A BLUE LAGOON FUNCTION

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Abstract. We consider a specific function of two variables whose graph surface resembles a blue lagoon. The function has a saddle point p, but when the function is restricted to any given straight line through p it has a *strict local minimum* along that line at p.

1. Definition and properties

A function $f(u,v)$ is defined in $\mathbb{R}^2$ as follows:

$$f(u,v) \;=\; \left(1-(u-1)^2-v^2\right)\left(4-(u-2)^2-v^2\right) \quad .$$

The function is zero along the two circles (the red circles in Figure 1):

$$(u-1)^2+v^2 \;=\; 1 \;\text{ and }\; (u-2)^2+v^2 \;=\; 4 \quad .$$

The point of interest is p, where the two red circles meet. This point has coordinates $p \;=\; (0,0)$. It is a stationary point for f:

$$\nabla f_{|(0,0)} \;=\; 0 \quad .$$

The Hessian of f is positive semi-definite at p:

$$\operatorname{Hess} f_{|(0,0)} \;=\; \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad .$$

In the disc domain shown in Figure 1 there are two subdomains, where the function is positive (the green subdomains), and one subdomain where the function is negative (the blue subdomain). Every straight line through p therefore only experiences positive values of f close to p - except precisely at p, where the value is 0. The point p is thence a strict local minimum along every one of these straight lines. The yellow circle marks the location of the local maxima along the respective straight lines through p. The function (considered as a function in $\mathbb{R}^2$) does not itself have a local minimum at p. For example, the function is decreasing from p along the blue circle through p in the blue subdomain in between the two red circles through p in Figure 1 (see the precise analysis on page 5). The point p is thus a saddle point in the sense that it is a stationary point with the property that every neighborhood around p contains points where f is strictly larger than $f(p) \;=\; 0$ as well as points where f is strictly smaller than 0.

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2. Figures

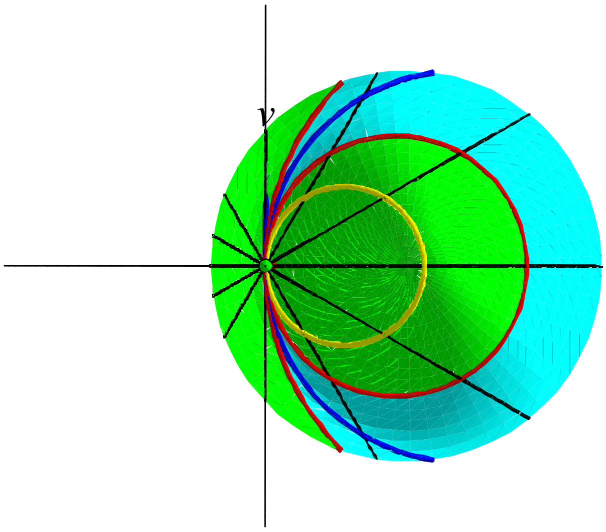

Figure 1. Straight lines through the stationary point and descriptive circles in the considered domain for the function f.

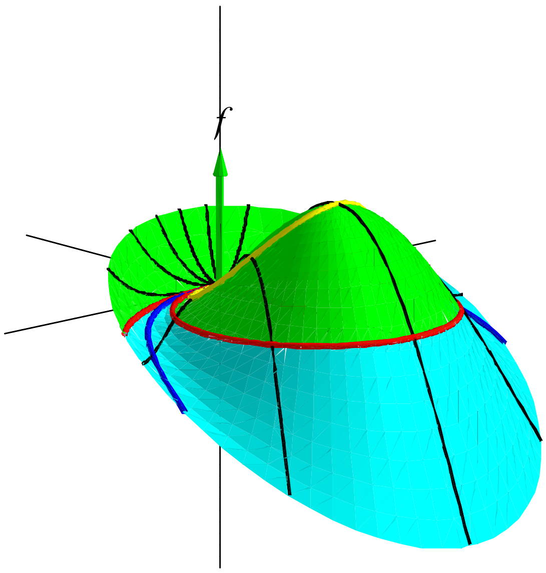

Figure 2. The graph surface of f looks roughly like the Blue Lagoon in Iceland, see the picture on page 4.

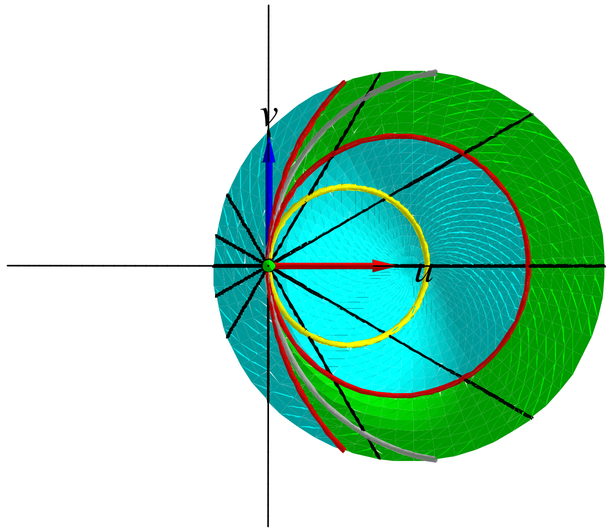

FIGURE 3. The function $-f$ unfolded with 'dual colors'.

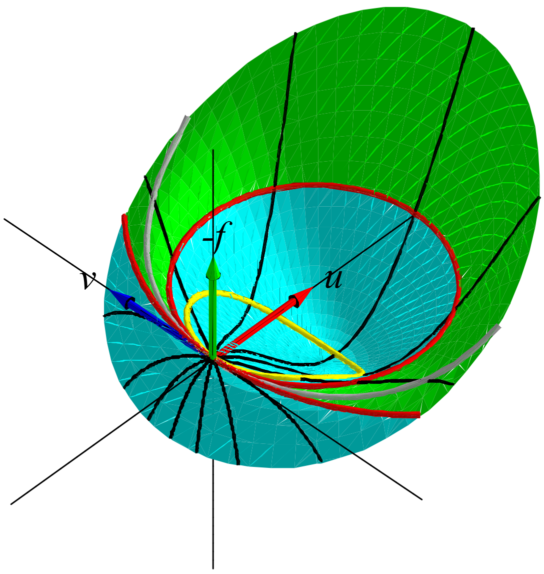

FIGURE 4. The graph surface of $-f$ (with 'dual' colors) looks roughly like the Blue Lagoon at Abereiddy in Wales, UK.

FIGURE 5. The Blue Lagoon in Iceland.

FIGURE 6. The Blue Lagoon at Abereiddy in Wales, UK.

3. Analysis

Any straight line through $p = (0, 0)$ may be parametrized as follows

$$L_w \,:\, r(t) = (\, t\,\cos(w), t\,\sin(w)\,) \text{ for } t \in \mathbb{R} \text{ and } w \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad .$$

When restricting f to L_w we get the restricted function:

$$g(t) = f(r(t)) = f(\, t\,\cos(w), t\,\sin(w)\,) = t^2(8\cos^2(w) - 6t\cos(w) + t^2) \;.$$

These restricted functions are displayed in Figure 7 for a couple of $w-$values. It is clear from this inspection, that at least for $w \neq \pm\pi/2$,

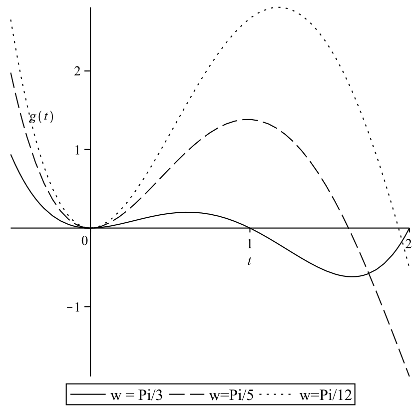

FIGURE 7. The line-restricted functions $g(t)$.

every g has a local minimum along L_w at p corresponding to $t = 0$. This also follows precisely from the derivatives of g at $t = 0$:

$$g'(0) = 0 \text{ and } g''(0) = 16\cos^2(w) > 0 \quad . \tag{3.1}$$

For the special values $w = \pm\pi/2$ (corresponding to L_w being the $v-$axis), we get $g(t) = t^4$. This shows that the restriction of f to the $v-$axis also has a strict local minimum at p.

The yellow circle in Figure 1 appears as the locus of local maxima (on the green 'island') of g along the straight lines L_w for $w \in [-\pi/2, \pi/2]$. The blue circle in Figure 1 is correspondingly the locus of local minima (in the blue 'lagoon') of g along the lines. Indeed, $g'(t) = 2t(-9t\cos(w) + 2t^2 + 8\cos^2(w))$. Thus $g'(t) = 0$ for $t = 0$, $t = (1/4)(9-\sqrt{17})\cos(w)$, and for $t = (1/4)(9+\sqrt{17})\cos(w)$. When inserted into $r(t)$ this gives the point p and the two circles, respectively.

In particular we note, that the values of f along the blue circle are, as a function of the direction angle $w \in [-\pi/2, \pi/2]$ from the point p:

$$h(w) = -\cos^4(w)(107 + 51\sqrt{17})/32 \quad .$$

This function is clearly negative, except at p - corresponding to $w = \pm\pi/2$ - and it is clearly decreasing when w moves away from these values of w - corresponding to walking (or rather diving) away from p along the blue circle in Figure 2, as claimed on page 1.

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