mu_0 = (\mu_2, \ldots, \mu_g), g = \text{dim} \Gamma \), to get a one-parameter family. We solve

\[
F^\mu(x, \xi, \lambda_\mu) = \begin{pmatrix}
F(x, \xi, \lambda_\mu) \\
J_2(x) - \tilde{\mu}_2 \\
\vdots \\
J_r(x) - \tilde{\mu}_r
\end{pmatrix} = 0.
\]
Then for any solution \( y = (x, \xi, 0) \) of (2.14) the \( x \)-component lies in
\[
\mathcal{X}^{\mu^g} = \{ x \in \mathcal{X}, \quad \mathbf{J}_j(x) = \mu_j, \quad j = 2, \ldots, g \}.
\]
To see this, first note that by identifying \( g \) with \( g^* \) by a \( \Gamma \)-invariant inner product we get \( \bar{\mu} \in g_\xi(\xi, \mu) \) and so \( \mu_j = 0, \quad j = r + 1, \ldots, g \). For \( \xi \in g_\xi \) with \( \xi \equiv \xi \) we have \( g_\xi \subseteq g_\xi, \quad g_\xi^* \subseteq g_\xi^* \). By construction \( \xi \in g_\xi \) at any solution \( y = (x, \xi, 0) \) of (2.14). As stated before (2.4), we have \( \text{ad}_{\xi}^* \mu = 0 \) for the drift velocity \( \xi \) and momentum \( \mu = \mathbf{J}(x) \) of the relative equilibrium given by \( y = (x, \xi, 0) \). From this we conclude that \( \mathbf{J}_j(x) = 0, \quad j = q + 1, \ldots, g, \) where \( (x, \xi, 0) \) solves (2.14). If \( q \neq r \) then \( \mu_j = 0, \quad j = r + 1, \ldots, g \) are subequations of \( F = 0 \), see (2.12). Hence, the solutions of (2.14) satisfy \( \mathbf{J}^g(x) = \bar{\mu}^g \). Moreover, we have:

**Corollary 2.10** Let \( \Gamma \bar{x} \) be a transversal relative equilibrium with regular velocity-momentum pair \((\xi, \bar{\mu})\) and define \( \chi(s) \) as in Theorem 2.6. Then the \((r - 1, r - 1)\)-matrix \( Q \) with \( Q_{ij} = \partial s_{i+1} \chi_j(s)_{s=0} \) is invertible if and only if \([D_{\chi_1}f, f_N(0), D_{w_1}f_N(0)] \) has full rank and if and only \( DF^\mu(y) = 0 \) has full rank. In this case there is a path \( x(e) \) of points on relative equilibria in \( \mathcal{X}^{\mu^g} \) with \( x(0) = \bar{x} \) which solves (2.14). We say that the relative equilibrium \( \Gamma \bar{x} \) is transversal with respect to \( C(x) := \mathbf{J}_1(x) \).

Under the above assumption, (2.14) can be solved by standard numerical methods for underdetermined systems, for example by the Gauss-Newton method, for initial values close to \( \bar{y} = (\bar{x}, \xi, 0) \). For tangential continuation methods, we choose nontrivial continuation tangent \( t = t(\bar{y}) \) in a solution point \( \bar{y} \) in the kernel of \( D F^\mu(\bar{y}) \) which is orthogonal to the group orbit, i.e., \((t, t^\xi) = 0, \quad i = 1, \ldots, r \). Here \( t^\xi \in \ker(D F^\mu(\bar{y})) \) is given by \( t^\xi = (t^\xi, t^\xi, t^\xi) \), \( t^\xi = e^\xi \bar{x} \), \( t^\xi = 0, \quad t^\xi = 0, \quad i = 1, \ldots, r \).

**Remark 2.11** In the case of dissipative ODEs
\[
\dot{x} = f(x, \lambda) \tag{2.15}
\]
which depend on a parameter \( \lambda \in \mathbb{R} \) and are equivariant under a compact group \( \Gamma \) the bundle equations (2.6) take the more general form
\[
\dot{\gamma} = \gamma f_\Gamma(v, \lambda), \quad \dot{v} = f_N(v, \lambda) \tag{2.16}
\]
and the relative equilibrium through \( \bar{x} = (id, 0) \) at parameter \( \bar{\lambda} \) satisfies \( f_N(0, \bar{\lambda}) = 0 \) and \( f_\Gamma(0, \bar{\lambda}) = \xi \). A relative equilibrium of a general non-Hamiltonian \( \Gamma \)-equivariant system (2.15) is called transversal if the matrix \([D_i f_N(0, \bar{\lambda}), D_{\lambda} f_N(0, \bar{\lambda})] \) has image \( N \) and nondegenerate if \( D_i f_N(0, \bar{\lambda}) \) is invertible. Near a transversal relative equilibrium there is a path of equilibria \( v(s) \) with parameter \( \lambda(s) \) of the slice equation (the \( \dot{v} \)-equation of (2.16)) which corresponds to a path \( x(s) \approx (id, v(s)) \) of relative equilibria of (2.15). The path of relative equilibria \((x(s), \lambda(s))\) can be computed by solving the following underdetermined system which is the analogue of (2.11) in the non-Hamiltonian case:
\[
F(x, \xi, \lambda) = f_H(x) - \sum_{i=1}^{q} \xi_i e_i^\xi x \tag{2.17}
\]
where \( q = \dim g_\xi \). Typically the drift velocity \( \xi \) of the relative equilibrium at \( \bar{x} \) is regular in which case \( q = r \) is the rank of \( \Gamma \) (the dimension of maximal tori of \( \Gamma \)).
2.2 Persistence of transversal RPOs

A point \( x \in X \) lies on a relative periodic orbit (RPO) if there exists \( t > 0 \) such that \( \Phi^t(x) = x \). The infimum \( \bar{t} \) of such \( t \) is called the relative period of the RPO and the element \( \sigma \in \Gamma \) such that \( \sigma \Phi^{\bar{t}}(x) = x \) is called a phase-shift symmetry, reconstruction phase or drift symmetry of the RPO. The relative periodic orbit \( P \) itself is given by

\[
P = \{ \gamma \Phi^\theta(x), \, \gamma \in \Gamma, \theta \in \mathbb{R} \}.
\]

We assume that \( \bar{t} > 0 \) so that \( P \) is a proper RPO (i.e., not a relative equilibrium). For \( \alpha \in \Gamma, \xi \in \mathfrak{g} \) let \( Z(\alpha) = \{ \gamma \in \Gamma, \, \gamma \alpha = \alpha \gamma \} \) and \( Z(\xi) = \{ \gamma \in \Gamma, \, \text{Ad}_\gamma \xi = \xi \} \) denote the centralizer of \( \alpha \) and \( \xi \) respectively.

An RPO of an compact group action becomes periodic in a comoving frame as the following lemma shows:

**Lemma 2.12** [21, 20, 25]

1. Any element \( \sigma \) of a compact group \( \Gamma \) can be decomposed as

\[
\sigma = \alpha \exp(-\xi),
\]

for some \( \xi \in \mathfrak{g} \) and \( \alpha \in \Gamma \) such that

\[
\alpha^\ell = \text{id} \quad \text{for some } \ell \in \mathbb{N}, \quad \text{Ad}_\alpha \xi = \xi,
\]

and

\[
Z(\sigma) = Z(\alpha) \cap Z(\xi).
\]

2. For any relative periodic orbit with drift-momentum pair \( (\bar{\sigma}, \bar{\mu}) \in (\Gamma \times \mathfrak{g}^+)^c \), trivial isotropy \( K \) and relative period \( \bar{t} \) there is a frame moving with velocity \( \xi \in \mathfrak{g}(\sigma, \mu) \), called drift velocity of the RPO with respect to \( x \), and some integer \( \ell \) such that in this comoving frame the RPO becomes a periodic orbit of period \( T = \ell \bar{t} \) and drift symmetry \( \bar{\alpha} \in \Gamma_\mu \). Moreover \( \bar{\sigma} = \alpha \exp(-\bar{t} \bar{\xi}) \).

**Remark 2.13** Note that the decomposition \( \sigma = \alpha \exp(-\xi) \) in Lemma 2.12 is in general not unique: assume that the group \( C \) generated by \( \sigma \) is continuous. Let \( \eta \) be an infinitesimal rotation in the Lie algebra of \( C \) which generates the rotation group \( \exp(\phi \eta) = R_\phi \in C, \phi \in [0, 2\pi] \). Then other possible choices for \( \alpha \) and \( \xi \) would be \( \bar{\alpha} = R_{2\pi j/\ell \alpha} \) for some \( \ell \in \mathbb{Z}, \) gcd\((\ell, j) = 1\), and \( \bar{\xi} = \xi + 2\pi(n + j/\ell) \eta, \) \( n \in \mathbb{N} \).

Let \( \bar{x} \) lie on a relative periodic orbit with drift symmetry \( \bar{\sigma} \), momentum \( \bar{\mu} = J(\bar{x}) \) at \( \bar{x} \), and relative period \( \bar{t} \). The following simple relation between the drift symmetry and momentum of an RPO

\[
\bar{\sigma} \bar{\mu} = \bar{\mu}, \quad (2.18)
\]

is analogous to the corresponding relation (2.4) for relative equilibria, and is crucial for the problem of persistence to nearby momentum values, see [20] and the sections below. It simply follows from the fact that \( J \) is preserved by the flow, and so \( \bar{\sigma} \bar{\mu} = \bar{\sigma} J(\bar{x}) = J(\bar{\sigma} \bar{x}) = J(\Phi^{-\tau}(\bar{x})) = J(\bar{x}) = \bar{\mu} \).

We will need the following definitions, which are analogous to the corresponding definitions 2.2 for relative equilibria.
**Definition 2.14** [20], [25]

(i) We call pairs $(\sigma, \mu) \in \Gamma \times \mathbf{g}^*$ satisfying (2.18) drift-momentum pairs and denote the space of drift-momentum pairs by

$$(\Gamma \times \mathbf{g}^*)^c := \{ (\gamma, \mu) \in \Gamma \times \mathbf{g}^*, \ \gamma \mu = \mu \}. \quad (2.19)$$

(ii) We define an action of $\Gamma$ on the space of drift-momentum pairs as follows:

$$\gamma(\sigma, \mu) = (\gamma \sigma \gamma^{-1}, (\text{Ad}_\gamma)^{-1} \mu), \quad \gamma \in \Gamma, \quad (\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c.$$  

For later purposes we define the isotropy subgroup $\Gamma_{(\sigma, \mu)}$ of $(\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c$ with respect to this action as $\Gamma_{(\sigma, \mu)}$, denote its Lie algebra by $\mathbf{g}_{(\sigma, \mu)}$ and let $r_{(\sigma, \mu)} = \dim \mathbf{g}_{(\sigma, \mu)}$. Moreover we define the isotropy subgroup of $\sigma \in \Gamma$ as $\Gamma_{\sigma} = \Gamma_{(\sigma, 0)}$, denote its Lie algebra by $\mathbf{g}_{\sigma}$ and let $r_{\sigma} = \dim \mathbf{g}_{\sigma}$.

(iii) We call a drift-momentum pair $(\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c$ regular if $r_{(\sigma, \mu)}$ is locally constant in the space of drift-momentum pairs (2.19).

(iv) We call $\sigma \in \Gamma$ regular if $r_{\sigma}$ is locally constant.

**Remark 2.15** [20, 25] Regular drift-momentum pairs $(\sigma, \mu)$ are generic in the set of drift-momentum pairs $(\Gamma \times \mathbf{g}^*)^c$. Regular elements $\sigma \in \Gamma$ are generic in $\Gamma$. The drift symmetry $\sigma$ of a regular drift-momentum pair $(\sigma, \mu)$ is typically regular in which case $\mathbf{g}_{\sigma} = \mathbf{g}_{(\sigma, \mu)}$, and typically the momentum $\mu$ of a regular drift-momentum pair $(\sigma, \mu)$ is regular in which case $\mathbf{g}_{\mu} = \mathbf{g}_{(\sigma, \mu)}$. Moreover the following conditions are equivalent:

(i) $(\sigma = \exp(-\xi), \mu)$ is a regular drift momentum pair;

(ii) $\mathbf{g}_{(\sigma, \mu)}$ is the Lie algebra of a Cartan subgroup;

(iii) $\mathbf{g}_{(\sigma, \mu)} = \mathbf{g}_\sigma \cap \mathbf{g}_{(\xi, \mu)}$ is abelian.

Let $\bar{x}$ lie on an RPO $\bar{\mathcal{P}}$ with relative period $\bar{\tau}$. We assume without loss of generality that the isotropy $K$ of the RPO is finite (if not, we restrict the dynamics to the fixed point space $\text{Fix}(K)$ of $K$ so that $K$ is trivial). We call a $K$-invariant section $\mathcal{S} := \mathcal{S}_x$ that is transverse to the RPO at $\bar{x}$, i.e., transverse to $\mathbf{g}\bar{x} \oplus \text{span}(f_H(\bar{x}))$, a Poincaré section at $\bar{x}$. As usual, we define the Poincaré map $\Pi : \mathcal{S} \to \mathcal{S}$ as follows. For $x \in \mathcal{S}$ close to $\bar{x}$ there are unique $\gamma(x) \in \Gamma$, $\gamma(x) \approx \sigma$, and $\tau(x) \approx \bar{\tau}$ such that $\gamma(x) \Phi^{\tau(x)}(x) \in \mathcal{S}$ (this follows from the implicit function theorem since we assume that the isotropy $K$ of the RPO is trivial). Now we set

$$\Pi(x) = \gamma(x) \Phi^{\tau(x)}(x). \quad (2.20)$$

Let $\tilde{E} = H(\bar{x})$ and $\tilde{\mu} = J(\bar{x})$ be the energy and momentum of the RPO at $\bar{x}$ and let $\mathcal{S}^{E, \tilde{\mu}} \subseteq \mathcal{X}^{E, \tilde{\mu}}$ be a Poincaré-section transverse to the time orbit and $\Gamma_{\mu}$-orbit through $\bar{x}$ within the energy momentum level set

$$\mathcal{X}^{E, \tilde{\mu}} = \{ x \in \mathcal{X}, \ \tilde{E} = H(x), \ \tilde{\mu} = J(x) \}.$$  

Denote the corresponding Poincaré-map by $\Pi^{E, \tilde{\mu}} : \mathcal{S}^{E, \tilde{\mu}} \to \mathcal{S}^{E, \tilde{\mu}}$. For the definition of transversality of an RPO we need the following lemma:

**Lemma 2.16** There is a choice of Poincaré-section $\mathcal{S}$ near an RPO at $\bar{x}$ with momentum $\tilde{\mu} = J(\bar{x})$, energy $\tilde{E}$, drift symmetry $\tilde{\sigma}$, and finite isotropy $K$ such that $\mathcal{S}^{E, \tilde{\mu}} = \mathcal{S} \cap \mathcal{X}^{E, \tilde{\mu}}$ and the following hold true:
a) The tangent space $T_xS$ to the Poincaré-section $S$ at $x$ takes the form

$$T_xS \simeq \mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2,$$

where $\mathcal{N}_0 \simeq \mathfrak{g}_\mu^*$, $\mathcal{N}_1 \simeq T_xS \cap \ker DH(\bar{x})$, $\mathcal{N}_2 \simeq \mathbb{R}$.

Here $\mathcal{N}_1 = T_xS^{E,\mu}$ is a symplectic vectorspace, $\mathcal{N}_0$ is isomorphic to a $\Gamma_\mu$-invariant section transverse to the momentum group orbit $\Gamma_\mu$ in $\mathfrak{g}^*$ and $\mathcal{N}_2$ parametrizes the energy level.

b) Let $\Pi : \mathcal{S} \to \mathcal{S}$ be the Poincaré map. Then there is a choice of coordinates $x \simeq (\nu, w, E) \in \mathcal{S}$ where $\nu \in \mathcal{N}_0$, $w \in \mathcal{N}_1$ and $E \in \mathcal{N}_2$, and the point $\bar{x}$ is identified with $\nu = 0$, $w = 0$, $E = \bar{E}$, such that the Poincaré map takes the form

$$\Pi(\nu, w, E) = (\Pi_{\mathcal{N}_0}(\nu, w, E), \Pi_{\mathcal{N}_1}(\nu, w, E), \Pi_{\mathcal{N}_2}(\nu, w, E)).$$

Here

$$\Pi_{\mathcal{N}_0}(\nu, w, E) = \gamma(\nu, w, E)\nu, \quad \Pi_{\mathcal{N}_1}(\nu, w, E) = E, \quad \Pi_{\mathcal{N}_2}(0, w, \bar{E}) = \Pi^{E,\mu}(w),$$

$$\gamma(\nu, w, E) \in \Gamma_\mu, \quad \gamma(0, 0, \bar{E}) = \bar{\sigma}, \quad \text{and the map } \nu \to \Pi_{\mathcal{N}_1}(\nu, w, E) \text{ is symplectic.}$$

In these coordinates the momentum map restricted to $\mathcal{N}$ takes the form

$$j|_{\mathcal{N}}(\nu, w) = \bar{\mu} + \nu.$$

\textbf{Proof.} Part a) is contained in [22, Theorem 3.1] and most of part b) is implicitly contained in [20]:

Theorem 3.3 and Remark 3.4 of [22] implies that a $\Gamma$-invariant neighbourhood $\mathcal{U}$ is symplectomorphic to $(\Gamma \times \mathbb{R}/2\pi\mathbb{Z} \times \mathcal{N})/\mathbb{Z}_n$ where $\mathcal{N}$ can be decomposed as specified above. The $\mathbb{Z}_n$-action is generated by the drift symmetry $\alpha$ of the RPO $\mathcal{P}$ in the comoving frame and acts as $(\gamma, v, \theta) \to (\gamma \alpha^{-1}, Qv, \theta + 1)$ where $Q$ acts $v = (\nu, w, E) \in \mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2$ as $Q(\nu, w, E) = (\text{Ad}_\nu^*\nu, Q\mathcal{N}_1, w)$ and $Q\mathcal{N}_1$ has order $n$. The symplectic form is $\mathbb{Z}_n$-invariant and given by

$$\omega_{\mathcal{N} \times \mathcal{N} \times \mathcal{N}} = \omega_{\mathcal{N} \times \mathcal{N}} + \omega_{\mathcal{N}_1} + \omega_{\mathcal{N}_2},$$

(2.21)

on $\Gamma \times \mathbb{R}/2\pi\mathbb{Z} \times \mathcal{N}$. Here $T_1^* \mathcal{N} \simeq \mathbb{R}/2\pi \times \mathcal{N}_2$, $\omega_{T_1^* \mathcal{N}}$ is the restriction to $\Gamma \times \mathfrak{g}_\mu^* \simeq \Gamma \times \ker (\text{ann}(\mathfrak{n}_\mu)) \subset \Gamma \times \mathfrak{g}^*$ of the symplectic form on $T^*\Gamma$ and $\omega_{\mathcal{N}_1}$ is the symplectic form on $\mathcal{N}_1$. As before, $\mathfrak{n}_\mu$ is $\Gamma_\mu$-invariant complement to $\mathfrak{g}_\mu$ in $\mathfrak{g}$. The energy of $(\gamma, \nu, w, \theta)$ is $E = h(\nu, w, \theta) + e$ and its momentum is $J(\gamma, \nu, w, \theta) = \gamma(\mu + \nu)$, see [22, Remark 3.4d)].

In these coordinates the Poincaré-section $\mathcal{S}$ is given by $\mathcal{S} = \{(\gamma, \nu, w, \theta) \in (\Gamma \times \mathcal{N})/\mathbb{Z}, \quad \theta = 0\}$, and the Poincaré-map $\Pi(x) = \gamma(x)\Phi^+(x)$ becomes a map $\Pi : \mathcal{N} \to \mathcal{N}$ which decomposes as form $\Pi = (\Pi_0, \Pi_1, \Pi_2)$, where $\Pi_0$ maps into $\mathcal{N}_0$, $\Pi_1$ into $\mathcal{N}_1$ and $\Pi_2(\nu, w, E)$ into $\mathcal{N}_2$. Due to the form of the momentum map in these coordinates and the conservation of momentum we have $\Pi_0(\nu, w, E) = \gamma(\nu, w, E)(\bar{\mu} + \nu) - \bar{\mu}$. Since $\text{ann}(\mathfrak{n}_\mu)$ is $\Gamma_\mu$-invariant and $\nu \in \text{ann}(\mathfrak{n}_\mu)$, $\nu \approx 0$, we have $\gamma(\nu, w, E) \in \Gamma_\mu$. Energy conservation implies that $\Pi_2(\nu, w, E) \equiv E$. Moreover the map $\Pi_1(\nu, w, E) : \mathcal{N}_1 \to \mathcal{N}_1$ is symplectic. \hfill \blacksquare

The relation between the linearization of the Poincaré-map and the full linearization $\bar{\sigma}D\Phi^+(\bar{x})$ at the RPO through $\bar{x}$ is given in follow proposition:

\textbf{Proposition 2.17} [22, Propositions 4.3 and 4.4] Let $\bar{x} = \bar{\sigma}\Phi^+(\bar{x}) \in \mathcal{N}$ lie on an RPO $\mathcal{P}$ of a compact group action $\Gamma$ with relative period $\bar{\tau}$, finite isotropy subgroup and momentum
\[\hat{\mu} = J(\bar{x}), \text{ and let } M = \sigma D\Phi^*(\bar{x}). \] Then the matrix \(M\) has the following structure with respect to the decomposition \(T_x\mathcal{N} = T_x\Gamma(x) \oplus \text{span}(f_H(\bar{x}))) \oplus \mathcal{N}:
\]
\[
M = \begin{pmatrix} \text{Ad}_\sigma & 0 & D_N \\
0 & 1 & \Theta_N \\
0 & 0 & M_N \end{pmatrix},
\]
(2.22)
and the block \(M_N\) in (2.22) has the following block structure with respect to the decomposition \(\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2:\n
\]
\[
M_N = \begin{pmatrix} M_0 & 0 & 0 \\
M_{10} & M_1 & M_{1,2} \\
0 & 0 & 1 \end{pmatrix},
\]
where \(M_0 = \text{Ad}_{\sigma}^{-1}|_{\mathfrak{g}_\mu}\). Note that \(M_N = D\Pi(\bar{x})\). Moreover
\[
D_N = \begin{pmatrix} D_0 & D_1 & D_2 \\
0 & D_1 & D_2 \\
0 & 0 & 0 \end{pmatrix}, \quad \Theta_N = (\Theta_0, \Theta_1, \Theta_2).
\]
and
\[
\text{Ad}_{\sigma}^{-1} = \begin{pmatrix} (\text{Ad}_{\sigma})^{-1}|_{\mathfrak{g}_\mu} & 0 & 0 \\
0 & P_{\mathfrak{n}_\mu}\text{Ad}_{\sigma}|_{\mathfrak{n}_\mu} & 0 \end{pmatrix}.
\]
Here \(P_{\mathfrak{n}_\mu}\) is a projection on \(\mathfrak{n}_\mu\) with kernel \(\mathfrak{n}_\mu\) and \(P_{\mathfrak{n}_\mu} = \text{id}_{\mathfrak{g}_\mu} - P_{\mathfrak{g}_\mu}\) a projection onto \(\mathfrak{n}_\mu\).

**Remark 2.18** In [22, Proposition 4.3] we have \(\Theta_N = 0\) in (2.22) and \(\bar{\tau} = 1\) since time is reparametrized such that \(\tau(x) \equiv 1\) in the definition of the Poincaré map \(\Pi\). The matrix \(D_N\) is the drift-derivative with respect to the variables \(v \in \mathcal{N}\).

As before let \(\bar{x}\) lie on an RPO with drift-momentum pair \((\bar{\sigma}, \bar{\mu})\) and let \(e_1^\lambda, \ldots, e^\lambda_{\ell}\) be a basis of \(\mathfrak{g}_{(\bar{\sigma}, \bar{\mu})}\) and let \(\mu_i = \mu(e_i^\lambda), J_i = J_{e_i^\lambda}, i = 1, \ldots, r\). We then have
\[
\text{Fix}_{\mathfrak{g}_\mu}(\text{Ad}_{\sigma}) = \text{ann}_{\mathfrak{g}_\mu}((\text{Ad}_{\sigma} - \text{id})|_{\mathfrak{g}_\mu}) \simeq \mathfrak{g}_{(\sigma, \mu)},
\]
and a complement of \(\mathfrak{g}_{(\sigma, \mu)}\) in \(\mathfrak{g}_\mu\) is given by \(\text{ann}_{\mathfrak{g}_\mu}(\mathfrak{g}_{(\sigma, \mu)})\). We decompose \(\nu \in \mathfrak{g}_\mu\) as \(\nu = (\chi, \zeta)\) where \(\chi \in \mathfrak{g}_{(\sigma, \mu)}\) and \(\zeta \in \text{ann}_{\mathfrak{g}_\mu}(\mathfrak{g}_{(\sigma, \mu)})\).

The following transversality condition for RPOs is a direct extension of Definition 2.4 to RPOs.

**Definition 2.19** A relative periodic orbit of a Hamiltonian system (2.1) through \(\bar{x} \simeq (0, 0, \bar{E}) \in \mathcal{N}\) is called transversal if it is not a relative equilibrium and if
\[
D_{(\chi, \nu, E)}(\Pi_{\mathcal{N}_1}(0, 0, \bar{E}) - \text{id}_{\mathcal{N}_1})
\]
has full rank where \(\chi \in \mathfrak{g}_{(\sigma, \mu)}\).

We call an RPO through \(\bar{x} \simeq (0, 0, \bar{E}) \in \mathcal{N}\) nondegenerate if \(D_{\bar{\mu}}\Pi_{\mathcal{N}_1}(0, 0, \bar{E}) = D_{\bar{\mu}}\Pi^E_{\bar{\mu}}(\bar{x})\) has no eigenvalue 1.

**Remark 2.20** If the RPO through \(\bar{x}\) has a regular momentum \(\bar{\mu} = J(\bar{x})\) at \(\bar{x}\) then \(\mathfrak{g}_\mu\) and hence also \(\mathcal{S}^{E, \bar{\mu} + \nu}\) has locally constant dimension and we can choose \(\mathcal{S}^{E, \bar{\mu} + \nu}\) to depend smoothly on \(E\) and \(\nu\). Moreover the RPO is transversal if
\[
D_{(\bar{x}, E, \nu)}(\Pi_{E, \nu}^{E, \bar{\mu} + \nu}(\bar{x}) - \text{id}_{\mathcal{S}^{E, \bar{\mu} + \nu}})|_{(E = \bar{E}, x = z, \nu = 0)}
\]
(2.23)
has full rank. If the momentum \(\bar{\mu}\) of the RPO is nongeneric then \(\mathcal{S}^{E, \bar{\mu}}\) changes dimension for \(\mu \neq \bar{\mu}, \mu \approx \bar{\mu}\), and this equivalence does not hold. The situation is analogous in the case of relative equilibria as we have seen in Remark 2.5.
Example 2.21 In the case of the symmetry group $\Gamma = \text{SO}(2)$, every momentum $\mu \in \text{so}(2)^*$ is regular. Hence (2.23) can be applied to check whether an RPO is transversal.

Note that the RPO is called nondegenerate in [20, 25] if $D_x \Pi^{E, \mu}_r : \mathcal{S}^{E, \mu} \to \mathcal{S}^{E, \mu} = D_x \Pi_{x_0}(0)$ does not have an eigenvalue 1 at a point $x$ of the relative periodic orbit. The following theorem is an extension of the persistence result [20, Theorem 4.2] for nondegenerate RPOs to transversal RPOs.

Theorem 2.22 Let $\Gamma$ be compact. Let $\tilde{x}$ lie on a transversal RPO $\tilde{P}$ of (2.1) with relative period $\bar{\tau}$ and regular drift-momentum pair $(\bar{\sigma}, \bar{\mu}) \in (\Gamma \times \mathfrak{g}^*)^\mathbb{C}$. Let $r = r_{(\sigma, \rho)}$, decompose $\sigma = \alpha \exp(-r\xi)$ as before, and let $\tilde{E} = H(\tilde{x})$ be the energy of the RPO. Then

a) there is an $(r+1)$-dimensional family $\mathcal{P}(s)$, $s \in \mathbb{R}^{r+1}$, of RPOs near $\tilde{P} = \mathcal{P}(0)$ such that $x(s) \in \mathcal{P}(s)$ is smooth, $x(0) = \tilde{x}$, with

- energy $E(s)$ close to $E(0) = \tilde{E}$,
- momentum $\tilde{\mu} + \chi(s)$, $\chi(s) \in \mathfrak{g}^*_{(\sigma, \rho)}$, where $\chi(0) = 0$,
- relative period $\tau(s)$ close to $\tau(0) = \bar{\tau}$,
- drift symmetry $\sigma(s)$ close to $\sigma(0) = \bar{\sigma}$, and
- drift velocity $\xi(s) \in \mathfrak{g}_\rho$ close to $\xi(0) = \xi \in \mathfrak{g}_{(\sigma, \rho)}$.

where $\sigma(s) = \alpha \exp(-\tau(s)\xi(s))$, $\text{Ad}_\alpha \xi(s) = \xi(s)$.

b) If the RPO $\tilde{P}$ is nondegenerate then we can choose $s = (E, \chi)$.

Note that by the above theorem transversal RPOs of compact group actions persist to nearby energy-level sets and to those nearby momenta which are fixed by the drift symmetry $\bar{\sigma}$ of the RPO at $\tilde{x}$ and lie in a section transverse to the momentum group orbit $\Gamma \tilde{\mu}$ (the last condition is only needed in order to guarantee that the RPOs parametrized by $s$ are not symmetry related). During the continuation the momentum $\mu(s)$, drift symmetry $\sigma(s)$ and drift velocity $\xi(s)$ of the RPOs $\mathcal{P}(s)$ vary, but the drift symmetry $\alpha$ of the RPOs in the comoving frame is fixed.

Proof. Since $\text{Ad}_\sigma$ does not have Jordan blocks, we can conclude from Proposition 2.17 that the RPO is still transversal when considered as an RPO for the symmetry group $\hat{\Gamma} = \Gamma_\sigma = Z(\bar{\sigma})$. From Remark 2.15 the isotropy algebra $\mathfrak{g}_{(\sigma, \rho)} \subseteq \mathfrak{g}_{(\sigma, \rho)}$ is abelian. This implies, by Lemma 2.16 b) applied to the symmetry group $\hat{\Gamma}$, that $\Pi_{\mathcal{N}_0}(\tilde{v}, \tilde{w}) \equiv \tilde{v}$. Here $\mathcal{N}_0 \cong \mathfrak{g}^*_{(\sigma, \rho)}$, $(\tilde{v}, \tilde{w}) \in \mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$, and $\mathcal{N}$ is the slice to the action of $\hat{\Gamma}$ from Lemma 2.16. Due to the definition of transversality (Definition (2.19)) we can solve $\Pi_{\mathcal{N}_1}(\tilde{v}, \tilde{w}, E) = \tilde{w}$ for $(\tilde{v}, \tilde{w}, E)(s), s \in \mathbb{R}^{r+1}$, by the implicit function theorem with $\tilde{v}(s) = \chi(s) \in \mathfrak{g}^*_{(\sigma, \rho)}$. This gives an $(r+1)$-dimensional family of RPOs of (2.1) with the properties required.

2.3 Numerical continuation of transversal RPOs

In this section we show that the numerical methods presented in [25] for nondegenerate RPOs of compact symmetry group actions with regular drift-momentum pair also converge for transversal RPOs of noncompact algebraic group actions. We restrict attention to the case of single shooting.

Note that by Remark 2.15 generically the drift symmetry $\bar{\sigma}$ of an RPO is regular, and we assume this in the following theorem.
Theorem 2.23 Let $\Gamma$ be compact, let $\bar{x}$ lie on a transversal RPO $\bar{\Phi}$ with regular drift symmetry $\bar{\sigma} \in \Gamma$, and momentum $\bar{\mu}$, and decompose $\bar{\sigma} = \alpha \exp(-r \bar{\xi})$ as in Lemma 2.12. We set $r = r(\bar{\sigma}, \bar{\mu})$ and denote a basis of $g(\bar{\sigma}, \bar{\mu})$ by $e_1^\xi, \ldots, e_r^\xi$. For $\xi \in g(\bar{\sigma}, \bar{\mu})$ let $\bar{\xi} = \sum_{i=1}^r \xi_i e_i^\xi$ and identify $\xi = (\xi_1, \ldots, \xi_r) \in \mathbb{R}^r$. As before, let $J_i = J_{12}^i$, $i = 1, \ldots, r$, and define

$$\dot{x} = f_H(x) + \lambda_E \nabla H(x) + \sum_{i=1}^r \lambda_{i,\mu} \nabla J_i(x) - \sum_{i=1}^r \xi_i e_i^\xi. \quad (2.24)$$

Denote the flow of (2.24) by $\Phi^T(x; \xi, \lambda_E, \lambda_\mu)$. Then the derivative $DF(\bar{y})$ of

$$F(x, T, \xi, \lambda_E, \lambda_\mu) = \alpha \Phi^T(x; \xi, \lambda_E, \lambda_\mu) - x = 0, \quad (2.25)$$

where $F : \mathcal{X} \times \mathbb{R}^{2+2r} \to \mathcal{X}$, has full rank at any solution $y = (x, T, \xi, \lambda_E, \lambda_\mu)$ of $F = 0$ close to $(\bar{x}, T, \bar{\xi}, 0, 0)$. Moreover any such solution satisfies $\lambda_E = 0$, $\lambda_\mu = 0$, and, hence, determines an RPO of (2.1).

**Proof.** As in the proof of Theorem 2.22 we replace $\Gamma$ by $\Gamma_\sigma = Z(\bar{\sigma})$. Since $g_\sigma = g(\sigma, \bar{\mu})$, we look for RPOs with drift velocity $\xi \in g_\sigma$. We have

$$DF(x, T, \xi, 0, 0) = [D_x F, D_T F, D_\xi F, D_{\lambda_E} F, D_{\lambda_\mu} F] = (\sigma D\Phi^T(x) - \mathrm{id}, f_H(x), e_1^\xi, \ldots, e_r^\xi, D_{\lambda_E} F, D_{\lambda_\mu} F).$$

The $(r + 1, r + 1)$-matrix $B$ with

$$B_{ij} = DJ_i(\bar{x})D_{\lambda_{i,\mu}} F(\bar{y}) = DJ_i(\bar{x})D_{\lambda_{i,\mu}} \Phi^T(\bar{x}; \bar{\xi}, 0, 0)|_{\lambda_{i,\mu}=0}, \quad i, j = 1, \ldots, r,$$

$$B_{i, r+1} = DJ_i(\bar{x})D_{\lambda_{n,\mu}} F(\bar{y}) = DJ_i(\bar{x})D_{\lambda_{n,\mu}} \Phi^T(\bar{x}; \bar{\xi}, \lambda_E, 0)|_{\lambda_E=0}, \quad i = 1, \ldots, r,$$

$$B_{r+1, i} = DH(\bar{x})D_{\lambda_{i,\mu}} F(\bar{y}) = DH(\bar{x})D_{\lambda_{i,\mu}} \Phi^T(\bar{x}; \bar{\xi}, 0, \lambda_\mu)|_{\lambda_\mu=0}, \quad i = 1, \ldots, r,$$

$$B_{r+1, r+1} = DH(\bar{x})D_{\lambda_{n,\mu}} F(\bar{y}) = DH(\bar{x})D_{\lambda_{n,\mu}} \Phi^T(\bar{x}; \bar{\xi}, \lambda_E, 0)|_{\lambda_E=0}$$

has full rank. This was shown in [25]. We sketch the proof: Since $(\bar{\sigma}, \bar{\mu})$ is regular the isotropy algebra $g(\bar{\sigma}, \bar{\mu})$ is abelian by Remark 2.15. Consequently, $\Phi^T(\cdot; \xi) = \exp(-t \sum_{i=1}^r \xi_i e_i^\xi) \Phi^T(\cdot)$ conserves the momenta $J_j$, $j = 1, \ldots, r$, and as shown in [25], we have

$$DH(\bar{x})D_{\lambda_{n,\mu}} \Phi^T(\bar{x}; \bar{\xi}) = \int_0^T \|DH(\Phi^T(\bar{x}; \xi))/dt\|^2 \, ds,$$

$$DJ_i(\bar{x})D_{\lambda_{i,\mu}} \Phi^T(\bar{x}; \bar{\xi}) = \int_0^T (\nabla J_i(\Phi^T(\bar{x}; \xi)), \nabla J_i(\Phi^T(\bar{x}; \xi))) \, ds,$$

$$DJ_i(\bar{x})D_{\lambda_{n,\mu}} \Phi^T(\bar{x}; \bar{\xi}) = \int_0^T (DH(\Phi^T(\bar{x}; \xi)), DJ_i(\Phi^T(\bar{x}; \xi))) \, ds,$$

$$DH(\bar{x})D_{\lambda_{n,\mu}} \Phi^T(\bar{x}; \bar{\xi}) = \int_0^T (DH(\Phi^T(\bar{x}; \xi)), DJ_i(\Phi^T(\bar{x}; \xi))) \, ds.$$
Hence any \( c = (c_{\mu,1}, \ldots, c_{\mu,r}, c_E) \) with \( c^T B c = 0 \) satisfies \( c_E D H(\bar{x}) + \sum_{i=1}^r c_{\mu,i} D J_i(\bar{x}) = 0 \), which contradicts the assumption that \( \bar{x} \) does not lie on a relative equilibrium. Therefore \( B \) has full rank and is positive definite.

From Proposition 2.17 and the above we see that \( DF(\bar{x}, \bar{T}, \bar{\xi}, 0, 0) \) has full rank iff the matrix \( P N_1 D_x F \) has image \( N_1 \). Here \( P N_1 \) is a projection on \( N_1 \) with kernel \( N_0 \) and \( g \bar{x} \). Note that

\[
P N_1 D_x F = [M_{10}, M_{11} - id_{N_1}, M_{12}] = [D_{\mu} \Pi N_1(0), D_{\nu} \Pi N_1(0) - id, D_{E} \Pi N_1(0)]
\]

where we used the notation of Proposition 2.17. Therefore, by the definition of transversality (Definition 2.19), the matrix \( DF(\bar{x}, \bar{T}, \bar{\xi}, 0, 0) \) has full rank and \( F \) has a \((2+2r)\)-dimensional family of solutions.

By Theorem 2.22 there is an \((r+1)\)-dimensional manifold \( \mathcal{P}(s) \) of RPOs of (2.1) near \( \bar{x} \) and \( x(s) \in \mathcal{P}(s) \) has drift symmetry \( \sigma(s) \in \Gamma_{(\sigma, \bar{\mu})} \) and an average drift velocity \( \xi(s) \in g_{(\sigma, \bar{\mu})} \) which commutes with \( \alpha \). Therefore \( y = (x(s), T(s) = t \tau(s), \xi(s), 0, 0) \), is a solution of \( F = 0 \) for \( s \approx 0 \). Moreover, since, by Remark 2.15, the group \( \Gamma_{(\sigma, \bar{\mu})}^{id} \) is abelian, for every \( \gamma \in \Gamma_{(\sigma, \bar{\mu})}, \gamma \approx id \) and \( t \approx 0 \), the point \( y = (\gamma \Phi^t(x(s)), T(s), \xi(s), 0, 0) \) is a solution of \( F = 0 \). This gives an \((2r+2)\)-dimensional manifold of solutions of \( F = 0 \). Hence the \((2r+2)\)-dimensional solution manifold of \( F = 0 \) consists of RPOs of (2.1) near \( \bar{x} \) and satisfies \( \lambda_E = \lambda_\mu = 0 \). \( \blacksquare \)

Under the assumptions of Theorem 2.23, the underdetermined system (2.25) is amenable to standard numerical methods; for example, the Gauss-Newton method applied to (2.25) converges for initial values \( y = (x, T, \xi, \lambda_E, \lambda_\mu) \) close to \( \bar{y} \).

**Remark 2.24** In Theorem 2.22 we only assume that the RPO \( \mathcal{P} \) has a regular drift-momentum pair \( (\sigma, \bar{\mu}) \) and not necessarily a regular drift symmetry \( \bar{\sigma} \) so that in general \( g_{\sigma} \neq g_{(\sigma, \bar{\mu})} \).

In this case let \( \{e^1_\xi, \ldots, e^q_\xi\} \) denote a basis of \( g_{\sigma} \) such that span\(\{e^{i+1}_\xi, \ldots, e^q_\xi\} = n_\mu \cap g_{\sigma} \) and span\(\{e^1_\xi, \ldots, e^i_\xi\} = g_{(\sigma, \bar{\mu})} \). Let, as before, \( J_{e^i_\xi} = J_i, \; i = 1, \ldots, q \). Note that \( \bar{\mu}(e^i_\xi) = 0, \; j = r+1, \ldots, q \). Because of Remark 2.15 we generically have \( q = r \), but when \( q > r \) at the RPO \( \mathcal{P} \) then the derivative \( DF(\bar{y}) \) of (2.25) does not have full rank, and convergence of a Gauss-Newton method applied directly to (2.25) is expected to be slow. In a manner similar to the case of relative equilibria, see Remark 2.8, it is advantageous to solve the system

\[
F(x, T, \xi, \lambda_E, \lambda_\mu) = \begin{pmatrix}
\alpha \Phi^t_{x}(x; \xi, \lambda_E, \lambda_\mu) - x \\
J_{r+1}(x) \\
\vdots \\
J_{q}(x)
\end{pmatrix} = 0,
\]

where

\[
F : \mathcal{X} \times \mathbb{R}^{2+q+r} \to \mathcal{X} \times \mathbb{R}^{q-r}
\]

and now \( \Phi^t(x; \xi, \lambda_E, \lambda_\mu) \) is the flow of

\[
\dot{x} = f_H(x) + \lambda_E \nabla H(x) + \sum_{i=1}^r \lambda_{\mu,i} \nabla J_i(x) - \sum_{i=1}^q \xi e^i_\xi.
\]

Then

\[
DF(\bar{x}, \bar{T}, \bar{\xi}, 0, 0) = [D_x F, D_T F, D_\xi F, D_{\lambda_E} F, D_{\lambda_\mu} F]
\]

\[
= \begin{pmatrix}
\bar{\sigma}DF_{x}(\bar{x}) - id & f_H(\bar{x}) & e^1_x & \ldots & e^q_x & D_{\lambda_E} F & D_{\lambda_\mu} F \\
D_{J_{r+1}}(\bar{x}) & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & 0 & 0 & 0 \\
D_{J_{q}}(\bar{x}) & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
has full rank and so the solution manifold of (2.26) is \((2r + 2)\)-dimensional. As in the proof of Theorem 2.23, for \(\gamma\) in the abelian group \(\Gamma^{id}_{(\sigma, \mu)}\), and \(t \in \mathbb{R}\), the points \(\gamma \Phi(x(s))\) have momentum \(\mu(s) = \mathbf{J}(\gamma \Phi(x(s)))\) satisfying \(\mu(s) - \bar{\mu} \in \mathbf{g}_t^{id}_{(\sigma, \mu)} \subseteq \text{ann}(\mathbf{n}_\mu)\). Therefore the set \(\{y = (\gamma \Phi(x(s)), T(s), \xi(s), 0, 0), \gamma \in \Gamma^{id}_{(\sigma, \mu)}, t \in \mathbb{R}\}\) defines an \((2r + 2)\)-dimensional solution manifold of (2.26), too. From this \(\lambda_E = 0\), \(\lambda_\mu = 0\) follows.

**Remark 2.25** In order to continue RPOs in numerically delicate situations, that is, when the single shooting method is ill-conditioned, we use multiple shooting rather than single shooting, cf. [25] and references in there: we compute \(k\) points on the RPO in the comoving frame by solving the underdetermined equation

\[
F(x_1, \ldots, x_k, T, \xi, \lambda_E, \lambda_\mu) = 0, \quad F: \mathcal{X}^k \times \mathbb{R}^{2r+2r} \to \mathcal{X}^k. \tag{2.27}
\]

Here \(x_j \in \mathcal{X}, j = 1, \ldots, k, T, \lambda_E \in \mathbb{R}, \lambda_\mu, \xi \in \mathbb{R}^r\). Moreover \(0 = s_1 < \ldots < s_{k+1} = 1\) is a partition of the unit interval, \(\Delta s_i = s_{i+1} - s_i\) for \(i = 1, \ldots, k\), and

\[
F_i(x_1, \ldots, x_k, T, \xi, \lambda_E, 1) = \left\{ \begin{array}{ll}
\Phi_T^{-\Delta s_i}(x_i; \xi, \lambda_E, \lambda_\mu) - x_{i+1} & \text{for } i = 1, \ldots, k-1,
\alpha \Phi_T^{-\Delta s_k}(x_k; \xi, \lambda_E, \lambda_\mu) - x_1 & \text{for } i = k.
\end{array} \right.
\tag{2.28}
\]

It is well-known, see, e.g., [25] and references therein, that the derivative \(DF\) of (2.27) has full rank in an RPO if and only if the corresponding derivative of the single shooting equation (2.25) has full rank. Therefore Theorem 2.23 can readily be applied to the multiple shooting context.

**Remark 2.26** The relationship

\[
D_x F(\bar{x}, \bar{\ell}, \bar{\xi}, 0, 0) = \sigma D\Phi^T(\bar{x}) - \text{id}
\]

between the derivative of the Poincaré-map on one hand, and the shooting equations (2.25) on the other hand will play an important role in the computation of bifurcations, see Sections 2.5 and 3. In the multiple-shooting case the matrix \(\sigma D\Phi^T(\bar{x})\) can be obtained from the linearization of (2.27), for example see [25].

**Remark 2.27** Muñoz-Almaraz et al [15] call a periodic orbit of (2.1) with period \(\bar{T}\) through \(\bar{x}\) normal if, in our notation

\[
\text{Im}(D_x \Phi^T(\bar{x}) - \text{id}) + \text{span}(f_H(\bar{x})) = \ker\mathbf{D}H(\bar{x}) \cap \ker\mathbf{D}H(\bar{x}), \tag{2.29}
\]

see [15, Definition 4]. By projecting both sides of (2.29) on \(\mathcal{N}_1 \subseteq \ker\mathbf{D}J(\bar{x}) \cap \ker\mathbf{D}H(\bar{x})\) we see that a normal periodic orbit is nondegenerate and in particular transversal. In the case of trivial symmetry \(\Gamma = \{\text{id}\}\), a periodic orbit of (2.1) is normal if and only if it is nondegenerate. For nontrivial symmetry groups a nondegenerate periodic orbit need not be normal as for a nondegenerate periodic orbit \(g_\mu \bar{x}\) lies in \(\mathbf{D}J(\bar{x}) \cap \ker\mathbf{D}H(\bar{x})\), but might not lie in the image of \((D_x \Phi^T(\bar{x}) - \text{id})\). Galan et al [4, 15] prove continuation results for normal periodic orbits and design numerical continuation methods which are similar to our shooting equations (2.25) in the case of trivial drift symmetry \(\sigma\), see [15, Theorem 13]. However they only compute periodic orbits, not RPOs and they require the solution to lie in a Poincaré-section to the periodic orbit by adding phase conditions. Moreover, they study persistence of periodic orbits when external parameters are varied, see [15, Theorems 7, 14], whereas we restrict attention to continuation in the conserved quantities, energy and momentum, of the system. In [15, Theorem 7] they continue normal periodic orbits in external parameters, in [15, Theorem 14] they instead require that the eigenvalue 1 of the linearization \(M = D\Phi_T(\bar{x})\) of the periodic
As above, we see that the solutions \( J \) get a one-parameter family, without loss of generality the case there is a path \( \gamma \) of points on RPOs with drift-momentum pair \((\bar{\sigma}, \bar{\mu})\) and energy \( \bar{E} \), define the Poincaré-map \( \Pi \) as before and choose coordinates as in Lemma 2.16. Then the matrix

\[
\left[ D_E \Pi_{\Lambda_1}(0), D_u \Pi_{\Lambda_1}(0) - \text{id}_{\Lambda_1} \right]
\]

has full rank if and only if the \((r, r)\)-matrix \( \partial_u \chi(0) \) from Theorem 2.22 is invertible. In this case there is a path \( x(\epsilon) \in J^{-1}(\bar{\mu}) \) of points on RPOs \( \mathcal{P}(\epsilon) \) with energy \( E(\epsilon) \) such that \( x(0) = \bar{x} \), \( \mathcal{P}(0) = \mathcal{P}, E(0) = \bar{E} \). If the RPO \( \mathcal{P} \) is nondegenerate, we can choose \( \epsilon = \bar{E} \). Moreover \( DF_\mu(x, (T, \xi, 0, 0)) \) has full rank and there is a two-dimensional solution manifold of (2.30) in \( \mathcal{X}^{\mu} \), which consists of points on the family \( \mathcal{P}(\epsilon) \).

2.4 Numerical path-following of RPOs

As before let \( \bar{x} \) lie on an RPO with drift-momentum pair \((\bar{\sigma}, \bar{\mu})\) and let \( \xi_1, \ldots, \xi_r \) be a basis of \( g_{(\bar{\sigma}, \bar{\mu})} \). For \( \mu \in g^* \) let \( \mu_j = \mu(\xi_j), j = 1, \ldots, g \).

We can fix the momentum value \( \bar{\mu} \) and continue in energy, i.e., solve the equation

\[
F^\mu(x, T, \xi, \lambda_E, \lambda_\mu) = \begin{pmatrix} \lambda_j - \bar{\mu}_j \end{pmatrix}
\]

The \( x \)-component of the solution \( y = (x, T, \xi, 0, 0) \) of (2.31) then lies in \( \mathcal{X}^{\mu} = J^{-1}(\bar{\mu}) \) which can be shown as in Section 2.1.3: First, note that by identifying \( g \) with \( g^* \) by a \( \Gamma \)-invariant inner product we get \( \mu_j = 0 \), \( j = r + 1, \ldots, g \). For \( \sigma \in \mathfrak{g}_{\sigma} \) close to \( \bar{\sigma} \) we have \( \mathfrak{g}_{\sigma} \subseteq \mathfrak{g}_{\bar{\sigma}}, \mathfrak{g}_{\sigma} \subseteq \mathfrak{g}_{\bar{\sigma}} \). By construction, \( \xi \in \mathfrak{g}_{\sigma} \) at any solution \( y = (x, T, \xi, 0, 0) \) of (2.31). Since \( \text{Ad}_{\bar{\sigma}} \mu = \mu \) for the drift symmetry \( \sigma = \alpha \exp(-r\xi), \tau = T/\ell \), and momentum \( \mu = J(x) \) of the RPO given by \( y = (x, T, \xi, 0, 0) \) (see (2.18)), we conclude that \( J_j(x) = 0, j = q + 1, \ldots, g \). If \( q \neq r \) then \( \mu_j = 0 \), \( j = r + 1, \ldots, q \) is one of the equations of \( F \), see (2.26). Hence the solutions of (2.30) satisfy \( J(x) = \bar{\mu} \).

Alternatively, we can fix the energy and \( g - 1 \) out of the first \( g \) momentum components to get a one-parameter family, without loss of generality the \( g - 1 \) components \( \bar{\mu}^{\prime} = (\bar{\mu}_2, \ldots, \bar{\mu}_g) \)

\[
F^{E, \bar{\mu}^{\prime}}(x, T, \lambda_E, \lambda_\mu) = \begin{pmatrix} \lambda_j - \bar{\mu}_j \end{pmatrix}
\]

As above, we see that the solutions \( y = (x, T, \xi, 0, 0) \) of (2.30) satisfy

\[
x \in \mathcal{X}^{E, \bar{\mu}^{\prime}} = \{ x \in \mathcal{X}, \ H(x) = \bar{E}, J_j(x) = \bar{\mu}_j, \ j = 2, \ldots, g \}.
\]

Corollary 2.28 Let \( \bar{x} \) lie on transversal RPO with regular drift-momentum pair \((\bar{\sigma}, \bar{\mu})\) and energy \( \bar{E} \), define the Poincaré-map \( \Pi \) as before and choose coordinates as in Lemma 2.16. Then the matrix

\[
[D_E \Pi_{\Lambda_1}(0), D_u \Pi_{\Lambda_1}(0) - \text{id}_{\Lambda_1}]
\]
If (2.32) is satisfied we say that the RPO \( \tilde{P} \) is transversal with respect to \( C(x) := H(x) \). Under this assumption, (2.30) can be solved by standard numerical methods, for example by the Gauss-Newton method, for initial values close to \( \tilde{y} = (\hat{x}, \hat{T}, \hat{\xi}, 0, 0) \).

**Corollary 2.29** Let \( \hat{x} \) lie on transversal RPO with regular drift-momentum pair \((\sigma, \tilde{\mu})\) and energy \( E \), define the Poincaré-map \( \Pi \) as before and choose coordinates as in Lemma 2.16. Then the matrix
\[
[D_{x_1} \Pi_{X_1}(0), D_{w} \Pi_{X_1}(0) - \text{id}_{X_1}]
\]
has full if and only if the \((r,r)\)-matrix
\[
\partial_s (E, \chi_2, \ldots, \chi_r)(s)|_{s=0}
\]
from Theorem 2.22 is invertible. In this case there is a path \( x(\epsilon) \in \mathcal{X}^E, \rho^j \) of points on RPOs \( \mathcal{P}(\epsilon) \) with energy \( E(\epsilon) \) such that \( x(0) = \hat{x} \), \( \mathcal{P}(0) = \mathcal{P} \), \( \mu_1(0) = \mu_1 \). If the RPO \( \mathcal{P} \) is nondegenerate then we can choose \( \epsilon = \mu_1 \). Moreover \( D\mathcal{F}^{E, \rho^j}(\tilde{y}) \) has full rank, and there is a two-dimensional solution manifold of (2.31), which consists of points on the family \( \mathcal{P}(\epsilon) \).

If (2.33) is satisfied we say that the RPO \( \mathcal{P} \) is transversal with respect to \( C(x) := J_1(x) \) (analogously to the case of relative equilibria, see Section 2.1.3). Under this assumption, (2.31) can be solved by standard numerical methods, for example by the Gauss-Newton method, for initial values close to \( \tilde{y} = (\hat{x}, \hat{T}, \hat{\xi}, 0, 0) \).

For continuation we use the tangent \( t(\tilde{y}) \) to the solution manifold of \( F^{E, \rho^j} = 0 \) or \( F^\mu(\hat{x}, \hat{T}, \hat{\xi}, 0, 0) \) and has an \( x \)-component \( t_x = t_x(\tilde{y}) \) orthogonal to \( g_x \hat{x} \) and \( f_H(\hat{x}) \).

**Remark 2.30** In the actual implementation it is more convenient to add the continuation parameter \( C = E \) resp. \( C = \mu_1 \) to the vector of unknowns \( y = (x, T, \xi, \lambda_E, \lambda_\mu, C) \) and to add the additional equation \( C(x) - C = 0 \) to \( F \). Then \( t_C = DC(x) t_x \) as required, and computing boundary points in \( C \) is greatly simplified.

**Remark 2.31** In the multiple shooting context, let \( F^\mu : \mathcal{X}^k \times \mathbb{R}^{2r+2} \to \mathcal{X}^k \times \mathbb{R}^r \) resp. \( F^{E, \rho^j} : \mathcal{X}^k \times \mathbb{R}^{2r+2} \to \mathcal{X}^k \times \mathbb{R}^r \) be given by (2.27) with \( r \) conserved quantities fixed as in (2.30) and (2.31)). At a transversal RPO the kernel of \( D\mathcal{F}(\tilde{y}) \) is \((r+1)\)-dimensional by Corollaries 2.28 resp. 2.29. We write vectors in the kernel as \( t = (t_1, \ldots, t_k, t_T, t_T, t_{\lambda_E}, t_{\lambda_\mu}) \), where \( t_j \in \mathcal{X}_j \), \( j = 1, \ldots, k \), \( t_T \in \mathbb{R}^{2r+2} \), \( t_{\lambda_E} \in \mathbb{R}^r \). Then the kernel of \( D\mathcal{F}(\tilde{y}) \) contains the vectors \( t^f_1, t^{\xi_1}, \ldots, t^{\xi_r} \) where
\[
t^f_i = f(x_i), \quad i = 1, \ldots, k, \quad t^f_T = 0 = t^f_{\lambda_E}, \quad t^f_{\lambda_\mu} = 0, \quad t^f_j = 0, \quad (2.34)
\]
and
\[
t^{\xi_j} = e^\xi_j x_i, \quad i = 1, \ldots, k, \quad t^{\xi_j} = t^{\xi_j}_{\lambda_E}, \quad t^{\xi_j}_{\lambda_\mu} = 0, \quad t^{\xi_j}_j = 0, \quad t^{\xi_j} = 0. \quad (2.35)
\]
We define the continuation tangent \( t = (t_1, \ldots, t_k, t_T, t_{\xi}, t_{\lambda_E}, 0, t_{\lambda_\mu} = 0) \) as the element of the kernel which is orthogonal to \( t^f \) and \( t^{\xi_1}, \ldots, t^{\xi_r} \), as in [25] for nondegenerate RPOs.

**Remark 2.32** In the case of dissipative parameter dependent \( \Gamma \)-equivariant ODEs (2.15) we call an RPO through \( \hat{x} \) with parameter \( \lambda \) transversal if \( [D_x \Pi(\hat{x}, \lambda) - \text{id}, D_x \Pi(\hat{x}, \lambda)] \) has rank \( N \) and nondegenerate if \( D_x \Pi(\hat{x}) \) has no eigenvalues one. Near a transversal RPO there is a path \((x(s), \lambda(s))\) of fixed points of \( \Pi \) corresponding to RPOs of (2.15). Numerically, this path can be computed by solving the underdetermined system \( F(x, T, \xi, \lambda_E, \lambda_\mu) = \alpha \Phi^f(x; \xi, \lambda) - x = 0 \) where \( \Phi^f(x; \xi, \lambda) \) is the flow of
\[
\dot{x} = f(x; \lambda) - \sum_{i=1}^q \xi_i e^i_{\xi}. \]

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Here $q = \dim g_{\sigma}$ and $\sigma$ is the drift symmetry at the RPO through $\bar{x}$. This case is similar to, but much simpler than in the Hamiltonian case, see (2.24), (2.25). Note that typically $r = \dim g_{\sigma}$ is locally constant, see Remark 2.15.

2.5 Turning points of relative equilibria and RPOs

In this section we deal with the most simple situation of a critical relative equilibrium RPO, namely we consider a transversal relative equilibrium/RPO with regular velocity-momentum pair / drift-momentum pair which ceases to be non-degenerate. As before, we restrict to the case of free group actions.

2.5.1 Turning points of Hamiltonian relative equilibria

Let $\bar{x}$ lie on a transversal relative equilibrium and regular velocity-momentum pair $(\tilde{\xi}, \tilde{\mu})$ and assume that the relative equilibrium is degenerate. By Theorem 2.6 it persists to an $r$-dimensional family $x(s)$ of relative equilibria nearby with velocity-momentum pairs $(\xi(s), \mu(s) = \tilde{\mu} + \chi(s))$, $\chi \in g^*_{(\tilde{\xi}, \tilde{\mu})} \simeq \mathbb{R}^r$. Since the relative equilibrium is degenerate the matrix $J_{\chi}h(0)$ has an eigenvalue 0. Therefore we see from the proof of Theorem 2.6 that the $(r,r)$-matrix $\frac{Dh}{Ds}(0)$ is singular. The eigenvalue 0 of $J_{\chi}D^s h(0)$ is of algebraic multiplicity two since $J_{\chi}D^s h(0)$ is infinitesimally symplectic and, generically, of geometric multiplicity one, see [13]. As discussed above, see (2.14), we numerically compute a one-parameter family of relative equilibria by fixing all components of the momentum map except for the first component, $C(x) := J_1(x)$.

**Proposition 2.33** Assume that the relative equilibrium through $\bar{x}$ is degenerate, but transversal with respect to the conserved quantity $C(x) := J_1(x)$ of (2.1). Let $x(\epsilon) \in X^0$ be the path of relative equilibria from Corollary 2.10, $c(\epsilon) = C(x(\epsilon))$. Then generically $\bar{x}$ is a turning point in $c(\epsilon) = C(x(\epsilon))$, i.e., $c'(0) = 0$ and $c''(0) \neq 0$, and the pair of eigenvalues of the linearization of the relative equilibria, which collide at 0 at the turning point, lies on the imaginary axis before the turning point is passed, and on the real axis after the turning point, or vice versa.

**Proof.** As we see from the proof of Theorem 2.6, the relative equilibria $x(s) \simeq (\bar{\nu}(s), \bar{\omega}(s)) \in \tilde{\mathcal{X}}_{\bar{x}} \oplus \tilde{\mathcal{N}}_1$ correspond to equilibria of the $\bar{\nu}$-dependent Hamiltonian system $\frac{dx}{dt} = f_{\chi}(\bar{\nu}, \bar{\omega})$ after symmetry reduction by $\tilde{\Gamma} = \Gamma_{\bar{x}}$. Here, $\bar{\nu} = \chi$ and so $\bar{\nu}_j = 0$, $j = 2, \ldots, r$, $\bar{\nu}_1 = c$. Define the parameter-dependent Hamiltonian vectorfield $f_{\chi}(\bar{\nu}, \bar{\omega})$ by choosing $\bar{\nu}$ in this way. By assumption $D_{\bar{\nu}}f_{\chi}(0,0)$ has a one-dimensional kernel and $[D_{\bar{\nu}}f_{\chi}(0,0), D_{\bar{\omega}}f_{\chi}(0,0)]$ has full rank. Denote the solutions of $f_{\chi}(\bar{\nu}, \bar{\omega}) = 0$ corresponding to $x(\epsilon)$ as $(c(\epsilon), \bar{\omega}(\epsilon))$ where $\bar{c} = c(0)$. Since $D_{\bar{\nu}}f_{\chi}(c(0),0)w'(0) + D_{\bar{\omega}}f_{\chi}(0,0)c'(0) = 0$, we infer from the above that $D_{\bar{\nu}}f_{\chi}(c(0),0)w'(0) = 0$, $D_{\bar{\omega}}f_{\chi}(c(0),0) \neq 0$, and so $c'(0) = 0$. Moreover, applying well-known results on turning points of Hamiltonian systems on the Hamiltonian vectorfield $f_{\chi}(\bar{\nu}, \bar{\omega})$, we see that typically $c''(0) \neq 0$ and that the pair of eigenvalues of $Df_{\chi}(c(\epsilon), \bar{\omega}(\epsilon))$, which collide at 0 at $\epsilon = 0$, is on the imaginary axis for $\epsilon < 0$ ($\epsilon > 0$) and on the real axis for $\epsilon > 0$ ($\epsilon < 0$).

2.5.2 Turning points of RPOs

Let $\bar{x}$ lie on a transversal RPO with relative period $\bar{T}$ and regular drift momentum pair $(\bar{\sigma}, \bar{\mu})$ and energy $E$, and assume that the RPO is degenerate. By Theorem 2.22 it persists to an $(r+1)$-dimensional family $x(s)$ of RPOs with drift-momentum pairs $(\sigma(s), \mu(s) = \bar{\mu} + \chi(s))$ and energy $E(s)$, where $\chi \in g^*_{(\bar{\sigma}, \bar{\mu})} \simeq \mathbb{R}^r$. Since the RPO is degenerate the matrix $D_{\chi}\Pi_{\chi}(0) = D_{\chi}\Pi^{E,\mu}(\bar{x})$ has an eigenvalue 1. Therefore we see from the proof of Theorem 2.22 that the $(r+1, r+1)$-matrix $\frac{D^2}{Ds^2}(E, \chi)(0)$ is singular.
As discussed above, see (2.31) and (2.30), we numerically compute a one-parameter family of RPOs and continue RPOs with respect to a component of the momentum map or the energy. Then we have:

**Proposition 2.34** Assume that the RPO through \( \bar{x} \) is degenerate, but transversal with respect to the conserved quantity \( C(x) := J_1(x) \) or \( C(x) := H(x) \) of (2.1). Denote by \( x(\epsilon) \in X^E, \mu' \) the path of RPOs from Corollary 2.28 and Corollary 2.29, and let \( c(\epsilon) = C(x(\epsilon)) \). Then generically \( \bar{x} \) is a turning point in \( c(\epsilon) = C(x(\epsilon)) \), i.e., \( c'(0) = 0 \) and \( c''(0) \neq 0 \), and the pair of eigenvalues of the linearization of the RPOs, which collide at 1 at the turning point, lies on the unit circle before the turning point is passed, and on the real axis after the turning point, or vice versa.

**Proof.** From the proof of Theorem 2.22 we see that \( \bar{x} \) lies on a turning point in the conserved quantity \( C(x) \) if \( \overline{\Pi}^{E, \mu}(\bar{x}) = \overline{\Pi}_{X_1}(0) \) has rank defect one, i.e., has a double eigenvalue 1 with geometric multiplicity 1, and if \( \overline{\Pi}_{(x, C)}(0) \) has full rank. Then \( c'(0) = 0 \). This can be seen as follows: Since \( \bar{x} \) is degenerate, but transversal, we have \( (\overline{\Pi}_{X_1}(0) - \text{id})w'(0) = 0 \) and \( \overline{\Pi}_{X_1}(0) \neq 0 \). Then \( (\overline{\Pi}_{X_1}(0) - \text{id})w'(0) + \overline{\Pi}_{C}(0)e(0) = 0 \) implies \( c'(0) = 0 \).

From the proof of Theorem 2.22 we see that the path of RPOs \( x(\epsilon) \simeq (\tilde{\nu}(\epsilon), \tilde{w}(\epsilon), E(\epsilon)) \) corresponds to fixed points of the \( (\tilde{\nu}, E) \)-dependent symplectic map \( \Pi_{X_1}(\tilde{\nu}, \tilde{w}, E) = \tilde{w} \) after reduction by \( \tilde{\Gamma} = \Gamma_{\sigma} \) such that \( x(0) = \bar{x} \simeq (0,0,E) \). Here \( \tilde{\nu} = \chi \in \mathbf{g}^*_{(\sigma, \mu)} \), so that in the case \( C(x) = H(x) \) we have \( \tilde{\nu} \equiv 0 \) and in the case \( C(x) = J_1(x) \) we have \( \tilde{\nu} \equiv 0, j = 2, \ldots, r, E(\epsilon) = E \). So we obtain a symplectic map \( \Pi_{X_1}(c, \bar{w}) \) which depends on one parameter \( c \), and \( \bar{w} = 0 \), the derivative \( \overline{\Pi}_{X_1}(0,0) \) has an eigenvalue 1 with geometric multiplicity one and algebraic multiplicity two such that \( \overline{\Pi}_{X_1}(0,0), D_x \overline{\Pi}_{X_1}(0,0) \) has full rank.

Applying the well-known results on turning points of parameter-dependent symplectic maps, see, e.g., [13], on the symplectic map \( \Pi_{X_1} \) then proves the proposition.

**2.5.3 Detection and computation of turning points of relative equilibria and RPOs**

Turning points along paths of RPOs continued in the conserved quantity \( C(x) \) can be detected by a sign change of

\[
 u(y) = \langle t_x, \nabla C(x) \rangle \tag{2.36}
\]

between two consecutively computed solutions \( y^{(0)} = (x^{(0)}, T^{(0)}, \xi^{(0)}) \) and \( y^{(1)} = (x^{(1)}, T^{(1)}, \xi^{(1)}) \) of (2.31) or (2.30) respectively. Here \( t_x \) is the \( x \)-component of the continuation tangent \( t(y) \) at \( y = (x, T, \xi) \). Analogously, turning points of paths of relative equilibria which are continued in \( C(x) = J_1(x) \) are detected by a sign change of (2.36) between two consecutively computed solutions of (2.14).

Once detected, turning points can be computed by a combination of Hermite interpolation and subdivision of (2.36) along the solution path of (2.31) or (2.30), see, e.g., [23] and references therein.

**Remark 2.35** In the case of non-Hamiltonian systems, see Remark 2.11 and 2.32, turning points are detected and computed in the same way, only that now \( c = \lambda \) and \( u(y) = t_\lambda \) is the \( \lambda \)-component of the continuation tangent \( t(y) \) at \( y \).
3 Hamiltonian relative period doubling bifurcations

In this section we first present a theorem on relative period doubling bifurcations of RPOs with regular drift momentum pair (\(\sigma, \mu\)) and energy \(E\). We say that \(\bar{x}\) is a relative period doubling bifurcation point if \(\text{DII}^{E, \bar{\mu}}(\bar{x}) = \text{DII}_{\mathcal{N}}(0,0,0)\) has an eigenvalue \(-1\). This eigenvalue has algebraic multiplicity \(\geq 2\) since, by Lemma 2.16 b), the map \(\Pi_{\mathcal{N}}\) is symplectic. We make the generic assumption that the eigenvalue \(-1\) of \(\text{DII}_{\mathcal{N}}(0)\) has algebraic multiplicity 2 and geometric multiplicity 1 and denote the eigenvector of \(\text{DII}_{\mathcal{N}}(0)\) to the eigenvalue \(-1\) by \(\bar{w}\).

Let \(x(E, \chi), \chi \in \mathfrak{g}^{*}_{(\sigma, \mu)}\), be the family of RPOs through \(\bar{x} = x(E, 0)\) whose existence was proven in Theorem 2.22. Denote its coordinates on \(\mathcal{N}\) as \((\nu(\chi, E), w(\chi, E))\). Let \(\lambda_1(E, \chi)\) and \(\lambda_2(E, \chi)\) be the eigenvalues of \(\text{D}_w \Pi_{\mathcal{N}}(\nu(E, \chi), w(E, \chi))\) which collide at the bifurcation: \(\lambda_1(E, 0) = \lambda_2(E, 0) = -1\) and denote the generalized eigenspace of \(\text{D}_w \Pi_{\mathcal{N}}(\nu(E, \chi), w(E, \chi), E)\) to the eigenvalues \(\lambda_1(E, \chi)\) and \(\lambda_2(E, \chi)\) by \(\mathcal{Y}(E, \chi)\). Then \(\mathcal{Y}(E, \chi)\) and the matrices

\[
M_1(E, \chi) := \text{D}_w \Pi_{\mathcal{N}}(\nu(E, \chi), w(E, \chi), E) |_{\mathcal{Y}(E, \chi)}
\]

are symplectic \((2, 2)\)-matrices and depend smoothly on \((E, \chi)\). If

\[
\psi(E, \chi) := \text{tr}(M(E, \chi)) + 2 = 0,
\]

then \(M(E, \chi)\) has a double eigenvalue \(-1\). We assume that

\[
\text{D}_1(E, \chi) \psi(E, 0) \neq 0.
\]

In [13] it is shown that this condition is generically satisfied in the space \(\text{SP}(2)\) of symplectic \((2, 2)\)-matrices. Under this condition the equation \(\psi(E, \chi) = 0\) determines a smooth hypersurface of codimension 1 in \((E, \chi)\)-space. We choose coordinates such that \(\text{D}_1(E, \chi) \psi(E, 0) \neq 0\).

Remark 3.1 At a transverse passing of the hypersurface \(\psi(E, \chi) = 0\) in \(\text{SP}(2)\) the eigenvalues are either on the unit circle before the collision and on the real axis after or vice versa: to see this let \(\lambda_1(E, \chi) = -1 + \delta(E, \chi)\) where \(\delta(E, 0) = 0\). Then \(\lambda_2(E, \chi) = 1/( -1 + \delta(E, \chi)) = -1 - \delta(E, \chi) - \delta^2(E, \chi) + h.o.t.,\) and so \(\psi(E, \chi) = -\delta^2(E, \chi) + h.o.t.\). As \(\psi(E, \chi)\) changes sign at the relative period-doubling hypersurface determined by (3.1), we see that \(\delta(E, \chi)\) is real before bifurcation and complex after bifurcation or vice versa.

In the following theorem we make the generic assumption that \(\text{D}_E \psi(E, 0) \neq 0\). We deal with the case \(\text{D}_E \psi(E, 0) = 0, \text{D}_x \psi(E, 0) \neq 0\) in Remark 3.3.

Theorem 3.2 Let \(\bar{x}\) be a relative period doubling bifurcation point with a regular drift-momentum pair \((\sigma, \mu)\) and energy \(E\). Assume that \((\tilde{\sigma}, \tilde{\mu})\) is also a regular drift-momentum pair and that \(r = r_{(\sigma, \mu)}(\Gamma) = r_{(\tilde{\sigma}, \tilde{\mu})}(\Gamma)\). Then, generically, a family \(\tilde{x}(\epsilon, \chi), \chi \in \mathfrak{g}^{*}_{(\sigma, \mu)}\), of points on RPOs \(\tilde{\mathcal{P}}(\epsilon, \chi)\) with relative period \(\tilde{\tau}(\epsilon, \chi)\), drift symmetry \(\tilde{\sigma}(\epsilon, \chi)\), momentum \(\tilde{\mu}(\epsilon, \chi) = \tilde{\mu} + \chi,\) and energy \(\tilde{E}(\epsilon, \chi)\) bifurcates from \(\bar{x}\) such that

\[
\tilde{x}(0, 0) = \bar{x}, \quad \tilde{\tau}(0, 0) = 2\tau, \quad \tilde{\sigma}(0, 0) = \tilde{\sigma}^2, \quad \tilde{\mu}(0) = \tilde{\mu}, \quad \tilde{E}(0) = \bar{E}.
\]
Moreover, 
\[ \bar{\sigma}(\epsilon, \chi) = \alpha^2 \exp(\bar{\tau}(\epsilon, \chi)\bar{\xi}(\epsilon, \chi)), \quad \text{with} \quad \bar{\xi}(s) \in \mathbf{g}_{(\sigma, \mu)}^t, \quad \text{Ad}_{\alpha} \bar{\xi}(\epsilon, \chi) = \bar{\xi}(\epsilon, \chi). \]

We have \( D_\epsilon \bar{x}(0, 0) = \bar{w}, \) and, with \( \bar{x}(0, -\epsilon) = \Pi(\bar{x}(\epsilon, \chi)), \) we obtain a smooth manifold of RPOs with relative period \( \bar{\tau}(0, -\epsilon, \chi) = \tau(\epsilon, \chi), \) energy \( \bar{E}(0, -\epsilon, \chi) = \bar{E}(\epsilon, \chi), \) momentum \( \bar{\mu}(0, -\epsilon, \chi) = \bar{\mu}(\epsilon, \chi) \) and drift symmetry \( \bar{\sigma}(-\epsilon, \chi) = \bar{\sigma}(\epsilon, \chi). \)

**Proof.** We reduce by \( \bar{\Gamma} = \Gamma_\sigma \) only. Since \( \Gamma \) is compact the matrices \( \text{Ad}_\sigma \) and \( \text{Ad}_{\alpha^2} \mathbf{g}_\mathbf{z}, \) do not have Jordan blocks. Therefore we conclude from (2.22) that the RPO through \( \bar{x} \) is nondegenerate when considered as an RPO for the symmetry group \( \bar{\Gamma} = \Gamma_\sigma. \) Then, as before, since \( \mathbf{g}_{(\sigma, \mu)}^t \) is abelian by Remark 2.15, we have \( \Pi_{\bar{\mathcal{N}}^0}(\bar{\nu}, \bar{w}, \bar{E}) \equiv \bar{\nu}, \) where \( \bar{\nu} = \chi \in \mathbf{g}_{(\sigma, \mu)}^t \simeq \mathbb{R}^r. \) Here \( \bar{\mathcal{N}}^0 \simeq \mathbf{g}_{(\sigma, \mu)}^t(\bar{\nu}, \bar{w}) \in \bar{\mathcal{N}} = \bar{\mathcal{N}}^0 \oplus \bar{\mathcal{N}}^1 \) is the slice to the action of \( \bar{\Gamma} \) from Lemma 2.16. Moreover, if we reduce by \( \bar{\Gamma} \) only, then \( D_\mu \Pi_{\bar{\mathcal{N}}^1}(0, 0, 0) \) still has an eigenvalue \(-1\) of multiplicity two, and the symplectic map \( \Pi_{\bar{\mathcal{N}}^1}(\bar{\nu}, \ldots, \bar{E}) \) undergoes a period-doubling bifurcation at \( \bar{E} = \bar{E}_0 \), \( \bar{\nu} = 0. \)

On the manifold of fixed points \( \bar{w}(\bar{E}, \bar{\nu}) \) of \( \Pi_{\bar{\mathcal{N}}^1}(\bar{r}, \bar{w}, \bar{E}), \) corresponding to RPOs \( \mathcal{P}(\bar{E}, \chi), \) \( \chi = \bar{\nu}, \) of (2.1) there is a codimension one manifold \( \bar{w}(\bar{\nu}, \chi) \) with parameters \( \bar{\nu} = \chi \) and \( \bar{E}(\chi) \) such that \( D_\mu \Pi_{\bar{\mathcal{N}}^1}(\bar{\nu}, \bar{w}, \bar{E}(\chi)) \) has an eigenvalue \(-1\), where \( \chi \in \mathbb{R}^r. \) Then, generically, the parameter dependent equation
\[ \Pi_{\bar{\mathcal{N}}^1}(\bar{\nu}, \bar{w}, \bar{E}) = \bar{w} \]
has a second solution manifold \( \bar{w}(\epsilon, \chi) \) to the parameters \( \bar{\nu}(\epsilon, \chi) = \chi \) and \( \bar{E} = \bar{E}(\epsilon, \chi) \) such that \( w(0, \chi) = w'(\chi) \) [13].

Hence, we obtain an \((r + 1)\)-dimensional family of fixed points \( \bar{\nu}(\bar{E}, \chi) = (\bar{\nu}, \bar{w}, \bar{E}(\epsilon, \chi)) \) of \( \Pi_{\bar{\mathcal{N}}^1}, \) which gives rise to a family \( \bar{\nu}(\epsilon, \chi) = 0 \) of points on RPOs \( \mathcal{P}(\epsilon, \chi) \) with relative period \( \bar{\tau}(\epsilon, \chi) \) such that \( \bar{\tau}(0, 0) = 2\bar{\tau}, \) drift symmetry \( \bar{\sigma}(\epsilon, \chi) \) with \( \bar{\sigma}(0, 0) = \bar{\sigma}^2, \) momentum \( \bar{\mu}(\epsilon, \chi) = \bar{\mu} + \chi, \) and energy \( \bar{E}(\epsilon, \chi) \) such that \( \bar{E}(0, 0) = \bar{E}. \)

From \( \bar{\tau}(\epsilon, \chi) \) \( \bar{\nu}(\epsilon, \chi) \) \( \bar{\mu}(\epsilon, \chi) \) (as we saw in (2.18)) we conclude that \( \bar{\sigma}(\epsilon, \chi) \in \bar{\Gamma}_{\bar{\nu}(\epsilon, \chi)}. \)

Since \( \bar{\mathcal{N}}^0 \simeq \mathbf{g}_{(\sigma, \mu)}^t \) is invariant under \( \bar{\Gamma} \) we know that \( \bar{\Gamma}_{\bar{\nu}(\epsilon, \chi)} \subseteq \bar{\Gamma}_{\bar{\nu}(\epsilon, \chi)} \) and, therefore, \( \bar{\sigma}(\epsilon, \chi) \in \bar{\Gamma}_{(\sigma, \mu)}. \)

By Remark 2.15 we know that \( \text{Ad}_{\alpha^2} \) is abelian and that \( \mathbf{g}_{(\sigma, \mu)}^t = \mathbf{g}_{\alpha^2} \cap \mathbf{g}_{(\xi, \bar{\mu})}. \) So we can decompose \( \bar{\sigma}(\epsilon, \chi) = \alpha^2 \exp(\bar{\tau}(\epsilon, \chi)\bar{\xi}(\epsilon, \chi)) \) and \( \bar{\xi}(\epsilon, \chi) \in \mathbf{g}_{(\sigma, \mu)}, \) Ad, \( \bar{\xi}(\epsilon, \chi) = \bar{\xi}(\epsilon, \chi), \) and \( \text{Ad}^\mathbf{g}_{(\sigma, \mu)} = \chi. \)

Moreover, from [13] we have \( D_\mu \bar{w}(0, 0) = \bar{w}, \) and with \( \bar{w}(0, -\epsilon) := \Pi_{\bar{\mathcal{N}}^1}(\bar{\nu}(\epsilon, \chi), \bar{w}(\epsilon, \chi)), \) we obtain a smooth manifold of RPOs at \( \epsilon = 0 \) and see that \( \bar{E}(\epsilon, \chi) = \bar{E}(\epsilon, \chi), \) \( \bar{\tau}(\epsilon, \chi) = \bar{\tau}(\epsilon, \chi). \)

Also, \( \bar{\sigma}(\epsilon, \chi) = \gamma(\bar{\tau}(\epsilon, \chi)) \gamma(\bar{\tau}(\epsilon, \chi)) = \gamma(\bar{\tau}(\epsilon, \chi)) \gamma(\bar{\tau}(\epsilon, \chi)) \) equals \( \bar{\sigma}(\epsilon, \chi) = \gamma(\bar{\tau}(\epsilon, \chi)) \gamma(\bar{\tau}(\epsilon, \chi)) \) as \( \gamma(\bar{\tau}(\epsilon, \chi)) \) is \( \alpha^2 \exp(\bar{\tau}(\epsilon, \chi)\bar{\xi}(\epsilon, \chi)) \). Furthermore, \( \chi = \bar{\nu}(\epsilon, \chi) = \alpha^2 \bar{\nu}(\epsilon, \chi). \)

**Remark 3.3** If \( D_\epsilon \bar{\psi}(\bar{E}, 0) = 0, D_\chi \bar{\psi}(\bar{E}, 0) \neq 0, \) then Theorem 3.2 remains valid if we change the parametrization from \( (\epsilon, \chi) \) to \( s = (\epsilon, \chi_2, \ldots, \chi_r, e) \) where \( e = E - \bar{E}. \) In this case the component \( \bar{\nu}(s) \in \mathbf{g}_{(\sigma, \mu)}^t \) of the bifurcating RPOs through \( \bar{x}(s) \simeq (\bar{\nu}(s), \bar{\bar{w}}(s), \bar{E}(s)) \) on the slice \( \bar{\mathcal{N}} \) and the momentum \( \bar{\mu}(s) \) of the RPO through \( \bar{x}(s) \) depend nonlinearly on \( s \) and \( \bar{E}(s) = \bar{E} + e. \)

**Remark 3.4** In the comoving frame \( \bar{\xi} \) a relative period doubling bifurcation becomes a flip-doubling bifurcation or a flip-pitchfork bifurcation of the corresponding \( \bar{T} = \ell \bar{E} \)-periodic orbit [5, 23]. The drift symmetry of the bifurcating periodic orbit in the comoving frame is \( \bar{\tau} = \alpha^2 \bar{\tau} \).

If \( \ell \) is even then its period in the comoving frame is \( \bar{T} = \ell \bar{E} \) and its spatio-temporal symmetry group is \( \mathbb{Z}_{2\ell} \) with \( \ell = 2\ell. \) This scenario is called a flip-pitchfork bifurcation in [5]. If \( \ell \) is odd then its period in the comoving frame is \( \bar{T} = 2\bar{T} \) and its spatio-temporal symmetry group remains \( \mathbb{Z}_{\ell} \) with \( \ell = \ell. \) This scenario is called flip-doubling bifurcation in [5].
Remark 3.5 The assumption \( r_{(\sigma, \mu)} = r_{(\sigma^2, \mu)} \) implies that the block \( M_0 = \text{Ad}^*_\sigma |_{g_2^*} \) in the linearization \( M = \sigma \Phi'(x) \) of the RPO at the bifurcation point does not have an eigenvalue \(-1\), see Proposition 2.17. Therefore, the eigenvalue \(-1\) of \( \text{DII}(x) \) has algebraic multiplicity two. Generically, the drift symmetry \( \tilde{\sigma}^2 \) of the bifurcating RPO at the bifurcation point is regular. In this case, \( \tilde{\sigma} \) is regular too, and the above assumption reads \( r_{\sigma^2} = r_\sigma \). Then \( \text{Ad}_\sigma \) does not have an eigenvalue \(-1\) either, and the eigenvalue \(-1\) of \( M \) has algebraic multiplicity two, too. If \( \tilde{\sigma}^2 \) is not regular then \( M \) might have additional eigenvalues \(-1\).

If we continue in energy while fixing the value of the momentum map as in (2.30), we get the following corollary:

Corollary 3.6 Under the assumptions of Theorem 3.2, if \( D_E \psi(E_0, 0) \neq 0 \), there is a smooth path \( \tilde{x}(\epsilon) \in X^0 \) with \( \tilde{x}(0) = \tilde{x} \), \( \epsilon \geq 0 \), such that \( \tilde{x}(\epsilon) \) lies on an RPO \( \tilde{\mathcal{P}}(\epsilon) \) of (2.1) with relative period \( \tilde{\tau}(\epsilon) \), drift symmetry \( \tilde{\sigma}(\epsilon) \) at \( \tilde{x}(\epsilon) \), energy \( \tilde{E}(\epsilon) \) and momentum \( \tilde{\mu}(\epsilon) \) where \( \tilde{\sigma}(0) = \tilde{\sigma}^2 \), \( \tilde{E}(0) = \tilde{E}, \tilde{\mu}(0) = 0 \).

Similarly, if we continue in a momentum component while fixing the other momentum components and the energy as in (2.31), we get the following corollary:

Corollary 3.7 Under the assumptions of Theorem 3.2, if \( D_{\xi_{1}} \psi(E, 0) \neq 0 \), there is a smooth path \( \tilde{x}(\epsilon) \in X^E \) with \( \tilde{x}(0) = \tilde{x} \), such that \( \tilde{x}(\epsilon) \) lies on an RPO \( \tilde{\mathcal{P}}(\epsilon) \) of (2.1), \( \epsilon \geq 0 \), with relative period \( \tilde{\tau}(\epsilon) \), drift symmetry \( \tilde{\sigma}(\epsilon) \) at \( \tilde{x}(\epsilon) \), energy \( \tilde{E} \) and momentum \( \tilde{\mu}(\epsilon) = (\tilde{\mu}_{1}(\epsilon), \tilde{\mu}_{\nu}) \) where \( \tilde{\sigma}(0) = \tilde{\sigma}^2 \), \( \tilde{\mu}_{1}(0) = \tilde{\mu}_{1}, \tilde{\mu}_{\nu}(0) = 0 \), \( \tilde{\sigma}(0) = 0 \).

### 3.2 Detection of relative period doublings

Assume as before that at \( \tilde{x} \) there is a relative period doubling bifurcation and that \( D_{w} \Pi_{X_{1}}(0) \) has an eigenvalue \(-1\) with multiplicity 2. Let \( \mathcal{P}(\epsilon) \) be a branch of RPOs with \( C(x) = J_{1}(x) \) or \( C(x) = H(x) \) as in Corollaries 3.6, 3.7 above. Choose coordinates on the Poincaré section \( \mathcal{S} \) at \( \tilde{x} \) as in Lemma 2.16 and let \( x(c) \simeq (\nu(c), w(c), E(c)) \in \mathcal{N} \) such that \( x(\tilde{c}) = \tilde{x} \).

Lemma 3.8 Generically, at a relative period doubling bifurcation point \( \tilde{x} \in \tilde{\mathcal{P}} \) along a path of RPOs \( x(c) \in \mathcal{P}(c) \), the determinant \( \det(D_{w} \Pi_{X_{1}}((\nu, w, E)(c)) + \text{id}_{X_{1}}) \) changes sign.

Proof. Let \( M_{1}(c) := D_{w} \Pi_{X_{1}}((\nu, w, E)(c)) \). Under the above assumptions, the pair \( \lambda_{1, 2}(c) \) of eigenvalues of \( M_{1}(c) \) with \( \lambda_{1, 2}(c) = -1 \) generically lies on the unit circle before collision and on the real axis after collision or vice versa as we saw in Remark 3.1. Let \( B(c) = D_{w} \Pi_{X_{1}}((\nu, w, E)(c)) \) and let \( \mathcal{V}(c) \) be the generalized eigenspace of \( M_{1}(c) \) to the eigenvalues \( \lambda_{1, 2}(c) \) with \( \lambda_{1, 2}(c) = -1 \). Then, \( \lambda_{1}(c) = -1 + a(c), \lambda_{2}(c) = 1/(-1 + a(c)) \) for \( c < \bar{c} \) with \( a(c) \in \mathbb{R}, a(\bar{c}) = 0 \), and \( \lambda_{1, 2}(c) = e^{\pm i \phi(c)} \) with \( \phi(c) \in \mathbb{R}, \phi(\bar{c}) = \pi \) or vice versa. Then, \( \det(B(c) + \text{id}_{2}) = -a(c)^{2} + O(a^{3}(c)) < 0 \) for \( c < \bar{c} \) and

\[
\det(B(c) + \text{id}_{2}) = (e^{i \phi(c)} + 1)(e^{-i \phi(c)} + 1) = 2(1 + \cos \phi(c)) > 0 \quad \text{for} \quad c > \bar{c},
\]
or vice versa. Since the other eigenvalues of \( M_{1}(c) \) are bounded away from \(-1\) near \( c = \bar{c} \), we conclude that \( \det(M_{1}(c) + \text{id}_{X_{1}}) \) also changes sign.

Under the assumptions of Theorem 3.2 also \( \det(\text{DII}(x(c)) + \text{id}_{X_{1}}) \) changes sign at \( c = \bar{c} \), as we saw in Remark 3.5; hence, in a manner similar to the case of dissipative systems with discrete symmetry groups [23], relative period doubling bifurcations can be detected by a sign change of

\[
d(y) = \det(P_{X_{1}} \alpha D_{y} \Phi^*(x; \xi)P_{X_{1}} + \text{id}_{X_{1}}).
\]

(3.2)
Here $P_N$ is the projection from the phase space $\mathcal{X}$ to the tangent space $\mathcal{N}$ of the Poincaré section at $\bar{x}$ with kernel $T_\bar{x}P$. Recall that $\alpha D_\bar{x} \Phi^T(x; \xi)$ is related to the derivative of the shooting equations, see Remark 2.26. The projection $P_N$ can be computed numerically as an orthogonal projection to $g_s x$ and $f_H(x)$ and $\ker D J_i(x)$, $i = 1, \ldots, r$, $r = r(\sigma, \mu)$.

**Remark 3.9** Note that the trace condition (3.1) can not be used to detect relative period-doubling bifurcations in general (of course this test works if $\dim \mathcal{X} = 2$). The determinant test based on (3.2) also works if instead of a projection on $\mathcal{N}$ a projection on in (3.2) is used. Since the eigenvalue $-1$ generically has algebraic multiplicity two at bifurcation, see Remark 3.5, we could remove the projection $P_N$ completely from the test function $d(y)$ or replace it a the projection onto $\tilde{N}_1$ with kernel $\tilde{N}_0 \oplus g_s \bar{x} \oplus \text{span}(f_H(\bar{x}))$. Here $\tilde{N} = \tilde{N}_0 \oplus \tilde{N}_1 \oplus \tilde{N}_2$ is the normal space for the symmetry group $\tilde{\Gamma} = \Gamma_x$. From a theoretical point of view, a projection on $\tilde{N}_1$ is possible as well. However, the dimension of $\tilde{N}_1$ may vary along the branch of RPOs, and this may cause numerical instability. Note that the dimension of $\mathcal{N}$ is constant along the branch of RPOs as the group is assumed to act freely. Moreover, since $\mu$ is a regular momentum for the group $\tilde{\Gamma} = \Gamma_x$, the dimension of $\tilde{N}_1$ is also locally constant.

### 3.3 Computation of relative period doubling points and branch switching

Relative period doubling bifurcations can be computed similarly to period doubling bifurcations in the dissipative case, see, e.g., [9, 23], by subdivision of (3.2) along a family of solutions $y(c)$ of (2.30) or (2.31) respectively.

Once a relative period doubling point $(\bar{x}, \bar{\tau}, \bar{\xi})$ on the original branch has been found, the starting point $\bar{\eta} = (\bar{x}, \bar{\tau}, \bar{\xi})$ for the bifurcating branch has to be computed. We set $\bar{T} = \bar{T}$ at the bifurcation point for a relative flip pitchfork bifurcation (as defined in Remark 3.4) and $\bar{T} = 0 \bar{T}$ otherwise, and $\bar{\xi} = -\bar{\xi}$.

In a manner similar to the case of dissipative systems with finite symmetry [23], we compute the tangent $\bar{t}$ for the bifurcating branch as follows: At a relative period doubling bifurcation the matrix $\bar{M}$ in (2.22) has an eigenvalue $-1$ which is generically of algebraic multiplicity two (Remark 3.5) and $\bar{M} = \sigma D \Phi^T(\bar{x})$ has an eigenvector $\bar{w}$ to the eigenvalue $-1$ corresponding to this eigenvalue of $M$. The fact that $\bar{w}$ lies in the kernel of $\bar{M} + \text{id}$ follows from Proposition 2.17, the form of $D_N$ and the fact that $\sigma_0 = \sigma_{D_1} \bar{g}_s$ has no eigenvalue $-1$. We compute that

$$DH(\bar{x}) \bar{w} = D\bar{H}(\bar{x}) M \bar{w} = -DH(\bar{x}) \bar{w} = 0.$$  

Moreover,

$$DJ_{E_j}(\bar{x}) \bar{w} = 0, \quad j = 1, \ldots, r,$$

since $\sigma_0 \bar{g}_s|_{E_j}$ has no eigenvalue $-1$. So, for the computation of the branch direction we only have to project $\bar{w}$ orthogonal to $f(\bar{x})$ and $g \bar{x}$ to obtain a projected vector $\bar{w}$ which lies in $\mathcal{N}_1$. This vector $\bar{w}$ is an eigenvector of $M_1$ to the eigenvalue $-1$. Let $\bar{t} = (\bar{t}_x, \bar{t}_T, \bar{t}_\xi, \bar{t}_{\lambda_x}, \bar{t}_{\lambda_T})$ be the continuation tangent for the bifurcating branch. Clearly, $\bar{t}_{\lambda_x} = 0$, $\bar{t}_{\lambda_T} = 0$. By Theorem 3.2 the bifurcating family of RPOs $\bar{P}(\epsilon, \chi)$ satisfies $D_\epsilon \tilde{\tau}(0) = 0$ and $D_\chi \tilde{\xi}(0) = 0$ so that $\bar{t}_T = 0$, $\bar{t}_\bar{\xi} = 0$. Moreover $D_\epsilon \tilde{x}(0) = \tilde{w}$ so that $\bar{t}_x = \bar{w}$.

**Remark 3.10** In the multiple shooting context, see Remark 2.31, it is natural to double the number of multiple-shooting points on the bifurcating branch. The second half of $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{2k})$ is computed as

$$\bar{x}_i = \bar{x}_i \quad \text{for} \quad i = 1, \ldots, k, \quad \bar{x}_{i+k} = \alpha^{-1} \bar{x}_i \quad \text{for} \quad i = 1, \ldots, k.$$
The continuation tangent \( \tilde{t} = (\tilde{t}_x, 0, 0, 0, 0, 0) \) of the bifurcating branch is computed in a similar way: we compute the eigenvector \( w \) to the eigenvalue \(-1\) of \( M = \sigma \Phi \tilde{x} \) (see Remark 2.26) and set

\[
t_1 = w, \quad t_{i+1} = G_i t_i, \quad i = 1, \ldots, k - 1, \quad t_{i+k} = -\alpha^{-1} t_i, \quad i = 1, \ldots, k - 1.
\]

Here the matrices

\[
G_i = D_x \Phi \frac{\Delta x_{\hat{r}}}{\Delta x_{\hat{r}}} (x_i; \lambda), \quad i = 1, 2, \ldots, k - 1, \quad G_k = \alpha D_x \Phi \frac{\Delta x_{\hat{r}}}{\Delta x_{\hat{r}}} (x_k; \lambda)
\]

are available as derivative of (2.27) if for example a Gauss-Newton method is used to solve (2.27). Let \( F : \mathcal{X}^k \times \mathbb{R}^{r+q+2} \to \mathcal{X}^k \times \mathbb{R}^q \) be given by (2.30) or (2.31). Define \( t^j, t_{\xi i}, \ldots, t_{\xi^r} \) as in (2.34) and (2.35), see Remark 2.31. Define \( \tilde{t}^j, \tilde{t}_{\xi i} \in \mathcal{X}^{2k} \times \mathbb{R}^{r+1}, j = 1, \ldots, r \), as

\[
\tilde{t}^j_i = t^j_i, \quad \tilde{t}_{\xi i} = t_{\xi i}, \quad i = 1, \ldots, k, \quad \tilde{t}^j_{i+k} = \alpha^{-1} t^j_i, \quad \tilde{t}_{\xi i+k} = \alpha^{-1} t_{\xi i}, \quad i = 1, \ldots, k,
\]

and set \( \tilde{t}^j_T = 0, \quad \tilde{t}_{\xi T} = 0, \quad \tilde{t}^j_T = 0, \quad \tilde{t}_{\xi T} = 0 \). Projecting \( t \) orthogonally to \( \tilde{t}^j, \tilde{t}_{\xi i}, \ldots, \tilde{t}_{\xi^r} \) we obtain a continuation tangent for the bifurcating branch.

### 3.4 Detection and computation of relative period halving bifurcations

In this section we show how relative period halving bifurcations, which occur along branches of RPOs defined by (2.30) or (2.31) and satisfy the assumptions of Corollary 3.6 or Corollary 3.7, respectively, can be detected and computed numerically.

As in the case of dissipative systems with finite symmetry group discussed in [23], we compute all choices of \( \tilde{\alpha} \) such that \( \tilde{\alpha}^2 = \alpha \), and at each solution \( y = (x, T, \xi) \) we compute for each choice of \( \tilde{\alpha} \),

\[
u(y) := \tilde{\alpha} \Phi \tilde{x} (x; \xi) - x.
\]

A relative period halving bifurcation is detected, if

\[
\langle u^{(0)}, u^{(1)} \rangle < 0, \quad u^{(0)} = u(y^{(0)}), \quad u^{(1)} = u(y^{(1)}),
\]

between two consecutively computed solutions \( y^{(0)} = (x^{(0)}, T^{(0)}, \xi^{(0)}) \) and \( y^{(1)} = (x^{(1)}, T^{(1)}, \xi^{(1)}) \) of (2.30) or (2.31), respectively. This can be seen as follows: by Theorem 3.2, the functions \( x(\epsilon), \xi(\epsilon) \) and \( \tau(\epsilon) = T(\epsilon)/\ell \) are smooth in \( \epsilon \). Moreover at the relative period halving point, \( u(y(\epsilon))|_{\epsilon = 0} = 0 \) and \( D_\epsilon u(y(\epsilon))|_{\epsilon = 0} = -2 \tilde{\alpha} \neq 0 \).

By Corollary 3.6 or Corollary 3.7 at the relative period halving bifurcation point we have \( D_{\xi} C(x(\epsilon))|_{\epsilon = 0} = 0 \). Therefore, once a relative period halving bifurcation has been detected between \( y^{(0)} \) and \( y^{(0)} \), an initial guess \( \hat{y} \) for the bifurcation point can be computed by using interpolation \( (y(\tau), c(\tau)) \) between \( (y^{(0)}, C(x^{(0)}) \) and \( (y^{(1)}, C(x^{(1)}) \) to find a critical point of \( c(\tau) \). Then the bifurcation point can be computed by switching to the branch of RPOs with halved relative period and using the methods for relative period doubling bifurcations from the previous section on this branch (similarly to case of dissipative systems with finite symmetry groups [23]).

Since we assume (see Theorem 3.2) that the relative period halving point is an RPO with regular drift momentum pair \((\hat{\sigma}, \hat{\mu})\), by Remark 2.15 the subalgebra \( g^{(\hat{\sigma}, \hat{\mu})} \) is abelian. Therefore \( \Gamma^{\hat{\sigma}} \) is isomorphic to \( \mathbb{T}^n \) and is a subgroup of \( Z(\alpha) \). Here, \( \mathbb{T}^n = (S^1)^n \) is an \( n \)-dimensional torus. The group \( L = \Gamma^{(\hat{\sigma}, \hat{\mu})} \cap Z(\Gamma^{\hat{\sigma}}) / \Gamma^{\hat{\sigma}} \) is finite because \( \Gamma \) is assumed to be compact. Since \( r^{(\sigma^2, \mu)} = r^{(\sigma, \mu)} \) in Theorem 3.2, all possible choices of square roots \( \tilde{\alpha} \) of \( \alpha \) lie in \( Z(\Gamma^{\hat{\sigma}}) \).

So after reduction by \( \Gamma^{\hat{\sigma}} \), we are back to the case of finite groups \( L \) treated in [23].

Assume that \( \alpha \) has a square root \( \tilde{\alpha} \in L; \tilde{\alpha}^2 = \alpha \). Then \( \alpha \) has indeed two square roots in \( Z_{2\ell}(\tilde{\alpha}) \subseteq L \), namely \( \tilde{\alpha}_1 = \tilde{\alpha} \) and \( \tilde{\alpha}_2 = \tilde{\alpha}^{\ell+1} \). In the case of continuous symmetry groups \( \Gamma \) the
test (3.3) for relative period halving bifurcations has to be used for all $\tilde{\alpha} = \tilde{\alpha}_{1,2} \exp(\xi)$ where $\xi \in t^n$ is such that $\xi_j = 0$ or $\xi_j = \pi$, $j = 1, \ldots, n$. Here $t^n$ is the Lie algebra of $\mathbb{T}^n$. For example, if $\Gamma_{(\sigma,t)} = \text{SO}(2)$, and $R_\sigma$ is a rotation by $\phi$, then $\exp(\xi) = \text{id}$ or $\exp(\xi) = R_\pi$.

**Remark 3.11** In the case of non-Hamiltonian systems, see Remark 2.32, relative period doubling points are detected and computed in the same way, with $c = \lambda$ in Section 3.3. In the case of relative period halving, the possible drift symmetries $\sigma$ in the comoving frame for the relative period halving bifurcation point can be found as in the Hamiltonian case above, if we assume that both, the drift symmetry $\sigma$ of the RPO on the branch with halved relative period, and the drift symmetries of the RPOs on the original branch near the bifurcation are regular and that $r_\sigma = r_\sigma^2$.

### 4 Application to rotating choreographies

In this section we apply our methods for numerical bifurcations of RPOs to rotating choreographies of the three-body problem. Using the software package SYMPERCON [19] we find that the type II rotating choreography undergoes a symmetry-increasing flip pitchfork bifurcation in the corotating frame to the type I rotating choreographies. We also report on several relative period doubling bifurcations and a turning point of the planar (type III) rotating choreographies.

#### 4.1 N-body problems and their symmetries

We consider the motion of $N$ identical bodies of mass 1 in $\mathbb{R}^3$ subject to internal forces they exert on each other. We assume that these forces are given by $\frac{1}{2}N(N-1)$ identical copies of a potential energy function $V$ (one for each pair of bodies), which depends only on the distance between the bodies. Writing $p_j$ for the momenta conjugate to the positions $q_j$, $q = (q_1, \ldots, q_N)$, $p = (p_1, \ldots, p_N)$, the Hamiltonian is

$$H(q, p) = \frac{1}{2} \sum_{j=1}^{N} |p_j|^2 + \sum_{i<j} V(r_{ij}), \quad \text{where} \quad r_{ij} = |q_i - q_j| \quad \text{and} \quad V(r) = -\frac{1}{r}. \quad (4.1)$$

Excluding collisions, the configuration space $Q$ is

$$Q = \{ q = (q_1, \ldots, q_N) \in \mathbb{R}^{3(N-1)}, \quad q_i \neq q_j \text{ for } i \neq j \}$$

and the phase space is $Q \times \mathbb{R}^{3N} \subset \mathbb{R}^{6N}$. The equations of motion are

$$\dot{q}_j = p_j, \quad \dot{p}_j = \sum_{i \neq j} \frac{q_i - q_j}{r_{ij}^3}, \quad j = 1, \ldots, N, \quad (4.2)$$

and the angular momentum is $J(q, p) = \sum_{j=1}^{N} q_j \wedge p_j$. Without loss of generality, the centre of mass of the systems can be assumed to be fixed at 0 restricting the configuration space to

$$Q^0 = \{ q \in Q : \sum_{j=1}^{N} q_j = 0 \}$$

with corresponding phase space $X = Q^0 \times \mathbb{R}^{3(N-1)} \subseteq \mathbb{R}^{6(N-1)}$.

The $N$-identical-body Hamiltonian (4.1) has the following symmetries:
1. **Rotations and reflections of** $\mathbb{R}^3$: These form the orthogonal group $O(3)$ which acts diagonally on the positions and velocities:

$$ R(q_1, \ldots, q_N, p_1, \ldots, p_N) = (Rq_1, \ldots, Rq_N, Rp_1, \ldots, Rp_N), \quad R \in O(3), \quad q_j, p_j \in \mathbb{R}^3. $$

We define the *symmetry axis* of a rotational symmetry to be its usual rotation axis and that of a reflectional symmetry to be the axis perpendicular to the reflection plane. In the following let $\kappa_i \in O(3)$ be the reflection with symmetry axis $e_i$, i.e., let $\kappa_i$ be such that $\kappa_i e_i = -e_i, \kappa_i e_j = e_j$ for $j \neq i, i, j = 1, 2, 3$. We denote by $R_j(\phi)$ a rotation around the $e_j$ axis by the angle $\phi$.

2. **Permutations of identical bodies**: Because we assume that all the bodies are identical the Hamiltonian is also invariant under the action of $S_N$, the group of all permutations of the integers $1, \ldots, N$:

$$ \pi(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N) = (q_{\pi(1)}, \ldots, q_{\pi(N)}, p_{\pi(1)}, \ldots, p_{\pi(N)}) \quad \pi \in S_N, \quad q_j, p_j \in \mathbb{R}^3. $$

We frequently use the notation $\pi = (\pi(1), \ldots, \pi(N))$.

Taken together these three symmetry groups give an action of $\Gamma = O(3) \times S_N$ on $\mathcal{X}$ and leaves the Hamiltonian (4.1) invariant.

**Remark 4.1** We call a matrix $\rho \in \text{GL}(n)$ of a general Hamiltonian system (2.1) a *time-reversing symmetry* of (2.1) if

$$ H(\rho x) = H(x), \quad x \in \mathcal{X}, \quad \rho\mathbb{I} = -\mathbb{I}\rho. $$

This implies that $f_H(\rho x) = -f_H(x), x \in \mathcal{X}$, and so with $x(t)$ also $\rho x(-t)$ is a solution of (2.1). In addition to the symmetries listed above, the $N$-body system (4.2) has the time-reversing symmetry

$$ \rho(q_1, \ldots, q_N, p_1, \ldots, p_N) = (q_1, \ldots, q_N, -p_1, \ldots, -p_N) \quad q_j, p_j \in \mathbb{R}^3 $$

which generates a group $\mathbb{Z}_2(\rho)$ of order 2. A periodic orbit through $x$ with drift symmetry $\alpha$, so that $\alpha \Phi(x) = x$ is transformed into an orbit through $\rho x = \rho x \rho^{-1} \Phi^{-\tau}(\rho x)$ and has drift symmetry $\rho \alpha^{-1}\rho^{-1} = \alpha^{-1}$; see [11] for a discussion of reversible RPOs.

### 4.2 Rotating choreographies of the three body problem

As in [3] we define:

**Definition 4.2** A periodic orbit of (4.2) is a choreography if all the bodies follow the same path in $\mathbb{R}^3$, separated only by a phase shift. This is equivalent to requiring that the spatio-temporal symmetry group $L$ of the periodic orbit contains an order $N$ cyclic permutation $\pi \in S_N$ which can always be taken to act on $Q$ by $\pi q = (q_2, q_3, \ldots, q_N, q_1)$. Similarly a relative periodic orbit of (4.2) with angular velocity $\xi$ is a rotating choreography if it is a choreography in coordinates rotating with velocity $\xi$.

The famous Figure Eight of Chenciner and Montgomery [2] is a choreography of the $N$-identical-body system (4.2) with $N = 3$.

Let $\{e_1, e_2, e_3\}$ be a fixed orthogonal set of axes in $\mathbb{R}^3$ and assume that the eight lies in the plane perpendicular to $e_3$ aligned along the $e_2$ axis with both $e_2$ axis and $e_1$ axis as symmetry
axis. As before, for \( i = 1, 2, 3 \) let \( \kappa_i \) denote the (time-preserving) reflection with reflection axis \( e_i \). The purely spatial symmetry group of the Figure Eight choreography is the group

\[
K = \mathbb{Z}_2 = \langle \kappa_3 \rangle
\]
generated by \( \kappa_3 \), a reflection about the \((x_1, x_2)\)-plane containing the Figure Eight. The drift symmetry \( \alpha := \kappa_1(231) \) of the eight is a reflection in the \( \{ e_1, e_2 \} \)-plane composed with a cyclic permutation of the bodies and has order \( \ell = 6 \).

As shown by Chenciner et al. \cite{3} three families of rotating choreographies bifurcate from the Figure Eight when momentum is switched on:

I. The type I family of rotating eights \( \mathcal{P}_I(E, \nu) \) rotates around the \( e_1 \) axis. Its drift symmetry in the rotating frame is \( \alpha_I = \kappa_1(231) \) and has order \( \ell_I = 6 \). The relative periods \( \tau_I(E, \nu) \) of the bifurcating RPOs are close to the relative period of the original Figure Eight, \( \tau_I(0, 0) = \tilde{\tau} \).

II. The type II rotating eights \( \mathcal{P}_{II}(E, \nu) \) rotate around the \( e_2 \)-axis. Its drift symmetry in the rotating frame is \( \alpha_{II} = \kappa_1 \kappa_3(231) \) and has order \( \ell_{II} = 6 \). The relative period \( \tau_{II}(E, \nu) \) of the bifurcating RPOs satisfies \( \tau_{II}(0, 0) = \tilde{\tau} \).

III. The type III rotating eights \( \mathcal{P}_{III}(E, \nu) \) are planar, i.e., have spatial symmetry \( K = \langle \kappa_3 \rangle \). The drift symmetry in the rotating frame is \( \alpha_{III} = (312) \) and has order \( \ell_{III} = 3 \). The relative period \( \tau_{III}(E, \nu) \) of the family of bifurcating RPOs has doubled at the bifurcation point: \( \tau_{III}(0, 0) = 2\tilde{\tau} \).

It is well-known that the type I rotating choreography ends at a Lagrange relative equilibrium, see, e.g., the discussion in \cite{3}.

### 4.3 Flip up pitchfork bifurcation of the type II rotating eight

Along the branch of rotating eights of type II with drift symmetry \( \sigma(s) = \exp(\tau(s)\xi(s))\alpha \), where \( \alpha = \alpha_{II} = (231)R_2(\pi) \) is of order \( \ell = 6 \) and the energy is fixed to the value \( E = -1.287 \) of the original Figure Eight, using SYMPERCON we found a symmetry-increasing pitchfork bifurcation in a corotating frame. The emanating RPOs \( \mathcal{P}(s) \) have halved relative period and drift symmetry \( \tilde{\sigma}(s) = \exp(\tilde{\tau}(s)\xi(s))\tilde{\alpha} \) such that, at the bifurcation point, \( \tilde{\tau}(0) = \tilde{\tau} \), \( \tilde{\alpha}^2 = \alpha \) and \( \tilde{\tau}(0) = \tilde{\tau}/2 \). Here, \( \tilde{\alpha} = (312)\kappa_2R_2(\pi/2) \) has order \( \ell = 12 \). This corresponds to a bifurcation increasing the spatio-temporal symmetry \( \mathbb{Z}_6(\alpha_{II}) \) in the corotating frame to \( \mathbb{Z}_{12}(\tilde{\alpha}) \). The bifurcation, marked as "PD" in the second plot of Figure 1, occurs at

\[
\begin{align*}
q_1 &= (0.5545, -0.3628, 0.2530), & p_1 &= (0.3616, 0.1919, -0.7436), \\
q_2 &= (-0.4921, 0.3094, 0.3679), & p_2 &= (0.5237, 0.3138, 0.6587),
\end{align*}
\]

the period in the frame rotating with velocity \( \omega_{\text{rot}} = 0.7305 \) around the \( e_2 \)-axis is \( T = 9.626 \), the second component of the angular momentum is \( \mu_2 = 1.576 \), and \( \mu_1 = \mu_3 = 0 \). The linearization \( \tilde{\sigma}D\Phi^*(x) \) at the bifurcation point has six eigenvalues 1 and the eigenvalues

\[
\lambda_1 = -2.10, \quad \lambda_2 = -0.477, \quad \lambda_{3,4} = 0.453 \pm 10.892, \quad \lambda_{5,6} = -0.553 \pm 0.833.
\]

The first plot of Figure 1 shows the continuation of the Figure Eight at zero momentum up to the bifurcation point. For each computed rotating eight solution the motion of the first body in the \((q_{1,1}, q_{1,3})\)-plane is depicted in the respective corotating frame in which the solution becomes periodic. The Figure Eight trajectory of the first body lies in the \((q_{1,1}, q_{1,2})\)-plane, therefore it
corresponds to a line in the plot. We can see from the picture that at the bifurcation point the solution is invariant under rotations by $\pi/2$.

As indicated by the label in Figure 1, the bifurcating family lies on the type I family $P_I(E, \nu)$ of rotating eights. The branch ends at a relative Lyapounov centre bifurcation in the Lagrange relative equilibrium at $\mu_2 = 1.869$, marked as "LC" in the second plot of Figure 1. At the relative Lyapounov centre bifurcation the relative period $T_{I}^{LC}$ of the type I family and the relative period $T_{II}^{LC}$ of the bifurcating family of RPOs are identical, but the periods $T_{I}^{LC}$ and $T_{II}^{LC}$ in their respective corotating frames satisfy $T_{I}^{LC} = 2T_{I}^{LC}$, and the rotating frames and drift-symmetries of both rotating choreographies in their corotating frames are also different. If we continue the bifurcating family backwards to momentum $\mu_2 = 0$ then we obtain the original Figure Eight solution (denoted "Fig.8 I" in the second plot of Figure 1), but with nonvanishing rotation frequency and period $2T_I$ in the corotating frame. Here $T_I$ and $T_{II}$ are the relative period and the period of the Figure Eight solution at momentum $\mu = 0$.

To understand this note that relative periods of RPOs are independent of the coordinate frame, but the period in the corotating frame and the drift-symmetry in the corotating frame differ for different corotating frames (Remark 2.13).

The drift-symmetry $\alpha_I = \kappa_I(231)$ of the type I family $P_I(E, \nu)$ of rotating eights in its corotating frame is transformed into the drift symmetry $\tilde{\alpha} = R_2(\pi/2)\kappa_2(312)$ of the family $P(s)$ of rotating choreographies bifurcating from the type II rotating eight by conjugation with the reversing symmetry $\rho$, and thus, inverting $\alpha_I$ (as discussed in Remark 4.1 and [11]), by conjugating with the symmetry element $R_3(\pi/2)$, so that $\alpha_I$ becomes $\kappa_2(312)$, and then by a change of the rotation frequency $\omega_I^{rot}(E, \nu)$ of the RPO $P_I(E, \nu)$ to the rotation frequency $\tilde{\omega}_I^{rot}(E, \nu)$ such that $(\tilde{\omega}_I^{rot}(E, \nu) - \omega_I^{rot}(E, \nu))\tau_I(E, \nu) = \pi/2$. This transforms the period $\tilde{T}_I(E, \nu) = \tilde{\ell}_I\tau_I(E, \nu) = 6\tau_I(E, \nu)$ of the type I rotating eight $P_I(E, \nu)$ in its corotating frame to $\tilde{T}(E, \nu) = \tilde{\ell}_I\tau_I(E, \nu) = 12\tau_I(E, \nu) = 2T_I(E, \nu)$ and explains the above observations.

If we continue the branch of type I rotating choreographies from the original Figure Eight in the momentum component $\mu_1$ fixing the energy, this connection between the type I and type II rotating eights corresponds to a relative period-doubling and a symmetry-breaking pitchfork bifurcation in the corotating frame at $\mu_1 = 1.576$. In this case the bifurcating branch $P_{II}(s)$ has drift symmetry $\tilde{\alpha}_{II} = \alpha_{II}^I = (312)$ of order $\tilde{\ell}_{II} = 3$ in the corotating frame. If we continue the bifurcating branch $P_{II}(s)$ back to the Figure Eight solution at $\mu_1 = 0$, then the Figure Eight on this branch (denoted "Fig.8 II" in the second plot of Figure 1) has nonvanishing rotation frequency and period $T/2 = \tilde{\ell}_{II}\tau$ in this corotating frame. Denote the rotation frequency of $P_{II}(s)$ by $\tilde{\omega}_{II}^{rot}(s)$. Applying the above transformation on the branch of type I rotating choreographies amounts to conjugating $\tilde{\alpha}_{II}$ with $\rho$ and changing the rotation frequency $\tilde{\omega}_{II}^{rot}(s)$ of the RPO $\tilde{P}_{II}(s)$ to $\omega_{II}^{rot}(s)$ where $(\omega_{II}^{rot}(s) - \tilde{\omega}_{II}^{rot}(s))\tau_{II}(s) = \pi$. Hence, we retrieve the drift
4.4 Turning points and relative period doubling bifurcations of type III rotating choreographies

As reported in [25], the type III rotating eights can be continued to negative momentum values \( \mu_3 < 0 \) at fixed energy \( E = -0.1287 \) and, after coming close to a collision, this family undergoes a relative period doubling bifurcation at momentum \( \mu_3 = -6.6383 \). A more detailed investigation revealed that along the primary branch further relative period-doubling bifurcations (at momentum \( \mu_3 = -6.4136 \) and \( \mu_3 = -6.5550 \)) and a turning point (at \( \mu_3 = -6.6627 \)) occur before the relative period doubling bifurcation point at \( \mu_3 = -6.6383 \) takes place, see Figure 2.

At the turning point (marked "TP" in the first panel of Figure 2) through

\[
q_1 = (1.509, -0.1663), \quad p_1 = (-9.321, 5.939), \quad q_2 = (0.017, 0.0999), \quad p_2 = (0.1616, 0.2359),
\]

the RPO has period \( T = 299.02 \) in the corotating frame rotating with frequency \( \omega^{\text{rot}} = 0.0029 \). In addition to the multipliers \( \lambda_{1,2,3,4,5,6} = 1 \) its linearization in the corotating frame \( \hat{\sigma}D\Phi_f(x) \) has eigenvalues \( \lambda_7 = -3.712, \lambda_8 = -0.2694 \).

Figure 2: Turning points and relative period doubling bifurcation of the type III rotating choreographies; the second picture shows the primary solution on the lower branch, the third picture the primary solution on the upper branch.

The type III rotating choreography, as continued from the Figure Eight, is on the lower part of the (unlabelled) primary solution branch in the first plot of Figure 2. The second plot of Figure 2 shows this solution at momentum \( \mu_3 = -6.2 \), and the third plot shows it after the turning point, on the upper branch of the bifurcation diagram, at momentum \( \mu_3 = -6.2 \). Note that the linearization at the RPO in the second plot has, in addition to four eigenvalues \( \lambda_{1,2,3,4} = 1 \), one pair on the unit circle and two real negative eigenvalues.

The linearization at the RPO in the third plot has, in addition to four eigenvalues \( \lambda_{1,2,3,4} = 1 \), two real positive eigenvalues and one pair on the unit circle.

When the turning point is approached from the lower part of the primary branch then one pair of eigenvalues of the linearization switches from a pair on the unit circle to a pair on the real line in a collision at \( \lambda_{5,6} = 1 \), cf. Proposition 2.34.

The first relative period doubling bifurcation (the emanating branch is marked "PD1" in the first plot of Figure 2) is at momentum \( \mu_3 = -6.4136 \), and the RPO passes through

\[
q_1 = (2.023, 0.1397), \quad p_1 = (-9.928, 6.417), \quad q_2 = (-0.0084, 0.1079), \quad p_2 = (0.1635, 0.2083)
\]

and has period \( T = 225.51 \) in a frame rotating with frequency \( \omega^{\text{rot}} = 0.0166 \). In addition to four eigenvalues \( \lambda_{1,2,3,4} = 1 \) and two eigenvalues \( \lambda_{5,6} = -1 \) its linearization has the eigenvalues...
In addition to four eigenvalues $eigenvalues$. In this paper we presented methods for detecting and computing certain critical points including
symmetry breaking and symmetry increasing bifurcations of Hamiltonian relative periodic orbits with regular drift-momentum pair in the case of compact symmetry groups. The bifurcations we analyzed occur generically during the pathfollowing of RPOs under the assumption that isotropy groups are trivial. We applied our results to rotating choreographies in the three-body problem.

The Floquet eigenvalues are
with regular drift-momentum pair in the case of compact symmetry groups. The bifurcations we
analyzed occur generically during the pathfollowing of RPOs under the assumption that isotropy groups are trivial. We applied our results to rotating choreographies in the three-body problem.

The second relative period doubling bifurcation (the emerging branch is marked "PD2" in
the first plot of Figure 2) is at momentum $q_1 = (1.766, -0.0071)$, $p_1 = (-9.684, 6.265)$, $q_2 = (-0.0025, 0.1019)$, $p_2 = (0.1646, 0.2206)$ and has period $T = 245.92$ in a frame rotating with frequency $\omega^{rot} = 0.0120$. In addition to four eigenvalues $\lambda_{1,2,3,4} = 1$ and two eigenvalues $\lambda_{5,6} = -1$ its linearization has the eigenvalues $\lambda_7 = -8.265, \lambda_8 = -0.1210$, and the pair of eigenvalues $-1$ passes from the real axis to the unit circle as $\mu_3$ is decreased.

The second plot of Figure 3 shows the solution on branch PD2 at momentum $\mu_3 = -6.2$. In addition to four eigenvalues $\lambda_{1,2,3,4} = 1$, the linearization at this solution has one positive real pair of eigenvalues and one pair on the unit circle.

The third relative period doubling bifurcation point (the emanating branch is marked "PD3" in
the first plot of Figure 2) at momentum $\mu_3 = -6.6383$, which we already reported in [25], passes through $q_1 = (1.482, -0.3477)$, $q_2 = (-9.178, 5.833)$, $p_1 = (0.0285, 0.1116)$, $p_2 = (0.1592, 0.2366)$, $T = 325.86, \omega^{rot} = -0.00049$.

The Floquet eigenvalues are
with regular drift-momentum pair in the case of compact symmetry groups. The bifurcations we
analyzed occur generically during the pathfollowing of RPOs under the assumption that isotropy groups are trivial. We applied our results to rotating choreographies in the three-body problem.

Figure 3: RPOs bifurcating from the type III rotating choreographies.

$\lambda_7 = -7.362, \lambda_8 = -0.1358$, and the pair of eigenvalues $-1$ passes from the unit circle to the
real axis as $\mu_3$ is decreased.

The first plot of Figure 3 shows the solution on branch PD1 at momentum $\mu_3 = -6.2$. In addition to four eigenvalues $\lambda_{1,2,3,4} = 1$, the linearization at this solution has four real positive eigenvalues.

The second relative period doubling bifurcation (the emerging branch is marked "PD2" in
the first plot of Figure 2) is at momentum $\mu_3 = -6.555$, and the RPO passes through $q_1 = (1.766, -0.0071)$, $p_1 = (-9.684, 6.265)$, $q_2 = (-0.0025, 0.1019)$, $p_2 = (0.1646, 0.2206)$ and has period $T = 245.92$ in a frame rotating with frequency $\omega^{rot} = 0.0120$. In addition to four eigenvalues $\lambda_{1,2,3,4} = 1$ and two eigenvalues $\lambda_{5,6} = -1$ its linearization has the eigenvalues $\lambda_7 = -8.265, \lambda_8 = -0.1210$, and the pair of eigenvalues $-1$ passes from the real axis to the unit circle as $\mu_3$ is decreased.

The second plot of Figure 3 shows the solution on branch PD2 at momentum $\mu_3 = -6.2$. In addition to four eigenvalues $\lambda_{1,2,3,4} = 1$, the linearization at this solution has one positive real pair of eigenvalues and one pair on the unit circle.

The third relative period doubling bifurcation point (the emanating branch is marked "PD3" in
the first plot of Figure 2) at momentum $\mu_3 = -6.6383$, which we already reported in [25], passes through $q_1 = (1.482, -0.3477)$, $q_2 = (-9.178, 5.833)$, $p_1 = (0.0285, 0.1116)$, $p_2 = (0.1592, 0.2366)$, $T = 325.86, \omega^{rot} = -0.00049$.

The Floquet eigenvalues are
with regular drift-momentum pair in the case of compact symmetry groups. The bifurcations we
analyzed occur generically during the pathfollowing of RPOs under the assumption that isotropy groups are trivial. We applied our results to rotating choreographies in the three-body problem.

The third plot of Figure 3 shows the solution on branch PD3 at momentum $\mu_3 = -6.2$. In addition to four eigenvalues $\lambda_{1,2,3,4} = 1$, the linearization at this solution has four negative eigenvalues.

5 Conclusion

In this paper we presented methods for detecting and computing certain critical points including
symmetry breaking and symmetry increasing bifurcations of Hamiltonian relative periodic orbits with regular drift-momentum pair in the case of compact symmetry groups. The bifurcations we analyzed occur generically during the pathfollowing of RPOs under the assumption that isotropy groups are trivial. We applied our results to rotating choreographies in the three-body problem.

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Note that a systematic theory for all generic bifurcations of Hamiltonian RPOs does not yet exist. Consequently, a systematic numerical treatment of all such bifurcations is yet to be developed. For dissipative differential equations generic bifurcations of symmetric periodic orbits have been classified by Lamb and Melbourne [10] and a general bifurcation theory for RPOs of dissipative systems has been developed in [21].

A next step would be to develop numerical methods for the computation of generic symmetry changing bifurcations of RPOs of dissipative systems, which break spatial as well as spatiotemporal symmetries, and then to extend these methods to bifurcations of Hamiltonian RPOs which break discrete spatial symmetries. Note that bifurcations of Hamiltonian RPOs breaking continuous isotropy are much more difficult to analyze as the momentum map $J$ is in general not surjective near such points. Preliminary results on bifurcations of Hamiltonian relative equilibria have been obtained for example in [8, 16], see also references therein.

In the bifurcations that we analyze in this paper we assume that the RPO at the bifurcation has a generic drift-momentum pair. The numerical treatment of bifurcations from RPOs with singular drift-momentum pair is as yet an open problem.

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