Chapter 2

Differential Geometry of Curves

2.1 Introduction

Intuitively a curve is a one-dimensional object, i.e., an object that can be described by a single parameter. A curve with a particular choice of parameter is called a *parameterized curve* and we have in the previous chapter already seen many examples of this concept.

In this chapter we will study *local* properties of abstract curves. The main result is that a plane curve is completely determined by a single real valued function, the *curvature*, and a space curve is completely determined by two real valued functions, the *curvature* and *torsion*. A curve in $\mathbb{R}^n$ is completely determined by $n-1$ functions, called the curvatures.

2.2 Parameterized Curves

Our study of curves will be restricted to a certain class of curves. First of all we want to use calculus in the analysis so a curve has to be described by a differential function[1]. If the derivative of a map $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ vanishes at some point then the image can have sharp corner or a cusp, see Problems 2.2.1 and 2.2.2, and we want to avoid that too. So we will only work with *regular curves*. Our main interest are plane curves or space curves so in the following you may think of $\mathbb{R}^n$ as $\mathbb{R}^2$ or $\mathbb{R}^3$.

Definition 2.1. A *regular parametrization* of class C^k, with $k \geq 1$, of a curve in $\mathbb{R}^n$ is a vector function $\mathbf{r} : I \to \mathbb{R}^n$ defined on an interval I which satisfies

[1] Besides the convenience of being able to use calculus there is a more severe reasons for insisting on differentiable functions. There exists continuous maps $[0, 1] \to [0, 1]^n$ whose image is all of $[0, 1]^n$ and we do not want to call them curves.

1. $\mathbf{r}$ is of class C^k.
2. $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

The variable t is called the *parameter* and I is called the *parameter interval*.

Example 2.1 The function $\mathbf{r}(t) = (t, t^2 - 1)$, $t \in \mathbb{R}$, is a regular parametrization because $\mathbf{r}$ is of class C^∞ and $\mathbf{r}'(t) = (1, 2t) \neq (0, 0)$ for all $t \in \mathbb{R}$. The image of $\mathbf{r}$ is the parabola shown in Figure 2.1

Example 2.2 The function $\mathbf{r}(t) = (r\cos t, r\sin t, ht)$, where $r, h > 0$, is a regular parametrization because $\mathbf{r}$ is of class C^∞ and $|\mathbf{r}'(t)|^2 = r^2\sin^2 t + r^2\cos^2 t + h^2 = r^2 + h^2 \neq 0$ for all $t \in \mathbb{R}$. The image of $\mathbf{r}$ is the *right circular helix* shown in Figure 2.1.

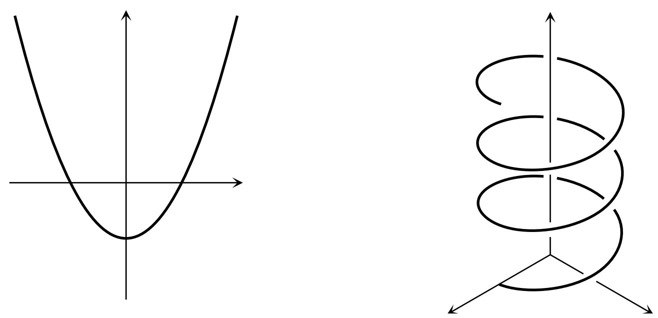


Figure 2.1: To the left a parabola to the right a circular helix.

Definition 2.2. An *allowable change of parameter* of class C^k is a real function $f : I_1 \to I$ such that

1. f is of class C^k.
2. $f'(t) \neq 0$ all $t \in I_1$.

As I is an interval we have either $f'(t) > 0$ for all $t \in I$, in which case we call f *orientation preserving*, or $f'(t) < 0$ for all $t \in I$, in which case we call f *orientation reversing*. If $f : I_1 \to I$ is an allowable change of parameter of class C^k then the condition $f'(t) \neq 0$ implies that the inverse exists and is an allowable change of parameter of class C^k. If $\mathbf{r} : I \to \mathbb{R}^n$ is a regular parameterization of a curve and $f : I_1 \to I$ is an allowable change of parameter and both are of class C^k then $\mathbf{r}_1 = \mathbf{r} \circ f : I_1 \to \mathbb{R}^n$ is of class C^k too, and it satisfies $\mathbf{r}_1'(t) = \mathbf{r}'(f(t))f'(t) \neq \mathbf{0}$, i.e., it is a regular parametrization, see Figure 2.2. We say that $\mathbf{r}_1$ is a *reparameterization* of $\mathbf{r}$, and this defines an equivalence relation on the set of parametrizations, cf. Problem 2.2.5. We will consider a regular parametrization of class C^k and any reparametrization as defining the same *curve*, that is

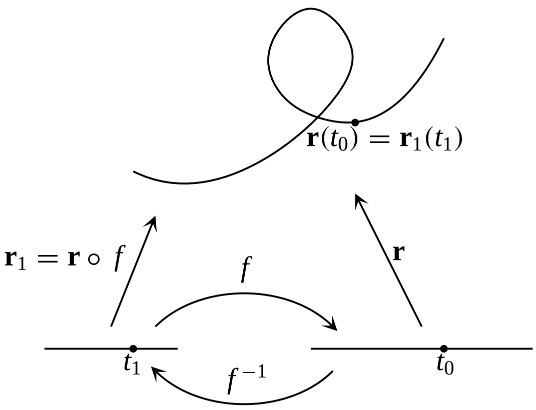


Figure 2.2: Reparametrization of a curve

Definition 2.3. A *regular curve* in $\mathbb{R}^n$ is a collection of regular parametrizations $\mathbf{r} : I \to \mathbb{R}^n$ of class C^k any two of which are reparametrizations of each other.

An *oriented regular curve* in $\mathbb{R}^n$ is a collection of regular parametrizations $\mathbf{r} : I \to \mathbb{R}^n$ of class C^k any two of which are orientation preserving reparametrizations of each other.

A regular parametrization $\mathbf{r} : I \to \mathbb{R}^n$ uniquely determines a curve and all other parametrizations are related to it by an allowable change of parameter. Thus we may say "the curve given by $\mathbf{r}(t)$...". However, a property of or a concept associated with the parametrization $\mathbf{r} : I \to \mathbb{R}^n$ need not be a property of the underlying curve. Any property of or concept associated with the curve must be common to all representations or, as we say, "independent of the parameter".

A regular curve given by $\mathbf{r} : I \to \mathbb{R}^n$ is said to be *simple* if there are no multiple points; that is, if $t_1 \neq t_2$ implies $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$. This is clearly a property of the curve, not of the parametrization. Locally, a regular curve is always simple, cf. Problem 2.2.6.

If we think of the curve as the path of a moving particle then $\mathbf{r}'(t_0)$ is the velocity of the particle at time $t = t_0$.

Definition 2.4. The *velocity vector* of a regular parametrization $\mathbf{r} : I \to \mathbb{R}^n$ at $t = t_0$ is the derivative $\mathbf{r}'(t_0)$. The *velocity vector field* is the vector valued function $\mathbf{r}' : I \to \mathbb{R}^n$. The *speed* of $\mathbf{r}$ at $t = t_0$ is the length of the velocity vector at $t = t_0$, $\left|\mathbf{r}'(t_0)\right|$. The *tangent vector* is the unit vector $\mathbf{t}(t_0) = \mathbf{r}'(t_0)/|\mathbf{r}'(t_0)|$, and the *tangent vector field* is the vector valued function $t \mapsto \mathbf{t}(t)$.

Observe that the regularity condition ensures that the instantaneous speed always is different from zero so we are able to divide by $|\mathbf{r}'|$ and define $\mathbf{t}$. When we have a vector field $\mathbf{v} : (a, b) \to \mathbb{R}^n$ along a curve $\mathbf{r}$ then we should think of the vector $\mathbf{v}(t)$ to be attached to the point $\alpha(t)$, see Figure 2.3. If $\mathbf{r}$ and $\mathbf{r}_1 = \mathbf{r} \circ f$

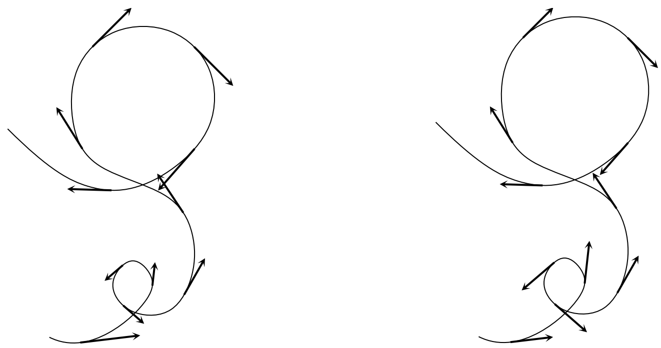


Figure 2.3: To the left the velocity vector field and to the right the tangent vector field.

are reparametrizations of each other then $\mathbf{r}_1'(t) = f'(t)\mathbf{r}'(f(t))$ so the velocity vector depends on the parametrization, but the tangent vectors satisfies $\mathbf{t}_1(t) = f'(t)/|f(t)|\mathbf{t}(f(t)) = \pm\mathbf{t}(f(t))$ so the tangent vector is a well defined property of an oriented curve, but is in general only defined up to a sign.

Definition 2.5. The straight line through a point $\mathbf{r}(t)$ on a regular curve parallel to the tangent vector is called the *tangent line* to the curve at $\mathbf{r}(t)$.

A more geometric way of defining the tangent line at a point $\mathbf{x}_0$ on a curve is as the limit position of a *secant*, i.e., a straight line through two points $\mathbf{x}_1 \neq \mathbf{x}_2$ on the curve when $\mathbf{x}_1, \mathbf{x}_2 \to \mathbf{x}_0$, see Figure 2.4.

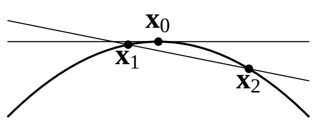


Figure 2.4: As $\mathbf{x}_1$ and $\mathbf{x}_2$ approaches $\mathbf{x}_0$ the secant approaches the tangent line.

That the limit position of such a secant indeed is the tangent is shown in Problem 2.2.7.

The tangent to a regular curve given by $\mathbf{r} : I \to \mathbb{R}^n$ at the point $\mathbf{r}(t_0)$ can be parameterized as

$$u \mapsto \mathbf{r}(t_0) + u\mathbf{t}(t_0) \qquad \text{or} \qquad u \mapsto \mathbf{r}(t_0) + u\mathbf{r}'(t_0) \tag{2.1}$$

2.2.1 Length of curves

An *arc* of a curve given by $\mathbf{r} : I \to \mathbb{R}^n$ is a curve given by the restriction of $\mathbf{r}$ to a *closed* interval $[a, b] \subseteq I$. The points $\mathbf{r}(a)$ and $\mathbf{r}(b)$ are called the *end points* of the arc.

Definition 2.6. If $\mathbf{r} : I \to \mathbb{R}^n$ is a regular parametrization of a curve and $[a, b] \in I$ then the *length* of the arc $\mathbf{r}_{|[a,b]}$ is

$$\int_a^b |\mathbf{r}'(t)|\, \mathrm{d}t.$$

The following proposition shows that the length of an arc is independent of the parametrization.

Proposition 2.7. *Let $f : I_1 \to I$ be a reparametrization of a curve $\mathbf{r} : I \to \mathbb{R}^n$, and let $\mathbf{r}_1 = \mathbf{r} \circ f$. If $f([a_1, b_1]) = [a, b]$, then*

$$\int_a^b |\mathbf{r}'(t)|\, \mathrm{d}t = \int_{a_1}^{b_1} |\mathbf{r}_1'(u)|\, \mathrm{d}u.$$

The proof is left as Problem 2.2.8.

Definition 2.8. If $\mathbf{r} : I \to \mathbb{R}^n$ is a regular parametrization of a curve and $t_0 \in I$ then the *arc length* measured from t_0 is the function

$$s(t) = \int_{t_0}^t |\mathbf{r}'(\tau)|\, \mathrm{d}\tau, \quad t \in I. \tag{2.2}$$

If $t \geq t_0$, then $s \geq 0$ and is equal to the length of the arc between $\mathbf{r}(t_0)$ and $\mathbf{r}(t)$. If $t \leq t_0$, then $s \leq 0$ and is equal to minus the length of the arc between $\mathbf{r}(t_0)$ and $\mathbf{r}(t)$.

If $\mathbf{r}$ is of class C^k then the velocity $\mathbf{r}'$ is of class C^{k-1} and as the velocity never vanishes the speed $|\mathbf{r}'|$ is of class C^{k-1} too. It now follows that the arc length s is of class C^k and that $s'(t) = |\mathbf{r}'(t)| > 0$ for all $t \in I$. Hence $s = s(t)$ is an allowable change of parameter and we can use s as a parameter on the curve. This, of course, is an abuse of notation, s denotes both the function defined by (2.2) and a parameter, i.e, a real number. Similarly, we will denote the inverse function of $t \mapsto s(t)$ by $s \mapsto t(s)$ so t will also denotes both a function and a parameter. The reparametrization of $t \mapsto \mathbf{r}(t)$ by arc length, i.e., $s \mapsto \mathbf{r}(t(s))$ will be denoted by the same symbol $\mathbf{r}$, the advantage of this abuse of notation is that we now can write identities like

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| \left|\frac{\mathrm{d}t}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| \Big/ \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| \Big/ \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| = 1. \tag{2.3}$$

A parametrization by arc length is called a *natural parametrization*, or more precise

Definition 2.9. A parametrization $\mathbf{r} : I \to \mathbb{R}^n$ is called a *natural parametrization* if $|\mathbf{r}'(s)| = 1$ for all $s \in I$.

We now have

Proposition 2.10. *If $\mathbf{r} : I \to \mathbb{R}^n$ is a natural parametrization, then*

1. *The length of the arc between $\mathbf{r}(s_1)$ and $\mathbf{r}(s_2)$ is $|s_2 - s_1|$.*
2. *If $s^* \mapsto \mathbf{r}^*(s^*)$ is another natural parametrization, then $s = \pm s^* + \text{constant}$.*
3. *If t is an arbitrary parameter, then $|\mathrm{d}s/\mathrm{d}t| = |\mathrm{d}\mathbf{r}/\mathrm{d}t|$.*
4. *The tangent vector is $\mathbf{t} = \mathrm{d}\mathbf{r}/\mathrm{d}s$.*

Proof. The proof of 1 and 2 is left as Problems 2.2.9 and 2.2.10. Now 3 follows from (2.3). Finally $\mathbf{t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \Big/ \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right| = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$, which proves 4. □

A more geometric definition of arc length is in terms of approximating polygons. Let an arc be given by a parametrization $\mathbf{r}(t)$ with $t \in [a, b]$ and consider a partition

$$a = t_0 < t_1 < \cdots < t_m = b \tag{2.4}$$

of the interval $[a, b]$. This determines a sequence of points in $\mathbb{R}^n$

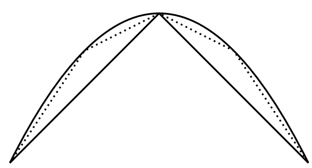


Figure 2.5: An approximating polygon P and a refinement P'.

$$\mathbf{x}_0 = \mathbf{r}(t_0), \quad \mathbf{x}_1 = \mathbf{r}(t_1), \quad \ldots \quad \mathbf{x}_m = \mathbf{r}(t_m).$$

The points form an *approximating polygon* P as shown in Figure 2.5 The length of P is clearly

$$\ell(P) = \sum_{i=1}^{m} |\mathbf{x}_i - \mathbf{x}_{i-1}| = \sum_{i=1}^{n} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|$$

If the partition is refined to give a better polygonal approximating P', see Figure 2.5 then we clearly have $\ell(P') \geq \ell(P)$ so we are lead to consider the quantity

$$\ell = \sup\left\{ \sum_{i=1}^{m} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| \;\middle|\; a = t_0 < t_1 < \cdots < t_m = b, m \in \mathbb{N} \right\}. \tag{2.5}$$

Observe that this makes sense even if $\mathbf{r}$ is only continuous, but we may have $\ell = \infty$.

Definition 2.11. The image of a map $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is called *rectifiable* with length ℓ if

$$\ell = \sup\Big\{ \sum_{i=1}^{m} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| \;\Big|\; a = t_0 < t_1 < \cdots < t_m = b, m \in \mathbb{N} \Big\} < \infty.$$

The following theorem shows that the two notions of arc length coincide.

Theorem 2.12. *Let $\mathbf{r} : [a, b] \to \mathbb{R}^n$ be of class C^1, then the image is a rectifiable arc with length*

$$\ell = \int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t.$$

Proof. Let $\mathbf{r}(t) = \big(x_1(t), \ldots, x_n(t)\big)$ and put $M = \max\big\{|\mathbf{r}'(t)| \;\big|\; t \in [a, b]\big\}$. If we have a partition (2.4) then

$$\begin{aligned}\sum_{i=1}^{m} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| &\le \sum_{i=1}^{m}\sum_{j=1}^{n} |x_j(t_i) - x_j(t_{i-1})| \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} |x_j'(\xi_{i,j})(t_i - t_{i-1})| \le \sum_{i=1}^{m}\sum_{j=1}^{n} |x_j'(\xi_{i,j})|(t_i - t_{i-1}) \\ &\le \sum_{i=1}^{m}\sum_{j=1}^{n} M(t_i - t_{i-1}) = \sum_{i=1}^{m} nM(t_i - t_{i-1}) = nM(b - a),\end{aligned}$$

where $t_{i-1} < \xi_{i,j} < t_i$. So ℓ is finite and the arc is rectifiable.

Now consider an arbitrary $\epsilon > 0$. As $\mathbf{r}$ is of class C^1 we can find $\delta_1 > 0$ such that $|x_j'(t) - x_j'(t')| < \epsilon/\big(3n(b - a)\big)$, $j = 1, \ldots, n$, if $|t - t'| < \delta_1$. Furthermore, we can find $\delta_2 > 0$ such that for a partition (2.4) with $t_i - t_{i-1} < \delta_2$ we have $\left|\int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t - \sum_{i=1}^{m} |\mathbf{r}'(t_i)|(t_i - t_{i-1})\right| < \epsilon/3$. Now let $\delta = \min\{\delta_1, \delta_2\}$, and choose a partition such that the corresponding approximating polygon has a length that satisfies $0 \le \ell - \ell(P) < \epsilon/3$. If we refine the partition then the inequalities are still satisfied so we may assume that the partition has $t_i - t_{i-1} < \delta$. For such a partition we have

$$\begin{aligned}\left|\ell - \int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t\right| &\le |\ell - \ell(P)| + \left|\ell(P) - \int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t\right| \\ &\le \frac{\epsilon}{3} + \left|\sum_{i=1}^{m} \big|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\big| - \int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t\right| \\ &= \frac{\epsilon}{3} + \left|\sum_{i=1}^{m} \left|\sum_{j=1}^{n} (x_j(t_i) - x_j(t_{i-1}))\mathbf{e}_j\right| - \int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t\right|\end{aligned}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis in $\mathbb{R}^n$, by the mean value theorem we get

$$= \frac{\epsilon}{3} + \left| \sum_{i=1}^{m} \left| \sum_{j=1}^{n} x_j'(\xi_{i,j})\mathbf{e}_j \right| (t_i - t_{i-1}) - \int_a^b |\mathbf{r}'(t)|\, \mathrm{d}t \right|$$

by adding and subtracting $\sum_i |\mathbf{r}(t_i)|(t_i - t_{i-1})$, we obtain

$$\leq \frac{\epsilon}{3} + \left| \sum_{i=1}^{m} \left(\left| \sum_{j=1}^{n} x_j'(\xi_{i,j})\mathbf{e}_j \right| - \left| \sum_{j=1}^{n} x_j'(t_i)\mathbf{e}_j \right| \right) (t_i - t_{i-1}) \right|$$
$$+ \left| \sum_{i=1}^{n} |\mathbf{r}(t_i)|(t_i - t_{i-1}) - \int_a^b |\mathbf{r}'(t)|\, \mathrm{d}t \right|$$

as $\big| |p| - |q| \big| \leq |p - q|$ we have

$$\leq \frac{\epsilon}{3} + \left| \sum_{i=1}^{m} \left| \sum_{j=1}^{n} \big(x_j'(\xi_{i,j}) - x_j'(t_i)\big)\mathbf{e}_j \right| (t_i - t_{i-1}) \right| + \frac{\epsilon}{3}$$
$$\leq \frac{2\epsilon}{3} + \sum_{i=1}^{m} \sum_{j=1}^{n} \big|x_j'(\xi_{i,j}) - x_j'(t_i)\big|(t_i - t_{i-1})$$
$$< \frac{2\epsilon}{3} + \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\epsilon}{3n(b-a)}(t_i - t_{i-1}) = \epsilon$$

as ϵ is arbitrary we see that $\left| \ell - \int_a^b |\mathbf{r}'(t)|\, \mathrm{d}t \right| = 0$. □

2.2.2 Curvature

Definition 2.13. Let $\mathbf{r} : I \to \mathbb{R}^n$ be a regular curve with arc length s and tangent vector $\mathbf{t}$. The *curvature vector* is $\boldsymbol{\kappa} = \dfrac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}$ and the *curvature* is a $\kappa = |\boldsymbol{\kappa}|$. If $\kappa \neq 0$ then the *radius of curvature* is is $\rho = 1/\kappa$, the *center of curvature* is the point $\mathbf{c} = \mathbf{r} + \rho^2\boldsymbol{\kappa}$, and the *circle of curvature* is the circle with center $\mathbf{c}$ and radius $|\rho|$.

As $\mathbf{t}\cdot\mathbf{t} = 1$ differentiation with respect to arc length shows that $\mathbf{t}\cdot\mathbf{t}' = 0$ and hence that $\mathbf{t} \perp \boldsymbol{\kappa}$.

In practise curves are rarely given by their natural parametrization, but the following lemma tells us how to determine the curvature of curve given in an arbitrary parametrization.

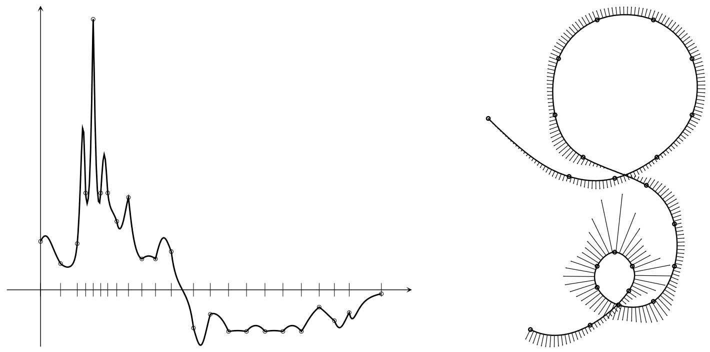


Figure 2.6: To the left a curvature plot and to the right a porcupine plot of a cubic B-spline curve. The endpoints of Bézier segments are indicated on both plots.

Theorem 2.14. *Let* $\mathbf{r} : I \to \mathbb{R}^n$ *be a regular parametrization of a curve. Then*

$$\boldsymbol{\kappa} = \frac{|\mathbf{r}'|^2\mathbf{r}'' - (\mathbf{r}' \cdot \mathbf{r}'')\mathbf{r}'}{|\mathbf{r}'|^4} \quad \textit{and} \quad \kappa = \frac{\sqrt{|\mathbf{r}'|^2|\mathbf{r}''|^2 - (\mathbf{r}' \cdot \mathbf{r}'')^2}}{|\mathbf{r}'|^3}.$$

Furthermore, if $\kappa \neq 0$, *then*

$$\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}t} = \frac{4(\mathbf{r}' \cdot \mathbf{r}'')^2 - |\mathbf{r}'|^2|\mathbf{r}''|^2 - |\mathbf{r}'|^2(\mathbf{r}' \cdot \mathbf{r}''')}{|\mathbf{r}'|^6}\mathbf{r}' - 3\frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|^4}\mathbf{r}'' - \frac{\mathbf{r}'''}{|\mathbf{r}'|^2}.$$

Proof. We have $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ so

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}t} = \frac{\mathbf{r}''}{|\mathbf{r}'|} - \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|^3}\mathbf{r}' = \frac{|\mathbf{r}'|^2\mathbf{r}'' - (\mathbf{r}' \cdot \mathbf{r}'')\mathbf{r}'}{|\mathbf{r}'|^3}$$

so

$$\boldsymbol{\kappa} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}t} \Big/ \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{|\mathbf{r}'|^2\mathbf{r}'' - (\mathbf{r}' \cdot \mathbf{r}'')\mathbf{r}'}{|\mathbf{r}'|^4},$$

and hence

$$\boldsymbol{\kappa} \cdot \boldsymbol{\kappa} = \frac{|\mathbf{r}'|^4|\mathbf{r}''|^2 + (\mathbf{r}' \cdot \mathbf{r}'')^2|\mathbf{r}'|^2 - 2|\mathbf{r}'|^2(\mathbf{r}' \cdot \mathbf{r}'')^2}{|\mathbf{r}'|^8} = \frac{|\mathbf{r}'|^2|\mathbf{r}''|^2 - (\mathbf{r}' \cdot \mathbf{r}'')^2}{|\mathbf{r}'|^6}.$$

Differentiating the equation for $\boldsymbol{\kappa}$ finishes the proof. □

The curvature is often used to assess the quality of a curve, either in the form of a *curvature plot* or a *porcupine plot*, see Figure 2.6. In a porcupine plot the curve is plotted along with the vector field '$-$scale $\times$ $\boldsymbol{\kappa}$'. A designer normally wants a slowly varying curvature plot without unnecessary undulations, so the curve above would not be satisfactory. The designer would then change the curve slightly either by changing the control points manually, or by an automatic procedure, eg. by minimizing $\int (\mathrm{d}\kappa/\mathrm{d}s)^2\,\mathrm{d}s$, under the side condition that the control points are only allowed to move a certain distance. This process is called *fairing* and the goal is to obtain a *fair* curve.

2.2.3 Contact

An important notion in geometry is the concept of contact between objects. First we need the distance between points and subsets.

Definition 2.15. Let p be a point in $\mathbb{R}^n$ and let A be a nonempty subset of $\mathbb{R}^n$. The *distance* between p and A is

$$d(p, A) = \inf\bigl\{|p - q| \bigm| q \in A\bigr\}.$$

Normally A will be a nice geometric object like a line, a plane, a circle, etc., but the definition makes sense for any subset of $\mathbb{R}^n$.

Example 2.3 Let L be the line parametrized as $t \mapsto \mathbf{x}_0 + t\mathbf{e}$ where $\mathbf{e}$ is a unit vector. Then

$$d(\mathbf{x}, L) = \sqrt{\mathbf{r}\cdot\mathbf{r} - (\mathbf{r}\cdot\mathbf{e})^2}, \quad \text{where } \mathbf{r} = \mathbf{x} - \mathbf{x}_0. \tag{2.6}$$

Let C be the circle parametrized as $t \mapsto \mathbf{x}_0 + r\cos t\,\mathbf{e}_1 + r\sin t\,\mathbf{e}_2$, where $\mathbf{e}_1, \mathbf{e}_2$ is an orthonormal pair of vectors. Then

$$d(\mathbf{x}, C) = \sqrt{\mathbf{r}\cdot\mathbf{r} + r^2 - 2r\sqrt{(\mathbf{r}\cdot\mathbf{e}_1)^2 + (\mathbf{r}\cdot\mathbf{e}_2)^2}} \quad \text{where } \mathbf{r} = \mathbf{x} - \mathbf{x}_0. \tag{2.7}$$

Definition 2.16. Let $\mathbf{r} : I \to \mathbb{R}^n$ be a regular parametrization of a curve and let A be a subset of $\mathbb{R}^n$. We say that the curve has *contact* with A of order k at $\mathbf{r}(t_0)$ if

$$\frac{d\bigl(\mathbf{r}(t), A\bigr)}{(t - t_0)^k} \to 0 \quad \text{for} \quad t \to t_0.$$

If $t = f(u)$ is a reparametrization with $f(u_0) = t_0$ then we have

$$\begin{aligned}\frac{d\bigl(\mathbf{r}(f(u)), A\bigr)}{(u - u_0)^k} &= \frac{d\bigl(\mathbf{r}(t), A\bigr)}{\bigl(f(u) - f(u_0)\bigr)^k}\frac{\bigl(f(u) - f(u_0)\bigr)^k}{(u - u_0)^k} \\ &= \frac{d\bigl(\mathbf{r}(t), A\bigr)}{(t - t_0)^k}\left(\frac{f(u) - f(u_0)}{u - u_0}\right)^k\end{aligned}$$

and as $\bigl(f(u) - f(u_0)\bigr) \bigm/ (u - u_0) \to f'(u_0) \neq 0$ for $u \to u_0$ we see that the notion of contact of order k is a property of the curve. It can in general be difficult to determine the order of contact, but when it comes to contact between curves the following theorem is helpful

Theorem 2.17. *Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be natural parametrizations of two regular curves of class C^k. Suppose $\mathbf{r}_2(s) \neq \mathbf{r}_1(s_0)$ if $s \neq s_0$, then $\mathbf{r}_1$ has contact of order k with $\mathbf{r}_2$ at $\mathbf{r}_1(s_0)$ if and only if $\mathbf{r}_1^{(l)}(s_0) = \mathbf{r}_2^{(l)}(s_0)$ for all $l = 0, \dots, k$.*

Proof. We only prove the 'if' part. As the Taylor expansions of $\mathbf{r}_1$ and $\mathbf{r}_2$ agree up to order k we have $\mathbf{r}_1(s - s_0) - \mathbf{r}_2(s - s_0) = \mathbf{o}\big((s - s_0)^k\big)$ and hence

$$\frac{d\big(\mathbf{r}_1(s), \mathbf{r}_2\big)}{(s - s_0)^k} = \frac{\inf_{s_1}\big|\mathbf{r}_1(s) - \mathbf{r}_2(s_1)\big|}{(s - s_0)^k} \leq \frac{\big|\mathbf{r}_1(s) - \mathbf{r}_2(s)\big|}{(s - s_0)^k} = \frac{o\big((s - s_0)^k\big)}{(s - s_0)^k} \to 0 \quad \text{for } s \to s_0.$$

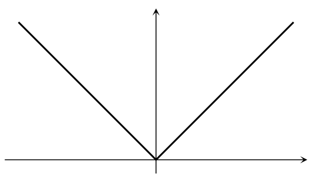
The 'only if' part is considerable more difficult, and we will return to a special case in Chapter 4 ☐

It can now be shown that the only straight line which has contact of order 1 with a curve at some point is the tangent line at that point, cf. Problem 2.2.18.

Problems

2.2.1 Show, that the vector function

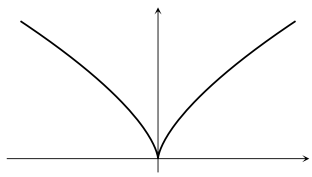
$$\mathbf{r}(t) = \begin{cases} (-e^{-1/t^2}, e^{-1/t^2}) & \text{for } t < 0 \\ (0, 0) & \text{for } t = 0 \\ (e^{-1/t^2}, e^{-1/t^2}) & \text{for } t > 0 \end{cases}$$



is of class C^∞, and that $\mathbf{r}'(0) = 0$.

2.2.2 Show that the vector function

$$\mathbf{r}(t) = \big(t^3, t^2\big), \quad t \in \mathbb{R}$$



is of class C^∞, and that $\mathbf{r}'(0) = 0$.

2.2.3 Prove that a Bézier curve is either constant or piecewise a regular curve.

2.2.4 Prove that a B-spline curve is piecewise a constant or a regular curve.

2.2.5 We say that two vector functions $\mathbf{r}_i : I_i \to \mathbb{R}^n$, $i = 1, 2$, of class C^k are *equivalent* and write $\mathbf{r}_1 \sim \mathbf{r}_2$ if there exists an allowable change of parameter $f : I_2 \to I_1$ of class C^k such that $\mathbf{r}_2 = \mathbf{r}_1 \circ f$. Show that $\sim$ is an *equivalence relation*, i.e., that

(a) $\mathbf{r} \sim \mathbf{r}$.

(b) $\mathbf{r}_1 \sim \mathbf{r}_2 \Rightarrow \mathbf{r}_2 \sim \mathbf{r}_1$.

(c) $\mathbf{r}_1 \sim \mathbf{r}_2 \wedge \mathbf{r}_2 \sim \mathbf{r}_3 \Rightarrow \mathbf{r}_1 \sim \mathbf{r}_3$.

2.2.6 Show that if $\mathbf{r} : I \to \mathbb{R}^n$ is of class C^1 and $\mathbf{r}'(t) \neq \mathbf{0}$ for a $t \in I$ then there exists an $\epsilon > 0$ such that $\mathbf{r}_{|(t-\epsilon,t+\epsilon)\cap I}$ is injective.

2.2.7 Let $\mathbf{r} : I \to \mathbb{R}^n$ be a regular parametrization, and let $t_0 \in I$. Show that if $t_1, t_2 \in I$ are different and sufficiently close to t_0 then there is a well defined *secant* through $\mathbf{r}(t_1)$ and $\mathbf{r}(t_2)$. Show that if $t_1 < t_2$ and $t_1, t_2 \to t_0$ then the unit vector in the direction $\mathbf{r}(t_2) - \mathbf{r}(t_1)$ converges to the tangent vector $\mathbf{t}(t_0)$.

2.2.8 Prove Proposition 2.7, p. 43.

2.2.9 Prove 1 in Proposition 2.10, p. 44.

2.2.10 Prove 2 in Proposition 2.10, p 44.

2.2.11 Find the arc length of the *helix* in Example 2.2, and determine a natural parametrization.

2.2.12 Determine a parametrization of the tangent line to the parabola in Example 2.1 at an arbitrary point.

2.2.13 Determine a parametrization of the tangent line to the helix in Example 2.2 at an arbitrary point.

2.2.14 Prove that if the curvature of a regular curve is zero then the curve is a straight line.

2.2.15 Prove that $\dfrac{\mathrm{d}\kappa}{\mathrm{d}t} = \dfrac{\boldsymbol{\kappa}}{\kappa} \cdot \dfrac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}t}$, $\dfrac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}s} = \dfrac{1}{|\mathbf{r}'|}\dfrac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}t}$, and $\dfrac{\mathrm{d}\kappa}{\mathrm{d}s} = \dfrac{1}{|\mathbf{r}'|}\dfrac{\mathrm{d}\kappa}{\mathrm{d}t}$.

2.2.16 Prove (2.6), p. 48.

2.2.17 Prove (2.7), p. 48.

2.2.18 Prove that if a curve has contact of order 1 with a straight line then the line is the tangent line.

2.2.19 Let a regular curve be given by a parametrization $\mathbf{r}(t)$ defined on the interval $[a, b]$, let $s(t)$ be the arc length function, let $a = t_0 < t_1 < \cdots < t_m = b$ be a sequence of parameter values, and put $s_i = s(t_i)$ and $v_i = s'(t_i) = |\mathbf{r}'(t_i)|$.

(a) Show that the inverse function $t(s)$ satisfies $t_i = t(s_i)$ and $t'(s_i) = w_i = 1/v_i$.

(b) Show that there is a unique B-spline function $f : [s_0, s_m] \to [a, b]$ of degree 3 with knot vector

$$s_0, s_0, s_0, s_0, s_1, s_1, s_2, s_2, \dots, s_{m-1}, s_{m-1}, s_m, s_m, s_m, s_m$$

such that $f(s_i) = t_i$, and $f'(s_i) = w_i$.

(c) Determine the control points for f (in this situation we consider a map into $\mathbb{R}$ so a control point is just a real number).

2.3 Plane Curves

We now specialize to curves in the plane, i.e., we consider a regular parametrization $\mathbf{r} : I \to \mathbb{R}^2$. If $\mathbf{t}$ is the tangent vector at some point then the *normal vector* is the vector $\mathbf{n}$ such that $(\mathbf{t}, \mathbf{n})$ is a positively oriented orthonormal basis of $\mathbb{R}^2$, see Figure 2.7. Just like the tangent vector, the normal vector is an invariant concept associated with an oriented curve. It changes sign if the orientation is reversed.

As the curvature vector $\boldsymbol{\kappa} = \mathrm{d}\mathbf{t}/\mathrm{d}s$ and the tangent vector $\mathbf{t}$ are orthogonal we see that $\boldsymbol{\kappa}$ and $\mathbf{n}$ are parallel. In other words we have $\boldsymbol{\kappa} \pm \kappa\mathbf{n}$. This means that we in the planar case can define a signed curvature, which we will denote with the same symbol κ, hopefully without causing to much confusion. We will in this case denote the previous defined curvature and radius of curvature with $|\kappa|$ and $|\rho|$ respectively.

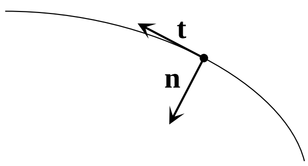


Figure 2.7: The tangent vector $\mathbf{t}$ and the normal vector $\mathbf{n}$ of a plane curve.

Definition 2.18. The *plane curvature* of a regular plane curve is $\kappa = \boldsymbol{\kappa} \cdot \mathbf{n}$ and the *plane radius curvature* is $\rho = 1/\kappa$.

If the orientation on a curve is reversed then both the tangent vector and the arc length changes sign, so the derivative $\mathrm{d}\mathbf{t}/\mathrm{d}s = \kappa\mathbf{n}$ is left unchanged and is thus a property of the curve. On the other hand $\mathbf{n}$ changes sign so κ and ρ changes sign too. All in all we have

Proposition 2.19. *For a regular plane curve we have that* $|\kappa|$, $|\rho|$, $\boldsymbol{\kappa} = \kappa\mathbf{n}$, $\rho\mathbf{n}$, *and the circle of curvature are invariant concepts associated with the curve. While* $\mathbf{t}$, $\mathbf{n}$, κ, *and* ρ *are concepts associated with an oriented curve and changes sign if the orientation is reversed. We furthermore have the* Frenet-Serret equations *for a plane curve:*

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa\mathbf{n}, \qquad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = -\kappa\mathbf{t}. \tag{2.8}$$

Proof. The only thing left to prove is the Frenet-Serret equations and we leave that as Problem 2.3.7. □

As the normal vector is to the left we easily see that if $\kappa > 0$ then the curve turns left, if $\kappa < 0$ then the curve turns right, and that if $\kappa = 0$ at some point and has different sign on each side of the point then the curve has an inflection point.

Example 2.4 Consider the circle with radius $r > 0$ given by the parametrization

$$\mathbf{r}(t) = (x_0 + r\cos t, y_0 + r\sin t).$$

We easily see that $\mathbf{r}'(t) = (-r\sin t, r\cos t)$, and $|\mathbf{r}'(t)| = r$, so the arc length measured from $t = 0$ is $s = \int_0^t r\,\mathrm{d}\tau = rt$. I.e., $t = s/r$ and we obtain a natural parametrization by $s \mapsto \mathbf{r}(s/r) = \big(x_0 + r\cos(s/r), y_0 + r\sin(s/r)\big)$. The tangent vector is $\mathbf{t} = \mathrm{d}\mathbf{r}/\mathrm{d}s = \big(-\sin(s/r), \cos(s/r)\big)$ and the normal vector is $\mathbf{n} = \big(-\cos(s/r), -\sin(s/r)\big)$ and $\kappa\mathbf{n} = \mathrm{d}\mathbf{t}/\mathrm{d}s = 1/r\big(-\cos(s/r), -\sin(s/r)\big) = \mathbf{n}/r$. From this we see that the curvature is constant $\kappa = 1/r$, the radius of curvature is $\rho = r$ and the circle of curvature is the circle itself.

A more geometric way of defining the circle of curvature at a point $\mathbf{x}_0$ on a curve is as the limit position of a circle through three distinct points $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ on the curve as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \to \mathbf{x}_0$, see Figure 2.8, and Problem 2.3.1.

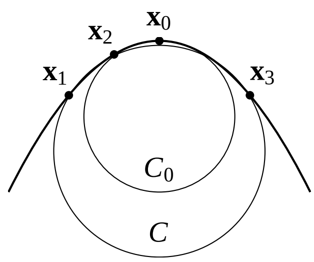


Figure 2.8: If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \to \mathbf{x}_0$ then $C \to C_0$.

It can also be shown that the circle of curvature is the only circle that has contact of order 2 with the curve, cf. Problem 2.3.2

Given a natural parametrization it is a simple matter to determine the curvature. It is in general impossible to determine a natural parametrization, but the following theorem tells how to calculate the curvature from an arbitrary regular parametrization.

Theorem 2.20. *Let $t \mapsto \mathbf{r}(t) = \big(x(t), y(t)\big)$ be a regular parametrization of class C^2. The curvature is then given by*

$$\kappa = \frac{\big[\mathbf{r}', \mathbf{r}''\big]}{\big|\mathbf{r}'\big|^3} = \frac{\begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}}{\big(x'^2 + y'^2\big)^{3/2}},$$

and the derivative with respect to arc length is

$$\frac{\mathrm{d}\kappa}{\mathrm{d}s} = \frac{|\mathbf{r}'\big|^2\big[\mathbf{r}', \mathbf{r}'''\big] - \mathbf{r}'\cdot\mathbf{r}''[\mathbf{r}', \mathbf{r}''\big]}{|\mathbf{r}'\big|^6}.$$

Proof. Let s denotes the arc length of the curve. We then have

$$\begin{aligned} \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathrm{d}s}{\mathrm{d}t}\mathbf{t}, \\ \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathbf{t} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathbf{t} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\kappa\mathbf{n}. \end{aligned}$$

Hence

$$\left[\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right] = \left[\frac{\mathrm{d}s}{\mathrm{d}t}\mathbf{t}, \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathbf{t} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \kappa\mathbf{n}\right]$$
$$= \frac{\mathrm{d}s}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 [\mathbf{t},\mathbf{t}] + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3 \kappa[\mathbf{t},\mathbf{n}] = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^3 \kappa.$$

The proof of last part of the theorem is left to the reader. □

As the tangent vector is a unit vector we can write it as $\mathbf{t} = (\cos\phi, \sin\phi)$ where ϕ is an angle determined up to a multiple of 2π, see Figure 2.9. If the tangent vector field is a continuous vector function $u \mapsto \mathbf{t}(u)$, then it is not hard to see that it is possible to make a continuous choice $u \mapsto \phi(u)$ of this angle. Such a choice is unique up to a (constant) multiple of 2π and it has the same degree of differentiability as $\mathbf{t}$.

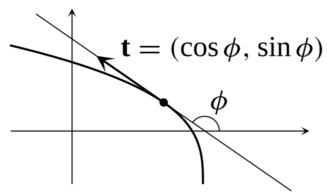


Figure 2.9: The tangent direction is the angle between the tangent and the x-axis.

Definition 2.21. Let $\mathbf{r} : I \to \mathbb{R}^2$ be a regular parametrization of class C^k. The *tangent direction* is a continuous choice of ϕ such that $\mathbf{t}(u) = \big(\cos\phi(u), \sin\phi(u)\big)$.

We immediately have the following results.

Proposition 2.22. *The tangent direction ϕ is a property of an oriented plane curve, but if the orientation is reversed then $\phi \mapsto \phi + \pi$. If κ is the curvature then $\mathrm{d}\phi/\mathrm{d}s = \kappa$. Furthermore, if $\kappa \neq 0$, then ϕ can be used as a parameter and $\mathrm{d}s/\mathrm{d}\phi = \rho$, where $\rho = 1/\kappa$ is the radius of curvature.*

Proof. As the tangent direction is a property of an oriented curve the same is true for the tangent direction. If the orientation is reversed $\mathbf{t}$ changes sign and that corresponds to adding π to the tangent direction. On one hand we have $\mathrm{d}\mathbf{t}/\mathrm{d}s = \kappa\,\mathbf{n}$ and on the other hand we have $\mathrm{d}(\cos\phi, \sin\phi)/\mathrm{d}s = \mathrm{d}\phi/\mathrm{d}s(-\sin\phi, \cos\phi) = \mathrm{d}\phi/\mathrm{d}s\,\mathbf{n}$, so $\mathrm{d}\phi/\mathrm{d}s = \kappa$. If $\kappa \neq 0$, then ϕ is a monotone function of s, so the inverse function exists and is differentiable with derivative $\mathrm{d}s/\mathrm{d}\phi = 1/\big(\mathrm{d}\phi/\mathrm{d}s\big) = 1/\kappa = \rho$. □

We can now prove that the curvature determines a plane curve uniquely up to a Euclidean motion, i.e., up to a rotation and a translation. We formulate it as the following theorem.

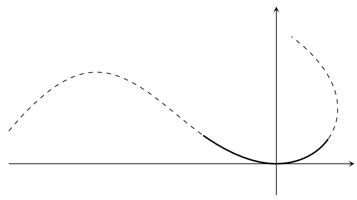


Figure 2.10: A plane curve is locally the graph of function "from" the tangent line "to" the normal line.

Theorem 2.23. *Let $\kappa : I \to \mathbb{R}$ be a continuous function. Then there exists a natural parametrization $\mathbf{r} : I \to \mathbb{R}^2$ of class C^2 such that κ is the curvature of $\mathbf{r}$. Furthermore, the curve is determined uniquely up to a Euclidean motion.*

Proof. Assume $\mathbf{r}$ is a curve with curvature κ and tangent direction ϕ. Then $\mathrm{d}\phi/\mathrm{d}s = \kappa$ so $\phi(s) = \phi_0 + \int_{s_0}^{s} \kappa(\tau)\,\mathrm{d}\tau$. The tangent vector is now $\mathbf{t}(s) = \big(\cos\phi(s), \sin\phi(s)\big)$ and as $\mathrm{d}\mathbf{r}/\mathrm{d}s = \mathbf{t}$ we must have $\mathbf{x}(s) = \mathbf{x}_0 + \int_{s_0}^{s} \mathbf{t}(\tau)\,\mathrm{d}\tau$. Different choices of ϕ_0 corresponds to rotations and different choices of $\mathbf{x}_0$ corresponds to translations. All that remains is to show that the curvature of $\mathbf{r}$ is κ, but by construction we have $\mathbf{t} = \mathrm{d}\mathbf{r}/\mathrm{d}s$ and $\mathrm{d}\mathbf{t}/\mathrm{d}s = \kappa\mathbf{n}$ which shows that κ indeed is the curvature of $\mathbf{r}$. □

By inspection of the proof above we realize that the function $s \mapsto \phi(s)$ determines the curve up to a translation. The equation $\phi = \phi(s)$ is called an *intrinsic equation* of the curve, but the equations $s = s(\phi)$ or $\mathrm{d}\phi/\mathrm{d}s = \kappa$ also determine the curve up to a translation and a Euclidean motion respectively. In fact any equation, including differential equations, that links the arc length and the tangent direction is called an *intrinsic equation* of the curve. In [14] the intrinsic equation $\mathrm{d}s/\mathrm{d}\phi = \rho$ was instrumental for the design of scroll compressors.

If $\mathbf{r}$ is a natural parametrization of a plane curve, and we put $\mathbf{r}_0 = \mathbf{r}(s_0)$, and let $\mathbf{t}$, $\mathbf{n}$, and κ be the tangent vector, the normal vector, and the curvature at s_0, respectively, then the Taylor expansion of $\mathbf{r}$ to second order at s_0 is

$$\mathbf{r}(s) = \mathbf{r}_0 + (s - s_0)\mathbf{t} + \frac{1}{2}(s - s_0)^2\kappa\mathbf{n} + \mathbf{o}\big((s - s_0)^2\big). \tag{2.9}$$

This expression is called the *canonical form* of a plane curve. It follows from Theorem 2.17 that every plane curve at a point with non vanishing curvature has second order contact with a unique parabola, cf. Problem 2.3.4. We also see that any plane curve is locally the graph of a function from the tangent line to the normal line, see Figure 2.10, and Problem 2.3.9.

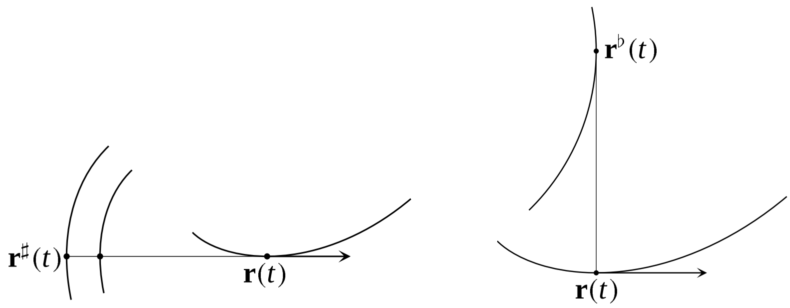


Figure 2.11: To the left two involutes of a curve and to the right the evolute.

We end this section by stating some properties of *involutes* and *evolutes*, all proofs are left as problems.

Definition 2.24. Let $\mathbf{r} : I \to \mathbb{R}^2$ be a regular parametrization of a regular curve with tangent $\mathbf{t}$, arc length s, and radius of curvature ρ. An *involute* of $\mathbf{r}$ is a curve given by

$$\mathbf{r}^\sharp(t) = \mathbf{r}(t) + (c - s(t))\mathbf{t}(t), \tag{2.10}$$

for a $c \in \mathbb{R}$. The *evolute* of $\mathbf{r}$ is the curve given by

$$\mathbf{r}^\flat(t) = \mathbf{r}(t) + \rho(t)\mathbf{n}(t), \tag{2.11}$$

see Figure 2.11

Different choices of the constant c in (2.10) lead to *parallel curves* and the distance between the curves is exactly the difference between the two constants, see Figure 2.11 and Problem 2.3.11. Furthermore the two constructions are the inverse of each other in the sense that the evolute of one of the involutes gives the original curve back, cf. Problem 2.3.13, while a curve is itself one of the involutes of its evolute, cf. Problem 2.3.14. An involute to an evolute is parallel to the original curve and any parallel curve is obtained this way. For other properties of involutes and evolutes, cf. Problems 2.3.10–2.3.15.

Problems

2.3.1 Let $\mathbf{r} : I \to \mathbb{R}^2$ be a natural parametrization of a regular curve. Let $\mathbf{t} : I \to \mathbb{R}^2$ be the tangent vector field and assume that $\mathbf{t}(s_0) \neq \mathbf{0}$. Show that if $s_1 < s_2 < s_3$ are sufficiently close to s_0 then there is a well defined circle through $\mathbf{r}(s_1)$, $\mathbf{r}(s_2)$, $\mathbf{r}(s_3)$. Show that if $s_1, s_2, s_3 \to s_0$ then the centre and radius converges to the centre of curvature and the absolute value of the radius of curvature respectively.

2.3.2 Show that if $\kappa(s_0) \neq 0$, then the curve has contact of order 2 with the circle of curvature at the point $\mathbf{r}(s_0)$. Show that the contact with any other circle is of lower order.

2.3.3 Show that if $\kappa(s_0) = 0$ then the curve has contact of order 2 with the tangent line at the point $\mathbf{r}(s_0)$.

2.3.4 Show that if $\kappa(s_0) \neq 0$ then the curve has contact of order 2 with a unique parabola.

2.3.5 Let $t \mapsto \mathbf{r}(t) = \big(x(t), y(t)\big)$ be a regular parametrization of class C^2. Show that the curvature vector is given by

$$\boldsymbol{\kappa}(t) = \kappa(t)\mathbf{n}(t) = \frac{\begin{vmatrix} x'(t) & x''(t) \\ y'(t) & y''(t) \end{vmatrix}}{\big(x'(t)^2 + y'(t)^2\big)^2}\big(-y'(t), x'(t)\big).$$

2.3.6 Find the curvature of the parabola in Example 2.1, p. 40 at an arbitrary point.

2.3.7 Prove the Frenet-Serret equations for a plane curve, cf. (2.8), p. 51.

2.3.8 Show that if a plane regular curve has constant curvature different from zero then the curve is a circle.

2.3.9 Show that a regular plane curve locally is the graph of a function "from" the tangent line "to" the normal line, cf. Figure 2.10, p. 54.

2.3.10 Let $\mathbf{r} : I \to \mathbb{R}^2$ be a natural parametrization of a regular curve and consider the involute $\mathbf{r}^\sharp(s) = \mathbf{r}(s) + (c - s)\mathbf{t}(s)$. Determine $\mathbf{r}^{\sharp\prime}$. For which values of s is $\mathbf{r}^{\sharp\prime}(s) \neq \mathbf{0}$? Determine the tangent vector, the normal vector, and the curvature of $\mathbf{r}^\sharp$, in terms of the corresponding quantities of $\mathbf{r}$.

2.3.11 Let $\mathbf{r} : I \to \mathbb{R}^2$ be a natural parametrization of a regular curve. Let $\mathbf{r}_i^\sharp(s) = \mathbf{r}(s) + (c_i - s)\mathbf{t}(s)$, $i = 1, 2$, be two different involutes of $\mathbf{r}$. Show that a tangent line of $\mathbf{r}$ is a normal line of both $\mathbf{r}_1^\sharp$ and $\mathbf{r}_2^\sharp$, and that $\mathbf{r}_1^\sharp - \mathbf{r}_2^\sharp = (c_2 - c_1)\mathbf{n}^\sharp$, where $\mathbf{n}^\sharp$ is the *common* normal of $\mathbf{r}_1^\sharp$ and $\mathbf{r}_2^\sharp$.

2.3.12 Let $\mathbf{r} : I \to \mathbb{R}^2$ be a natural parametrization of a regular curve and let $\mathbf{t}$, $\mathbf{n}$, and ρ be the tangent vector, the normal vector, and the radius of curvature respectively. Consider the involute $\mathbf{r}^\flat(s) = \mathbf{r}(s) + \rho(s)\mathbf{n}(s)$. Determine $\mathbf{r}^{\flat\prime}$. For which values of s is $\mathbf{r}^{\flat\prime}(s) \neq \mathbf{0}$? Determine the tangent vector, the normal vector, and the curvature of $\mathbf{r}^\flat$. Show that a normal line of $\mathbf{r}$ is a tangent line of $\mathbf{r}^\flat$.

2.3.13 Show that a curve is the evolute of any one of it's involutes.

2.3.14 Show that the a curve is an involute of it's evolute.

2.3.15 Consider a regular curve with non vanishing curvature and let ϕ be the tangent direction. Consider the intrinsic equation $\mathrm{d}s/\mathrm{d}\phi = \rho(\phi)$ where ρ is the radius of curvature. Show that the radius of curvature for the evolute is $\mathrm{d}\rho/\mathrm{d}\phi$. What is the radius of curvature for an involute? Find the intrinsic equation for the evolute and the involutes.

2.4 Space Curves

We now consider curves in space. Here we return to the original definition of the curvature, cf. Definition 2.13

Definition 2.25. Let $s \to \mathbf{r}(s)$ be a natural parametrization of class C^3 of a space curve. The *normal plane* of the curve at a point $\mathbf{r}(s)$ is the plane through $\mathbf{r}(s)$ orthogonal to the tangent vector.

If $\kappa(s) \neq 0$, then the *principal normal vector* is $\mathbf{n}(s) = \boldsymbol{\kappa}(s)/\kappa(s) = \mathbf{t}'(s)/|\mathbf{t}'(s)|$. The *binormal vector* is $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$, and the *torsion* is $\tau(s) = -\mathbf{b}'(s) \cdot \mathbf{n}(s)$. The *osculating plane* is the plane through $\mathbf{r}(s)$ orthogonal to the binormal vector and the *rectifying plane* is the plane through $\mathbf{r}(s)$ orthogonal to the principal normal vector.

Notice that the two normal vectors only are defined if $\kappa \neq 0$. In this case $\mathbf{t}, \mathbf{n}, \mathbf{b}$ is a positively oriented orthonormal frame called the *Frenet-Serret frame*. If the

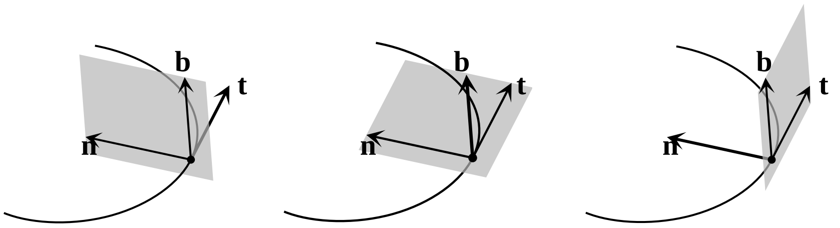


Figure 2.12: The Frenet-Serret frame at a point of a curve. The normal, osculating, and rectifying plane is spanned by $(\mathbf{n}, \mathbf{b})$, $(\mathbf{t}, \mathbf{n})$, and $(\mathbf{t}, \mathbf{b})$ respectively.

orientation on a curve is reversed then both the tangent vector and the arc length changes sign, so the derivative $\mathrm{d}\mathbf{t}/\mathrm{d}s = \kappa\mathbf{n}$ is left unchanged and is an invariant property of the curve, as is κ, ρ, and $\mathbf{n}$. On the other hand $\mathbf{t}$ changes sign so $\mathbf{b}$ and τ change sign too. All in all we have

Proposition 2.26. *For a regular space curve we have that* κ, ρ, $\boldsymbol{\kappa} = \kappa\mathbf{n}$, $\mathbf{n}$, *and the circle of curvature are invariant concepts associated with the curve. And* $\mathbf{t}$, $\mathbf{b}$, *and* τ *are invariant concepts associated with the oriented curve. They change sign if the orientation is reversed.*

Just as for plane curves the circle of curvature at a point $\mathbf{x}_0$ on a space curve can be defined as the limit of a circle through three distinct points $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ on the curve as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \to \mathbf{x}_0$, see Figure 2.8. Likewise, the osculating plane at a point $\mathbf{x}_0$ on a space curve can be defined as the limit position of a plane through three distinct points $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ on the curve as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \to \mathbf{x}_0$.

It can also be shown that the circle of curvature is the only circle that has contact of order 2 with the curve, and the osculating plane is the only plane that has contact of order 2 with the curve.

The derivative of $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ are given by the *Frenet-Serret equations* (2.12) in the following theorem.

Theorem 2.27. *Let $s \mapsto \mathbf{r}(s)$ be a natural parametrization of a space curve with non vanishing curvature $\kappa(s) \neq 0$, torsion $\tau(s)$ and Frenet-Serret frame $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$. Then*

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix} \tag{2.12}$$

Proof. The equation $\mathbf{t}'(s) = \kappa(s)\,\mathbf{n}(s)$ is the equation that defines κ and $\mathbf{n}$. As $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ is an orthonormal frame we have

$$\mathbf{b}' = (\mathbf{b}' \cdot \mathbf{t})\mathbf{t} + (\mathbf{b}' \cdot \mathbf{n})\mathbf{n} + (\mathbf{b}' \cdot \mathbf{b})\mathbf{b}$$

As $|\mathbf{b}(s)|$ is constant $\mathbf{b}' \cdot \mathbf{b} = 0$. Similar, $\mathbf{t}(s) \cdot \mathbf{b}(s) = 0$ is constant too, so

$$0 = \frac{\mathrm{d}(\mathbf{t} \cdot \mathbf{b})}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} \cdot \mathbf{b} + \mathbf{t} \cdot \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} \tag{2.13}$$

and we see that $\mathbf{b}' \cdot \mathbf{t} = -\mathbf{b} \cdot \mathbf{t}' = -\kappa\, \mathbf{b} \cdot \mathbf{n} = 0$. The definition of τ tells us that $\mathbf{b}' \cdot \mathbf{n} = -\tau$ so all in all we have the equation $\mathbf{b}'(s) = -\tau(s)\,\mathbf{n}(s)$. Finally,

$$\begin{aligned} \mathbf{n}'(s) &= \big(\mathbf{n}'(s) \cdot \mathbf{t}(s)\big)\mathbf{t}(s) + \big(\mathbf{n}'(s) \cdot \mathbf{n}(s)\big)\mathbf{n}(s) + \big(\mathbf{n}'(s) \cdot \mathbf{b}(s)\big)\mathbf{b}(s) \\ &= -\big(\mathbf{t}'(s) \cdot \mathbf{n}(s)\big)\mathbf{t}(s) + 0\,\mathbf{n}(s) - \big(\mathbf{b}'(s) \cdot \mathbf{n}(s)\big)\mathbf{b}(s) \\ &= -\kappa(s)\,\mathbf{t}(s) + \tau(s)\mathbf{b}(s). \end{aligned}$$

□

As $\mathbf{t}' = \kappa\mathbf{n}$ and $\mathbf{b}' = -\tau\mathbf{n}$ we see that the curvature is a measure for how fast the tangent line turns around $\mathbf{b}$, and the torsion is a measure for how fast the osculating plane turns around $\mathbf{t}$.

For practical calculations we need a formula that expresses the curvature, torsion and Frenet-Serret frame in terms of an arbitrary parametrization, this is the content of the next theorem.

Theorem 2.28. *Let $t \mapsto \mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$ be a regular parametrization of class C^3. The curvature is then given by*

$$\kappa(t) = \frac{\left|\mathbf{r}'(t) \times \mathbf{r}''(t)\right|}{\left|\mathbf{r}'(t)\right|^3}. \tag{2.14}$$

The torsion is given by

$$\tau(t) = \frac{\left[\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)\right]}{\left|\mathbf{r}'(t) \times \mathbf{r}''(t)\right|^2}. \tag{2.15}$$

The binormal vector is given by

$$\mathbf{b}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\left|\mathbf{r}'(t) \times \mathbf{r}''(t)\right|} \tag{2.16}$$

The principal normal vector is given by

$$\mathbf{n}(t) = \mathbf{b}(t) \times \mathbf{t}(t). \tag{2.17}$$

The derivative of the torsion is

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{[\mathbf{r}', \mathbf{r}'', \mathbf{r}^{(4)}]|\mathbf{r}' \times \mathbf{r}''|^2 - 2[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''](\mathbf{r}' \times \mathbf{r}'') \cdot (\mathbf{r}' \times \mathbf{r}''')}{|\mathbf{r}' \times \mathbf{r}''|^4} \tag{2.18}$$

Proof. Let s denotes the arc length of the curve. We then have

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathrm{d}s}{\mathrm{d}t}\mathbf{t},\\
\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathbf{t} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathbf{t} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \kappa\mathbf{n}.\\
\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3} &= \frac{\mathrm{d}^3 s}{\mathrm{d}t^3}\mathbf{t} + \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} + 2\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\frac{\mathrm{d}s}{\mathrm{d}t}\kappa\mathbf{n} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3 \frac{\mathrm{d}\kappa}{\mathrm{d}s}\mathbf{n} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3 \kappa\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\\
&= \left(\frac{\mathrm{d}^3 s}{\mathrm{d}t^3} - \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3 \kappa^2\right)\mathbf{t} + \left(3\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\frac{\mathrm{d}s}{\mathrm{d}t}\kappa + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3 \frac{\mathrm{d}\kappa}{\mathrm{d}s}\right)\mathbf{n} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3 \kappa\tau\mathbf{b}.
\end{aligned}$$

Hence

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3 \kappa\mathbf{t} \times \mathbf{n} = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^3 \kappa\mathbf{b}$$

which implies (2.14) and (2.16). We also have

$$\left[\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}, \frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\right] = \left[\frac{\mathrm{d}s}{\mathrm{d}t}\mathbf{t}, \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathbf{t} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \kappa\mathbf{n}, A\,\mathbf{t} + B\,\mathbf{n} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3 \kappa\tau\mathbf{b}\right]$$
$$= \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^6 \kappa^2\tau[\mathbf{t}, \mathbf{n}, \mathbf{b}] = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|^2 \tau$$

which implies (2.15). Finally, (2.17) simply follows from the fact that $\mathbf{t}, \mathbf{n}, \mathbf{b}$ is a positively oriented orthonormal basis and (2.18) follows by differentiation of (2.15). □

Just as in the case of a plane curve a designer will often use a curvature plot or a porcupine plot, see Figure 2.6 to asses the quality of a curve. And in an automatic fairing procedure it is again usually the integral $\int (\mathrm{d}\kappa/\mathrm{d}s)^2\,\mathrm{d}s$ that is minimized, under some suitable side conditions. One may (and should?) take the torsion into acount too, but there is no universally accepted way of doing this.

Just as the plane curvature determines a plane curve up to a Euclidean motion, the curvature and torsion determine a space curve up to a Euclidean motion.

Theorem 2.29. *Let I be an interval, let $\kappa : I \to \mathbb{R}$ be a strictly positive C^1 function and let $\tau : I \to \mathbb{R}$ be a C^0 function. Let furthermore $s_0 \in I$, let $\mathbf{x}_0$ be a fixed point of $\mathbb{R}^3$ and let $\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0$ be fixed positively oriented orthonormal basis of $\mathbb{R}^3$. Then there exists a unique regular natural parametrization $\mathbf{r} : I \to \mathbb{R}^3$ of class C^3 such that the curvature is κ, the torsion is τ, $\mathbf{r}(s_0) = \mathbf{x}_0$, and the Frenet-Serret frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ satisfies $\mathbf{t}(s_0) = \mathbf{t}_0$, $\mathbf{n}(s_0) = \mathbf{n}_0$, and $\mathbf{b}(s_0) = \mathbf{b}_0$.*

Proof. The Frenet-Serret equations (2.12) is a *linear* system of ordinary differential euations (in $\mathbb{R}^9$). It follows that there is unique solution $\mathbf{t}, \mathbf{n}, \mathbf{b}$ defined on all of I, with $\mathbf{t}(s_0) = \mathbf{t}_0$, $\mathbf{n}(s_0) = \mathbf{n}_0$, and $\mathbf{b}(s_0) = \mathbf{b}_0$. The set $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ is a positively oriented orthonormal frame for $s = s_0$, we want to show that it is a positively oriented orthonormal frame for all $s \in I$. To that end we define six functions $f_i : I \to \mathbb{R}$ by

$$f_1(s) = \mathbf{t}(s)\cdot\mathbf{t}(s) \qquad f_2(s) = \mathbf{t}(s)\cdot\mathbf{n}(s) \qquad f_3(s) = \mathbf{t}(s)\cdot\mathbf{b}(s)$$
$$f_4(s) = \mathbf{n}(s)\cdot\mathbf{n}(s) \qquad f_5(s) = \mathbf{n}(s)\cdot\mathbf{b}(s) \qquad f_6(s) = \mathbf{b}(s)\cdot\mathbf{b}(s)$$

As $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ is a solution to the Frenet-Serret equations (2.12) we have

$$\begin{aligned}
f_1' &= 2\mathbf{t}' \cdot \mathbf{t} = 2\kappa\, \mathbf{n} \cdot \mathbf{t} = 2\kappa\ f_2 \\
f_2' &= \mathbf{t}' \cdot \mathbf{n} + \mathbf{t} \cdot \mathbf{n}' = \kappa\, \mathbf{n} \cdot \mathbf{n} - \kappa\, \mathbf{t} \cdot \mathbf{t} + \tau\, \mathbf{t} \cdot \mathbf{b} = -\kappa\ f_1 + \tau\ f_3 + \kappa\ f_4 \\
f_3' &= \mathbf{t}' \cdot \mathbf{b} + \mathbf{t} \cdot \mathbf{b}' = \kappa\, \mathbf{n} \cdot \mathbf{b} - \tau\, \mathbf{t} \cdot \mathbf{n} = -\tau\ f_2 + \kappa\ f_5 \\
f_4' &= 2\mathbf{n} \cdot \mathbf{n}' = -2\kappa\, \mathbf{n} \cdot \mathbf{t} + 2\tau\, \mathbf{n} \cdot \mathbf{b} = -2\kappa\ f_2 + 2\tau\ f_5 \\
f_5' &= \mathbf{n}' \cdot \mathbf{b} + \mathbf{n} \cdot \mathbf{b}' = -\kappa\, \mathbf{t} \cdot \mathbf{b} + \tau\, \mathbf{b} \cdot \mathbf{b} - \tau\, \mathbf{n} \cdot \mathbf{n} = -\kappa\ f_3 - \tau\ f_4 + \tau\ f_6 \\
f_6' &= 2\mathbf{b} \cdot \mathbf{b}' = -2\tau\, \mathbf{b} \cdot \mathbf{n} = -2\tau\ f_5
\end{aligned}$$

We see that $(f_1, \dots, f_6)$ is a solution to the following linear system of ordinary differential equations

$$\begin{bmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \\ f_6' \end{bmatrix} = \begin{bmatrix} 0 & 2\kappa & 0 & 0 & 0 & 0 \\ -\kappa & 0 & \tau & \kappa & 0 & 0 \\ 0 & -\tau & 0 & 0 & \kappa & 0 \\ 0 & -2\kappa & 0 & 0 & 2\tau & 0 \\ 0 & 0 & -\kappa & -\tau & 0 & \tau \\ 0 & 0 & 0 & 0 & -2\tau & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}, \qquad \begin{bmatrix} f_1(s_0) \\ f_2(s_0) \\ f_3(s_0) \\ f_4(s_0) \\ f_5(s_0) \\ f_6(s_0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

We can immediately see that the constant function $s \mapsto (1, 0, 0, 1, 0, 1)$ also is a solution. By uniqueness we have that $\big(f_1(s), \dots, f_6(s)\big) = (1, 0, 0, 1, 0, 1)$ for all $s \in I$, i.e., $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ is an orthonormal frame for all $s \in I$. Then we have $[\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)] = \pm 1$ for all $s \in I$ and as $[\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)] = 1$ continuity shows that $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ is positively oriented for all $s \in I$.

We have in particular that $\mathbf{t}(s)$ is a unit vector for all $s \in I$ so if we put

$$\mathbf{r}(s) = \mathbf{x}_0 + \int_{s_0}^{s} \mathbf{t}(u)\, \mathrm{d}u, \qquad \text{for } s \in I$$

then $\mathbf{r} : I \to \mathbb{R}^3$ is a natural parametrization with $\mathbf{r}(s_0) = \mathbf{x}_0$. As $\mathbf{r}' = \mathbf{t}$ and $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ is a solution to the Frenet-Serret equations (2.12) we see that κ and τ is the curvature and torsion respectively of $\mathbf{r}$ and that $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ is the Frenet-Serret frame.

We have now established the existence of $\mathbf{r}$, but the definition of $\mathbf{r}(s)$ was forced if $\mathbf{r}$ was to solve the problem. □

If $\mathbf{r}$ is a natural parametrization of a space curve, and we put $\mathbf{r}_0 = \mathbf{r}(s_0)$, and let $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, κ, and τ be the tangent vector, the principal normal vector, the binormal vector, the curvature, and the torsion at s_0, respectively, then the Taylor expansion of $\mathbf{r}$ to third order at s_0 is

$$\begin{aligned}
\mathbf{r}(s) = \mathbf{r}_0 + (s - s_0)\mathbf{t} &+ \frac{1}{2}(s - s_0)^2 \kappa \mathbf{n} \\
&+ \frac{1}{6}(s - s_0)^3 \big(-\kappa^2 \mathbf{t} + \kappa' \mathbf{n} + \kappa\tau \mathbf{b}\big) + \mathbf{o}\big((s - s_0)^3\big). \qquad (2.19)
\end{aligned}$$

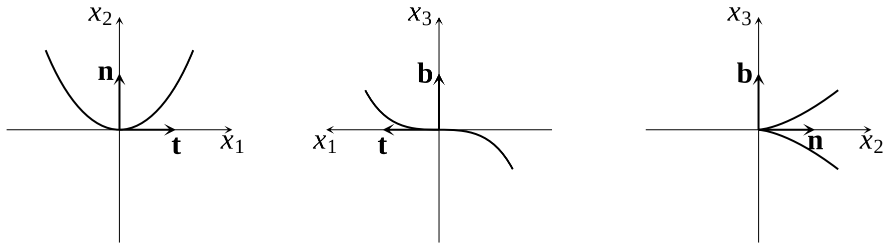


Figure 2.13: The projection of a curve into the osculating plane, the rectifying plane, and the normal plane.

This expression is called the *canonical form* of a space curve. In a neighbourhood of $\mathbf{r}_0$ the projection into the osculating plane looks like the parabola $x_2 = \frac{1}{2}\kappa\, x_1^2$, the projection into the rectifying plane looks like the cubic $x_3 = \frac{1}{6}\kappa\tau\, x_1^3$, and projection into the normal plane looks like the curve $x_3^2 = \frac{2}{9}\big(\tau^2/\kappa\big)\, x_2^3$, see Figure 2.13.

Problems

2.4.1 Find the curvature, the torsion and the Frenet-Serret frame of the *helix* in Example 2.2, p. 40.

2.4.2 Show that if a regular space curve has non vanishing curvature and constant torsion equal to zero then the curve is contained in a plane. Hint: first show that the binormal $\mathbf{b}$ is constant. Then consider the quantity $\mathbf{r}(t) \cdot \mathbf{b}$, where $\mathbf{r} : I \to \mathbb{R}^3$ is a regular parametrization of the curve.

2.4.3 Show that if a regular space curve has constant curvature different from zero and constant torsion equal to zero then the curve is a circle.

2.4.4 Consider the curve given by the parametrization

$$\mathbf{r}(t) = \begin{cases} (t, t^4, 0) & \text{for } t < 0 \\ (0, 0, 0) & \text{for } t = 0 \\ (t, 0, t^4) & \text{for } t > 0 \end{cases}$$

Show that this is a regular parametrization of class C^3. Let κ be the curvature and show that $\kappa(t) = 0$ if and only if $t = 0$. Show that the torsion is zero for all $t \neq 0$.

2.4.5 Show that if a regular space curve has constant curvature and torsion, both different from zero, then the curve is a circular helix, cf. Problem 2.4.1.

2.4.6 Let $\mathbf{r} : I \to \mathbb{R}^3$ be a regular parametrization and assume that $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are linearly independent. Show that the Gram-Schmidt orthonormalization procedure of $\big(\mathbf{r}'(t), \mathbf{r}''(t)\big)$ gives $\big(\mathbf{t}(t), \mathbf{n}(t)\big)$.

2.5 Curves in higher dimensional spaces

In this section we introduce the generalization of the Frenet-Serret frame and the Frenet-Serret equations to higher dimensions.

Theorem 2.30. *Let* $\mathbf{r} : I \to \mathbb{R}^n$ *be a natural parametrization of a regular curve in* $\mathbb{R}^n$ *of class* C^n *with tangent vector* $\mathbf{t}(s)$*. If the first* $n-1$ *derivatives* $\mathbf{r}'(s), \mathbf{r}''(s), \ldots, \mathbf{r}^{(n-1)}(s)$ *are linearly independent then there exists* $n-1$ normal vectors $\mathbf{n}_1(s), \ldots, \mathbf{n}_{n-1}(s)$ *and* $n-1$ curvatures $\kappa_1, \ldots, \kappa_{n-1}$ *such that* $\kappa_i(s) > 0$ *for* $i = 1, \ldots, n-2$ *and* $\mathbf{t}(s), \mathbf{n}_1(s), \ldots, \mathbf{n}_{n-1}(s)$ *is a positively oriented orthonormal frame that satisfies the* Frenet-Serret equations:

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_1' \\ \mathbf{n}_2' \\ \vdots \\ \mathbf{n}_{n-2}' \\ \mathbf{n}_{n-1}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & \ldots & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \ldots & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -\kappa_{n-2} & 0 & \kappa_{n-1} \\ 0 & \ldots & 0 & 0 & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_1 \\ \mathbf{n}_2 \\ \vdots \\ \mathbf{n}_{n-2} \\ \mathbf{n}_{n-1} \end{bmatrix} \tag{2.20}$$

Furthermore, for $m = 1, \ldots, n-1$

$$\operatorname{span}\{\mathbf{r}'(s), \mathbf{r}''(s), \ldots, \mathbf{r}^{(m)}(s)\} = \operatorname{span}\{\mathbf{t}(s), \mathbf{n}_1(s), \ldots, \mathbf{n}_m(s)\} \tag{2.21}$$

Proof. To ease notation a bit we put $\mathbf{n}_0 = \mathbf{t}$. We now use induction to prove that for $k = 1, \ldots, n-2$ we can find $\mathbf{n}_1, \ldots, \mathbf{n}_k$ and $\kappa_1, \ldots, \kappa_k$ such that

1. $\mathbf{n}_0, \ldots, \mathbf{n}_k$ are orthonormal.
2. (2.21) holds for $m = 1, \ldots, k$.
3. $\mathbf{n}_0' = \kappa_1 \mathbf{n}_1$ and $\mathbf{n}_m' = -\kappa_m \mathbf{n}_{m-1} + \kappa_{m+1}\mathbf{n}_{m+1}$ for $m = 1, \ldots, k-1$

As $\mathbf{r}'$ and $\mathbf{r}''$ are linearly independent we have in particular that $\mathbf{t}' = \mathbf{r}'' \neq \mathbf{0}$. So we can put $\kappa_1 = |\mathbf{t}'|$ and $\mathbf{n}_1 = \mathbf{t}'/\kappa_1$. Then we have

$$\mathbf{t}' = \kappa_1 \mathbf{n}_1, \qquad \text{and} \qquad \operatorname{span}\{\mathbf{r}', \mathbf{r}''\} = \operatorname{span}\{\mathbf{t}, \mathbf{n}_1\}.$$

Furthermore $\mathbf{t}$ is a unit vector so $\mathbf{t} \cdot \mathbf{n}_1 = 0$, i.e., $\mathbf{n}_0, \mathbf{n}_1$ are orthonormal. This proves the case $k = 1$.

Now assume we have proved the statement for some k. By hypothesis 1, a calculation like (2.13), and hypothesis 3 we have that

$$\mathbf{n}_m \cdot \mathbf{n}_k' = -\mathbf{n}_m' \cdot \mathbf{n}_k = \begin{cases} -\kappa_k & \text{for } m = k-1 \\ 0 & \text{otherwise} \end{cases}$$

By hypothesis 2 we have that

$$\mathbf{r}^{(k+1)} = \sum_{m=0}^{k} a_m \mathbf{n}_m.$$

Differentiation then gives

$$\begin{aligned}\mathbf{r}^{(k+2)} &= \sum_{m=0}^{k} a'_m \mathbf{n}_m + \sum_{m=0}^{k} a_m \mathbf{n}'_m \\ &= \sum_{m=0}^{k} a'_m \mathbf{n}_m + \sum_{m=0}^{k} a_m(-\kappa_m \mathbf{n}_{m-1} + \kappa_{m+1}\mathbf{n}_{m+1}) + a_k \mathbf{n}'_k\end{aligned}$$

As $\mathbf{r}^{(k+2)} \notin \operatorname{span}\{\mathbf{r}', \dots, \mathbf{r}^{(k+1)}\}$ we see that $\mathbf{n}'_k \notin \operatorname{span}\{\mathbf{n}_0, \dots, \mathbf{n}_k\}$. Furthermore the orthogonal projection of $\mathbf{n}'_k$ on $\operatorname{span}\{\mathbf{n}_0, \dots, \mathbf{n}_k\}$ is

$$\sum_{m=0}^{k} (\mathbf{n}'_k \cdot \mathbf{n}_m)\mathbf{n}_m = -\sum_{m=0}^{k} (\mathbf{n}_k \cdot \mathbf{n}'_m)\mathbf{n}_m = -\kappa_{k-1}\mathbf{n}_{k-1}.$$

If we now put $\kappa_{k+1} = \left|\mathbf{n}'_k + \kappa_{k-1}\mathbf{n}_{k-1}\right|$ then $\kappa_{k+1} > 0$ so we can define $\mathbf{n}_{k+1} = \left(\mathbf{n}'_k + \kappa_{k-1}\mathbf{n}_{k-1}\right)/\kappa_{k+1}$. By construction we have $|\mathbf{n}_{k+1}| = 1$ and $\mathbf{n}_{k+1} \cdot \mathbf{n}_m = 0$ all $m = 0, \dots, k$, so $\mathbf{n}_0, \dots, \mathbf{n}_{k+1}$ are orthonormal. We also have that $\mathbf{n}'_k = -\kappa_k \mathbf{n}_{k-1} + \kappa_{k+1}\mathbf{n}_{k+1}$ by construction. Finally

$$\operatorname{span}\{\mathbf{r}', \dots, \mathbf{r}^{(k+2)}\} = \operatorname{span}\{\mathbf{n}_0, \dots, \mathbf{n}_k, \mathbf{n}'_k\} = \operatorname{span}\{\mathbf{n}_0, \dots, \mathbf{n}_k, \mathbf{n}_{k+1}\}.$$

This completes the induction.

We have now found $\mathbf{n}_0, \dots, \mathbf{n}_{n-2}$ and there is a unique unit vector $\mathbf{n}_{n-1}$ such $\mathbf{n}_0, \dots, \mathbf{n}_{n-1}$ is a positively oriented frame. We put $\kappa_{n-1} = \mathbf{n}'_{n-2} \cdot \mathbf{n}_{n-1}$ and then we have

$$\begin{aligned}\mathbf{n}'_{n-2} &= \sum_{k=0}^{n-1} \left(\mathbf{n}'_{n-2} \cdot \mathbf{n}_k\right)\mathbf{n}_k = -\sum_{k=0}^{n-3} \left(\mathbf{n}_{n-2} \cdot \mathbf{n}'_k\right)\mathbf{n}_k + \left(\mathbf{n}'_{n-2} \cdot \mathbf{n}_{n-1}\right)\mathbf{n}_{n-1} \\ &= -\kappa_{n-2}\mathbf{n}_{n-2} + \kappa_{n-1}\mathbf{n}_{n-1}\end{aligned}$$

and

$$\mathbf{n}'_{n-1} = \sum_{k=0}^{n-1} \left(\mathbf{n}'_{n-1} \cdot \mathbf{n}_k\right)\mathbf{n}_k = -\sum_{k=0}^{n-1} \left(\mathbf{n}_{n-1} \cdot \mathbf{n}'_k\right)\mathbf{n}_k = -\kappa_{n-1}\mathbf{n}_{n-2}$$

This completes the proof. □

The proof of Theorem 2.29 generalizes to curves in $\mathbb{R}^n$ and give us the following theorem.

Theorem 2.31. *Let I be an interval, let for $k = 1, \dots, n-2$, $\kappa_k : I \to \mathbb{R}$ be a strictly positiv C^{n-k-1} function and let $\kappa_{n-1} : I \to \mathbb{R}$ be a C^0 function. Let furthermore $s_0 \in I$, let $\mathbf{x}_0$ be a fixed point of $\mathbb{R}^3$ and let $\mathbf{t}_0, \mathbf{n}_{1,0}, \dots, \mathbf{n}_{n-1,0}$ be a fixed positively oriented orthonormal basis of $\mathbb{R}^n$. Then there exists a unique natural parametrization $\mathbf{r} : I \to \mathbb{R}^n$ of class C^n such that the curvatures are $\kappa_1, \dots, \kappa_{n-1}$ and the Frenet-Serret frame $(\mathbf{t}, \mathbf{n}_1, \dots, \mathbf{n}_{n-1})$ satisfies $\mathbf{t}(s_0) = \mathbf{t}_0$ and $\mathbf{n}_k(s_0) = \mathbf{n}_{k,0}$ for $k = 1, \dots, n-1$.*

The following theorem tells us how to find the normals and curvatures from an arbitrary parametrization of a curve in $\mathbb{R}^n$.

Theorem 2.32. *Let $\mathbf{r} : I \to \mathbb{R}^n$ be a parametrization of class C^n such that the first $n-1$ derivatives are linearly independent. The normals $\mathbf{n}_1, \dots, \mathbf{n}_{n-2}$ can be found by the Gram-Schmidt orthonormalization procedure.*

Step 1:

$$\mathbf{v}_0 = \mathbf{r}', \qquad \mathbf{n}_0 = \mathbf{t} = \frac{\mathbf{r}'}{|\mathbf{r}'|}.$$

Loop: for $m = 1, \dots, n-2$ do

$$\mathbf{v}_m = \mathbf{r}^{(m+1)} - \sum_{k=0}^{m-1} \big(\mathbf{r}^{(m+1)} \cdot \mathbf{n}_k\big)\mathbf{n}_k \qquad \mathbf{n}_m = \frac{\mathbf{v}_m}{|\mathbf{v}_m|}.$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis in $\mathbb{R}^n$.

For $m = 1, \dots, n$ we put

$$\mathbf{w}_m = \mathbf{e}_m - \sum_{k=0}^{n-2} \big(\mathbf{e}_m \cdot \mathbf{n}_k\big)\mathbf{n}_k \quad \text{and if } \mathbf{w}_m \neq \mathbf{0} \text{ then} \quad \mathbf{n}_{n-1} = \pm\frac{\mathbf{w}_m}{|\mathbf{w}_m|}$$

where "$+$" is used if $[\mathbf{n}_0, \dots, \mathbf{n}_{n-2}, \mathbf{e}_m] < 0$ otherwise "$-$" is used.

The curvatures $\kappa_1, \dots, \kappa_{n-1}$ are now given by

$$\kappa_m = \frac{\mathbf{r}^{(m+1)} \cdot \mathbf{n}_m}{|\mathbf{r}'||\mathbf{v}_{m-1}|}, \quad m = 1, \dots, n-1. \tag{2.22}$$

Proof. By Problem 2.5.2 the Gram-Schmidt orthonormalization procedure gives the first $n-2$ normals. For at least one m we have that $\mathbf{n}_0, \dots, \mathbf{n}_{n-2}, \mathbf{e}_m$ is a basis. For such a m $\mathbf{n}_0, \dots, \mathbf{n}_{n-2}, \mathbf{w}_m/|\mathbf{w}_m|$ is an orthonormal basis so $\mathbf{n}_{n-1} =$

$\pm\mathbf{w}_m/|\mathbf{w}_m|$. The sign is determined by the requirement that $\mathbf{n}_0, \ldots, \mathbf{n}_{n-1}$ is positively oriented, i.e., by the requirement that $[\mathbf{n}_0, \ldots, \mathbf{n}_{n-1}] = 1$. So the sign is the same as the sign of $\big[\mathbf{n}_0, \ldots \mathbf{n}_{n-1}, \mathbf{w}_m/|\mathbf{w}_m|\big] = \big[\mathbf{n}_0, \ldots \mathbf{n}_{n-1}, \mathbf{w}_m\big]/|\mathbf{w}_m|$ which has the same sign as $\big[\mathbf{n}_0, \ldots \mathbf{n}_{n-1}, \mathbf{w}_m\big] = \big[\mathbf{n}_0, \ldots \mathbf{n}_{n-1}, \mathbf{e}_m\big]$.

For $m = 1, \ldots, n-1$ we now have

$$|\mathbf{v}_{m-1}|\mathbf{n}_{m-1} = \mathbf{r}^{(m)} - \sum_{k=0}^{m-2}\big(\mathbf{r}^{(m)} \cdot \mathbf{n}_k\big)\mathbf{n}_k$$

so differentiation with respect to s gives

$$\begin{aligned}
\frac{\mathrm{d}|\mathbf{v}_{m-1}|}{\mathrm{d}s}\mathbf{n}_{m-1} &+ |\mathbf{v}_{m-1}|\frac{\mathrm{d}\mathbf{n}_{m-1}}{\mathrm{d}s} \\
&= \frac{\mathrm{d}\mathbf{r}^{(m)}}{\mathrm{d}s} - \sum_{k=0}^{m-2}\frac{\mathrm{d}\big(\mathbf{r}^{(m)} \cdot \mathbf{n}_k\big)}{\mathrm{d}s}\mathbf{n}_k - \sum_{k=0}^{m-2}\big(\mathbf{r}^{(m)} \cdot \mathbf{n}_k\big)\frac{\mathrm{d}\mathbf{n}_k}{\mathrm{d}s} \\
&= \frac{\mathbf{r}^{(m+1)}}{|\mathbf{r}'|} - \sum_{k=0}^{m-2}\frac{\mathrm{d}\big(\mathbf{r}^{(m)} \cdot \mathbf{n}_k\big)}{\mathrm{d}s}\mathbf{n}_k - \sum_{k=0}^{m-2}\big(\mathbf{r}^{(m)} \cdot \mathbf{n}_k\big)\frac{\mathrm{d}\mathbf{n}_k}{\mathrm{d}s}.
\end{aligned}$$

If we take the inner product with $\mathbf{n}_m$ then we obtain

$$|\mathbf{v}_{m-1}|\kappa_m = \frac{\mathbf{r}^{(m+1)} \cdot \mathbf{n}_m}{|\mathbf{r}'|}$$

This is the same as (2.22). □

Problems

2.5.1 Let $\mathbf{r} : I \to \mathbb{R}^n$ be a natural parametrization of a regular curve of class C^n and assume that the derivatives $\mathbf{r}'(s), \ldots, \mathbf{r}^{(n-1)}(s)$ are linearly independent. Show that if the Gram-Schmidt orthonormalization procedure is used on $\big(\mathbf{r}'(s), \ldots, \mathbf{r}^{(n-1)}(s)\big)$ then we get $\big(\mathbf{n}_0(s), \ldots, \mathbf{n}_{n-2}(s)\big)$.

2.5.2 Let $\mathbf{r} : I \to \mathbb{R}^n$ be a regular parametrization of class C^n and assume that the derivatives $\mathbf{r}'(t), \ldots, \mathbf{r}^{(n-1)}(t)$ are linearly independent. Show that the Gram-Smidth orthonormalization procedure of $\big(\mathbf{r}'(t), \ldots, \mathbf{r}^{(n-1)}(t)\big)$ gives $\big(\mathbf{n}_0(t), \ldots, \mathbf{n}_{n-2}(t)\big)$.

2.5.3 Check that the procedure in Theorem 2.32 for $n = 2$ gives the same result as Theorem 2.20.

2.5.4 Check that the procedure in Theorem 2.32 for $n = 3$ gives the same result as Theorem 2.28.

2.5.5 What simplifications can be made to the procedure in Theorem 2.28 if we only want $\kappa_1, \ldots, \kappa_{n-12}, |\kappa_{n-1}|$, i.e., if we ignore the sign of κ_{n-1}.