COMPUTING CMC AND SPHERICAL SURFACES BY THE DPW METHOD

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This is an introduction on how to use Matlab to compute constant mean curvature surfaces and their parallel spherical surfaces using the DPW method [4]. An accessible introduction to DPW can be found in [5].

Spherical surfaces: Use the functions *K1surf* for a rectangular region and *K1surfpolar* for a disc around the origin.

CMC surfaces Can also be computed from the same functions (with H=1/2), or you can use the function *ecmch* and *ecmchpolar*.

The matlab functions can, at the time of writing, be found at: http://davidbrander.org/software.html.

1. CMC SURFACES

A surface is computed given a *potential*, which is the analogue for CMC surfaces of the Weierstrass data of minimal surfaces. Potentials for examples of non-minimal CMC surfaces that are deformations of minimal surfaces are readily supplied by using the formula (1.1) below, which comes from [2] (preprint). Solutions of Bj”orling's problem for non-minimal CMC surfaces (solved in [3]) can also easily be computed using the formulae in the appendix of [1].

1.1. Using the function ecmch.

The function ecmch computes the surface corresponding to the potential A. (A faster alternative to ecmch, that uses c++-complied mex functions is *ecmchX*). This is a loop valued function handle. The general form is described below in Section 1.4, but first we look at some examples.

One example of a potential is the matrix-valued function handle:

```
A=@(z,h)[0,-h,0,h-1,0,0; 0,0,1-h,0,h,0];
```

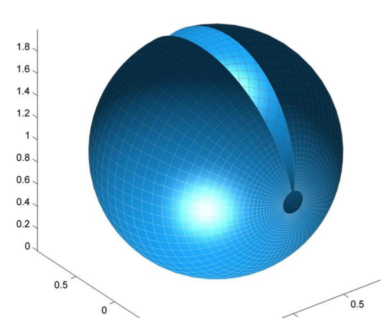
This is the boundary potential for the Björling problem where the initial curve is a circle and the prescribed normal along the curve is the curve's own normal (see Section 3.1 of [3]. The solution will be an unduloid, a cylinder, a sphere or a nodoid, depending on the value of h. The command:

```
f = ecmch(A, eye(2), [0, 0.05, 30, 30], [0, 0.05, 30, 30] , 3, 1);
```

produces a figure and some text output, a sample of which are displayed below:

```
Row 51 Max Error 1.8e-15. Errors: 6.3e-
Row 61 Max Error 2.5e-15. Errors: 6.2e-

Max error:3.9e-15. Mean error: 5.2e-16.
6.2e-16  6.2e-16
6.2e-16  6.2e-16
```

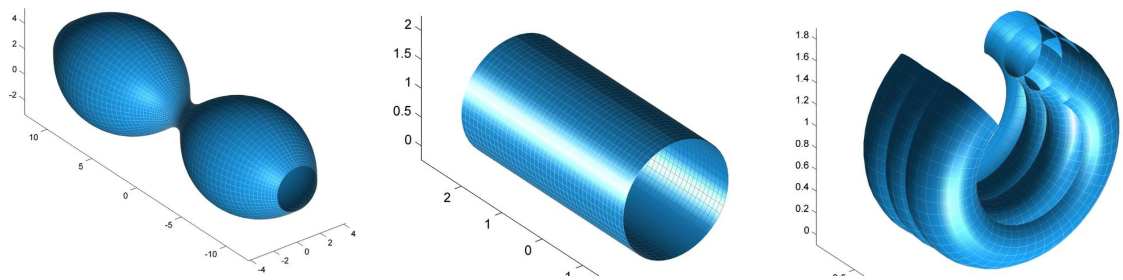


Date: June 5, 2015.

It is a sphere because we chose $h = 1$. The important figures: Max error:3.9e-15. Mean error: 5.2e-16, are near the bottom of the text output. The maximum error estimate is of the order 10^{-15}. These estimates are computed by checking that the matrix corresponding to the solution is in $\mathfrak{su}(2)$. Normally, if the maximum error is less than 10^{-1} and the mean error is less than 10^{-2} then the image is accurate, .i.e computing to higher accuracy will result in an image that is indistinguishable. To increase the accuracy, choose a higher order of polynomial approximation for the loops (3 was used here).

Similarly, choosing $h = 1/2$, $h = 2$ and $h = 0.2$, we get examples of the other types of surfaces of revolution (Figure 1.1), with:

```
f = ecmch(A, eye(2), [0, pi/50, 25, 25], [0, 0.055, 50, 50] , 5, 0.2);
f = ecmch(A, eye(2), [0, pi/50, 25, 25], [0, 0.05, 20, 20] , 4, 1/2);
f = ecmch(A, eye(2), [0, pi/50, 20, 20], [0, 0.05, 30, 30] , 6, 2);
```



$h = 0.2$ $h = 1/2$ $h = 2$

FIGURE 1. CMC surfaces of revolution.

Given a potential A, and an initial condition IC (usually the 2×2 identity matrix eye(2) in Matlab), to compute the corresponding CMC surface, enter:

```
f = ecmch(A, IC,  Ix , Iy, n, H);
```

The rectangle of integration is given by Ix and Iy. Ix is of the form *[x_0, stepsize, pointsleft, pointsright]*, representing the interval $[x_0 - pointsleft \times stepsize, x_0 + pointsright \times stepsize]$ and Iy is similar. The integration is done over the rectangle corresponding to $Ix \times Iy$, starting from the middle, the point (x_0, y_0). H is the mean curvature of the surface that will be computed, and can be any non-zero real number. The parameter n is the maximum order of polynomial approximation used. If the initial condition is the identity, the true solution has order 0 at the center point, and this grows higher as one moves away from this point. How quickly this grows depends on the potential A of course. The computation time increases with n. Depending on the problem, it is best to start with something like $n = 4$ and a 20×20 grid, (which takes about 1 second to integrate) and then modify this according to the output error estimates and image.

1.2. **Non-minimal surfaces associated to minimal surfaces.** Given the Weierstrass data (μ, ν) for a minimal surface,

$$f = 2\Re \int_{z_0}^{z} f_z \, \mathrm{d}z, \qquad f_z \, \mathrm{d}z = \big((1-\nu^2)e_1 - i(1+\nu^2)\,e_2 - 2\nu\, e_3\big)\, \mathrm{d}z,$$

with the coordinates chosen such that $\nu(z_0) = 0$ at some basepoint z_0, we show in [2] that a non-minimal CMC surface with the same Hopf differential is given by the potential with function handle:

$$\text{A= @(z,h) [0 ,-h } \mu(z)\text{, 0,0,0,0;} \quad \text{0,0, } -\tfrac{\partial \nu(z)}{\partial z}\text{,0,0,0].} \tag{1.1}$$

The potential is integrated with z_0 as the center point and the initial condition $IC =$ eye(2). More generally, if $\nu(z_0)$ is non-zero, one can instead use the potential

A= @(z,h) [0 ,-h $\mu(z)\Gamma_0(\bar{\nu}_0\nu+1)^2$, 0,0,0,0;   0,0, $-\frac{1}{\Gamma_0(\bar{\nu}_0\nu+1)^2}\frac{\partial\nu}{\partial z}$,0,0,0].

where

$$\Gamma_0 := \frac{\bar{\mu}(z_0)}{|\mu(z_0)|(|\nu(z_0)|^2+1)}.$$

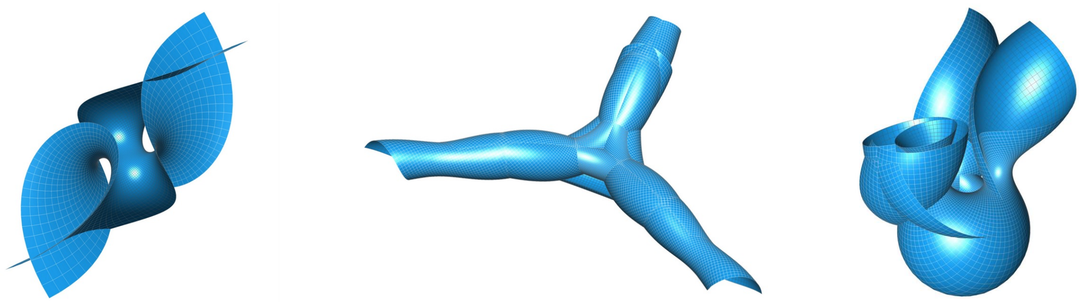
Example 1.1. *Enneper's surface:* Enneper's surface of order $k \geq 1$ is given by $\mu = 1$, $\nu = z^k$ on $\mathbb{C}$. So the non-minimal CMC h surfaces associated have potential:

A= @(z,h) [0 ,-h, 0,0,0,0;   0,0, -k*z∧(k-1),0,0,0].

For the case $k = 1$ and $h = 1$ we obtain a round cylinder. The other cases (for $H \neq 0$) are known as Smyth surfaces or $(k+1)$-legged Mister Bubbles.

We compute the case $k = 2$, for the values of $h = 0.000001$, 1 and 10:

```
A= @(z,h) [0 ,-h, 0,0,0,0; 0,0, -2*z, 0,0,0].
f = ecmch(A, eye(2), [0, 0.04, 30, 30], [0, 0.04, 30, 30] , 3,0.000001);
f = ecmch(A, eye(2), [0, 0.02, 80, 80], [0, 0.02, 80, 80] , 6,1);
f = ecmchX(A, eye(2), [0, 0.008, 60, 60], [0, 0.008, 60, 60] , 4,10);
```



$h = 0.000001$ $h = 1$ $h = 10$

FIGURE 2. Enneper surface of order 2 and 3-legged mister bubbles.

1.3. **Using *ecmchpolar*.** For surfaces with a point of symmetry, like Enneper's, a better image is obtained by using polar coordinates and computing a disc in the coordinate domain, using *ecmchpolar*. In polar coordinates, one replaces z with $r * exp(i * t)$, so the potential for the Enneper surface of order 2 is

```
A = @ (r,t,h)[0,-h,0,0,0,0;0,0,-2*r*exp(i*t),0,0,0];
```

The input data for ecmchpolar is of the form:

```
f = ecmchpolar(A, IC, Ir, It, n,H);
```

The integration is done in only one direction, radially from the middle, so Ir is expected to be of the form [0, stepsize, points], and It of the form [t_0, stepsize, points]. Loops are approximated to the same order n along each ray.

Given A as above, the following commands produce the images in Figure 3:

```
f=ecmchpolar(A, eye(2), [0,0.08,15], [0 pi/50, 100], 2,0.000001);
f=ecmchpolar(A, eye(2), [0,0.04,40], [0, pi/60, 120], 5,1);
f=ecmchpolar(A, eye(2), [0,0.012,40], [0, pi/100, 200], 3,10);
```

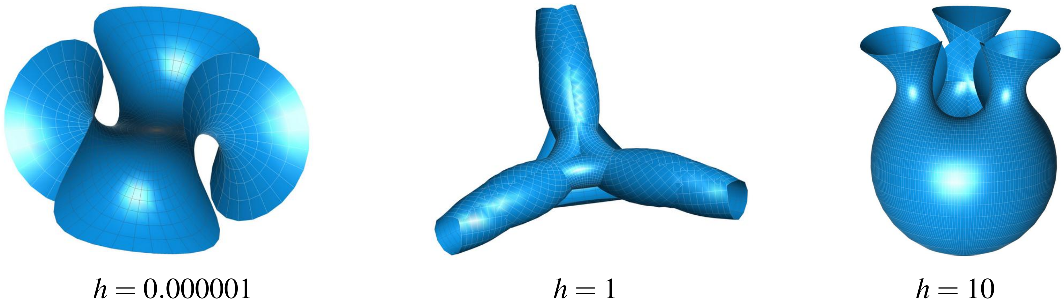


FIGURE 3. Enneper surface of order 2 and 3-legged mister bubbles computed on a polar region.

1.4. **The function handles for general DPW potentials.** To get a CMC-surface as output from ecmch, the function handle A has to have the appropriate properties for the DPW potential for a CMC surface [4]. The exact form, including the relationship with the Weierstrass data for minimal surfaces, can be found in [3] (see Theorem 2.6). Briefly, the twisted loop expression for A must be of the form:

$$A(z) = \sum_{n=-1}^{\infty} A_n(z)\lambda^n,$$

$$A_n(z) = \begin{pmatrix} d_n(z) & 0 \\ 0 & -d_n(z) \end{pmatrix}, \quad n \text{ even}, \qquad A_n(z) = \begin{pmatrix} 0 & a_n(z) \\ b_n(z) & 0 \end{pmatrix}, \quad n \text{ odd}, \tag{1.2}$$

where all functions are holomorphic. The surface is regular at points where $a_0(z) \neq 0$.

For ecmch, all loops are entered untwisted and as Laurent polynomials of the form $\sum_{-n}^{n} A_i \lambda^i$. (Coefficients of λ^{n+k}, for $k > 0$, are discarded). For example:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \lambda = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \lambda$$

and this is entered as a 2×6 matrix

$$\begin{pmatrix} 0 & 0 & a_{11} & a_{12} & b_{11} & b_{12} \\ 0 & 0 & a_{21} & a_{22} & b_{21} & b_{22} \end{pmatrix} \quad \leftrightarrow [0,0,\ a_{11},a_{12},\ b_{11},b_{12};\quad 0,0,\ a_{21},a_{22},\ b_{21},b_{22}] \quad \text{in Matlab.}$$

In most of the literature on DPW, *twisted* loops are used, i.e. they satisfy the conditions at (1.2) on even and odd coefficients. Untwisting a twisted loop is done as follows:

$$\begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix} \mapsto \begin{pmatrix} a(\sqrt{\lambda}) & B_{-1}(\sqrt{\lambda}) \\ C_{+1}(\sqrt{\lambda}) & d(\sqrt{\lambda}) \end{pmatrix}, \quad B_{-1}(\lambda) := \lambda^{-1} b(\lambda), \quad C_{+1}(\lambda) := \lambda c(\lambda).$$

For example if A is the twisted potential $\begin{pmatrix} 0 & -h\lambda^{-1} + (h-1)\lambda \\ (1-h)\lambda^{-1} + h\lambda & 0 \end{pmatrix} \mathrm{d}z$, the matrix untwists to

$$\begin{pmatrix} 0 & -h\lambda^{-1} + (h-1) \\ (1-h) + h\lambda & 0 \end{pmatrix} = \begin{pmatrix} 0 & -h \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} 0 & h-1 \\ 1-h & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} \lambda,$$

The potential would then be entered as: A=@(z,h)[0,-h,0,h-1,0,0; 0,0,1-h,0,h,0]; the potential for the Delaunay surfaces used above in Section 1.1.

2. Spherical Surfaces

This section describes how to compute spherical surfaces, as described in [1]. A lot of the details, for example the way a loop is represented as a function handle, are the same as for CMC surfaces, so in this section, we will mainly show how to use *K1surf* with examples. The mean curvature H is not included as a parameter in *K1surf*. It computes a CMC $1/2$ surface and the parallel spherical surface.

We use the functions *K1surfX.m, K1surfpolarX.m, loop2SMatC.mexw64, rloopEval.mexw64, gengcp.m, plotfp.m, sgcp.m, sing.m*. If these are not compatible with your system, then use *K1surf.m* and K1surfpolar.m instead, which should work on any system.

2.1. **Computing a surface from a normalized potential.** A normalized potential is of the form:
X=@(z)[0,a(z),0,0,0,0; 0,0,b(z) ,0,0,0],
where a and b are holomorphic functions. For example, the first image in Figure 4 is produced by the commands:

```
X=@(z)[0,1+z∧4,0,0,0,0; 0,0, z∧2,0,0,0];
[f,g] = K1surfX(X, eye(2), [0 0.02 40 40], [0 0.02 40 40], 5);
```

The surface f is plotted automatically. To plot the parallel CMC surface g, we can enter `plotfp(g);` and this produces the second image. We can compute the same surface on a polar disc with the commands:

```
Y=@(r,t)[0,1+(r*exp(1i*t))∧4,0,0,0,0; 0,0,(r*exp(1i*t))∧2,0,0,0];
[f,g]= K1surfpolarX(Y, eye(2), [0 0.02 45], [0 pi/100 200],4);
plotfp(g);
```

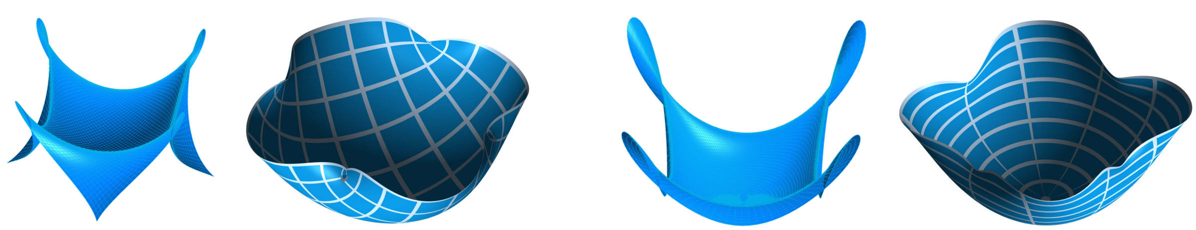


FIGURE 4. Symmetric spherical surface and the parallel CMC surface, computed on both rectangular and polar coordinate patches

2.2. **Solutions of the geometric Cauchy problem.** The function *gengcp.m* produces the potential for the regular geometric Cauchy problem (Theorem 4.4 in [1]). For example a non-orientable cylinder is computed as Example 4.5 in [1]. The data is $\kappa_n(s) = -\sin(s/2)$, $\kappa_g(s) = \cos(s/2)$ and $\mu(s) = 1/2$. We compute the solution with:

```
X=gengcp(@(t)-sin(t/2), @(t)cos(t/2), @(t)1/2);
[f,g] = K1surfX(X, eye(2), [0 pi/50 50 50], [0 0.04 30 30], 4);
```

This produces the first image in Figure 5.

We can create a plot showing the middle x coordinate line of the patch in red using *plotfp(f, numx, numy, middlelinewidth)*. This plots *numx* x-coordinate strips, *numy* y-coordinate strips and, if included it will plot a strip around the middle x coordinate line of width *middlelinewidth*. For example `plotfp(f,4,0,1);` produces the second image in Figure 5.

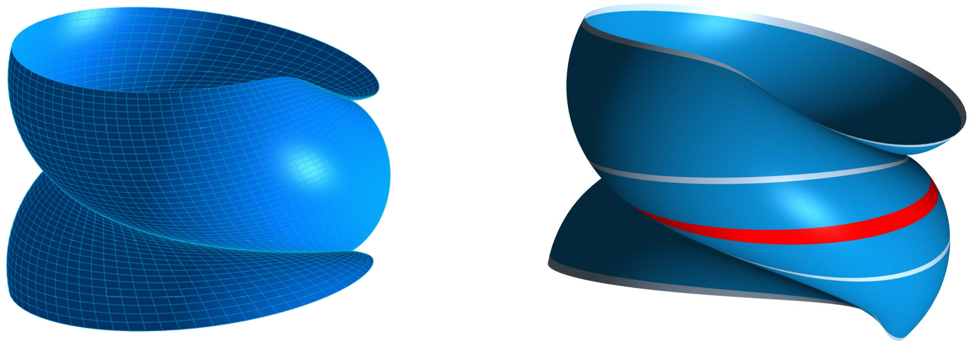


FIGURE 5. Non-orientable cylinder of constant Gauss curvature 1.

2.3. **The singular geometric Cauchy problem.** There are two types of potentials for producing surfaces with prescribed singular curves. Theorem 4.6 of [1] takes the curvature κ and the torsion τ of an arc-length parameterized curve and produces the spherical surface that contains this curve as a cuspidal edge. If the curvature vanishes to first order at a point where the torsion function is non-zero, we get a cuspidal beaks. The potential is produced by the function *sgcp.m*. For example a cuspidal beaks can be computed with:

```
X=sgcp(@(t)t, @(t)cos(t));
[f,g] = K1surfX(X, eye(2), [0 0.02 50 50], [0 0.02 50 50], 4);
```

And we can plot it showing a red band around the x-axis with the command plotfp(f,0,0,6). We need a wide band (here 6 steps in each y direction) for the strip to be visible, because the x parameter lines are very close together around a cuspidal edge. The result is displayed in Figure 6.

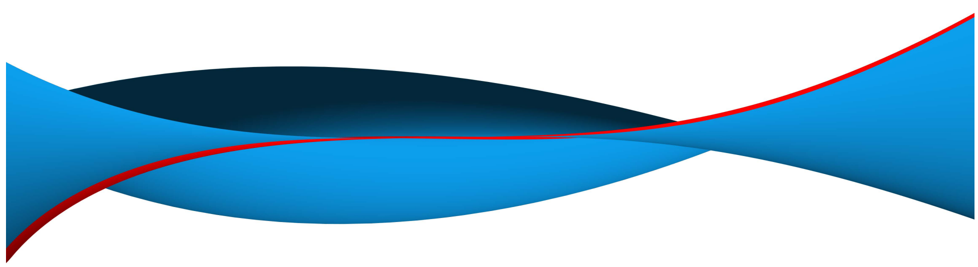


FIGURE 6. Cuspidal beaks singularity

To compute examples where the singular curve is non-degenerate but does not have a regular image in $\mathbb{R}^3$, such as a swallowtail, one can use Theorem 4.8 of [1]. The input is a pair of functions $b(t)$ and $c(t)$, where c is actually the geodesic curvature function for the curve in $\mathbb{S}^2$ corresponding to the normal of the solution surface along the curve $y = 0$. The non-degeneracy condition is $c \neq 0$. Given this, different choices of b give the following types of singularity at $(0,0)$:

(1) A cone point if $b \equiv 0$.
(2) Swallowtail if $b(0) = 0$ and $b'(0) \neq 0$.
(3) A cuspidal butterfly if $b(0) = b'(0) = 0$ and $b''(0) \neq 0$.
(4) A cuspidal edge if $b(0) \neq 0$.

The potential is produced by the function *sing.m*. An example with a swallowtail at $(k\pi, 0)$ for $k \in \mathbb{Z}$ is:

```
X=sing(@(t)sin(t), @(t)cos(t));
[f,g] = K1surfX(X, eye(2), [0 pi/100 110 110], [0 0.018 68 68], 5);
```

plotfp(f,10,20,5);

shown in Figure 7. It has degenerate singularities at $((2k+1)\pi/2, 0)$ for $k \in \mathbb{Z}$, where $c = 0$. These are apparently cuspidal beaks (although this question is not discussed in [1]).

FIGURE 7. Swallowtail and cuspidal beaks

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